Topology, Dynamics and Multifractal Analysis of Turbulence near a Turbulent/Non-Turbulent Interface

Rui Manuel Marques Jaulino

Thesis to obtain the Master of Science Degree in

Aerospace Engineering

Supervisor(s): Prof. Carlos Frederico Neves Bettencourt da Silva
Dr. Rodrigo Miguel Ribeiro Taveira

Examination Committee
Chairperson: Prof. Filipe Szolnoky Ramos Pinto Cunha
Supervisor: Prof. Carlos Frederico Neves Bettencourt da Silva
Member of the Committee: Prof. José Carlos Fernandes Pereira

December 2014
Acknowledgments

I would like to thank my supervisor Prof. Carlos B. da Silva and my co-supervisor Rodrigo Taveira for their daily availability to help, teach and discuss. Without them all this work would not be possible. All the people from LASEF also deserve a mention with special attention to Pedro Valente for his opportune comments and to Duarte Albuquerque for the time spent fixing my computer. Since with this work I finish my course, all my friends deserve a special note. Their friendship made my path along university one of the best period of my life. During my university time I was blessed with the privilege of having the best of friends on my side. Our mutual support in the good and tough moments was one of the most important contribution to the successful conclusion of this period. Finally a word for my family which always stood with me especially to my mother who supported me all these years. Repay all that she gave me so warmly and foremost make her proud makes me try to overcome myself.
Resumo

Dois aspectos universais da turbulência, a distribuição dos invariantes do tensor gradiente de velocidade e o espectro multifractal da dissipação, são analisados perto da interface turbulento/não-turbulento utilizando simulações numéricas directas.

Os invariantes do tensor gradiente de velocidade contêm apenas informação local, ao contrário das estruturas turbilhonares que necessariamente involvem informação de uma determinada região do escoamento. Observa-se que na interface turbulento/não-turbulento como aquela que existe na periferia de esteiras e jactos, os invariantes apresentam a marca dos turbilhões de pequena escala da zona turbulenta mais próxima, e podem ser descritos por uma modelo simples que relaciona a topologia, dinâmica e estrutura turbilhonar do escoamento unificando os modelos anteriores para a sobre-camada viscosa e a sub-camada viscosa dentro da camada de interface.

A não conservação da cascata multiplicativa que descreve a dissipação de energia recentemente observada, é verificada usando o método Wavelet Leaders ao analisar o escoamento na zona turbulenta e na interface turbulento/não-turbulento. Os resultados obtidos para o espectro multifractal da dissipação implicam que os resultados presentes na literatura sobre a dimensão fractal da interface turbulento/não-turbulento necessitam de ser revistos.

Palavras-chave: Turbulência, T/NT Interface, Análise Multifractal
Abstract

Two universal aspects of turbulence, the distribution of the invariants of the velocity gradient tensor and the multifractal spectrum of dissipation, are analysed near a Turbulent/Non-Turbulent Interface using Direct Numerical Simulations of shear-free turbulence. The invariants of the velocity gradient tensor provide only local information, in contrast to eddy structures which necessarily involve data from a given flow region. We show that in a turbulent/non-turbulent interface, such as exists at the edges of wakes and jets, the invariants show the imprint of the small scale eddies from the nearby turbulent region, and can be described by a simple model relating local flow topology, dynamics and eddy structure, thus unifying the previous models for the viscous superlayer and the turbulent sublayer within the Turbulent/Non-Turbulent Interface.

The recent result of non-conservativeness of the multiplicative cascade which describes the energy dissipation is obtained with the Wavelet Leaders method by analysing the flow both in the turbulent region and in the Turbulent/Non-Turbulent Interface. The present results on the multifractal spectrum of dissipation imply that the previous results in the literature on the fractal dimension of the Turbulent/Non-Turbulent Interface need to be revisited.

Keywords: Turbulence, T/NT Interface, Multifractal Analysis
# Contents

Acknowledgments ......................................................... iii  
Resumo ................................................................. v  
Abstract .............................................................. vii  
List of Tables .......................................................... xiii  
List of Figures ......................................................... xvii  
Nomenclature .......................................................... xx  
Glossary ................................................................. xxi

1 Introduction .......................................................... 1  
1.1 The Equations of Fluid Motion .................................... 2  
1.1.1 Symmetries of the Navier-Stokes .............................. 3  
1.2 Turbulence Overview ............................................... 3  
1.2.1 Deviations from Kolmogorov’s Theory .......................... 7  
1.2.2 The Invariants of the Velocity Gradient Tensor ............... 8  
1.3 Turbulent Interfaces ............................................... 11  
1.3.1 The Flux of Quantities Across the Interface ................. 13  
1.4 Objectives .......................................................... 13  
1.5 Outline ............................................................. 13

2 Multifractal Analysis .................................................. 15  
2.1 Introductory Concepts .............................................. 15  
2.1.1 Fractals ........................................................ 15  
2.1.2 Holder exponent of a function .................................. 22  
2.2 Multifractals ....................................................... 22  
2.3 The Multifractal Formalism ....................................... 23  
2.4 Multifractal Processes .............................................. 26  
2.4.1 Binomial Cascade .............................................. 26  
2.4.2 Random Binomial Cascade ..................................... 27  
2.4.3 Canonical Mandelbrot Cascades (CMC) ......................... 28  
2.5 Turbulence Energy Cascade as a Multiplicative Process .......... 29  
2.5.1 Fluxes Across the Interface ................................... 31
2.6 Practical Multifractal Analysis .................................................. 32
  2.6.1 Box Counting ................................................................. 32
  2.6.2 Wavelet Methods .......................................................... 33

3 Direct Numerical Simulations and Post-Processing Tools ............... 40
  3.1 Shear Free Turbulence ......................................................... 40
    3.1.1 Interface Detection ....................................................... 40
    3.1.2 Conditional Statistics .................................................. 41
  3.2 Wavelet Leaders ............................................................... 41
    3.2.1 Discrete Wavelet Transform ........................................... 41
    3.2.2 Leaders and Partition Function Calculation ....................... 44
    3.2.3 Determination of $\tau_q$ ................................................ 44
    3.2.4 Determination of Multifractal Spectrum ............................. 45

4 Flow Topology near a Turbulent/Non-Turbulent Interface ............... 46
  4.1 Results ................................................................................. 46
    4.1.1 Conditional Profiles ....................................................... 46
    4.1.2 The Teardrop Formation .................................................. 48
    4.1.3 Mean Trajectories ......................................................... 49
  4.2 The Burguers Vortex Model .................................................. 50
    4.2.1 Analytical Radial Profiles of Enstrophy Production and Diffusion for a Burger Vortex 50
    4.2.2 Invariants of the Velocity Gradient Tensor ......................... 51
    4.2.3 Relation to Flow at Interface ........................................... 52

5 Multifractal Analysis of Turbulence ........................................... 54
  5.1 Trial Functions ...................................................................... 54
    5.1.1 Brownian Motion ........................................................... 54
    5.1.2 Weierstrass Function ...................................................... 57
    5.1.3 Deterministic Binomial Cascade ....................................... 58
    5.1.4 Random Binomial Cascade .............................................. 60
    5.1.5 CMC-LN ........................................................................ 60
  5.2 Fractional Integration .......................................................... 61
  5.3 Dissipation in Fully-Developed Turbulence ............................... 63
    5.3.1 Statistical Convergence ................................................... 64
    5.3.2 Wavelet Influence .......................................................... 66
    5.3.3 Results for Fully-Developed Turbulence ............................. 67
  5.4 Dissipation near the TNT Interface ........................................ 69
    5.4.1 Data Treatment .............................................................. 69
    5.4.2 Results .......................................................................... 70
6 Conclusions

6.1 Achievements .................................................. 73
6.2 Future Work .................................................. 73

Bibliography ...................................................... 78

A The Wavelet Transform ........................................ 79

A.1 Properties of Wavelets .................................... 79
A.2 Continuous Wavelet Transform ......................... 79
A.3 Discrete Wavelet Transform ............................ 80
  A.3.1 Multilevel Representation of a Function .......... 80
  A.3.2 Multiresolution Analysis Using the Mallat Transform 85
List of Tables

4.1 Symbols used to denote several points across the TNTI layer, which is divided into the viscous superlayer (VSL) and the turbulent sublayer (TSL). $D\omega^2 / Dt$ indicates the dominating mechanisms for vorticity generation: viscous diffusion ($D\omega$) or vortex stretching ($P$).

Point G is deep inside the T region where production and dissipation roughly balance. . . 47
## List of Figures

1.1 A plume of smoke in a forest fire. .......................................................... 3
1.2 Vorticity modulus in a numerically simulated jet (with permission of Carlos B. da Silva). 5
1.3 Typical energy spectrum in turbulent flows. ........................................... 5
1.4 Structure function scaling exponents $\zeta_p$ plotted versus $p$. Circles and triangles correspond to experimental data and the solid line corresponds to Kolmogorov scaling $p/3$. ............. 8
1.5 Typical signals of the local dissipation: (a) was obtained in a laboratory boundary layer and (b) in the atmospheric surface layer. .................................................. 9
1.6 Interpretation of the $(Q, R)$ map (with permission of Carlos B. da Silva). .... 11
1.7 Typical $(Q, R)$ map observed in turbulence. .......................................... 11
1.8 Schematic showing the several regions, length scales, and main physical processes that take place inside a free shear layer. Included are intense vorticity structures (IVS; worms, red); large-scale vortices (LVS; yellow); the thickness of the viscous superlayer, $\delta_\nu$; and the thickness of the turbulent sublayer (or vorticity interface), $\delta_\omega$. The turbulent/nonturbulent (T/NT) interface with coordinate $Y_i$ (direction inwards and normal to the layer) is defined by the line separating these two sub layers. ............... 12
1.9 Schematic of the evolution of enstrophy in the TNTI (with permission of Carlos B. da Silva). 13
2.1 Lightning event. ....................................................................................... 16
2.2 The Romanesco broccoli superficially resembles a cauliflower, but it has a visually striking fractal form. ............................................................. 16
2.3 Measuring the coast of Britain with different rulers. It turns out that the length of the coastline is bigger for smaller ruler sizes. ........................................ 17
2.4 Log-log plot of the length of land frontiers of several countries versus the measuring length. 18
2.5 Number of elements needed to cover an area as function of the linear size of the decomposition elements. ................................................................. 19
2.6 The Koch curve. This object is generated iteratively by replacement of the straight lines with the original object (at each iteration). ..................... 20
2.7 Cantor Set. ............................................................................................. 20
2.8 Single realization of BM with $h = 0.5$ ($N = 2^{16}$). ............................. 21
2.9 Typical example of a multifractal spectrum. .............................................. 23
2.10 The generation of the binomial measure with \((m_0 = 0.25), \mu_i\), where \(i\) is the number of the iteration - from The Wolfram Project .................................... 26
2.11 Random Binomial Cascade \((m_0 = 0.25)\). ..................................................... 28
2.12 Illustration of the CMC cascade generation. ................................................. 29
2.13 One-dimensional version of a cascade model of eddies, each breaking down into two new ones. The flux of kinetic energy to smaller scales is divided into non equal fractions \(m_0\) and \(m_1\). .......................................................... 31
2.14 The Box Counting method. One counts the number of boxes of size \(l\) (varying \(l\)) which cover the object (in this case is the coastline). .................................................. 33
2.15 Procedure for the WTMM method. In the top figure the signal (Brownian Motion with \(h = 1/3\)) is presented. In the middle figure the Continuous Wavelet Transform (CWT) in space-scale is presented. The partition function is constructed based on the local maxima of the CWT in the space-scale representation in the bottom figure. ......................... 38
2.16 Definition of the Wavelet Leaders ................................................................. 39

3.1 Probability density function of the vorticity (in modulus) for DNS of shear free turbulence. The minimum value defines the vorticity threshold were the TNT interface is defined. .... 41
3.2 Local coordinate systems used to compute the conditional mean profiles, in relation to the distance from the turbulent/non-turbulent interface (TNTI), separating the turbulent (T) and non-turbulent (NT) or irrotational flow regions and comprising two (sub)layers: the viscous superlayer (VSL) with thickness \(\delta_v\) and the turbulent sublayer (TSL) with thickness \(\delta_\omega\): 1D (along the \(y\) direction); 2D (normal to the interface in the \((x, y)\) plane); 3D (normal to the local interface - not shown). ......................................................... 42
3.3 Illustration of the one level cascade algorithm (from Matlab). ....................... 43
3.4 Example of the mirroring procedure applied to the data (Binomial Cascade with \(p = 0.25\)). .......................................................... 43
3.5 Space-scale representation of the wavelet coefficients of a irregular function (Generated in Matlab). .......................................................... 44

4.1 Mean conditional profiles (as function of the distance from the TNTI) of vorticity magnitude \(\omega\), enstrophy production \((\omega^2\ Prod.)\) and enstrophy viscous diffusion \((\omega^2\ Diff.)\). ....................... 46
4.2 Joint probability density function of \(R, Q\) across the viscous-superlayer (VSL) region in the flow: start of the viscous superlayer (A), point of maximum mean enstrophy diffusion (B), end of the viscous superlayer (C). ......................................................... 48
4.3 Joint probability density function of \(R, Q\) across the turbulent sublayer (TSL) region in the flow: point of zero diffusion (D), point of minimum diffusion (E), point of maximum enstrophy (F). ......................................................... 48
4.4 Trajectory of the mean values of \(R\) and \(Q\) across the TNTI. \(D_A = 27/4R^2 + Q^4\) is the discriminant of the eigenvalues of \(A_{ij}\). ......................................................... 49
4.5 Sketch of the Burger vortex created by a balance between vortex stretching in the axial direction and radial viscous diffusion. ......................................................... 50
5.1 Multifractal analysis of Brownian Motion for $H = 0.5$. The superscripts $teo$ and $obt$ relate to the theoretical and obtained quantities, respectively and the analysing wavelet is $db3$. 55

5.2 Multifractal analysis of Brownian Motion with $h = 0.4$. The superscripts $teo$ and $obt$ relate to the theoretical and obtained quantities, respectively and the analysing wavelet is $db3$. 56

5.3 Multifractal analysis of Brownian Motion with $h = 0.7$. The superscripts $teo$ and $obt$ relate to the theoretical and obtained quantities, respectively and the analysing wavelet is $db3$. 57

5.4 Multifractal analysis of the Weierstrass function. The superscripts $teo$ and $obt$ relate to the theoretical and obtained quantities, respectively and the analysing wavelet is $db3$. 58

5.5 Integrated version (in the sense of equation 2.24) of the Binomial Cascade. 59

5.6 Multifractal analysis of the binomial cascade. The superscripts $teo$ and $obt$ relate to the theoretical and obtained quantities, respectively and the analysing wavelet is $db1$. 59

5.7 Multifractal analysis of the random binomial cascade. The superscripts $teo$ and $obt$ relate to the theoretical and obtained quantities, respectively and the analysing wavelet is $db1$. 60

5.8 Multifractal analysis of the CMC-LN. The superscripts $teo$ and $obt$ relate to the theoretical and obtained quantities, respectively and the analysing wavelet is $db1$. 61

5.9 Scaling exponent obtained with the fractionally integrated version of the binomial cascade. 62

5.10 Multifractal spectrum obtained with the fractionally integrated version of the binomial cascade. Note that the shift to the right was subtracted and we sum $d = 1$ to have an analogy with the previous results. 62

5.11 Location of the planes where the data is analysed. 63

5.12 Example of a dissipation signal in the turbulent zone from DNS simulation. 63

5.13 Behaviour of the partition function with scale. The analysing wavelet is the $db3$ and the data is taken in plane $0260$. 64

5.14 Behaviour of the scaling function for different numbers points (samples) used. The analysing wavelet is the $db3$ and the data is taken in plane $0260$. 65

5.15 Behaviour of the multifractal spectrum for different numbers points (samples) used. The analysing wavelet is the $db3$ and the data is taken in plane $0260$. 65

5.16 Influence of the wavelet on the scaling function. 66

5.17 Influence of the wavelet on the multifractal spectrum. 66

5.18 Scaling exponents obtained in different planes. 67

5.19 Multifractal spectrum obtained in different planes in comparison with several models proposed. 68

5.20 Example of a dissipation signal in the interface from DNS simulation. 69

5.21 Behaviour of the partition function with scale. The analysing wavelet is the $db3$. 70

5.22 Scaling exponents obtained for different distances. 70

5.23 Multifractal spectrum obtained for different distances. 71
Nomenclature

Greek symbols

\( \eta \)  
Kolmogorov scale.

\( \lambda \)  
Taylor micro-scale.

\( \mu \)  
Dynamic viscosity.

\( \nu \)  
Kinematic viscosity.

\( \tau_q \)  
Scaling Function (Parisi-Frisch).

\( \tau_q^M \)  
Scaling Function (Halsey).

\( \varepsilon \)  
Mean energy dissipation per unit time per unit mass.

Roman symbols

\( A_{ij} \)  
Velocity Gradient Tensor \( \partial u_i / \partial x_j \).

\( D(h) \)  
Multifractal spectrum.

\( d_f \)  
Fractal dimension.

\( D_q \)  
Generalized dimensions.

\( h \)  
Holder exponent.

\( H^* \)  
Fractional integration parameter.

\( L \)  
External scale.

\( N \)  
Number of points.

\( P \)  
First Invariant of \( A_{ij} \).

\( Q \)  
Second Invariant of \( A_{ij} \).

\( R \)  
Third Invariant of \( A_{ij} \).

\( Re_k \)  
Reynolds Number based on length \( k \).

\( S(l,q) \)  
Structure function.
\( S_{ij} \)  Rate of Strain Tensor.

\( Z_c(a,q) \)  Partition function based on CWT.

\( Z_d(j,q) \)  Partition function based on DWT.

\( Z_L(j,q) \)  Partition function based on Wavelet Leaders.

\( Z_{WTMM}(a,q) \)  Partition function based on WTMM.
**DNS**  Direct Numerical Simulation is a simulation in computational fluid dynamics in which the Navier–Stokes equations are numerically solved without any turbulence model.

**TNTI**  Turbulent/non-Turbulent Interface is a thin layer which separates irrotational from turbulent fluid.
Chapter 1

Introduction

In the real world turbulence is omnipresent, since it is found in almost all natural flows, environmental phenomena, industrial processes and engineering applications, and is known to be the most challenging problem of fluid mechanics. Turbulence is Nature’s way of speeding up diffusion, transport and mixing in fluids by the generation of self-similar multiscale motions.

In turbulent shear flows, such as jets, turbulence maintains itself, by means of the shear layer growth, promoted by the interaction between turbulent and non-rotational flow lying at opposite sides of the turbulent shear layer boundary, in a process denominated as turbulent entrainment. This entrainment process is characterized by significant exchanges of matter, momentum, passive or active scalars, like concentration or temperature, and energy. These processes do take place at a thin layer of turbulent flow known as T/NT interface.

