Tensor products of modules

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Abstract

Support notes for the MMAC course "Modules and Representations" of IST in the academic year 2024/2025.

Throughout these notes all rings are unital. Given a ring R, an expression like ${}_RM$ always denotes a left R-module, and M_R denotes a right R-module.

§1. Definition. Let R be a ring, and M_R and ${}_RN$ modules. Given an abelian group A, a map

$$\varphi : M \times N \to A$$

is *bi-additive* if for all $m, m' \in M$ and $n, n' \in N$ we have

$$\varphi(m+m', n) = \varphi(m, n)+\varphi(m', n) \quad \text{and} \quad \varphi(m, n+n') = \varphi(m, n)+\varphi(m, n'),$$

and *middle R-linear* if for all $m \in M$, $n \in N$ and $r \in R$ we have

$$\varphi(mr, n) = \varphi(m, rn).$$

§2. Definition. Let R be a ring, and M_R and ${}_RN$ modules. Letting $F_{\mathbb{Z}}(M \times N)$ be the free $\mathbb{Z}$-module generated by $M \times N$, where the latter is regarded as just a set, consider its subgroup K generated by the subset $A \subset F_{\mathbb{Z}}(M \times N)$ that contains the differences

$$\delta_{(m+m',n)} - \delta_{(m,n)} - \delta_{(m',n)}, \quad \delta_{(m,n+n')} - \delta_{(m,n)} - \delta_{(m,n')}, \quad \delta_{(mr,n)} - \delta_{(m,rn)}$$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. The quotient group $F_{\mathbb{Z}}(M \times N)/K$ is called the *tensor product* of M and N, and it is denoted by $M \otimes_R N$. The elements of this module are called *tensors*, and each class $\delta_{(m,n)} + K$ is called a *simple tensor* and is denoted by $m \otimes n$.

§3. Proposition. *The following properties of simple tensors are immediate from the above definition:*

$$(m+m')\otimes n = m\otimes n+m'\otimes n, \quad m\otimes(n+n') = m\otimes n+m\otimes n', \quad mr\otimes n = m\otimes rn,$$

where in the latter equation we omit brackets in $(mr) \otimes n$ and $m \otimes (rn)$.

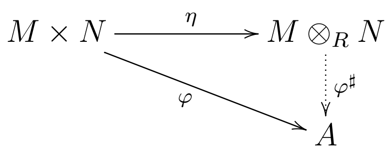
§4. Theorem. *Let R be a ring, and M_R and ${}_R N$ modules. The assignment $(m, n) \mapsto m \otimes n$ defines a bi-additive and middle R-linear map*

$$\eta : M \times N \to M \otimes_R N.$$

Moreover, for all abelian groups A and all bi-additive and middle R-linear maps $\varphi : M \times N \to A$ there is a unique homomorphism of abelian groups

$$\varphi^\sharp : M \otimes_R N \to A$$

such that the triangle commutes:



Proof. Exercise (direct application of the universal property of group quotients). ∎

§5. Note. $M \otimes_R N$ is generated by the simple tensors. In order to define a homomorphism to an abelian group A

$$\varphi : M \otimes_R N \to A$$

it suffices to define it on the simple tensors and verify that the following conditions are respected:

$$\begin{aligned}
\varphi((m + m') \otimes n) &= \varphi(m \otimes n) + \varphi(m' \otimes n) \\
\varphi(m \otimes (n + n')) &= \varphi(m \otimes n) + \varphi(m \otimes n') \\
\varphi(mr \otimes n) &= \varphi(m \otimes rn).
\end{aligned}$$

§6. Definition. Let R and S be rings. By an (R, S)-*bimodule*, or R-S-*bimodule*, is meant an abelian group M equipped with both a structure of left R-module and right S-module, satisfying the following additional associativity law for all $r \in R$, $s \in S$ and $m \in M$:

$$(rm)s = r(ms).$$

The abbreviated notation ${}_R M_S$ will always indicate that M is an (R, S)-bimodule. By a homomorphism of (R, S)-bimodules $\varphi : M \to N$ is meant a homomorphism of groups that is both a homomorphism of R-modules and a homomorphism of S-modules.