Therefore, it seems obvious that a deep understanding of turbulence is of utmost importance to a large number of natural flows, namely in domains like geophysics, meteorology and oceanography. Perhaps as important as in natural flows, entrainment processes remain of central importance to industry, in particular combustion, combustion chambers, mixing processes, aircraft high lifting devices, chemical reactions and polluting contaminants dispersion, among many other. This enormous relevance can be fairly attested through the great amount of work which has been devoted to turbulence, and focused particularly on turbulent entrainment dynamics, during the past century and recent years. Even small developments in our knowledge of this turbulence science can lead to huge improvements in engineering and thereby contribute to a better understanding about natural phenomena and allowing for the forecast of natural catastrophic events.

The present dissertation is devoted to the investigation of fundamental features present in turbulence. Special focus will be given to the universal turbulent characteristics within free shear layer plane jets, which constitute themselves as canonical prototypes for turbulent flows and thereby for fundamental turbulence research.
1.1 The Equations of Fluid Motion

The equations of motion for incompressible fluids are the Navier-Stokes equations [1],

\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \vec{u} \tag{1.1}
\]

and the conservation of mass:

\[
\nabla \cdot \vec{u} = 0 \tag{1.2}
\]

where constant \( \mu \left[ \text{Kg} \cdot \text{m} \cdot \text{s} \right] \) is the dynamic viscosity and the ratio \( \frac{\mu}{\rho} = \nu \left[ \text{m}^2 \cdot \text{s}^{-1} \right] \) is kinematic viscosity.

These equations must be supplemented by an appropriate boundary condition. For both gases and liquids this is the no-slip boundary condition that the relative velocity of fluid and solid vanishes at a solid surface. We take this boundary condition as phenomenologically given, recognizing that a full molecular understanding of its origin is far from trivial.

The presence of viscosity results in the dissipation of energy, which is transformed into heat. The total kinetic energy of an incompressible fluid is:

\[
E_{\text{kin}} = \frac{1}{2} \rho \int u^2 dV \tag{1.3}
\]

while the energy dissipated per unit time in the whole fluid body (using [1.1 to manipulate]) is:

\[
\dot{E}_{\text{kin}} = -\frac{1}{2} \mu \int \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)^2 dV \tag{1.4}
\]

A complete analytical treatment of the equations of motion for turbulent flows is impossible to the date, i.e, the demonstration of existence and uniqueness of solutions is lacking. When studying the motion of viscous fluids one can obtain a number of important results from simple arguments concerning the dimensions of various physical quantities. Of the parameters which characterize the fluid itself, only the kinematic viscosity appears in equation [1.1]; the unknown functions which are to be determined are the velocity field, \( \vec{u} \), and the ratio \( \frac{\mu}{\rho} \). A general solution will depend also on the boundary conditions.

For example, consider the motion of a body through a fluid; this object should have a characteristic dimension \( L \) and the flow around it a characteristic velocity \( U \). Now imagine the same body but with a different characteristic dimension and a different velocity. The fluid flow should be determined by three parameters: \( \nu \), \( L \) and \( U \).

It's easy to verify that the only dimensionless quantity that can be formed from the above parameters is \( \frac{U \cdot L}{\nu} \) - this is called the Reynolds Number.

\[
Re = \frac{U \cdot L}{\nu} \tag{1.5}
\]

It is possible to make some predictions for the dynamics of turbulence without solving the full Navier-Stokes equations. First we expect the fluid to be a dissipative dynamical system with a very large number of effective degrees of freedom. Thus, we expect chaotic behaviour of some kind as a natural conse-
quence of the dynamics. Second, the statistical properties of the velocity field should be similar if the Reynolds number is the same and the flow geometry is similar.

1.1.1 Symmetries of the Navier-Stokes

The Navier-Stokes equations [1.1] are invariant under some groups of transformations. The known symmetries are [2],[3]:

- **Space Translations**: \( t, \vec{r}, \vec{u} \rightarrow t, \vec{r} + \vec{\rho}, \vec{u} \)
- **Time Translations**: \( t, \vec{r}, \vec{u} \rightarrow t + \tau, \vec{r}, \vec{u} \)
- **Galilean Translations**: \( t, \vec{r}, \vec{u} \rightarrow t, \vec{r} + \vec{U}t, \vec{u} + \vec{U} \)
- **Parity**: \( t, \vec{r}, \vec{u} \rightarrow t, -\vec{r}, -\vec{u} \)
- **Rotations**: \( t, \vec{r}, \vec{u} \rightarrow t, [A]\vec{r}, [A]\vec{u} \)
- **Scaling**: \( t, \vec{r}, \vec{u}, \nu \rightarrow \lambda^{1-h}t, \lambda^{h}\vec{r}, \lambda^{h}\vec{u}, \lambda^{h+1}\nu \)

There seems to exist a certain tendency for the flow to break the symmetries as the Reynolds Number is increased but when this parameter is high enough, the symmetries are restored in a statistical way. All the listed symmetries are just a consequence of basic symmetries of Newton’s equation governing microscopic molecular motion - except for the scaling symmetry [Statistical Mechanics]. Some links and analogies can be made between this theoretical aspects and theories for turbulence (eg. K41 theory). This matter will be discussed further below.

1.2 Turbulence Overview

The image of water flowing or a plume of smoke must be in everyone’s mind. When we look at this events all we can see is a richness of motion hard to explain and very unpredictable.

Figure 1.1: A plume of smoke in a forest fire.
Turbulence is present in the almost totality of the flows encountered in nature and industrial applications - it's the natural state of fluid motion. This is one of the most universal, complex and challenging problems within fluid mechanics and physics.

We can say that turbulence arises in flows at high Reynolds numbers by means of non-linear amplification of small scale disturbances superimposed to the flow although there isn't a complete theory for turbulence and the complete understanding of this phenomenon (physically and mathematically) probably will have a huge impact in all areas of science. For example, back in the 18\textsuperscript{th} century, the Fourier series where developed to solve heat problems (the idea was that heat propagates in waves an so a way to a solution was discovered) and then this knowledge was used latter to understand electronic devices and improve communications. Imagine the impact to our daily life when humankind understands the complex interactions of non linear problems.

With such complexity, the emphasis should be in studying which aspects of turbulence in fluids can be universal.

Defining turbulence in fluid motion is a difficult task (how can we define something that we don't understand completely) but there are some well defined characteristics of this type of motion:

- **Space and time irregularity** - The intensive variables of the flow show great irregularity both in space and time;

- **High Reynolds number** - Turbulent flows occur at high Reynolds number; inertial effects dominate over the ability of the viscosity to stabilize the flow;

- **Unpredictability** - Turbulent flows appear to be very unpredictable; even two similar experiments can have different outcomes - the prototype of a chaotic system, where the extreme dependency on the initial conditions is felt. There isn't yet a proof of uniqueness for the Navier-Stokes equation nor ergodic results;

- **Large scale range** - there are important motions that occur at a great variety of scales (check figure 1.1. What is the scale that matters?).

- **Great mixture capability** - As a consequence of the chaotic motion, diffusion and mixing are enhanced.

The idea of large scale range can be further explored. Let's consider the following picture which represents the instantaneous vorticity field in a jet.

We can see that there are structures, or *eddies*, of various sizes and there is difficulty in finding a length scale, $l$, that is characteristic of the motion.

The following figure shows the distribution of energy as a function of the wavelength ($k = \frac{1}{l}$).

This can be interpreted in the following way: low wavelengths (higher eddies size) are associated with more energy and, as the wavelength increases (smaller eddies) the energy is reduced - larger eddies contain most of the energy.
Another aspect that can be observed is the following: if one defines $a$ as the characteristic size of an eddy and $\alpha_a$ the order of magnitude of the velocity variation over the distance $a$ it is possible to conclude that for large Reynolds number (characteristic of turbulence) the Reynolds number based on the length $a$, $R_a \sim \frac{\alpha_a u_a}{\nu}$ is also large for the larger eddies in the flow - so the viscosity becomes unimportant for the larger structures present in the turbulent motion. The viscosity only becomes important for the smallest motions, where it acts and dissipates energy.

An interesting geometric picture of turbulence was proposed by Richardson (1922):

Big whorls have little whorls,
which feed on their velocity;
and little whorls have lesser whorls
as so on to viscosity.

This idea combined with the argument that for the larger eddies the viscosity is unimportant leads to the idea of the energy cascade: the energy continuously passes from the larger eddies to the smaller ones by an inviscid process.

Let us apply these arguments to determine the order of magnitude of the energy dissipation in turbulent flow. Let $\varepsilon \left[ \frac{m^2}{s^3} \right]$ be the mean energy dissipation per unit time per unit mass. Although the energy transferred is ultimately dissipated, we have seen that this energy is derived from the larger eddies and then is transferred to the smaller ones, so the order of magnitude of $\varepsilon$ can be determined by the physical
quantities which characterize the larger structures. These are the density $\rho$, the dimension $l$ (related to the space where the flow takes place - the external scale of turbulence) and the variation of velocity $\Delta u$ over this distance $l$. From these quantities we can only form one with the dimensions of $\varepsilon$; so we have:

$$\varepsilon \sim \frac{(\Delta u)^3}{l}$$  \hspace{1cm} (1.6)

Let us now consider the properties of turbulence with regard to eddies of size $r$ which are small compared with $L$, the external scale. These are the local properties of turbulence. These heuristics are valid far from boundaries (at least at a distance larger than $L$ itself) and assuming that, statistically, turbulence is homogeneous and isotropic - this is a natural assumption since far from boundaries and at large Reynolds numbers there is a tendency for the flow to have these properties (it’s difficult to distinguish what happens in a place compared to other in a complete turbulent flow).

The results of local properties of turbulence are due to A. N. Kolmogorov and to A. M. Obukhov (1941) and are based in similarity arguments. Let’s determine which physical parameters can have meaning for these local properties; because of the idea that energy is passed from scale to scale and then its dissipated at the smaller scales it is possible to conclude that the mean energy dissipation $\varepsilon$ should be of interest (although it is produced initially in the larger scales it is then passed down scale to scale). These local properties do not depend of the manner at which turbulence is generated: so $\Delta u$ and $l$ are unimportant. The viscosity cannot matter also because, as seen, the energy is transferred mainly by an inviscid process.

With these arguments we can determine the order of magnitude of velocity variation over distance of order $r$; this variation should be determined by $\varepsilon$ and by the distance $r$ itself. From these two quantities the only one which have the dimensions of velocity is the combination

$$u_r \sim (\varepsilon r)^{\frac{1}{3}}$$  \hspace{1cm} (1.7)

This is the Kolmogorov law. It states that velocity variation over small distances is proportional to the cube root of the distance. This can also be interpreted as some form of self-similarity of the flow: the flow looks the same (statistically) independently of the scale of observation.

The law [1.7] can be written with a more formal definition:

$$< |u(x + r) - u(x)| > \sim (\varepsilon r)^{\frac{1}{3}}$$  \hspace{1cm} (1.8)

where $< |u(x + r) - u(x)| >$ is called structure function of the velocity. Kolmogorov law makes it possible to define a lower bound for its validity; it was said that this law must be valid for a length scale much smaller that the external (associated with the boundary conditions) scale of the flow but for small scales the viscosity becomes important. If the viscosity is an important parameter, the Reynolds number based on eddy size $R_e = \frac{\Delta u r}{\nu}$ is of the order of unity. Since $u_r \sim (\varepsilon r)^{\frac{1}{3}}$ we have that the length scale associated with viscous processes is:

$$\eta = \left( \frac{\nu}{\varepsilon} \right)^{\frac{1}{2}}$$  \hspace{1cm} (1.9)
This is the Kolmogorov micro-scale.

The heuristics developed until now are valid for high Reynolds number but the question of how high this parameter may be for it to be valid can be raised. This can be solved by introducing the Taylor scale:

\[ \lambda^2 \sim \frac{u'^2}{(\frac{\partial u}{\partial x})^2} \]  \hspace{1cm} (1.10)

This scale is not associated with the larger eddies, where the characteristic velocity is \( \Delta u = \sqrt{u'^2} \), nor to the smallest eddies, where its characteristic velocity is associated with the intensity of velocity gradients \( \frac{\partial u}{\partial x} \). The Taylor scale is associated both large and small scales so it is an intermediate scale of the eddies present in turbulence.

Kolmogorov law can be expressed in the Fourier space. If the scales \( r \) are replaced by wave numbers \( k \sim \frac{1}{r} \) of the eddies and if \( E(k) \) is the kinetic energy per unit mass in eddies with \( k \) values in the range \( dk \), the only dimensional combination possible is

\[ E(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \]  \hspace{1cm} (1.11)

The equivalence between [1.11] and [1.7] can be seen by noting that \( u'^2 r \) is the order of magnitude of the kinetic energy per unit mass in eddies of size \( r \) or less. If one performs an integration of [1.11]:

\[ \int_k^{\infty} E(k)dk \sim \varepsilon^{\frac{2}{3}} k^{\frac{5}{3}} \sim (\varepsilon r)^{\frac{5}{3}} \sim u_r^2 \]  \hspace{1cm} (1.12)

Kolmogorov’s law is implying some form of statistical scale invariance. This is related to the scale invariance of the Navier-Stokes equation: Kolmogorov theory implies that the exponent \( h \) in the scale invariance symmetry is \( \frac{1}{3} \) (in section 1.1.1).

### 1.2.1 Deviations from Kolmogorov’s Theory

Kolmogorov’s ideas can be generalized for other moments of the structure function [2], as

\[ < |u(x + r) - u(x)|^p > \sim (\varepsilon r)^{\zeta_p} \sim r^{\zeta_p} \]  \hspace{1cm} (1.13)

where \( \varepsilon \) is the mean energy dissipation and \( \zeta_p = p/3 \).

Experimentally, equation 1.13 is not observed for higher order of \( p \) (\( p > 4 \)) i.e, there is considerable deviations to equation 1.13 [2] (see figure 1.4).

In 1962 Obukhov proposed that Kolmogorov’s 1941 theory should be generalized to include the fluctuations in \( \varepsilon \). In particular, he suggested that the velocity structure functions should consider the manner at which \( \varepsilon \) varies locally, yielding

\[ S_p(r) = < |u(x + r) - u(x)|^p > \sim \varepsilon^{p/3} r^{p/3} \]  \hspace{1cm} (1.14)
Figure 1.4: Structure function scaling exponents $\zeta_p$ plotted versus $p$ from [3]. Circles and triangles correspond to experimental data and the solid line corresponds to Kolmogorov scaling $p/3$.

where $\varepsilon_r(x)$ is the mean dissipation over a ball of size $r$.

Figure 1.5 shows the time series of $\varepsilon$ in a turbulent flow. This signal is observed to be highly intermittent, with bursts of intense activity alternating with inactive periods.

this suggests that the mean dissipation is not the appropriate quantity to use in equation 1.13 since we cannot characterize the local dissipation with the mean value.

The underlying scale invariance implicit in Kolmogorov’s law implies that the structure functions $S_p(r)$ go as some power of $r$, but this power can be modified by the fluctuations of the spatially averaged dissipation in [1.14]). Therefore we assume that (as in 1.13)

$$S_p(r) \sim r^{\zeta_p}$$  \hspace{1cm} (1.15)

where the exponents $\zeta_p$ remain to be determined. This parameter will be highly influenced by the way $\varepsilon_r^q$ varies with different exponents $q$. This has geometrical implications in terms of the process by which the dissipation is distributed in the physical space. This aspects will be further analysed in the next chapter with the help of fractal geometry.

1.2.2 The Invariants of the Velocity Gradient Tensor

Until now, everything said about turbulence remained in statistical ideas which have geometric interpretation that will be discussed in the next section. The study of the invariants of the velocity gradient tensor present a clear geometric interpretation and results based on these quantities shown some universal characteristic of turbulence. The discussion goes as follows.

Imagine an instantaneous velocity field in a turbulent flow. This field will be very complex and rich and
very difficult to describe. But now let us consider one point \( \vec{x} = (x_0, y_0, z_0) \) in this instantaneous velocity field and let us move with this point. In this reference of frame the velocity of this point is zero. The velocity vector at a small distance \( \vec{r} \) from this point can be described by a Taylor series expansion of the velocity field about the point \( \vec{x} \). In the first approximation we have:

\[
\dot{u}_i = \frac{\partial u_i}{\partial x_j} x_j
\]

(1.16)

Since \( u_i = \frac{\partial x_i}{\partial t} \) then the previous equation can be written as:

\[
\dot{x} = [A] \vec{x}
\]

(1.17)

where \( [A] = \frac{\partial u_i}{\partial x_j} \). Equation [1.17] is a system of first order differential equation which can be solved analytically if the matrix \([A]\) is defined, but the important point is that the solution of [1.17] will describe the trajectories of fluid elements surrounding point \( \vec{x} \) in his reference frame. Since the solution of equation [1.17] is completely determined by the invariants of the tensor \([A]\) we can conclude that, by calculating the invariants of \( \frac{\partial u_i}{\partial x_j} \) locally in any flow we can get local information regarding the geometry because we have information of how the elements near a certain point are moving. This ideas can be found with more detail in reference [5]; the main conclusions about this type of analysis will be presented in a more
As will be shown below, the characteristic equation of the velocity gradient tensor $A_{ij}$, equation 1.18, determines the type of trajectories.

$$\lambda_3^2 + P\lambda_3^2 + Q\lambda_3 + R = 0$$  \hspace{1cm} (1.18)

where $\lambda_i$ are the eigenvalues of $A_{ij}$. The parameters $P$, $Q$ and $R$ are the first, second and third invariants of the velocity gradient tensor, respectively. Their expressions are:

$$P = -A_{ii} = -S_{ii},$$  \hspace{1cm} (1.19)

$$Q = -\frac{1}{2}A_{ij}A_{ji} = \frac{1}{4}(\Omega_i\Omega_i - 2S_{ij}S_{ij}),$$  \hspace{1cm} (1.20)

and

$$R = -\frac{1}{3}A_{ij}A_{jk}A_{ki} = -\frac{1}{3}(S_{ij}S_{jk}S_{ki} + \frac{3}{4}\Omega_i\Omega_jS_{ij}),$$  \hspace{1cm} (1.21)

where $\Omega_i = \varepsilon_{ijk}\partial u_j/\partial x_k$ is the vorticity field ($P = 0$ in incompressible flow).

Equation (1.18) can have one real and two complex-conjugate roots or two real distinct roots. The line defined by $D_A = 27/4R^2_A + Q^3_A = 0$ (in a two-dimensional space $Q, R$) divides the map into two regions where one of this two hypothesis is verified. If $D_A > 0$ equation (1.18) has one real and two complex-conjugate roots (and then the trajectories of the surrounding fluid elements approach or diverge locally), while if $D_A < 0$ the equation has two real distinct roots (and the trajectory remains stable).

The physical meaning (associated with the trajectories geometry) can be interpreted in terms of the invariants of $A_{ij}$. Starting with $Q$, the trajectories depends on the sign of $Q$. If $Q > 0$ then the enstrophy $$(\Omega^2/2 = \Omega_i\Omega_i/2)$$ dominates over strain product $$(S^2/2 = S_{ij}S_{ij}/2),$$ whereas if $Q < 0$ the opposite occurs.

The meaning of $R$ depends on the sign of $Q$. If $Q \gg 0$ then $R \sim -\Omega_i\Omega_jS_{ij}/4$ and $R < 0$ implies a predominance of vortex stretching over vortex compression, and if $R > 0$ vortex compression dominates.

On the other hand, if $Q \ll 0$ then, if $R > 0$ a sheet like structure is present whereas if $R < 0$ is associated with a tube like structure.

These two invariants are usually analysed in a joint probability distribution map ($Q, R$ map). Here, depending on the location, the ideas presented can be associated with zones in this map (figure 1.6). The $(Q, R)$ map (figure 1.6) allows to infer about the relation between the local flow topology (enstrophy or strain dominated) and the enstrophy production term (vortex stretching or vortex compression).

In many turbulent flows the $(R, Q)$ map displays a correlation between $R$ and $Q$ in the region $R < 0, Q > 0$ associated with a predominance of vortex stretching, and also in the region $R < 0, Q > 0$ a strong (anti) correlation between $R$ and $Q$ is present in the region $R > 0, Q < 0$ associated with sheet like structures.
This gives the \((Q, R)\) map its characteristic "teardrop" shape that has been observed in a great variety of different turbulent flows such as isotropic turbulence [6], mixing layers [7] and channel flows [8].

### 1.3 Turbulent Interfaces

In many natural and engineering flows, two distinct regions can be identified:

- the **turbulent region**, which is characterized by high vorticity levels and the turbulent fluid motion, and

- a **irrotational region** where the flow is irrotational.
In a turbulent jet the shear layer continuously draws and plunges irrotational fluid from the non-turbulent ambience into its turbulent core. This mass exchanges between between the non-turbulent and the turbulent flow regions take place at the Turbulent/Non-turbulent Interface (TNTI) and are accomplished through the contribution of both viscous and inviscid mechanisms. The process by which fluid acquires vorticity and becomes turbulent is denominated *Turbulent Entrainment*. The following figure represents the physical mechanisms present in turbulent entrainment.

Figure 1.8: Schematic showing the several regions, length scales, and main physical processes that take place inside a free shear layer. Included are intense vorticity structures (IVS; worms, red); large-scale vortices (LVS; yellow); the thickness of the viscous superlayer, $\delta_\nu$; and the thickness of the turbulent sub-layer (or vorticity interface), $\delta_\omega$. The turbulent/nonturbulent (T/NT) interface with coordinate $Y$ (direction inwards and normal to the layer) is defined by the line separating these two sub layers (from [9]).

Fluid elements outside the interface, which are initially irrotational, may acquire vorticity in one of two ways (figure [1.8]): either locally at selected zones, in which there are large scale fluctuations of the interface with negative curvature pointing inward (*engulfment*) or along the entire interface by a viscous diffusion process (*nibbling*). It was recently shown that the principal mechanism for entrainment is by diffusion of enstrophy [10].

The process of entrainment occurs at the TNTI. This is a thin layer that surrounds the eddies in the frontier of the turbulent zone. It is known that this layer is composed of two regions [9]:

- the so called, *viscous super-layer* (VSL), which is an external layer, close to the irrotational zone,
where viscous processes dominate the entrainment by diffusion of enstrophy;

- and the turbulent sub-layer (TSL), where enstrophy production and vortex stretching drives the entrainment.

![Schematic of the evolution of enstrophy in the TNTI](image)

Figure 1.9: Schematic of the evolution of enstrophy in the TNTI (with permission of Carlos B. da Silva).

This sharp interface is continually deformed over a wide range of scales and the flow dynamics in its vicinity determines many of the most important flow features: the growth and spreading rate of wakes, the exchanges of mass across mixing layers, and the mixing and reaction rates in jets are some of the flow features that are largely determined by the characteristics of the TNTI and the flow dynamics in its vicinity.

1.3.1 The Flux of Quantities Across the Interface

In [11] a order of magnitude estimation for the fluxes of quantities across the interface is proposed. This estimations have implications in the geometry of the interface; this subject will be further analysed after fractal geometry is introduced.

1.4 Objectives

The goal of this work to investigate the evolution of the small scale universal features at Turbulent/Non-Turbulent interfaces, as exists in jets, wakes, mixing layers and boundary layers. This will be accomplished by analysing the invariants of the velocity gradient tensor and the multifractal spectrum of dissipation. For the latter, all the tolls necessary to perform a multifractal analysis will be developed.

1.5 Outline

The present work can be divided in two main parts: the study of the dynamics and topology in turbulent/non-turbulent interfaces and the study of multifractal aspects of turbulence. These are related by the study
of multifractal aspects confined to the interface.

In Chapter 2 the theoretical background and numerical methods for multifractal analysis are explored; this is a vast subject and needs a proper introduction since review and methods for this type of analysis are the core of this work.

In Chapter 3 the simulations and procedures used for the present study will be presented. Chapter 4 contains the results for the study of the invariants of the velocity gradient tensor in the turbulent/non turbulent interface and Chapter 5 presents the results for the algorithms developed for multifractal analysis and their application to turbulence.

The thesis ends with the main conclusion and proposals of future work that can be developed with the knowledge obtained.
Chapter 2

Multifractal Analysis

2.1 Introductory Concepts

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line [12]

2.1.1 Fractals

A fractal can be defined as fragmented geometric shape which can be split into parts that are a scaled copy of the whole. Although the study of fractals have existed as early as the $17^{th}$ century the term fractal was only coined in 1975 by Benoit Mandelbrot. The word derives from the Latin word *fractus*, which means broken or fractured. While a fractal is a strictly mathematical construct, it is found in various non-mathematical models such as natural systems.

To understand fractals, it is important to know first what their characteristics are. Its first characteristic is that its structure is defined by fine and small scales and substructures. Another characteristic it has is that its shape cannot be defined with Euclidean geometry. The next is that it is recursive and shows iteration to some degree. In addition, fractals are informally considered to be infinitely complex as they appear similar in all levels of magnification. There are a lot of natural phenomena that can be defined and predicted using fractals. Some of these shapes include clouds, vegetables, color patterns, lightning, and snowflakes.

Fractals started to be considered mathematical when Leibniz considered recursive self similarity. However, a graph that can be considered fractal did not exist until 1872, when Karl Weierstrass was able to create a function that is "continuous everywhere yet differentiable nowhere" (this function will be described below).

Fractals are considered to be important because they define images that cannot be defined by Euclidean geometry. Fractals are described using algorithms and deals with objects that don't have integer dimensions. Some of the more prominent examples of fractals are the Cantor set, the Koch curve, the Sierpinski triangle, the Mandelbrot set, and the Lorenz model. Contrary to its complicated nature, fractals do have a lot of uses in real life applications. For example, a lot of objects in nature are composed
of complex figures that are otherwise not possible to be defined by Euclidean shapes. Most natural objects, such as clouds and organic structures, resemble fractals. As such, fractals can be used to capture images of these complex structures. In addition, fractals are used to predict or analyze various biological processes or phenomena such as the growth pattern of bacteria, the pattern of situations such as nerve dendrites, etc. Also one of the most important uses of fractals is with regards to image compressing. A pretty controversial process, it takes an image and expresses it into an iterated system of functions. This image is displayed quickly and is expressed in detail in any magnification. All in all, studying fractals is both a complicated yet interesting branch of mathematic study. And yet despite all its intricacies, it still proves to be a useful tool.

Figure 2.1: Lightning event.

Figure 2.2: The Romanesco broccoli superficially resembles a cauliflower, but it has a visually striking fractal form.
The Length of a Coastline

Mandelbrot [12] began his treatise on fractal geometry by considering the question: “How long is the coast of Britain?” The coastline is a geometric shape like any other, so a measure with a straight ruler, as in the next figure, provides an estimate. The estimated length, \( L \), equals the length of the ruler, \( l \), multiplied by the \( N(l) \), the number of such rulers needed to cover the measured object. It is

\[
\text{Figure 2.3: Measuring the coast of Britain with different rulers. It turns out that the length of the coastline is bigger for smaller ruler sizes.}
\]

possible to notice from figure [2.3] that the length of the coastline increases as the ruler is reduced. In general, as the ruler gets smaller, the length gets infinitely large. Obviously there is a scale where the image resolution does not allow to make a more precise estimation - and there is a physical limit for the measurements (and this length will stabilize). The points here is that coastlines (and other irregular objects) do not present a behaviour, in terms of the stabilization of the length measure, as for a regular object; this can be observed in the following image where the length \( L \) is plotted versus the resolution used to measure it. For the circle the slope stabilizes very quickly in a value close to zero whereas for the coastline the behaviour is different - Mandelbrot called the slope \( 1 - d_f \), where \( d_f \) is the fractal dimension of the object.

Features in nature like this anomalous behaviour led to a generalization of geometry where this type of objects can be characterized. The principal main conclusion is that the dimension needs not to be an integer.

An Heuristic View of Dimension

Let us take a square of side \( L \). We can divide the square into 4 equal sub squares of side \( L/2 \), 16 sub squares of side \( L/4 \) and so on. So one has \( 4^n = 2^{2n} \) sub squares of side \( 2^{-n}L \). The area of the square is then given by:

\[
A = 4^n L \left( \frac{1}{2^n} \right)^2
\]  

(2.1)
Figure 2.4: Log-log plot of the length of land frontiers of several countries versus the measuring length (from [12]).

The 2 in exponent is the Euclidean dimension of the square. Indeed, this can also work with a cube, which volume will be $V = 8^n L (\frac{1}{2^n})^3$. This is illustrated in the following figure (the length of the square is taken as the unity).

In general, if $N(l)$ is the number of sub-parts obtained after each step and $l$ is the linear length of each sub-part, one has that the quantity $V \sim N(l)^d$ (or $V = C \cdot N(l)^d$) has to be non vanishing and finite, i.e, it’s volume must be defined (note that, in the previous example, the linear length of the sub-parts is $2^{-n}$).

Taking the logarithm, one then writes:

$$\log(V) = \log(C) + \log(N(l)) + \log(l^d)$$  \hspace{1cm} (2.2)

with $0 < V < \infty$.

Thus one obtains:

$$\log(N(l)) = -d \cdot \log(l) + b$$  \hspace{1cm} (2.3)

So the concept of dimension can be associated with the slope of the function that relates the logarithm of the number of elements of a given linear size with the logarithm of this size. Mathematically this makes sense when the length tends towards zero; so the concept of dimension can be defined as:

$$d_f = \lim_{l \to 0} \frac{\log(N(l))}{\log(l)}$$  \hspace{1cm} (2.4)

The main point of this is that $d$ need not be an integer, as in Euclidean geometry. It can be a fraction, as it is in fractal geometry. This generalized treatment of dimension is named after the German mathematician, Felix Hausdorff (but popularized by Mandelbrot). It has proved useful for describing natural objects.
Some Fractal Objects

Here some examples of objects with non integer dimension will be given.

The first one is the Koch curve; this curve is the limit of the iterative process represented in figure [2.6]. One can notice that at each iteration, the operation of replacement of the straight lines by the initial curve is made - so the curve looks the same at all scales of magnification - it is self-similar. To determine it’s fractal dimension one only has to notice that, in the top of the figure (the piece that is always reproduced) are needed 4 elements with length $\frac{1}{3}$. In the next iteration it is composed of $16 = 4^2$ elements of length $\frac{1}{3^2}$. In some iteration $k$, this idea can be generalized and so one has $4^k$ elements with length $\frac{1}{3^k}$. Then by equation [2.4] is possible to obtain that the fractal dimension of the Koch curve is just $d_f = \lim_{l \to 0} -\frac{\log(N(l))}{\log(l)} = \lim_{k \to \infty} -\frac{\log(4^k)}{\log(1/3^k)} = \frac{\log(4)}{\log(3)} = 1.26$. This value is between 1 and 2; since the curve has an infinite length, it cannot fit in one dimension although it cannot fill two.

Another example is the Cantor set. As shown in figure [2.7], it is constructed by removing the middle third of the interval $[0, 1]$ at the beginning and then this procedure is iterated to the remaining intervals, and so on. Following the same idea as for the Koch curve, the fractal dimension of the Cantor set can be
Figure 2.6: The Koch curve. This object is generated iteratively by replacement of the \textit{straight} lines with the original object (at each iteration).

Figure 2.7: Cantor Set.

obtained. A straightforward calculation yields \( d_f = \frac{\log(2)}{\log(3)} = 0.63 \). This value means that the set cannot fill one dimension but is also non empty.

Fractal Functions

After introducing the geometrical concept of fractal and given some examples of objects with non integer dimensions, some functions that present fractal properties will now be introduced.

\textbf{Weierstrass Function} \hspace{1em} The first example is the Weierstrass function. Historically this function was created to give an example of a function that is non differentiable everywhere but is continuous [13]. Let \( g: [0, 1] \rightarrow \mathbb{C} \) be a complex function defined as:

\[
g(x) = \sum_{n=-\infty}^{+\infty} \frac{(1 - e^{ib^n x})e^{i\phi_n}}{b^{(2-d)n}}
\]  

\hspace{2em} (2.5)
The graphs of the real and imaginary part are fractals with fractal dimension $d$.

If the phases $\phi_n$ are set to zero one has that $g$ satisfies the exact scaling property:

$$g(bx) = b^{2-d}g(x)$$

(2.6)

this means that when the function is magnified by a factor of $b$, the result is the same.

This function can be observed in figure 5.4(a).

**Brownian Motion**  To introduce the Brownian motion and give some insights about it, the concept of random walk will be presented first. One can think of a random walk as a stochastic process where (in 1D) one starts at zero and then, in discrete steps, a value $\pm k$ (where $k$ is some number and the probability of being positive or negative is $\frac{1}{2}$) is added to the previous value; for example, if $g(t)$ (where $t = i\Delta t$) is the random walk, then $g(0) = 0$, $g(\Delta t) = g(0) \pm k$, $g(2\Delta t) = g(\Delta t) \pm k$ and so on. The limit process when $\Delta t \to 0$ is called a brownian motion, i.e, a continuous version of the random walk. More mathematical detail of this can be found in [14] and [15].

An example of the Brownian Motion can be observed in figure 2.8. The motion is very irregular and, from

![Brownian Motion (H=0.5)](image)

**Figure 2.8:** Single realization of BM with $h = 0.5$ ($N = 2^{16}$).

it's definition, it should be clear that the Brownian motion will be continuous but not differentiable. It can be shown that there is an exponent (called the holder exponent $H$ - discussed below) which describes all the dynamics of the function. For this process is $H = 0.5$. Moreover in reference [16], [17] and [13] a relation between the holder exponent and the fractal dimension can be obtained

$$d_f = 2 - H$$

(2.7)

so the Brownian Motion (the graphic of the function) will have a fractal dimension of 1.5.
Fractional Brownian Motion

The previous example of Brownian motion can be generalized to a motion called fractional Brownian motion where the holder exponent can be any value $H \in [0, 1]$. A full description of stochastic processes is far beyond the scope of this work since this functions will be used only for running tests in the algorithms. Further details can be found in [14] and [15].

2.1.2 Holder exponent of a function

The concept of holder regularity is at the heart of the concept of multifractal, so this concept will be introduced in a simple way.

If $f(x)$ is differentiable, one (always) has that:

$$|f(x) - f(x_0)| = C|x - x_0|$$

when $t$ tends to $t_0$ (where $C \in \mathbb{R}$). If $f(x)$ is not differentiable, one tries to write

$$|f(x) - f(x_0)| = C|x - x_0|^h$$

when $t$ tends to $t_0$. One defines then the Holder exponent of $f$ in $t_0$ as the maximal exponent $h$ such that the previous relation is satisfied. For instance, if $f$ is exactly (only) once differentiable in $t_0$, then one has $h = 1$. The interesting point is that there are functions for which $0 < h < 1$. This is, for instance, the case of the Brownian motion, where $h = 0.5$ in all the domain.

This concept can also be interpreted as a generalization of the notion of Taylor series. Since any function can be expanded as a Taylor series where the exponent $h$ is present in the higher order term.

$$f(x) = a_0 + a_1 x + \cdots + a_h x^h$$

More information can be found in reference [18].

2.2 Multifractals

Previously, when the Brownian motion was introduced, it could be concluded that a single holder exponent is enough to characterize its dynamics. Now, systems where a single exponent is not enough to characterize it will be introduced - these are called multifractals (the reader may remember that, the Hurst - or constant holder - exponent is related to the fractal dimension of the graph of the self-similar functions).

Multifractal systems are common in nature. They include fully developed turbulence, stock market time series, real world scenes, the Sun’s magnetic field time series, heartbeat dynamics and natural luminosity time series.

In a multifractal system, the behaviour near any point $x$ can be described as [15]:

$$f(x + l) - f(x) \sim l^h(x)$$
i.e, the holder exponent is not constant (it depends on the local coordinate $x$).

The fundamental quantity that characterizes a multifractal is the *multifractal spectrum*, $D(h)$. A typical multifractal spectrum $D(h)$ is bell-shaped [2.9 and it can be interpreted in the following way.

![Figure 2.9: Typical example of a multifractal spectrum.](image)

A geometric interpretation of $D(h)$ is that it represents the fractal dimension of the set of intervals where the holder exponent is $h$; since $h$ can take any value between $h_{\min}$ and $h_{\max}$ there are an infinite number of fractal sets residing within the same set - hence the term *multifractal*.

Another interpretation is a probabilistic one; this is related to the theory of large deviations. A full discussion of this interpretation is beyond the scope of this work but basically the idea is that $D(h)$ is related with the statistical distribution of the singularities: at the peak of $D(h)$ we have the expected value of singularity and the other points represents the deviation from the mean (as in the large deviation theory); this is an important conclusion although it can be seen in a intuitive way (in the author's opinion) - if the fractal dimension is larger the set *fills* better the space so it's normal that the singularities are more common. A full discussion can be found in [19] and [15].

Now the problem resides in the determination of the multifractal spectrum; this is the topic of the next section.

### 2.3 The Multifractal Formalism

Characterizing the multifractal spectrum from it's definition is an almost impossible task - it would involve the determination of the point wise singularity exponent for every point in the function from the definition. To overcome this issue the multifractal formalism was developed, in the beginning, from heuristic arguments (that will be presented in this work) and later it was put in a more formal ground. Basically the multifractal formalism allows to determine the multifractal spectrum from quantities that are relatively easy to compute.