§7. THEOREM. *Let R, S, and T be rings, and consider bimodules ${}_RM_S$ and ${}_SN_T$. Then $M \otimes_S N$ is an (R,T)-bimodule whose module structures are defined on the simple tensors, for all $r \in R$ and $t \in T$, by*

$$r(m \otimes n) = (rm \otimes n) \quad \textit{and} \quad (m \otimes n)t = m \otimes nt.$$

Proof. This was done in the lectures. The main point is to apply the previous note (considering the actions $r(-)$ and $(-)t$ as group endomorphisms on $M \otimes N$) in order to show that the above actions are well defined. For the R-action this means proving the following conditions:

$$\begin{aligned} r((m+m') \otimes n) &= r(m \otimes n) + r(m' \otimes n) \\ r(m \otimes (n+n')) &= r(m \otimes n) + r(m \otimes n') \\ r(ms \otimes n) &= r(m \otimes sn). \end{aligned}$$

Only the third condition requires the bimodule associativity:

$$r(ms \otimes n) = r(ms) \otimes n = (rm)s \otimes n = rm \otimes sn = r(m \otimes sn).$$

Exercise: verify the other two conditions, and prove that the pair of actions by R and T indeed make $M \otimes_S N$ an (R,T)-bimodule. ∎

§8. DEFINITION. Let R, S and T be rings, and ${}_RM_S$, ${}_SN_T$, ${}_RP_T$ bimodules. A map $\varphi : M \times N \to P$ is (R,T)-*bilinear* if it is bi-additive and for all $m \in M$, $n \in N$, $r \in R$ and $t \in T$ we have

$$\varphi(rm, n) = r\varphi(m, n) \quad \text{and} \quad \varphi(m, nt) = \varphi(m, n)t.$$

§9. EXERCISES

1. Show that any bi-additive map is middle $\mathbb{Z}$-linear.
2. Prove that $\mathbb{Z}$ cannot be made a $\mathbb{Q}$-module whose action extends that of $\mathbb{Z}$ on itself.
3. Prove that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong 0$.
4. Prove that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.
5. Let R, S and T be rings, and ${}_RM_S$, ${}_SN_T$, ${}_RP_T$ bimodules. Prove that for any (R,T)-bilinear map $\varphi : M \times N \to P$ which is also middle S-linear there is a unique homomorphism of (R,T)-bimodules $\varphi^\sharp : M \otimes_S N \to P$ such that for all simple tensors we have $\varphi^\sharp(m \otimes n) = \varphi(m, n)$.

6. For a bimodule ${}_R M_S$, prove that $R \otimes_R M$ and $M \otimes_S S$ are both isomorphic to M as (R, S)-bimodules.

7. Let M and N be (R, R)-bimodules. Prove that $M \otimes_R N$ and $N \otimes_R M$ are isomorphic as (R, R)-bimodules.

8. Let R, S, T and U be rings, and ${}_R M_S$, ${}_S N_T$, ${}_T P_U$ and ${}_R Q_U$ bimodules. Prove that in order to define a homomorphism of (R, U)-bimodules

$$\varphi : (M \otimes_S N) \otimes_T P \to Q$$

it suffices to define it on simple tensors of the form $(m \otimes n) \otimes p$. Show that the conditions that need to be satisfied for φ to be well defined are: (i) additivity in each of the three variables m, n and p; (ii) two middle linearity conditions with respect to the actions of S and T; (iii) bilinearity with respect to the actions of R and U.

9. For the same modules as above, prove that $(M \otimes_S N) \otimes_T P$ and $M \otimes_S (N \otimes_T P)$ are isomorphic as (R, U)-bimodules.