Parisi and Frisch in [20] developed a multifractal formalism to study the behaviour of velocity increments in fully-developed turbulence (a key quantity in Kolmogorov's theory). The idea was to study asymptotic power law behaviour of moments of velocity increments, when the incremental distance $l \to 0$ (here it
should be considered the velocity signal as a 1D function):

\[ \langle |f(x) - f(x + l)|^q \rangle \sim \text{Const} \cdot |l|^\tau_q \]  

From the quantity \( \tau_q \) it is possible to recover the multifractal spectrum \( D(h) \). The relation between these two parameters can be explained with simple arguments. The discussion goes as follows.

Let us define \( E(h) \) as the set (contained in the domain of the function \( f \)) of the points where the holder exponent of \( f \) is \( h \) and \( D(h) \) as the fractal dimension of \( E(h) \). In the same fashion of equation [2.12], the structure function \( S(l, q) \) is defined as

\[ S(l, q) = \frac{1}{n(l)} \sum_{x_0} |f(x_0 + l) - f(x_0)|^q \]  

where \( n(l) \) is the number of points separated by a distance \( l \). From the definition of holder exponent is possible to write

\[ f(x_0 + l) - f(x_0) \sim l^h \]  

Replacing in equation [2.13] one get

\[ S(l, q) = \frac{1}{n(l)} (N(l)l^{qh}) \]  

where \( N(l) \) is the number of points separated by a distance \( l \) which holder exponent is \( h \). If these points live in a Cantor-like set with dimension \( D \) then is possible to conclude that \( N(l) \sim l^{-D(h)} \) (from our notions of fractals). Furthermore one can notice that \( n(l) \sim l^{-1} \) because single variable functions are being considered.

Therefore, one can write,

\[ S(l, q) \sim l^{-D(h) +qh} \]  

Since this relation should hold when \( l \rightarrow 0 \) (from the definition of holder exponent), the dominant contribution from the exponent is when it has it's minimal value. Thus one has the following relation between \( \tau_q \) and \( D(h) \):

\[ \tau_q = \min_h (1 - D(h) + qh) \]  

The quantity \( \tau_q \) is called scaling function and, from equation [2.16] and [2.17] (in analogy with [2.12]) it is defined as

\[ \tau_q = \lim_{l \to 0} \frac{\log S(l, q)}{\log(l)} \]  

The converse relation also holds

\[ D(h) = \min_q (1 - \tau_q + qh) \]  

In reference [21], the multifractal spectrum is associated with a measure \( \mu(x) \) which is a function that assigns a value to an interval of the space - for example, the Lebesgue measure assigns the size to a set of the euclidean space, on some subset of \( \mathbb{R}^n \). For each point \( x \) a local singularity exponent \( \alpha(x) \) is
defined as (in analogy with the holder exponent)

$$\alpha(x) = \lim_{l \to 0} \frac{\log(\mu(I_l))}{\log(l)}$$  \hspace{1cm} (2.20)

where $I_l$ is the interval where the measure is defined. In the same vein as in the definition of the multifractal spectrum, it is possible to define the so called singularity spectrum $f(\alpha)$ as the fractal dimension of the set where the singularity exponent is $\alpha$. An analogue to the structure function defined in [2.13] is

$$S_m(l, q) = \int \mu(B_l(x))^q d\mu(x)$$  \hspace{1cm} (2.21)

where $B_l(x)$ is the ball of radius $l$ centred in $x$. A scaling exponent $\tau_q$ can be defined as (in analogy with $\tau_q$)

$$\tau_q^M = \lim_{l \to 0} \frac{\int \mu(B_l(x))^q d\mu(x)}{\log(l)}$$  \hspace{1cm} (2.22)

An analogy between $\tau_q$ and $\tau_q^M$ can be made [22]; if one works in one dimension, is easy to note that $B_l(x) = [x - l, x + l]$, and then $\tau_q$ can be rewritten as

$$\tau_q = \lim_{l \to 0} \frac{\log(\int_\mathbb{R} |\psi(x + l) - \psi(x - l)|^q d\mu(x))}{\log(l)}$$  \hspace{1cm} (2.23)

where

$$\psi(x) = \int_{-\infty}^{x} d\mu(y)$$  \hspace{1cm} (2.24)

Then $\tau_q$ and $\tau_q^M$ have the same meaning if one considers the cumulative integral of the measure under study. Another point of interest is the following: if one considers the holder exponent (from the definition) and the scaling exponent is possible to conclude from the previous analogy that $\alpha = h + 1$; by integration of the measure trough equation 2.24 we are increasing the exponent in one (for one dimension) and so $\tau_q^M = \tau_q + dq$. Also, the same reasoning that made possible the deduction of equation 2.17 can be applied for the scaling exponent in the sense of equation 2.21; this multifractal spectrum (usually denominated $f(\alpha)$, for example in [4]) it’s the same this as our definition of $D(h)$ but it will be shifted to the right (since $\alpha = h + 1$, $f(\alpha) = D(h + 1)$).

One can also notice that there is a difference in the definitions 2.13 and 2.21; in the latter there isn’t an average - a reason for this is just keep the notation more similar as possible to the literature on the subject although there must be an average over all boxes of radius $l$ to keep the analogy correct. The definition 2.21 will be used for then deduction in the binomial cascade but, since the method we will use employs the concept of holder exponent the theoretical values must be corrected (this is done in the results). The results derived in this section are the basis of multifractal analysis. In the next section, by studying multifractal processes, these ideas will be put into practice and they will become more clear.
2.4 Multifractal Processes

In this section, some of the most commonly used multifractal processes will be introduced and their properties discussed.

2.4.1 Binomial Cascade

The binomial cascade is one of the most (if not the most) simple processes that exhibit multifractal properties. Basically this iterative process starts with the unit interval \( I_0 = [0, 1] \), where a constant mass is assigned, divides it into smaller and smaller pieces and at the same time divides the mass attributed at each smaller interval.

The iteration begins with a uniform distribution \( \mu_0[0, 1] = m_0 \) (with \( 0 < m_0 < 1 \)), subdivides it into a distribution with \( \mu_1[0, \frac{1}{2}] = m_0 \) and \( \mu_1[\frac{1}{2}, 1] = m_1 = 1 - m_0 \), further subdivides it into \( \mu_2[0, \frac{1}{4}] = m_1m_1 \), \( \mu_2[\frac{1}{4}, \frac{1}{2}] = m_0m_1 \), \( \mu_2[\frac{1}{2}, \frac{3}{4}] = m_0m_1 \) and \( \mu_2[\frac{3}{4}, 1] = m_0m_0 \) and so on. Additional iteration of this procedure gives a multiplicative cascade that generates an infinite sequence of measures; the limit of the measures is the binomial measure. The iterative process can be observed in figure 2.10. One should notice that, by the way the binomial cascade is constructed, it resembles the definitions given in equations [2.21] and [2.22]. Using this definitions the multifractal spectrum of the binomial cascade can be calculated. The discussion goes as follows.

Since the binomial measure is the limit of the process described above, to determine it’s multifractal
spectrum, the structure function of the measure at the consecutive iterations can be obtained and then take it's limit.

In the first iteration ($\mu_1$ in the notation used in this document), equation [2.21] takes the form

$$S^1_m(l, q) = m_0^q + m_1^q$$  \hspace{1cm} (2.25)

In the next step ($\mu_2$)

$$S^2_m(l, q) = m_0^q + m_1^q + 2 \cdot (m_0 m_1)^q$$  \hspace{1cm} (2.26)

It is already clear that, in a iteration $k$, the following result should hold

$$S^k_m(l, q) = (m_0^q + m_1^q)^k$$  \hspace{1cm} (2.27)

where the binomial expansion was used (thus the name binomial cascade). To calculate the scaling exponent by equation [2.22] note that the ball sizes $l$ go as $l = 2^{-k}$ and then we have

$$\tau^M_q = - \log_2 (m_0^q + m_1^q)$$  \hspace{1cm} (2.28)

The determination of the multifractal spectrum is not so trivial as it is for the scaling exponent. First one can determine the holder (scaling) exponent through the relation $h(q) = \frac{d\tau^M_q}{dq}$. This yields [23],

$$h(q) = - \frac{m_0^q \log_2(m_0) + m_1^q \log_2(m_1)}{m_0^q + m_1^q}$$  \hspace{1cm} (2.29)

As a function of $q$, $D(h(q))$ is [23]:

$$D(h(q)) = \log_2 \left( \frac{m_0^q + m_1^q}{m_0^q} \right) - q \frac{m_0^q \log_2(m_0) + m_1^q \log_2(m_1)}{m_0^q + m_1^q}$$  \hspace{1cm} (2.30)

If one defines $h_{\text{min}} = - \log_2(m_1)$ and $h_{\text{max}} = - \log_2(m_0)$, then is possible to write equation 2.30 as:

$$D(h) = - \left( \frac{h_{\text{max}} - h}{h_{\text{max}} - h_{\text{min}}} \right) \log_2 \left( \frac{h_{\text{max}} - h}{h_{\text{max}} - h_{\text{min}}} \right) - \left( \frac{h - h_{\text{min}}}{h_{\text{max}} - h_{\text{min}}} \right) \log_2 \left( \frac{h - h_{\text{min}}}{h_{\text{max}} - h_{\text{min}}} \right)$$  \hspace{1cm} (2.31)

It should be emphasized that, in equations 2.30 and 2.31, $h$ should be interpreted as the scaling exponent.

### 2.4.2 Random Binomial Cascade

The idea developed previously for the binomial cascade can be made a bit more sophisticated by introducing some randomness in the process.

Let $a$ be a random variable with the following properties (in analogy with the masses defined previously):

- $a = m_0$ with a probability $\frac{1}{2}$ and $a = m_1$ with a probability $\frac{1}{2}$. So, in analogy, the masses $m_i$ are now replaced by $a$ at all iterations. This process is illustrated in figure 2.11.
It can be proved that the (multifractal) properties of this process are the same as for the binomial cascade [24].

2.4.3 Canonical Mandelbrot Cascades (CMC)

A more general class of multifractal processes can be defined with ideas similar to the binomial cascade (BC).

Like the BC, CMC are the limit of an iterative procedure; the construction starts from a uniform unit mass on the interval \([0, 1]\). First, the interval is cut into two intervals of equal size, \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\). Then a random mass is attributed to each of the intervals by multiplying the original masses with random multiplier \(W_{11}, W_{12}\) identically and independently distributed according to a probability distribution \(a_W\). The procedure is iterated on the subintervals, and so on and so forth. The construction of CMC is illustrated in Fig. [2.12].

The multipliers have to be strictly positive random variables, \(W > 0\), and they have to satisfy the constraint \(E W = 1\), ensuring that the cascade conserves mass in average.

Let the multipliers for the \(j\)th iteration be denoted by \(W_{jk}, k = 1, \cdots, 2^j\). The measure \(Q_r(t)\) at resolution \(r = 2^{-j}\) (i.e., after \(J\) iterations) is defined as:

\[
Q_r(t) = \prod_{j=1, \cdots, J, \{k : \epsilon \in [2^{j}k, 2^{j}(k+1)])} W_{j,k}
\]  

(2.32)

The process

\[
A(t) = \lim_{r \to 0} \int_0^t Q_r(u)du
\]  

(2.33)

is called the CMC motion [25]. One can notice the similarities between equation [2.33] and the definition of the function \(\psi\) in [2.24].
Log-Normal Multipliers

When the CMC was discussed above, a specific form for the multipliers $W$ was not given. One can consider Log-Normal multipliers $W = 2^{-U}$, where $U \sim N(m, \sigma^2)$ is a Gaussian distribution with mean $m$ and variance $\sigma^2$. The condition for conservation of mass, $E W = 1$, implies $\sigma^2 = \frac{2m}{\ln(2)}$ and, in the sense of equation [2.18] one have [25]:

$$r_q^M = mq(1 - q) + q$$  \hspace{1cm} (2.34)

and

$$D(h)_{A(t)} = 1 - \frac{(h - d - m)^2}{4m}$$  \hspace{1cm} (2.35)

2.5 Turbulence Energy Cascade as a Multiplicative Process

In the previous chapter, when the topic of turbulence was introduced it was stressed that Kormogorov law needs to be revisited in the form of equation [1.14]. Experimentally it was observed that, for higher order moments, Kolmogorov law was not verified - people realized that $\langle \varepsilon_r \rangle$ is itself a random variable and highly intermittent and the average in Kolmogorov law cannot be done at the same time. The key issue should be to understand how $\varepsilon_r$ behaves and our focus will be on this aspect. The point is to understand the moments $\langle \varepsilon_r^q \rangle$ with some kind of mechanism. Evidences of multifractal behaviour of $\varepsilon_r$ (i.e, deviation from a pure fractal, like the previous examples) where shown in [4]. This picture of the energy viscous dissipation as a multiplicative process originated from the argument of the energy cascade.

The basic ingredient of multiplicative processes in turbulence is that large eddies or fluid pieces transform
or break down into smaller ones; the fragmented pieces themselves yield even smaller ones, and so on. This then defines pieces of different generations; the generation step will be denoted by $n$. To each piece is associated a characteristic linear dimension $r$. We assume that the characteristic scale of a piece of the $n^{th}$ generation, $r(n)$, will be given by the product of $n$ numbers (called length multipliers $l_j$, $1 < j < n$), each of which is the ratio of consecutive length scales. In other words,

$$ r(n) = r(0) \prod_{j=1}^{n} \frac{r(j)}{r(j-1)} = r(0) \prod_{j=1}^{n} l_j $$

(2.36)

Another vital ingredient is the concept of a measure density which, in the present context, is the rate of dissipation per unit volume $\varepsilon_r$. Of particular interest is the total dissipation $E_r$, in a certain piece $\Omega$ of size $r$. This will be given by the integral of $\varepsilon(x)$ over the piece $\Omega$ as:

$$ E_r = \int \varepsilon(x) dx $$

(2.37)

When a piece $\Omega$ decays into smaller ones, each smaller piece can be thought of as receiving a fraction of $E_r$. Analogous to length scales, the total dissipation on a certain piece of size $r(n)$ of the $n^{th}$ generation will be given by the product of $n$ numbers (called measure multipliers $M_j$), each of which is the ratio of consecutive measures. That is,

$$ E_{r(n)} = E_{r(0)} \prod_{j=1}^{n} \frac{E_{r(j)}}{E_{r(j-1)}} = E_{r(0)} \prod_{j=1}^{n} M_j $$

(2.38)

It is then clear that Richardson’s (1922) picture of turbulence cascade, in which blobs or whorls of turbulent fluid break down into smaller pieces - each feeding on their velocity, i.e. receiving a certain fraction of the flux of kinetic energy from larger scales - is a multiplicative process - The discussion until now clearly remarks the multiplicative cascade models defined previously.

If dissipation behaves in this way then, in Kolmogorov’s theory, since the only relevant quantity is the mean flux $< \varepsilon >$ (or $< E_r >$) - since the dissipation is equal to the rate of energy transfer from large to small eddies we have a trivial multiplicative process where all the multipliers are the same.

With this picture of dissipation (in analogy with a multiplicative process) we can define the local singularity exponent in agreement with equation 2.20 in the following way:

$$ \frac{E_r}{E_L} \sim \left( \frac{r}{L} \right)^\alpha $$

(2.39)

or,

$$ \frac{\varepsilon_r}{\varepsilon_L} \sim \left( \frac{r}{L} \right)^{\alpha-d} $$

(2.40)

where $d$ is the euclidean dimension of the support of the measure (e.g. $d = 1$ for a line of dissipation data). Notice that $\alpha - d = h$, the holder exponent of the function $\varepsilon(x)$.

Following the same reasoning as for the study of multifractals, we can describe the process by it’s
moments,

\[
\sum E_r^q \sim E_L^q \left( \frac{r}{L} \right)^{\tau^M_q}
\]  

(2.41)

Additionally we can define the [Hent procc] generalized dimensions \( D_q \) as

\[
D_q = \frac{\tau^M_q}{q - 1}
\]  

(2.42)

One should notice the similarity of equation 2.41 with equation 2.21 - they display the same information. With this picture, reference [4] and [26] observed that a binomial cascade with \( m_0 = 0.3 \) agrees with the behaviour of dissipation in several flows and conjectured that \( D_q \) and \( D(h + 1) \) should be universal parameter in turbulence. This idea can be understood in the following way. Suppose that an eddy size \( r \) breaks down into, let's say, \( 2^d \) eddies of equal size \( r/2 \), \( d \) being the euclidean dimension of the space. Furthermore, suppose that the flux of energy to these smaller eddies proceeds unequally: some of then receive a different fraction of the original energy in the mother eddy as in figure 2.13.

![Figure 2.13: One-dimensional version of a cascade model of eddies, each breaking down into two new ones. The flux of kinetic energy to smaller scales is divided into non equal fractions \( m_0 \) and \( m_1 \) (from [26]).](image)

This is just the prototype of the Binomial Cascade presented previously. It is remarkable how simple ideas like a binomial cascade can be so close to our idealization of the phenomena.

2.5.1 Fluxes Across the Interface

In reference [27] an estimation of fluxes of several quantities across the interfaces given. The discussion goes as follows. It can be observed that interfaces in turbulence show very detailed and interesting geometric characteristics. A description using fractal geometry is proposed in reference [27] and a rough estimation of it’s fractal dimension is given: let's consider the flux of quantities which the diffusivity is equal to the viscosity (momentum, enstrophy, etc.). An estimation of the flux should be given by the
product of the diffusivity ($\mu$), gradient of the quantity ($\Delta_{AB}/\Delta x$) and the area of the surface ($S$). From physical arguments, the thickness of the interface should be of the Kolmogorov scale $\eta$. An estimation of the area can be given by the number of \textit{squared} of side $r$ multiplied by the area of each \textit{square}: $S = N(r)r^2$. If the surface is a fractal then $N \sim r^{-d_f}$ and so $S \sim r^{2-d_f}$. As a function of the Kolmogorov scale $\eta$ the area is $S(\eta) \sim L^2 (\eta^2)^{2-d_f}$ ($L$ is the integral scale). The total flux $\Phi_{AB}$ is therefore,

$$\Phi_{AB} \sim \Delta_{AB}L^2u'R_{\infty}^{3/4(d_f-7/3)}$$ \hspace{1cm} (2.43)

Since the flux should be independent of the Reynolds number this yields $d_f = 7/3$.

A more refined estimation that takes into account local fluctuation of the Kolmogorov scale (because of the intermittent behaviour of the dissipation) is present in reference [11]. Without giving much details of this deduction the major result is that

$$d_f = \frac{7}{3} + \frac{2}{3}(3 - D_{1/3})$$ \hspace{1cm} (2.44)

where $D_{1/3}$ is the generalized dimension at $q = 1/3$. This result depends on the multifractal spectrum of the dissipation in the interface (in the original work it is assumed that the spectrum in the interface is equal to the spectrum in fully-developed turbulence).

### 2.6 Practical Multifractal Analysis

There are several methods to perform a multifractal analysis; the aim of this section is to provide an insight about the main ones and take into account their properties. Just for note all of them were programmed and the conclusions about them is also a result of the knowledge obtained with this experience although only the results with the method chosen will be shown - it would be a very long discussion for this work.

#### 2.6.1 Box Counting

The Box Counting method is the simplest method (and the first to be developed) to determine the fractal dimension and the multifractal spectrum. From equation 2.4 is possible to build a simple algorithm which uses this definition. This is illustrated in figure 2.14 where one varies the length of the box $l$ and for each value of $l$ the number of boxes needed to cover the object is considered. With that is possible to determine the fractal dimension of the object with the slope of $\log(N(l))$ versus $\log(1/l)$.

For multifractal, this method calculates $\tau_q^M$ with the following definition [28]

$$\tau_q^M = \lim_{l \to 0} \frac{\log \left( \sum_i K_i(l)^q \right)}{\log(l)}$$ \hspace{1cm} (2.45)

where $K_i(l) = \frac{N_i(l)}{N}$ with $N_i(l)$ being the number of points in the box $i$ (not the number of boxes) and $N$ the total number of boxes. Several algorithms to perform this method are proposed (for example in reference [28]) but this method has some inherent limitations and problems. One of them is how to
Figure 2.14: The Box Counting method. One counts the number of boxes of size \( l \) (varying \( l \)) which cover the object (in this case is the coastline).

define the box sizes and, by construction, the measure in a certain box is the sum of the measures in smaller non overlapping boxes, which implies that, for example, in multiplicative cascades this method yield always a conservative cascade even if it is not the case [29].

2.6.2 Wavelet Methods

Wavelets have been used for a long time to study scaling behaviour and irregularity of functions [30]. To introduce the wavelet methods to study regularity of function, first we describe why is expected for it (wavelets) to work (with heuristic arguments) and after the principal methods (WTMM and WL) will be explained. A somewhat detailed explanation of the wavelet transform can be found in the Appendix and from here on is supposed that the reader understands the wavelet transform.

The wavelet transform of some function \( f(x) \) is defined as the projection of \( f(x) \) in some analysing wavelet \( \psi(x) \) [31]:

\[
T_{\psi}[f](x_0, a) = \frac{1}{a} \int f(x) \psi(\frac{x-x_0}{a}) dx
\]  

(2.46)

Usually some conditions are imposed on the wavelet \( \psi \); \( \psi \) has to have zero mean and, for the purpose of analysis of singularities, it is of interest that \( \psi \) is orthogonal to polynomials up to a certain order \( n_\psi \):

\[
\int x^n \psi(x) dx = 0
\]  

(2.47)

Now, if one remembers the definition of holder exponent and considers that, near a point \( x_0 \), the function behaves as \( f(x + l) = a_0 + a_1 x + a_2 x^2 + \cdots a_h x^h \) (where \( h \) is the local holder exponent), is possible to write equation [2.46] as (it will only make sense in the limit of small scales \( a \) - we are considering a generalized Taylor expansion):

\[
T_{\psi}[f](x_0, a) = \frac{1}{a} \int (a_0 + a_1 x + a_2 x^2 + \cdots a_h x^h) \psi(\frac{x-x_0}{a}) dx
\]  

(2.48)

it becomes obvious that, if \( n_\psi > h(x_0) \), then the terms for low order terms become zero (by equation [2.47]) and the transform becomes

\[
T_{\psi}[f](x_0, a) = C a \int x^{h(x_0)} \psi(\frac{x-x_0}{a}) dx
\]  

(2.49)
by a change of variable one gets that

$$T_\psi[f](x_0, a) \sim a^{h(x_0)}$$  \hspace{1cm} (2.50)$$

when \( a \to 0 \). This is the principal reasoning behind the study of regularity of functions with wavelets - one can recover the holder exponent from the transform. The presented results can be found with mathematical formalism in [32].

If one looks back at equations [2.12], [2.13] and [2.15], it’s natural to associate the function increments with the wavelet transform of the function at a point. In the same vein, a partition function (analogous to the structure function) can be defined as:

$$Z_c(q, a) = \int |T_\psi[f](x, a)|^q dx$$ \hspace{1cm} (2.51)

This partition function scales with \( a \) (the scale) as

$$Z_c(q, a) \sim a^{\tau_q}$$ \hspace{1cm} (2.52)

and so a reasoning identical to 2.17 is valid [32].

For the discrete transform it is possible to follow a similar idea. The discrete wavelet transform is a decomposition of the function \( f \) on the orthogonal basis \( \{\psi_{j,k}\} \) composed of discrete wavelets \( \psi_{j,k} \):

$$d(j, k) = \int_{\mathbb{R}} dx \psi_{j,k}(x)f(x)$$ \hspace{1cm} (2.53)

Wavelets are space-shifted and scale dilated templates of a mother wavelet \( \psi_0 \):

$$\psi_{j,k} = (const) \psi_0(2^j x - k)$$ \hspace{1cm} (2.54)

and define a basis distributed according to a dyadic basis in the space-scale plane (every wavelet \( \psi_{j,k} \), and then every \( d(j, k) \) can be associated to the dyadic interval \( I(j, k) \)) (for each pairs of integers \( j, k \) define a interval as \( I(j, k) = [2^{-j}k, 2^{-j}(k + 1)] \) - this is a dyadic interval).

The value of the constant can be define by choosing a unitary norm for the wavelet to conserve. This can be expressed as:

$$1 = \left( \int |const \times \psi(2^{-j}x - k)|^p dx \right)^{\frac{1}{p}}$$ \hspace{1cm} (2.55)

Simplifying the above expression (using a change of variable) yields

$$const = 2^{-j/p}$$ \hspace{1cm} (2.56)

Using this generalized form of the wavelet coefficient with arbitrary norm one can determine the scaling behaviour of the wavelet coefficients (this is a generalization if the results present in [33]; this is a rather
heuristic derivation but for the $L^1$ norm an rigorous proof is present in [33] but this follows the same logic and the result is correct for the degenerate case of $L^1$ norm.

If the function $f$ has local holder exponent $h$ then the following relation is valid:

$$|f(x) - P_N(x - x_0)| \sim |x - x_0|^h \tag{2.57}$$

As seen previously, this is a generalization of the Taylor expansion of a function:

$$f(x + x_0) = f(x) + ... + (x - x_0)^h \tag{2.58}$$

If the wavelet has enough vanishing moments one can replace $f(x)$ in [2.53] by its expansion [2.58] and obtain:

$$d(j,k) \sim \int \frac{dx \psi(2^{-j}x - k)[x - x_0]^h}{2^{j/p}} \tag{2.59}$$

With a suitable change of variable one can arrive at:

$$d(j, k) \sim C 2^{j(h(x_0)+1-\frac{1}{p})} \left(1 + |2^{-j}x_0 - k|^{h(x_0)}\right) \tag{2.60}$$

Equation [2.60] is a generalization of the results present in [[33], [34],...] and collapses into their results for the norm $L^1$. This result is valid for any type of singularities present in the function (chirp or oscillating): furthermore this relation can be simplified for the case of chirp singularities:

$$d(j,k) \sim C 2^{j\left(h(x_0)+1-\frac{1}{p}\right)} \tag{2.61}$$

As for the continuous transform, with equation [2.61] is possible to build a partition function $Z_d(j,q)$ and build a multifractal formalism:

$$Z_d(j,q) = 1 \frac{1}{2^j} \sum_k d(j,k)^q \tag{2.62}$$

In the same fashion was for the continuous transform this partition function scales with $j$ (a measure of the scale) as

$$Z_d(j,q) \sim 2^{j\tau_q} \tag{2.63}$$

and so a reasoning identical to 2.17 is valid [33].

$Z_c(q,a)$ and $Z_d(j,q)$ should have the same properties of $S(p,l)$ defined previously and the multifractal spectrum can be recovered from it in the same way was shown; also, a very interesting point is that, the structure function approach can be described as a wavelet approach when the analysing wavelet is Dirac function [35].

An inherent problem with $S(p,l)$, $Z(q,a)$ and $Z_d(j,q)$ is the fact that there is nothing that prevents the values at a single point to be zero (or very close); then, with negative powers of $q$ (or $p$) the values of $S(p,l)$, $Z(q,a)$ and $Z_d(j,q)$ can diverge unrealistically. To overcome this problem two methods were developed - this is why one should use a more sophisticated method (with wavelets) rather than a simple
structure function based on increments of the function under study [35].

**Structure Function Method**

As described in [31], the structure function method (which use a direct application of equation 2.13) is a particular case of wavelet methods (although for long time this was a method of its own) when the analysing wavelet is a Dirac function. Besides the divergences in negative exponents (more rigorously in for $q < -1$) a Dirac function is only orthogonal to constants and so, for higher $h$ this function cannot predict the scaling behaviour.

**Wavelet Transform Modulus Maxima (WTMM)**

To solve the problems of divergences for negative exponents, Arneodo proposed a method called Wavelet Transform Modulus Maxima in [32]. In this method, a partition function analogous to equation 2.51 is constructed but now one does not consider the whole wavelet transform. The local maxima in the space-scale representation are determined and these local maxima are chained. The maximum values are the only one considered in this method (so the problem with lower values which cause the unrealistic divergence for negative exponents is eliminated). The partition function takes the form:

$$Z_{WTMM}(q,a) = \sum_{l \in L} \left( \sup_{x \in l} |T_{\psi}[f](x,a)| \right)^q$$  \tag{2.64}

So, in other words, when one is calculating $Z_{WTMM}(q,a)$ for a given $a$, one considers the sum of the maximum value of the transform along each line (until the smallest scale) which cross the scale $a$ in observation. This is illustrated in figure 2.15. This method was programmed in this work but it was not used because the procedure of tracking the maxima lines is difficult (if the CWT as some noise is necessary to filter the transform to determine the local maxima and the method becomes computationally heavy), generalization of this method to more dimensions (although this is not done here for future work it could be a problem) is difficult (as in reference [29]) and following [33] there are inherent mathematical limitations regarding independence of the analysing wavelet and other technical aspects which beyond the scope of this work.

**Wavelet Leaders (WL)**

Probably inspired in the WTMM, the Wavelet Leaders method was developed by S. Jaffard [33]. It uses the discrete transform and it is a way to overcome the problem of the divergence for the negative moments (like the WTMM).

Instead of building the structure function [2.62] with the discrete wavelet coefficients, those are replaced by a quantity derived from them called the **Wavelet Leaders**.

The Wavelet Leaders, $L(j,k)$, are defined as the maximum of the wavelet coefficients, $d(j,k)$, in a certain modified dyadic interval $I_k = [(k-1)2^j, (k+2)2^j]$, where all the scales below the one in study are considered. This concept is better understood in the following picture: It is proved [33] that the same
multifractal formalism for $Z_d(q,j)$ hold for $Z_L(q,j)$ - the partition function based on the leaders. This method has been used in several works [23].
Figure 2.15: Procedure for the WTMM method. In the top figure the signal (Brownian Motion with $h = 1/3$) is presented. In the middle figure the Continuous Wavelet Transform (CWT) in space-scale is presented. The partition function is constructed based on the local maxima of the CWT in the space-scale representation in the bottom figure. From [31].
Figure 2.16: Definition of the Wavelet Leaders.

\[ L(j,k) = \sup \{ |d(j,k)|, (j,k) \in I3 \} \]
Chapter 3

Direct Numerical Simulations and Post-Processing Tools

3.1 Shear Free Turbulence

We use direct numerical simulations (DNS) of shear free turbulence in a periodic box with sizes $2\pi$ with $512^3$ collocation points[36]. The simulation starts by instantaneously inserting a velocity field from a previously run DNS of forced isotropic turbulence into a field of quiescent fluid. The initial isotropic turbulence region then spreads into the irrotational region in the absence of mean shear, developing a distinct TNTI layer. In the present shear free simulation the Reynolds number based on the Taylor micro-scale is equal to $Re_\lambda \approx 115$ and the resolution is $\Delta x/\eta \approx 1.5$. Details can be found in [36] and references therein.

3.1.1 Interface Detection

The TNTI can be seen as a (zero thickness) surface separating turbulent from irrotational flow and can be defined in terms of vorticity/no-vorticity content of the flow. Several methods have been used to detect the TNTI in several different flows e.g. [37], Westerweel et al. [38]. Many of these methods consist in looking for a low vorticity-magnitude threshold $\omega_{tr}$, below which flow regions can be considered to be (approximately) irrotational. A difficulty arises due to the existence of perturbations within the irrotational flow (in experimental data) or numerical noise (in numerical simulations) that prevent using a straightforward approach to detect this threshold. A useful observation that there is a vorticity magnitude range where statistics of the interface layer, e.g. conditional vorticity profiles in relation to the distance from the TNTI, as well as the geometric shape of the interface layer itself, are weakly dependent on the vorticity magnitude threshold used to detect the TNTI.

In the present work, to detect $\omega_{tr}$ we build a probability density function of the vorticity and then we define $\omega_{tr}$ as the minimum value of this PDF. Since we are dealing with shear free turbulence there will be a domain without vorticity and another with a higher value of this quantity. This is illustrated in figure

40
3.1. Conditional Statistics

Conditional statistics of several quantities with respect to the distance from the TNTI have been employed in recent works [9] and the procedure to obtain them is only briefly described here. The location of the outer edge of the TNTI is defined by the surface where the vorticity magnitude is equal to a certain threshold \( \omega = \omega_{tr} \), whose level is obtained from topological considerations as described in reference [9] and above. The conditional statistics are then computed as function of the distance \( y_I \) to the TNTI layer using one of three possible orientations to the interface (see Fig. 3.2): \( i) \) 'vertical' to the TNTI i.e. parallel to the \( y \)-axis (1D), normal to the TNTI projected into the \( (x,y) \) plane (2D), or normal to the TNTI (3D). In the resulting conditional mean profile the TNTI is by definition located at \( y_I = 0 \), while the irrotational and turbulent regions are defined by \( y_I < 0 \) and \( y_I > 0 \), respectively, where \( y_I \) is normalised by the Kolmogorov micro-scale in the turbulent region \( \eta = \eta(y_I \gg 0) \) [36].

3.2 Wavelet Leaders

Although several methods for multifractal analysis were presented in the previous section, the results present were obtained using the Wavelet Leaders method (presented in 2.6.2). The algorithmic form of the method will be described here.

All the algorithms were developed using Matlab.

3.2.1 Discrete Wavelet Transform

Wavelet leaders make use of the discrete wavelet coefficients and a detailed introduction to the mathematical basis of this transform is present in appendix A.

In this work only one dimension analysis were performed.
Figure 3.2: Local coordinate systems used to compute the conditional mean profiles, in relation to the distance from the turbulent/non-turbulent interface (TNTI), separating the turbulent (T) and non-turbulent (NT) or irrotational flow regions and comprising two (sub)layers: the viscous superlayer (VSL) with thickness $\delta_v$ and the turbulent sublayer (TSL) with thickness $\delta_\omega$: 1D (along the $y$ direction); 2D (normal to the interface in the $(x,y)$ plane); 3D (normal to the local interface - not shown).

**Cascade Algorithm**

To perform the discrete wavelet transform (DWT), the standard method is the cascade algorithm [[39]. This method uses the mathematical properties of multiresolution analysis (see appendix A).

In a simple way the algorithm works in the following way: given the set of data to analyse with $N = 2^k$ number of points and the wavelet and scaling function filters (see appendix A)) - which, depending on the wavelet chosen, have a number of points $n$ - a convolution with dyadic decimation (basically we are performing the convolution in two in two points of the data independently of the number of the number of points in the filters) is performed with the wavelet and scaling function filters over the data. The convolution with the scaling function gives the *approximation coefficients* and with the wavelet gives the *detail coefficients*. In the next step, the same procedure (convolution with this two filters) is performed over the approximation coefficients of the previous level. This is performed until there is only coefficient left. This is illustrated in the following figure (for one level only). In figure 3.3 LoD and HiD are the scaling function and wavelet coefficients, respectively. When this two filters are applied to the decimated (this is the *down sampling*) signal $s$ (in the figure) by a convolution then one gets the $cA1$ and the $cD1$ (approximation and detail coefficients, respectively). The approximation coefficient are the input in the next level. A particular example of the previous procedure for a set with 4 points with the Haar wavelet check appendix A).

This is a rather standard algorithm present in many textbooks (for example reference [39] and [40]) and can be found already programmed in Matlab, but it had to be programmed for this particular work.
because of a main problem with the tails of the data: since we have to treat data with a number of points that is a power of 2 ($N_{\text{data}} = 2^k$), if one want to make de DWT with a wavelet different from the Haar Wavelet ($n = 2$), there isn’t enough data to make all the convolutions - a typical solution for this that is present in most algorithms is to copy the data in the beginning of the series to the end and make this procedure for every scale of the algorithm but, since we are making a regularity study of a function there will be large error by making this procedure.

To overcome this problem the data is mirrored and so there will be no problems with regularity statistics. This is a solution proposed by the present author. An illustration of the procedure is in figure 3.4.

With this procedure, the transform will be applied to $2^{k+1}$ data points.

For regularity study of functions, only the detail coefficients are of interest and so these can be presented in a matrix of $2^{(k+1)-1} \text{ per } (k + 1) - 1$. A typical example of this scale/space representation of a signal can be observed in figure 3.5.
3.2.2 Leaders and Partition Function Calculation

With the wavelet coefficients determined and organized in a space/scale form the calculation of the wavelet leaders is straightforward from their definition 2.16. The same is valid for the partition function $Z_L(j, q)$ (the same as equation 2.63 but with the Wavelet Leaders instead of the Wavelet coefficients).

3.2.3 Determination of $\tau_q$

This is by far the most sensible step of the method. The determination of $\tau_q$ will have influence in all the results and conclusion and it can be a very ambiguous process (one just has to read carefully texts on the subject to understand that there are lots of results that are clearly biased). In this work we will keep a consistent procedure for the determination of $\tau_q$.

After $Z_L(j, q)$ is calculated, the procedure consists in writing the slopes of $\log(Z_L(j, q))$ versus $\log(2^j)$. The scaling range where the slopes are determined is chosen by observation although for similar phenomena (in turbulence) they are kept the same for all the analysis (this is our form of consistency to have at least correct results in comparison). Recently several authors claim to have a method to chose an universal and optimum scaling range as in [41] but this methods where not applied because they are very recent (2014) and so there isn’t more literature about their use (and there wasn’t time to test algorithms that are not very developed and documented) and also because we want to keep the same parameters for all the data and with an automatic method this could not happen. Nevertheless it should be noticed that this methods are being developed and in the future this can be a viable solution.

With the scaling range chosen, a linear fit in the sense of least-squares is performed over the data. This slopes determine $\tau_q$. 

Figure 3.5: Space-scale representation of the wavelet coefficients of a irregular function (Generated in Matlab).
3.2.4 Determination of Multifractal Spectrum

The direct way to determine the multifractal spectrum from $\tau_q$ is by its definition (equation 2.19): it is possible to perform a Legendre transform with $\tau_q$ and determine $D(h)$.

There is a method proposed in [42] where a direct determination of $D(h)$ without the Legendre transform is used. We do not use this procedure because we obtained correct results for trial functions with a direct determination of $D(h)$.

In this work we perform a direct determination of the multifractal spectrum from its definition and by a method of our own. The two methods give the same results as will be shown below. There is no great advantage of one method relatively to the other because this is an operation with few data, but we can say our proposal is more efficient computationally.

Our proposal to determine $D(h)$ is the following: inspired in the cumulant analysis present in [25] one can write $\tau_q$ in the following way:

$$\tau_q = \sum_{b=1}^{\infty} c_p \frac{q^p}{b!} \tag{3.1}$$

The coefficients $c_p$ can be determined by a polynomial interpolation of the $\tau_q$ curve obtained. Then we have the relations:

$$h(q) = \frac{\partial \tau_q}{\partial q} \tag{3.2}$$

and

$$D\left(h(q)\right) = d + q \frac{\partial \tau_q}{\partial q} - \tau_q \tag{3.3}$$

By performing a polynomial interpolation (least-squares) until a certain degree (in this work we used polynomial up to the $10^{th}$ order) of $\tau_q$ one can determine $D(h)$ from equations [3.2] and [3.3]. This procedure was tested versus the direct Legendre transform and the results are equivalent, Therefore, throughout this work our newly developed method was used.
Chapter 4

Flow Topology near a Turbulent/Non-Turbulent Interface

In this Chapter we investigate the enstrophy build up and the invariants of the velocity gradient tensor across a Turbulent/non-Turbulent Interface with data from DNS of shear free turbulence from the interface (as described in 3).

4.1 Results

4.1.1 Conditional Profiles

Figure [4.1] illustrates the enstrophy build up mechanisms across the TNTI by plotting conditional mean profiles of vorticity magnitude $\omega$, enstrophy viscous diffusion $\nu \partial^2 (\omega_i \omega_i / 2) / \partial x_j \partial x_j$, and enstrophy production $\omega_i \omega_j s_{ij}$.

![Figure 4.1: Mean conditional profiles (as function of the distance from the TNTI) of vorticity magnitude $\omega$, enstrophy production ($\omega^2$ Prod.) and enstrophy viscous diffusion ($\omega^2$ Diff.).](image)
Several letters (A-G) are assigned to specific locations within the TNTI (table 4.1), where A denotes the start or the ‘outer edge’ of the TNTI i.e. the origin of the local reference frame \( y_I = 0 \). The viscous diffusion exhibits a characteristic shape with positive/negative maxima at \( y_I/\eta = 2.4 \) (B) and \( y_I/\eta = 5.5 \) (E) associated with gain/loss of enstrophy, respectively, as previously reported by several authors[9], and it is clear that this is the first mechanism driving the observed enstrophy rise inside the TNTI[36]. The diffusive transport switches signal between the two extrema crossing zero at \( y_I/\eta = 3.9 \) (D). On the other hand the enstrophy production starts to be important after \( y_I/\eta \approx 2 \) but by \( y_I/\eta \geq 3.1 \) (C) is the main responsible for the enstrophy amplification because at this point enstrophy production surpasses the viscous diffusion. The conditional mean enstrophy exhibits a sharp rise between \( 0 \leq y_I/\eta \leq 6.3 \) (A-F) until by \( y_I/\eta = 6.3 \) (F) the maximum enstrophy is attained. At \( y_I/\eta = 23.5 \) (G) the flow exhibits all the characteristics of fully developed turbulence, with no sign of the presence of the TNTI.

Therefore, in the present case the VSL, associated with the viscous diffusion of vorticity towards the irrotational flow region [36], extends from \( 0 \leq y_I/\eta \leq 3.1 \) (A to C), i.e. with a mean thickness (defined by the region in Fig. 4.1 where diffusion exceeds production), equal to \( \langle \delta_\nu \rangle \approx 3\eta \), while the TSL (associated with the rapid vorticity rise by vorticity production) lays between \( 3.1 \leq y_I/\eta \leq 6.3 \) (D to F), with an estimated mean thickness (region where production exceeds diffusion culminating in the maximum vorticity - figure 4.1) equal to \( \langle \delta_\omega \rangle \approx 3\eta \). Thus in the present flow both \( \langle \delta_\omega \rangle \sim \langle \delta_\nu \rangle \sim \eta \) in agreement with reference [36]: the mean thickness of the VSL \( \langle \delta_\nu \rangle \) was estimated by Corrsin and Kistler to be of the order of the Kolmogorov micro-scale with the following reasoning: since the physical process within this layer is the viscous diffusion of vorticity from the turbulent core into the irrotational region this process should be solely controlled by the amount of vorticity in the turbulent region \( \omega' \) and by the molecular viscosity \( \nu \). On dimensional grounds it follows that the characteristic length scale for this process, defined as the thickness of the VSL, is \( \delta_\nu = \delta_\nu (\nu, \omega') \), leading to \( \delta_\nu \sim (\nu/\omega')^{1/2} \sim (\nu^3/\varepsilon)^{1/4} \sim \eta \), where \( \varepsilon \) is the mean rate of viscous dissipation in the core of the turbulent region.

Symbols used to denote several points across the TNTI layer, which is divided into the viscous superlayer (VSL) and the turbulent sublayer (TSL)[36]. \( D\omega^2/Dt \) indicates the dominating mechanisms for vorticity generation: viscous diffusion (D) or vortex stretching (P). Point G is deep inside the T region where production and dissipation roughly balance.

<table>
<thead>
<tr>
<th>symbol</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_I/\eta )</td>
<td>0.0</td>
<td>2.4</td>
<td>3.1</td>
<td>3.9</td>
<td>5.5</td>
<td>6.3</td>
<td>23.5</td>
</tr>
<tr>
<td>( D\omega^2/Dt )</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>-</td>
</tr>
<tr>
<td>(sub)layer</td>
<td>VSL</td>
<td>VSL</td>
<td>VSL</td>
<td>TSL</td>
<td>TSL</td>
<td>TSL</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 4.1: Symbols used to denote several points across the TNTI layer, which is divided into the viscous superlayer (VSL) and the turbulent sublayer (TSL). \( D\omega^2/Dt \) indicates the dominating mechanisms for vorticity generation: viscous diffusion (D) or vortex stretching (P). Point G is deep inside the T region where production and dissipation roughly balance.
4.1.2 The Teardrop Formation

The development of the 'teardrop' shape across the TNTI can be appreciated by plotting the joint pdfs of $R$ and $Q$ at several fixed distances from the TNTI layer. Such evolution describes how an initially NT fluid element becomes T. Fig. 4.2 shows the pdfs in the VSL (at $y_I/\eta = 0, 2.4$ and $3.1$ - A-B-C) while Fig. 4.3 shows the pdfs in the TSL (at $y_I/\eta = 3.9, 5.5$, and $6.3$ - D-E-F).

Inside the VSL is observed essentially the formation of the 'teardrop' shape in the 4th quadrant, associated with a predominance of sheet structures. Inside the VSL, $\omega_i \approx 0$, yields $Q \approx -s_{ij}s_{ij}/2 < 0$ and $R \approx -\alpha_S\beta_S\gamma_S$. Since the VSL is at the edge of the eddies from the T region[36], strain dominates over enstrophy, and $R > 0$ (sheet structures) is more frequent than $R < 0$ (tube structures). In contrast in the TSL the formation of the 'teardrop' shape occurs in the 2nd quadrant with the enstrophy now overcoming strain, thus $Q \approx \omega_i\omega_j/2 > 0$ and $R \approx -\omega_i\omega_j s_{ij}/4$. The predominance of the non-linear over the viscous mechanisms in the flow implies that the only way the enstrophy can grow is when $\omega_i$ and $s_{ij}$ are correlated so that $\omega_i\omega_j s_{ij} > 0$, i.e. more frequent events of $R < 0$ (vortex stretching) than $R > 0$ (vortex compression). The increasing intensity of $R$ and $Q$ as one moves from A to F, naturally reflects the increasing intensity of the fluctuating fields of enstrophy and strain. The results are in agreement with the conditional profiles displayed in Fig. 4.1. It is noteworthy that in contrast with the jet (where the...
teardrop forms in $\langle \delta \omega \rangle \sim \lambda$, here, where the thickness of the TSL is $\langle \delta \omega \rangle \sim \eta$, the 'teardrop' shape' forms in a much shorter distance, requiring only $\approx 5.5 \eta$ to form completely. Indeed, the joint pdf of $(R, Q)$ at $E$ is virtually identical to $F$ and also $G$ (not shown).

### 4.1.3 Mean Trajectories

A very interesting perspective of the formation of the teardrop is provided by the trajectory of the (conditional average of the) invariants of the velocity gradient tensor in the $(R, Q)$ map, obtained from the conditional mean profiles using the three different orientations (1D, 2D and 3D), presented in Fig. 4.4.

![Figure 4.4: Trajectory of the mean values of $R$ and $Q$ across the TNTI. $D_A = 27/4R^2 + Q^3$ is the discriminant of the eigenvalues of $A_{ij}$.](image)

Only in the 2nd quadrant do we observe some differences in the statistics which shows the robustness of the conditional statistics. The NT region and the VSL are represented only in the 4th quadrant, while the TSL develops in the 3rd quadrant, with some points appearing in the 2nd quadrant. The T region is mainly in the 2nd but also in the 1st quadrants. There is an increasing tendency for generating a sheet topology as the fluid particles enter the VSL ($A$ to $C$), followed by a sharply increased predominance of vortex stretching and formation of tube structures as the flow evolves inside the TSL ($C$ to $F$), in agreement with the model for the VSL proposed in reference [36]. Interestingly, the trajectory connecting the VSL and TSL regions ($B$ to $E$) consists in a straight line with a (constant) slope which is the same for all the three orientations used in the conditional statistics (1D, 2D and 3D).
4.2 The Burguer’s Vortex Model

The Burguer’s Vortex is an analytical solution of the Navier-Stokes equations where the vortex tube is intensified by straining so that the radial diffusion of enstrophy is exactly balanced by the enstrophy production by axial stretching, thus conserving the vortex radius constant. This idea can be represented by an axissimetric strain field \( U_z = z\gamma \) and \( U_r = -\frac{r\gamma}{2} \) which is divergence free, where \( \gamma \) is the rate of strain, as sketched in figure 4.5.

![Figure 4.5: Sketch of the Burger vortex created by a balance between vortex stretching in the axial direction and radial viscous diffusion.](image)

From the general expression for the vorticity in polar coordinates, it is possible to conclude that there is only vorticity in the \( z \) direction. From the transport equation of this quantity, it is possible to obtain

\[
\omega_z = \omega_0 e^{-\frac{r^2}{R^2}} \quad \text{and} \quad U_\theta = \frac{R^2 \omega_0}{2r} \left( 1 - e^{-\frac{r^2}{R^2}} \right),
\]

where \( \omega_0 \) is the vorticity at \( r = 0 \) and \( R = 2\sqrt{\frac{\gamma}{\nu}} \) is the constant Burger’s vortex radius.

4.2.1 Analytical Radial Profiles of Enstrophy Production and Diffusion for a Burger Vortex

The enstrophy \((\omega^2/2 = \omega_i\omega_i/2)\) evolves according to equation 4.1 (in a non-frame dependent form).

\[
\frac{D(\omega^2/2)}{Dt} = \omega_i S_{ij} \omega_j + \nu \nabla^2 (\omega^2/2) - \nu \nabla \omega_i \cdot \omega_i
\]  

(4.1)

The first and second terms on the LHS are the enstrophy production and diffusion, respectively. For a steady Burger vortex:

\[
Prod = \gamma \omega_0^2 e^{-\frac{r^2}{R^2}}
\]  

(4.2)

\[
Diff = -\gamma \omega_0^2 e^{-\frac{r^2}{R^2}} \left( R^2 - 2r^2 \right) / R^2
\]  

(4.3)
It can be observed that, at $r = R$, the production is equal to the diffusion. This can also be concluded due to the definition of $R$: it is the point where viscous diffusion os vorticity is in equilibrium with vorticity generation by stretching.

### 4.2.2 Invariants of the Velocity Gradient Tensor

The invariants of the velocity gradient tensor, $A_{ij}$, characterize the local topology of the flow through the instantaneous paths of fluid particles (or streamlines). In polar coordinates, the tensor $\frac{\partial u_i}{\partial x_j}$ has the form:

$$A_{ij} = \begin{pmatrix}
\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} & \frac{\partial u_r}{\partial \theta} & \frac{\partial u_r}{\partial z} \\
\frac{\partial u_\theta}{\partial r} & \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r} \right) & \frac{\partial u_\theta}{\partial z} \\
\frac{\partial u_z}{\partial r} & \frac{\partial u_z}{\partial \theta} & \frac{1}{r} \left( \frac{\partial u_z}{\partial z} \right)
\end{pmatrix}$$

The three invariants of this tensor are:

$$P = A_{ii},$$
$$Q = \frac{1}{2} P^2 - \frac{1}{2} A_{ij} A_{ji},$$
$$R = \det(A_{ij})$$

respectively. The first invariant ($P$) is zero, due to incompressibility. The others can be calculated using the Burger vortex analytical profiles yielding,

$$Q = \frac{1}{4} \frac{R^2}{r^4} \left( 2R^2 \omega_0^2 e^{\frac{2}{7} r^2 (r^2 + R^2)} - e^{\frac{2}{7} r^2 (3r^4 + R^4)} \right)$$

$$R = \frac{1}{4} \frac{R^2}{r^4} \left( -2R^2 \omega_0^2 e^{\frac{2}{7} r^2 (r^2 + R^2)} + R^2 \omega_0^2 (2r^2 + R^2) + e^{\frac{2}{7} r^2 (R^4 - r^4)} \right)$$

In general, $A_{ij}$ can be written as the sum of a symmetric and a skew-symmetric matrix; This yields $A_{ij} = S_{ij} + \Omega_{ij}$, with $S_{ij} = \frac{1}{2} (A^T + A)$ and $\Omega_{ij} = \frac{1}{2} (A^T - A)$. The invariants of $S_{ij}$ can also be calculated in the same fashion which gives:

$$R_S = \frac{1}{4} \frac{R^2}{r^4} \left( e^{\frac{2}{7} r^2 (r^2 + R^2)} (r^2 + R^2) + \omega_0 (r^2 + R^2) \right)$$

$$Q_S = \frac{1}{4} \frac{R^2}{r^4} \left( \omega_0^2 (r^2 + R^2) + 2R^2 \omega_0^2 e^{\frac{2}{7} r^2 (r^2 + R^2)} - \frac{R^2}{r^4} \omega_0^2 (3r^4 + R^4) \right)$$

For this particular flow, the tensor $A_{ij}$ can be written in the following form:
\[ A_{ij} = \begin{pmatrix} -\frac{\gamma}{2} & f_1(\omega_0, R, \gamma) & 0 \\ f_2(\omega_0, R, \gamma) & -\frac{\gamma}{2} & 0 \\ 0 & 0 & \gamma \end{pmatrix} \]

where the invariants can be written as:

\[ Q = -f_1 f_2 - \frac{3\gamma^2}{4} \]
\[ R = f_1 f_2 \gamma - \frac{\gamma^3}{4} \]

Solving for \( Q \) and \( R \) one can gets:

\[ Q = -\frac{R}{\gamma} - \gamma^2 \]

For the tensor \( S_{ij} \), a similar result can be obtained:

\[ Q_s = -\frac{R_s}{\gamma} - \gamma^2 \]

The analytical stemming from the Burger vortex have been rediscovered numerous times. This also happens here since the previous relations for the Burger vortex were obtained independently in this work but were discussed before in reference [43].

### 4.2.3 Relation to Flow at Interface

It is possible to explain the result presented in figure 4.4 by recalling that the small scale eddies ('worms') determine the TNTI characteristics. Recently, using the Burgers vortex which is a good model for the small scale eddies in turbulent flows, it has was that the periphery and core of the eddies existing near the TNTI determines the thickness of both the VSL and TSL [9],[36]. Recalling that a typical small scale eddy from inside the turbulent region has a vortex core radius equal to \( R_{ivs}/\eta \approx 5 \) [44], the strain rate imposed on this eddy can be estimated as \( \gamma_{ivs} = 4\nu/R_{ivs}^2 = 5.9 \). On the other hand, by measuring the slope of the straight line in the \((R,Q)\) trajectory from B to E in Fig. 4.4 we arrive at a stretching rate of \( \gamma = 5.5 \), which agree remarkably with the value obtained assuming the flow in the VSL and TSL is described by a Burgers vortex. Thus, the \((R,Q)\) trajectory shows the imprint of the small scale eddies existing near the TNTI, and unifies the existing models for the VSL and TSL within a TNTI. The obtained results also provides an explanation for the small differences obtained with the three orientations in the 2nd quadrant. The 1D orientation will less likely align with the radial direction of the eddies forming the TNTI than the 3D orientation, thus explaining why the straight line linking the VSL and TSL is longer for the 3D than for the 1D orientation. Both the 'tear-drop' formation and mean trajectory of \((R,Q)\) are expected to hold in other flows e.g. fully developed wakes and jets, since the VSL forms at the edge of similar small scale eddies in these flows[36]. However, the Burgers vortex model is probably unable to explain the detailed 'tear-drop' shape (Fig. 4.3 c) because it does not include non-linear effects e.g. extreme dissipation events \((Q \ll 0)\) tend to occur when two eddies come closer [43].
The invariants of the velocity gradient tensor show the specific topology of the flow in the VSL and TSL within the TNTI and can be described by a simple model unifying the previous models for each separate (sub)layer. With this model knowledge of the eddy characteristics near the TNTI e.g. eddy radius $R_{\text{eddy}}$ allows the determination of the background strain $\gamma_{\text{eddy}}$ which permits estimating the (mean) values of the invariants across the VSL and TSL, therefore opening the door to the development of new models for the turbulent entrainment mechanism based on the small scale eddies, associated with the ‘nibbling’ mechanism [45], and to mixing models for free shear flows, which are often based on the dissipation rate, which is simply $\varepsilon \approx -4\nu'Q'$ in the VSL. The TNTI provides also an interesting example where the coherent features of the flow are directly linked to the geometrical and dynamical features of the flow.
Chapter 5

Multifractal Analysis of Turbulence

In this chapter the results for the multifractal analysis of several test functions and turbulence quantities are presented. The method used here is the wavelet leaders and the parameters are determined as explained in Chapter 3.

5.1 Trial Functions

The validation and assessment of the robustness of the algorithm implemented is an important task of the present work. We will use the theoretical functions presented in Chapter 2 to extensively test to the algorithms developed. Because it is difficult to present here all the studies regarding the influence of the number of points and analysing wavelet those results will only be presented when the method is used to analyse the multiplicative process of the dissipation in turbulent flow.

5.1.1 Brownian Motion

The Brownian Motion was simulated using the function \textit{wfbm} available in Matlab. For the simulation of this random process one chooses the holder exponent $h$ and the desired number of points.

The multifractal spectrum from the Brownian Motion can be predicted from heuristic arguments; from the definition of multifractal spectrum, since the holder exponent is constant in all the domain the fractal dimension where the holder exponent is $h$ is just 1 and so $D(h)$ reduces to a single point ($D(h = H) = 1$).

It should be clear that for any mono fractal function the scaling exponent should be a linear function of $q$ ($\tau_q = qH$). Another point of interest is that, by adapting the result presented in [31] the fractal dimension of the graph of the function is just $d_f = 2 - \tau(1)$.

Setting $H = 0.5$ we have the classical Brownian Motion. A single realization of this function can be seen in figure 5.1(a). It should be noticed that, in contrast with the usual procedure (for example [46]) of considering several realizations of the movement, we only use one realization noticing that with the present method a single realization exhibits a good statistical convergence. The analysing wavelet is \textit{db3}. 

54
Brownian Motion (H=0.5)

(a) Single realization of BM with $h = 0.5$ ($N = 2^{16}$).

(b) Leaders partition function.

(c) Scaling exponent $\tau_q$.

(d) Multifractal spectrum $D(h)$.

Figure 5.1: Multifractal analysis of Brownian Motion for $H = 0.5$. The superscripts $\text{teo}$ and $\text{obt}$ relates to the theoretical and obtained quantities, respectively and the analysing wavelet is $db3$.

Analysing the results one observes a linear behaviour (in log scale) of the partition function with respect to the scale (figure 5.1(b)); the scaling range where the slopes are calculated, for all the Brownian Motion considered, is between $j = 3$ and $j = 10$. Figure 5.1(c) shows a clear linear dependency of $\tau_q$ in $q$, so $D(h)$ should be a point. However one can observe in figure 5.1(d) that there is some oscillation of $D(h)$ around the theoretical value - this is caused by some small deviations from pure linearity.

With $H = 0.4$, we are now considering a fractional Brownian Motion. The fractal dimension of this function is higher than for the classic Brownian Motion ($d_f = 1.6$). The results are present in figure 5.2.
Brownian Motion (H=0.4)

(a) Single realization of BM with \( h = 0.4 \) \((N = 2^{16})\).

(b) Leaders partition function.

(c) Scaling exponent \( \tau_q \).

(d) Multifractal spectrum \( D(h) \).

Figure 5.2: Multifractal analysis of Brownian Motion with \( h = 0.4 \). The superscripts \( \text{teo} \) and \( \text{obt} \) relates to the theoretical and obtained quantities, respectively and the analysing wavelet is \( \text{db}3 \).

It can be observed that there are some deviation from the theoretical value, however the error is small. This problem arises because, since the motion is more irregular (the fractal dimension is higher), we need more points to reach statistical convergence. One way to go around this problem is to consider more realizations of similar Brownian Motions and average their partition functions.

The last motion is considered with a higher \( H \) \((H = 0.7)\), and so a lower fractal dimension. Using the same argument as for \( h = 0.4 \) we can conclude that, because the motion more regular we do not require many point to reach statistical convergence and thereby the results exhibit a very good agreement with the theoretical predictions as illustrated in figure 5.3.
Brownian Motion (H=0.7)

(a) Single realization of BM with $h = 0.7 \ (N = 2^{16})$.

(b) Leaders partition function.

(c) Scaling exponent $\tau_q$.

(d) Multifractal spectrum $D(h)$.

Figure 5.3: Multifractal analysis of Brownian Motion with $h = 0.7$. The superscripts $\text{teo}$ and $\text{obt}$ relates to the theoretical and obtained quantities, respectively and the analysing wavelet is $db3$.

5.1.2 Weierstrass Function

The Weierstrass Function was implemented with the parameters $b = 2$ and $h = 2 - d_f = 0.4$ (see Chapter 2). The results obtained using our algorithm are presented in the following figures.
Figure 5.4: Multifractal analysis of the Weierstrass function. The superscripts theo and obt relates to the theoretical and obtained quantities, respectively and the analysing wavelet is db3.

In figure 5.4(b) the scaling of the partition functions with the scale are represented for several values of $p$. The plots show a linear scaling behaviour (as expected) although, for the first two scales, one can observe an anomalous behaviour caused by the influence that the small scales have on the leaders. This effect was previously reported in [25] and as consequence the range of scales to take the slope is $j = 4$ and $j = 10$.

The scaling exponents in figure 5.4(c) show a clear linear behaviour and thus the multifractal spectrum, figure 5.4(d), should be reduced to the point $D(h = 0.4) = 1$. The somewhat strange behaviour of $D(h)$ is due to the small deviations from the linear behaviour in the scaling exponents. For mono fractal functions is preferable to take the average slope and consider this value in the multifractal spectrum.

5.1.3 Deterministic Binomial Cascade

We now turn our attention to testing functions exhibiting a multifractal character. The scaling exponents and multifractal spectrum deduced for the binomial cascade in Chapter 2 resemble the picture given by equation 2.22. To make possible the use of Wavelet Leaders the analogy introduced in equation 2.23 has to be used. The analysis below uses the integrated version of the binomial cascade given by equation 2.24 and is shown in figure 5.5.
Figure 5.5: Integrated version (in the sense of equation 2.24) of the Binomial Cascade.

When deducing the scaling exponents, the number of measures at a given scale was not considered and so the scaling exponent with this detail becomes (this is a straightforward calculation and easy to understand):

$$\tau_q = 1 - \log_2(m_0^q + m_1^q)$$ (5.1)

The binomial cascade was programmed in this work and the results of it's analysis are presented in figure 5.6.

Figure 5.6: Multifractal analysis of the binomial cascade. The superscripts \textit{teo} and \textit{obt} relates to the theoretical and obtained quantities, respectively and the analysing wavelet is \textit{db1}.
The results exhibit a remarkable agreement with the theoretical predictions. Additionally, it was observed also that these results do not change with the wavelet used although the Haar wavelet is privileged to analyse this function since it mimics it’s own construction.

5.1.4 Random Binomial Cascade

For the random binomial cascade the same procedure used for the deterministic case (in the sense of considering a integrated version of it) is applied. This function was also programmed in this work and the results of it’s analysis can be seen in figure 5.7.

![Random Binomial Cascade](image)

(a) Random Binomial Cascade with \( m_0 = 0.25(N = 2^{16}) \).

![Leaders partition function](image)

(b) Leaders partition function.

![Scaling exponent \( \tau_q \)](image)

(c) Scaling exponent \( \tau_q \).

![Multifractal spectrum \( D(h) \)](image)

(d) Multifractal spectrum \( D(h) \).

Figure 5.7: Multifractal analysis of the random binomial cascade. The superscripts \( \text{teo} \) and \( \text{obt} \) relates to the theoretical and obtained quantities, respectively and the analysing wavelet is \( \text{db}1 \).

Once more, is observed a very good agreement between the theoretical predictions and the results obtained attesting the robustness and reliability of the implemented method even with a random cascade.

5.1.5 CMC-LN

The Canonical Mandelbrot Cascade with log normal multipliers was also implemented in this work. The results of the multifractal analysis of this function are depicted in figure 5.8.
Figure 5.8: Multifractal analysis of the CMC-LN. The superscripts \textit{teo} and \textit{obt} relates to the theoretical and obtained quantities, respectively and the analysing wavelet is \textit{db1}.

As in the previous cases, the results display a remarkable agreement between the theoretical predictions and the results obtained. This gives one more example of robustness with a random multiplicative process.

5.2 Fractional Integration

In the previous analysis of pre-programmed cascades the analogy with equation 2.24 was done to shown the validity of our reasoning. If one does not make such analogy the multifractal spectrum theoretically should be shifted to the left by a factor of one (since it would not be a integrated version of the process). The problem with this is that the Wavelet Leaders method cannot deal with $h < 0$ [25] and thus considering an integrated version of the cascade yields the correct result (where $h_{\text{min}} > 0$). An alternative procedure could be one where the cascade is not integrated and then make a shift to the right (by a factor of 1) in $D(h)$. However, because we cannot deal with negative holder exponents the results would be incorrect. A procedure to overcome this problem is by fractionally integrate the function to be analysed (if some negative exponents are expected) and then make the correction in $D(h)$ because of this procedure [25] - this is just like and integration but fractional integration allows to have integrations
of fractional order (1.5\textsuperscript{th} primitive of the function for example) and consequently fine tune the method without requiring an integration of high order.

The numerical procedure to do this is the following: after the wavelet coefficients \( d(j, k) \) are determined, for every scale we multiply them by \( 2^{\alpha j} \), proceeding in the same manner with the Leaders (now based on this new coefficients). Doing this one must expect that \( \tau_{p}^\alpha = \tau_{p} + \alpha p \) and in \( D(h) = D_{\alpha}(h + \alpha) \) so the multifractal spectrum becomes shifted to the right by \( \alpha \). For the remaining of this work this procedure is employed.

To illustrate the reliability and feasibility of this procedure works we will use the binomial cascade without the cumulative integration in the sense of equation 2.24 but with a fractional integration of order \( \alpha \).

The function under analysis now is the original binomial cascade (figure 5.6(a)).

![Figure 5.9: Scaling exponent obtained with the fractionally integrated version of the binominal cascade.](image)

![Figure 5.10: Multifractal spectrum obtained with the fractionally integrated version of the binomial cascade. Note that the shift to the right was subtracted and we sum \( d = 1 \) to have an analogy with the previous results.](image)

The results illustrates that this procedure can deal with the negative holder exponents; since after the appropriate shift, i.e, subtraction of \( \alpha \) in abscissae of \( D(h) \) and the sum of \( d = 1 \) (to match the previous results), we have excellent agreement with the theoretical predictions.
5.3 Dissipation in Fully-Developed Turbulence

In this section, dissipation signals from fully-developed turbulence in shear-free turbulence are analysed. The data is taken from DNS shear free turbulence described in Chapter 3. The data is taken from planes (shown in figure ??) and is converted to 1 dimensional signals as can be observed in figure 5.12.

![Figure 5.11: Location of the planes where the data is analysed.](image)

![Figure 5.12: Example of a dissipation signal in the turbulent zone from DNS simulation.](image)
A study on the influence of the parameters that define the method will be performed, including the number of points that assure statistical convergence of the results, the analysing wavelet influence and the scaling range. The parameter $\alpha$ in the $D(h)$ plots stands for the exponent of fractional integration (we make this shift to have the real spectrum) and $d$ stands for the euclidean dimension of the set of data - we are analysing the data in the sense of equation 2.13.

5.3.1 Statistical Convergence

Is important to define the standard parameters used throughout all the analysis in order to ensure some invariance across all the analysis and by so compare the value obtained without these type of bias.

Scaling Range

In general it was verified that the partition functions, for virtually all data analysed, behave in a way similar to the one illustrated in figure 5.13. Accordingly we choose the range of scales to take the slopes between $j = 4$ and $j = 8$. In this range one observes a linear evolution. We did not consider smaller values of $j$ because, as mentioned in [25], there is some influence of the small scales in the Leaders and from the previous trials on functions in scaling ranges of this type correct values were obtained.

![Figure 5.13: Behaviour of the partition function with scale. The analysing wavelet is the $db3$ and the data is taken in plane $0260$.](image)

Influence of the Number of Samples

A crucial aspect of this analysis is to determine whether there is enough data to reach statistical convergence. As noticed in [29], a common procedure is to average the partition function over several instants (or planes). Here we could have done the same thing, averaging to assess the distinct planes separately and then compare the obtained results in that way. Statistical convergence is verified while avoiding the
masking of statistical irregularities.

The influence of the number of points considered can be observed in figures 5.14 and 5.15. These values are taken in the scaling range defined above.

Figure 5.14: Behaviour of the scaling function with number of points used. The analysing wavelet is the \textit{db3} and the data is taken in plane 0260.

Figure 5.15: Behaviour of the multifractal spectrum with number of points used. The analysing wavelet is the \textit{db3} and the data is taken in plane 0260.

As the number of points employed increases both the scaling function and the multifractal spectrum changes slightly, however, for more than $2^{18}$ points both curves are virtually unchanged. It will be considered $2^{18}$ points for the subsequent analysis.
5.3.2 Wavelet Influence

Until now the results for the influence of number of points were obtained with a single analysing wavelet $db3$. Our choice can be justified by analysing the following results (see figures 5.16 and 5.17).

![Figure 5.16: Influence of the wavelet on the scaling function.](image)

![Figure 5.17: Influence of the wavelet on the multifractal spectrum.](image)

It can be seen that using either $db2$ and $db3$ there aren’t significant differences in the scaling function or the multifractal spectrum. According to [25] experiments showed that that one should use the lower wavelet possible with which the results do not change. Based on this reasoning our choice falls over the wavelet $db3$. 
5.3.3 Results for Fully-Developed Turbulence

With the settings defined previously (regarding number of points, wavelet and scaling range) the dissipation was analysed using DNS data from fully-developed turbulence at several planes of the simulation. In figure 5.18 is possible to observe that the scaling exponents are very close to each other - an indication that our procedure of considering the planes separately do not influence the results: because the partition function is additive, considering all the data at the same time one would obtain an average of the results and since they are similar the found result would remain unchanged.

The obtained multifractal spectrum is shown in figure 5.19. In the same figure several models proposed to describe the dissipation multiplicative process are represented for comparison: the model in reference [4] - binomial cascade with $m_0 = 0.3$ - is represented by a dash-dotted red line and the model in reference [29] - binomial cascade with $m_0 = 0.36$ and $m_1 = 0.78$ - is represented by a dashed blue line. In addition, a proposed model of our own to fit the obtained data is represented by a solid black line - binomial cascade with $m_0 = 0.4$ and $m_1 = 0.75$.

Figure 5.18: Scaling exponents obtained in different planes.
Figure 5.19: Multifractal spectrum obtained in different planes in comparison with several models proposed.

The first evident result is that for all the planes in the turbulent zone analysed, the multifractal spectrum data collapses. Another important aspect is that the cancellation exponent (see [29]) is different from zero. This is an indication that the main conclusion of reference [29] is observed: if the dissipation is to be modelled as a multiplicative cascade, this cascade is not conservative - \( (m_0 + m_1) \neq 1 \). Nevertheless it can be observed that the model proposed in [29] does not agree with our results at least for the negative exponents (right part of \( D(h) \)) - since the negative \( q \) values are associated with the weakest singularities and since in [29] the data is filtered we can conjecture that this is the reason for the difference in the negative exponents (and they need to filter the data in order to make their method work. We obtain this same behaviour with a method different from the WTMM that is used in [29]). Since there is also some ambiguity in the choice of the scaling range we can say that, only by biasing the results, one can obtain the same results but the ones obtained are in relatively good agreement.

Another important remark about non conservativeness of the cascade is that if one considers a conservative binomial cascade and fractionally integrates it by a factor of \( H^\ast \), say, \( H^\ast = \log_2(0.4 + 0.75) \approx 0.35 \) will have a multifractal spectrum equal to the one obtained with the non conservative model, however it will be shifted to the left by a factor of \( H^\ast = \log_2(0.4 + 0.75) \approx 0.2 \). This result can have a very simple interpretation: since the Reynolds number is finite the function will have some degree of regularity below a given scale (in this case the Kolmogorov micro-scale) and so it must be expected that, if the process is a pure multiplicative one (in the limit of infinite Reynolds) it must exhibit some kind of behaviour of this type. It is also noteworthy that this result cannot be achieved with box counting method [29].

The generalized dimensions for our data (according to the model which agrees better with the results) are given by

\[
D_q = \frac{-\log_2(0.4^q + 0.75^q)}{q - 1}
\]  

(5.2)
5.4 Dissipation near the TNT Interface

In reference [11] a possibility to calculate fluxes of quantities across the interface is proposed. However some corrections have to be done due to the multifractal nature of energy dissipation, even though although there was no proof if the dissipation behaves in such way in the interface. An example of a dissipation signal in the interface can be observed in figure 5.20.

![Dissipation signal at the TNTI]

Figure 5.20: Example of a dissipation signal at the interface from DNS simulation.

In order to employ the algorithms developed to study the multifractal spectrum of dissipation in the TNTI one must bear in mind that there in a inherent limitation: by taking dissipation data in the interface using conditional statistics the data points are not equally spaced and so the method is not exact. One way to overcome this problem is by generalization of the method for three dimensions. This was not done due to lack of time but we can argue that, in the points right on the interface, the error due to non uniformity of spacing between points is very small and therefore the multifractal analysis can be carried out.

5.4.1 Data Treatment

The data is collected using statistics conditioned to the interface position and the parameters are set as for the previous analysis carried out for fully-developed turbulence. In figure 5.21 the partition function is represented and it can be seen that the behaviour is very similar to the one observed before for the fully developed turbulent region.
In addition to the data collected at the TNTI, the multifractal analysis is performed at \( \pm 1.5 \eta \) from the interface outer edge.

### 5.4.2 Results

With the same parameters as for the analysis in the turbulent zone, in figure 5.22 is represented the scaling functions obtained; it is also shown for comparison a scaling function for one the planes in the turbulent zone.

![Scaling exponents obtained for different distances.](image)

**Figure 5.22:** Scaling exponents obtained for different distances.

It can be observed that the scaling functions in the interface are very similar to those in the turbulent zone, although some small, but not negligible, fluctuations are present for negative values of \( q \).

The respective multifractal spectra are presented in figure 5.23, where the same models as in 5.19 are
illustrated.

Figure 5.23: Multifractal spectrum obtained for different distances.

These results suggest that the multifractal spectrum in the interface is largely equal to the one observed in the turbulent zone and thus the same comments apply to this case.

Recalling equation 2.44 and establishing an analogy, since we determined the multifractal spectrum for one dimensional signals, we have

\[
d_f = \frac{7}{3} + \frac{2}{3} \left(1 - D_{1/3}\right)
\]

(5.3)

From equation 5.2 one can calculate that \(D_{1/3} \approx 1.0776\) and predict, with this results, \(d_f \approx 2.28\). This value is lower than the value predicted from [4] where \(d_f \approx 2.36\). We can state that if the cascade is non conservative, mathematically, the value of the fractal dimension will be lower (in comparison with the conservative case) because \(r_q^M(1) < 0\).

As shown in reference [27], experimentally it was obtained that the fractal dimension of the interface was \(d_f = 2.37\) consistent with a value slightly higher than \(7/3\) - as it should be in the case of a conservative cascade. With this results on the multifractal spectrum of the dissipation (together with the one found in [29]) the fractal dimension is lower than \(7/3\).
Chapter 6

Conclusions

Universality of the topology, geometry and dynamics of turbulence at the edges of jets, wakes, mixing layers and boundary layers is analysed by using direct numerical simulations (DNS) of shear free turbulence, where a sharp interface separates a turbulent from a non-turbulent region in the absence of mean shear. The large scale flow features of turbulence are affected by the initial and boundary conditions of the problem and therefore, change from flow to flow. On the other hand the small scales of turbulence are approximately isotropic and according to Kolmogrov’s self-similarity hypothesis, are universal and characteristic of all turbulent flows, provided the Reynolds number is high enough. Therefore, in the present work we assess the universality of the strongly inhomogeneous layer which exists at the edge of a shear free turbulent/non-turbulent interface, given that the small scale features of this interface will be representative of any free shear flow or a boundary layer.

First, the work addresses the invariants of the velocity gradient tensor and the detailed formation of the so called ‘tear-drop’ shape in the \((R, Q)\) invariants, which is present in virtually all known turbulent flows and which combines information on the geometry and vorticity dynamics at each point within the turbulent flow. The generation of the ‘tear-drop’ shape was analysed across the viscous superlayer (VSL) and the turbulent sublayer (TSL), and it was observed that while the VSL is responsible for the formation in the 4th quadrant, associated with a generation of sheet structures, the TSL is responsible for the formation of the ‘tear-drop’ in the 3rd quadrant associated with enstrophy production.

Remarkably, the evolution of this map shows the ‘finger print’ of the intense eddies from the turbulent region near the interface, and presents the first example where local geometric and dynamical information is linked with the flow coherent structures. Moreover, the observed evolution is well represented by an analytical model based on the Burgers vortex. The new model opens the door to a detailed explanation of the nibbling mechanism and to new models for the turbulent entrainment mechanism based on the small scale eddies and to the modelling of the velocity and scalar dynamics near the TNTI.

The multifractal characteristics of the TNTI were also studied. After comparing and testing several algorithms a new code for the computation of the multifractal spectrum was developed and tested based in the Wavelet Leaders. Fully-developed turbulence data from the DNS was assessed and it was observed that the regular region of the multifractal spectrum is well described by the model proposed in
[29], but not the singularity region. Finally, the multifractal spectrum of the dissipation at the TNTI is similar to deep inside the shear layer, which further supports the idea that the small scales are indeed universal even right at the TNTI. Finally, the results yield a fractal dimension of the interface of about \( d_f \approx 2.28 \) which is lower than the actual estimations.

### 6.1 Achievements

The major achievements of the present work can be summarized as follows:

- Development of a reliable method to determine the multifractal spectrum by programming a discrete wavelet transform with modifications to deal with tails of the data;
- Implementation and validation of the developed tools by means of analysis of multifractal characteristics of several trial functions;
- Application of the developed method to turbulence data which lead to the verification of the results present in [29] using a different method;
- Application of the developed method to the dissipation at the interface which lead to the conclusion that the multifractal spectrum is largely similar to the one found in fully-developed turbulence and to a prediction of the fractal dimension of the interface.
- Analysis of the teardrop formation in the VSL and TSL within the TNTI;
- Development of an analytical for \( Q \) and \( R \) model which unifies the existing theories for the turbulent/non turbulent interface;

### 6.2 Future Work

The analysis of flow with the invariants of the velocity gradient tensor should be extended for other types of flows (jets, boundary layers).

The method developed for multifractal analysis should be expanded to 3 dimension to analyse the multifractal spectrum of dissipation across the interface without any limitations and to determine it's fractal dimension. This will yield a complete study on the subject which can be done for several flows.

The multifractal analysis can be extended to dissipation in visco-elastic turbulence where the energy cascade is different to the classical one.
Bibliography


Appendix A

The Wavelet Transform

The wavelet transform is a fundamental part of this work so, with some detail, the theory of the transformation will be discussed.

A.1 Properties of Wavelets

The wavelet transform is a tool that cuts up data, functions or operators into different frequency components, and then studies each component with a resolution matched to its scale.

When one analyses a signal, the wavelet transform allows to study a time history of its frequency components - it performs in the same fashion as the Fourier transform but the latter looses all information related to the time location of a particular frequency. To achieve time-localization one can window the signal and take its Fourier transform; the problem with this procedure is the fact that the slice of the signal that is extracted is always the same length - the time(space) slice used to resolve the high and low frequencies have the same number of points.

In contrast to this procedure, the wavelet transform the width of its time(space) slice according to to the frequency components being extracted.

The general form of a wavelet transform is the following.

\[ T[f](a,b) = \int f(t) \psi \left( \frac{t-b}{a} \right) dt \]  

(A.1)

where the position of the slice of the signal that the wavelet samples in time(space) is controlled by \(b\) and the extent of the slice by the scaling parameter \(a\).

A.2 Continuous Wavelet Transform

The continuous wavelet transform follows directly from equation A.1. Since the core of this work uses the discrete wavelet transform, the continuous version will not be detailed here. More informations can be found in [40].
A.3 Discrete Wavelet Transform

Since the Wavelet Leaders method relies on the discrete wavelet transform this method will be introduced with some detail.

A.3.1 Multilevel Representation of a Function

The goal of multi-resolution analysis is to develop representation of a function \( f(x) \) at various levels of resolution. To this end is desirable to expand \( f(x) \) in terms of basis function \( \psi(x) \) which can be scaled to give multiple resolution of the original function.

The theory to achieve this expansion relies on the construction of a sequence of subspaces (of functions) \( V_j \) such that (in the following of this section it will be assumed that the function in study is in \( L^2(\mathbb{R}) \):

\[
\{0\} \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \subset L^2(\mathbb{R}) \quad (A.2)
\]

with the following properties:

- \( \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L^2(\mathbb{R}) \)
- \( \cap_{j \in \mathbb{Z}} V_j = \{0\} \)
- \( g(x) \in V_j \iff g(2x) \in V_{j+1} \)
- \( g(x) \in V_0 \iff g(x+1) \in V_0 \)

A sequence of function spaces with this properties is adequate to perform a multiresolution analysis of a function. Now one needs to find a function \( \phi \) (called scaling function) that translations and dilation of the latter are basis of the functions spaces \( V_j \) i.e:

\[
V_0 = \text{span}\{\phi(x-k)\} \quad (A.3)
\]
\[
V_1 = \text{span}\{\phi(2x-k)\} \quad (A.4)
\]
\[
V_j = \text{span}\{\phi(2^jx-k)\} \quad (A.5)
\]

One note can be made here: the fact that \( V_0 \subset V_1 \) (and so on) can be understood in a simple way; since the base functions of the subspace \( V_1 \) are more detailed versions of the base function of \( V_0 \), the span of \( V_1 \) is larger than \( V_0 \) and naturally include the latter.

Taking this in account is possible to write any function in \( V_0 \) (including its base functions) in terms of the base functions of \( V_1 \) (and so on):

\[
\phi(x) = \sum a_k \phi(2x-k) \quad (A.6)
\]

Equation [A.6] will be called dilation equation.
Example (The Box Function)

Now it’s pertinent to give a concrete example on how the previous conditions can define a way to represent a function (in $L^2(\mathbb{R})$).

For this end it will be considered the simplest basis function (the box function). This function is just unity between 0 and 1 and zero otherwise.

$$\phi(x) = 1, \quad 0 < x < 1$$  \hspace{1cm} (A.7)

It is evident that the set of functions $\{\phi(x-k), k \in \mathbb{Z}\}$ is orthonormal.

If one considers some function in $L^2(\mathbb{R})$ it will be always possible to make the projection of it in this set; the projection of $f$ in $V_0$ will be called $P_0f$:

$$P_0f = \sum c_{0,k} \phi(x-k)$$  \hspace{1cm} (A.8)

In general, for some space $j$ is always possible to approximate $f$ as:

$$P_jf = \sum c_{j,k} \phi(2^j x - k)$$  \hspace{1cm} (A.9)

If can be proved that $P_jf$ approaches $f$ as $j \to \infty$.

It is also evident that the box function verify the dilation equation with coefficients $a_0 = a_1 = 1$ (this should not be very difficult for the reader to imagine),

$$\phi(x) = \phi(2x) + \phi(2x-1)$$  \hspace{1cm} (A.10)

From this type of construction is also easy to observe that

$$V_0 \subset V_1 \subset V_2 \ldots$$  \hspace{1cm} (A.11)

With this simple example was shown that the box function can be used to develop a sequence of spaces (embedded hierarchically) with each space being spanned by translation of the box function at that scale.

Wavelets

To develop further the idea of the previous section one can look at how the subspaces $V_{j-1}$ and $V_j$ are related. To this end one can consider a new subspace, defined as $W_{j-1}$, such that it is the orthogonal complement of $V_{j-1}$ in $V_j$; this means that the subspace $W_{j-1}$ contains all the elements in $V_j$ that are orthogonal to $V_{j-1}$ (recall from the previous section that $V_{j-1} \subset V_j$ so this idea is valid). With this definition one can say that the space $V_j$ is the direct sum of the spaces $V_{j-1}$ and $W_{j-1}$, i.e.
\[ V_j = V_{j-1} \bigoplus W_{j-1} \]  
(A.12)

In the same fashion as done for the spaces \( V_j \), it will be defined a function \( \psi \) such that combinations of \( \psi(x - k) \) form a basis of \( W_0 \). Then

\[ \psi_{j,k} = \psi(2^j x - k) \]  
(A.13)

is a basis of \( W_j \). Being \( Q_j f \) the projection of \( f \) in \( W_j \), from equation [A.12] we have

\[ P_j f = P_{j-1} f + Q_{j-1} f \]  
(A.14)

This can be interpreted in the following way: \( Q_j f \) represents the detail needed to be added in to get from one level of approximation to the next finer level of approximation.

Since the space \( W_0 \) is contained in the space \( V_1 \) one can express the wavelet function in terms of the scaling function at the next higher scale.

\[ \psi(x) = \sum_k b_k \phi(2x - k) \]  
(A.15)

Now, if the previous example is considered (the box function) is easy to see that the coefficients are \( b_0 = 1 \) and \( b_1 = -1 \):

\[ \psi(x) = \phi(2x) - \phi(2x - 1) \]  
(A.16)

This is just the Haar wavelet.

**Construction of Wavelet Systems**

In the previous section the example of the box function with the Haar wavelet was given. The construction of other scaling function and wavelets will be addressed here.

In general, a scaling function is the solution to a *dilation* equation of the form.

\[ \phi(x) = \sum_k a_k \phi(2x - k) \]  
(A.17)

The constant coefficients \( a_k \) are called *filter coefficients*; these are derived by imposing conditions on the scaling function (next section).

Since the wavelet is orthogonal to the scaling function it can be defined as:

\[ \psi(x) = \sum_k (-1)^k a_{N-1-k} \phi(2x - k) \]  
(A.18)

where \( N \) is an even integer. It should be easy for the reader to note that with this definition of the wavelet, it is orthogonal with the scaling function.

The sets of coefficients \( \{a_k\} \) and \( \{-1^k a_{N-1-k}\} \) are said to form a pair of *quadrature mirror filters*.
Filter Coefficients

Previously, conditions on the scaling functions was mentioned. Now the conditions needed are developed and its reflection in the scaling coefficients will be determined.

- To uniquely define all scaling functions of a given shape, the area under the scaling function is normalized to unity:
  \[ \int \phi(x)dx = 1 \]  
  (A.19)

  In the coefficients this condition imposes that (to derive this just notice the construction of the scaling function at one level as a linear combination of scaling functions at the above level and make a substitution of variable):
  \[ \sum_k a_k = 2 \]  
  (A.20)

- The scaling function and translation of it should form an orthonormal set:
  \[ \int \phi(x)\phi(x + l) = \delta_{0,l} \]  
  (A.21)

  In the same fashion as done in the first condition, this leads to:
  \[ \sum_k a_k a_{k+2l} = 2\delta_{0,l} \]  
  (A.22)

- Equations [A.20] and [A.22] yield a total of \( \frac{N^2}{2} + 1 \) conditions (in an \( N \) coefficient system). Therefore \( \frac{N^2}{2} + 1 \) more relations are needed. One way to obtain more equation is by requiring the scaling function to be able to exact represent polynomials of order up to (but not greater) than, for example, \( p \). This requirement is equivalent to say that any function of the form:
  \[ f(x) = \alpha_0 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \cdots + \alpha_{p-1} \cdot x^{p-1} \]  
  (A.23)

  can be exactly represented by an expansion of the form:
  \[ f(x) = \sum_k c_k \phi(x - k) \]  
  (A.24)

  This condition can be translated into a condition on the wavelet. Taking the inner product of equation [A.24] with \( \psi \) gives:
  \[ < f(x), \psi(x) > = \sum_k c_k < \phi(x - k), \psi(x) > = 0 \]  
  (A.25)

  Thus, from equation [A.23],
  \[ \alpha_0 \int \psi(x)dx + \alpha_1 \int \psi(x)x dx + \cdots + \alpha_{p-1} \int \psi(x)x^{p-1}dx = 0 \]  
  (A.26)
for this relation to be valid one has the condition that

\[ \int \psi(x) x^l dx = 0 \]  
\[ \text{(A.27)} \]

for every \( l = 0, 1, 2, \cdots, p - 1 \). This means that the first \( p \) moments of the wavelet must be zero.

If one substitutes equation [A.18] in equation [A.27] and simplifies the expression one can get:

\[ \sum_{k} (-1)^k a_k k^l = 0 \]  
\[ \text{(A.28)} \]

This last equation yields \( p - 1 \) new equations and the problem of determining the coefficients is solved.

**Scaling Functions**

With the coefficients determined now the problem is to determine the actual scaling function from the filter coefficients.

In general scaling functions do not have a closed form solution; they have to be generated recursively from the *dilation* equation. This aspect will not be discussed here with more detail because in the author’s opinion this is not fundamental for the understanding of the algorithm.

**The Daubechies 4 Coefficient Wavelet System**

The preceding discussion now will be applied to a concrete example - the Daubechies 4 coefficient (D4) scaling function and wavelet. the conditions described can be written as the following system of equations:

\[ a_0 + a_1 + a_2 + a_3 = 2 \]  
\[ \text{(A.29)} \]

\[ a_0^2 + a_1^2 + a_2^2 + a_3^2 = 2 \]  
\[ \text{(A.30)} \]

\[ a_0 - a_1 + a_2 - a_3 = 0 \]  
\[ \text{(A.31)} \]

\[ -a_1 + 2a_2 - 3a_3 = 0 \]  
\[ \text{(A.32)} \]

This system has the solution:

\[ a_0 = \frac{1 + \sqrt{3}}{4} \]  
\[ \text{(A.33)} \]

\[ a_1 = \frac{3 + \sqrt{3}}{4} \]  
\[ \text{(A.34)} \]

\[ a_2 = \frac{3 - \sqrt{3}}{4} \]  
\[ \text{(A.35)} \]

\[ a_3 = \frac{1 - \sqrt{3}}{4} \]  
\[ \text{(A.36)} \]
A.3.2 Multiresolution Analysis Using the Mallat Transform

The Mallat transform provides a simple means of transforming data from one level of resolution \( j \) to the next coarser level of resolution \( j - 1 \).

Multiresolution Decomposition

Multiresolution decomposition takes the expansion coefficients of the approximation, \( P_j f \), to a function at a scale \( j \) and decomposes them into:

- the expansion coefficients, \( c_{j-1,k} \), of the approximation, \( P_{j-1} f \), at the next coarser scale, \( j - 1 \);
- the expansion coefficients, \( d_{j-1,k} \), of the detail component, \( Q_{j-1} f = P_j f - P_{j-1} f \), at the next coarser scale.

Consider a function \( f \). Let \( P_j f \) denote the projection of \( f \) onto the subspace \( V_j \) and \( Q_j f \) denote the projection of \( f \) onto the subspace \( W_j \) (as defined previously). Thus:

\[
P_j f = \sum_k c_{j,k} \phi_{j,k}(x)
\]

(A.37)

with \( c_{j,k} = \langle f, \phi_{j,k} \rangle \), and

\[
Q_j f = \sum_k d_{j,k} \psi_{j,k}
\]

(A.38)

with \( d_{j,k} = \langle f, \psi_{j,k} \rangle \).

Since \( W_{j-1} \) is the orthogonal complement of \( V_{j-1} \) in \( V_j \),

\[
P_{j-1} f = P_j f - Q_{j-1} f
\]

(A.39)

Substituting this in

\[
c_{j-1,k} = \langle P_{j-1} f, \phi_{j-1,k} \rangle
\]

(A.40)

leads to the following result:

\[
c_{j-1,k} = \sum_m c_{j,m} a_{m-2k}
\]

(A.41)

In the same fashion is possible to show that the following relation also holds.

\[
d_{j-1,k} = \sum_m c_{j,m} (-1)^m a_{N-1-m+2k}
\]

(A.42)

This hierarchical relations are the base for the construction of the mallat algorithm for the wavelet transform.
Example

A simple example of a practical implementation of the discrete wavelet transform will be given. Consider the vector $\vec{v} = (2, 6, 4, 1)$ with $N = 2^2 = 4$.

If one considers the Haar wavelet, the scaling coefficients are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and the wavelet coefficients are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

In the first step a projection of $\vec{v}$ with the scaling function is performed. This yields a new vector $\vec{v}_1^c = \left(\left(\frac{1}{\sqrt{2}} \cdot 2 + \frac{1}{\sqrt{2}} \cdot 6\right), \left(\frac{1}{\sqrt{2}} \cdot 4 + \frac{1}{\sqrt{2}} \cdot 1\right)\right) = \frac{1}{\sqrt{2}} (8, 5)$. The projection of $\vec{v}$ with the wavelet function yields a vector $\vec{v}_1^d = \left(\left(\frac{1}{\sqrt{2}} \cdot 2 - \frac{1}{\sqrt{2}} \cdot 6\right), \left(\frac{1}{\sqrt{2}} \cdot 4 - \frac{1}{\sqrt{2}} \cdot 1\right)\right) = \frac{1}{\sqrt{2}} (-4, 3)$.

In the next step, the same procedure is done with the coarse coefficients vector $\vec{v}_1^c$. This gives $\vec{v}_2^c = \frac{1}{2} \cdot (8 + 5) = \frac{1}{2} \cdot (13)$. In the same fashion with the wavelet function $\vec{v}_2^d = \frac{1}{2} \cdot (8 - 5) = \frac{1}{2} \cdot (3)$.

In a space-scale form the wavelet coefficients can be written as a matrix.

$$\begin{bmatrix}
2 & 6 & 4 & 1 \\
-4 \cdot \frac{1}{\sqrt{2}} & -4 \cdot \frac{1}{\sqrt{2}} & 3 \cdot \frac{1}{\sqrt{2}} & 3 \cdot \frac{1}{\sqrt{2}} \\
3 \cdot \frac{1}{2} & 3 \cdot \frac{1}{2} & 3 \cdot \frac{1}{2} & 3 \cdot \frac{1}{2}
\end{bmatrix}$$  (A.43)