Instituto Superior Técnico / Universidade de Lisboa

Departament of Bioengineering

Master on Biomedical Engineering

Signals and Systems in Bioengineering

1st Semester 2017/2018

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Test 2 / Exam 1

January 15, 2018

Name : Number:

The duration of the Test is 1h30m and the Exam is 3h. The score of each item is 2 when right and -0.5 if wrong in the Test and 1 when right and -0.25 if wrong in the Exam. Only one option can be selected in each question.

Test 1

1. What is the impulse response of the filter $H(z) = [1 + az^{-1}]^{-1}$?

- ☐ a) $a^n u(n)$
- ☐ b) $-a^n u(n)$
- ■ c) $(-a)^n u(n)$
- ☐ d) None

$H(z) = [1 + az^{-1}]^{-1} \Rightarrow Y(z)(1 + az^{-1}) = X(z) \Rightarrow y(n) = x(n) - ay(n-1)$
$\Rightarrow h(n) = \delta(n) - ah(n-1) \Rightarrow$
$h(0) = 1$
$h(1) = -a$
$h(2) = a^2$
...
$h(n) = (-a)^n h(n)$

2. Consider the following optimization problem,

$$\mathbf{c}^* = \arg\min_{\mathbf{c}} \|A\mathbf{c} - \mathbf{x}\|_1 \qquad (1)$$

where A is a $N \times L$ matrix, $\mathbf{x}$ is a N length column vector of observations and $\mathbf{c}$ is a L length column vector of coefficients to be estimated. $\|e\|_1 = \sum_{k=1}^{L} |e_k|$ is called $\mathcal{L}_1$ norm where $|.|$ denotes the absolute value function. This optimization problem can solved using the *Iteratve Reweighted Least Squares* (IRLS) algorithm as follows

$$\mathbf{c}^{t+1} = \arg\min_{\mathbf{c}} \mathbf{e}^T W(\mathbf{c}^t)\mathbf{e} \tag{2}$$

where $\mathbf{e} = A\mathbf{c} - \mathbf{x}$ and $W(\mathbf{c}^t) = diag(w_i)$ is a diagonal matrix with elements

- ☐ a) $w_i = 1$
- ☐ b) $w_i = |e_i^t|$
- ■ c) $w_i = 1/|e_i^t|$
- ☐ d) None of the above.

3. Consider the *Linear Time Invariant* (LTI) system described by the following transfer function

$$H(z) = \frac{1}{1 + (4/3)z^{-1} + (8/9)z^{-2}}, \tag{3}$$

where the roots of the denominator are $p_{1,2} = 2/3 \pm j2/3$. What type of filter is this system?

- ☐ a) High-pass filter .
- ■ b) Band-pass filter.
- ☐ c) Low-pass filter.
- ☐ d) None

4. Let $x(n) = [1, -1, 1, 0\ ]$. What is the value of the 4 dimension DFT coefficient $X_4(1)$?

- ☐ a) 0.
- ☐ b) 1.
- ■ c) j.
- ☐ d) None

5. Let $x(n)$ and $y(n)$ two discrete sequences of length 16 with *DFT* coefficients $X_{16}(k)$ and $Y_{16}(k)$ respectively, where $Y(k) = \begin{cases} X(k) & \text{for } k \text{ even} \\ -X(k) & \text{for } k \text{ odd} \end{cases}$.

What is the right option?

- ■ a) $y(2) = x(10)$.

- ☐ b) $y(1) = x(11)$.
- ☐ c) $y(0) = x(12)$.
- ☐ d) None

6. What is the period of the signal $\sin(\pi n/2) + 0.25\cos(\pi n/3)$?

- ☐ a) 6
- ■ b) 12
- ☐ c) 24
- ☐ d) None

7. Consider a space with inner-product, $< \mathbf{x}, \mathbf{y} >$ and induced norm $\|x\| = \sqrt{< \mathbf{x}, \mathbf{x} >}$. Which condition is true?

- ☐ a) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\|$.
- ■ b) $| < \mathbf{x}, \mathbf{y} > | \leq \|\mathbf{x}\|.\|\mathbf{y}\|$
- ☐ c) $\|\mathbf{x} + \mathbf{y}\| \geq \|\mathbf{x}\| + \|\mathbf{y}\|$.
- ☐ d) None

8. The *Fast Fourier Transform* (FFT) optimizes the computation of the DFT by removing completely redundant computations. The core of the FFT algorithm, called butterfly, is a structure that computes a 2-length DFT vector of Fourier coefficients $DFT_2(\mathbf{x}) = \mathbf{X} = [X(0), X(1)]^T$ from 2 length sequences, $\mathbf{x} = [x(0), x(1)]^T$. Using matrix notation

$$\mathbf{X} = W\mathbf{x}$$

where W is one of the following 2×2 square matrices. What is that matrix?

- ☐ a) $W = [1, 0; 0, 1]$.
- ■ b) $W = [1, 1; 1, -1]$.
- ☐ c) $W = [1, 0; 1, 0]$.
- ☐ d) None

Problem (T=4/Ex=2)

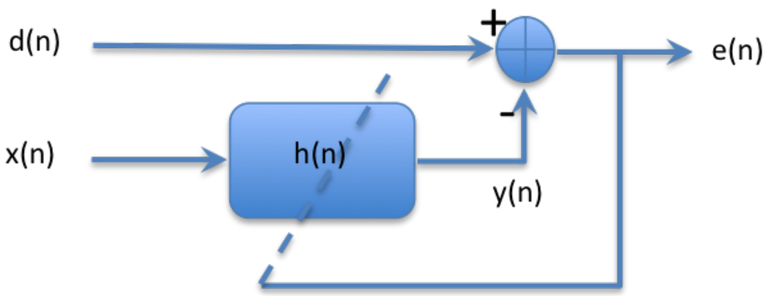

Figura 1: Adaptive filter.

Consider the FIR filter $h(n)$ in Fig.1,

$$y(n) = \sum_{k=0}^{p} h_k(n)x(n-k) \tag{4}$$

where $\mathbf{h}(n) = \{h_0(n), h_1(n), ...h_p(n)\}^T$ are the $p+1$ coefficients of the filter to be estimated at each n^{th} sample, according the following criterion

$$\mathbf{h}(n) = \arg\min_{\mathbf{h}} J(\mathbf{h}) \tag{5}$$

where

$$J(\mathbf{h}) = \sum_{i=0}^{N-1} (d(n-i) - y(n-i))^2 \tag{6}$$

with N is the length of a window.

(a) $\mathbf{y}(n) = [y(n), y(n-1), ..., y(n-N)]^T$ can be expressed as $\mathbf{y}(n) = A\mathbf{h}(n)$. **Define A.**

(b) Express $J(\mathbf{h})$ by using matrix notation.

(c) Derive a closed-form solution for the optimization problem described in (5).

(d) What is the optimal filter $h(n)$ if $p = 10$ and $y(n) = x(n-2)$?

Test 2

1. Consider the following procedure to change the sampling rate of a discrete signal $x(n)$ by a factor of $R = 2/3$,

$$y(n) = T_{\downarrow 3}[h(n) * T_{\uparrow 2}[x(n)]]$$

where $h(n)$ is an ideal low-pass filter with a cut-off frequency $\omega_c = \pi/3$. If $x(n) = sin(\pi n/3) + sin(5\pi/6)$ what is $y(n)$?

- ☐ a) $y(n) = sin(\pi n/2) + sin(15\pi/12)$
- ☐ b) $y(n) = sin(15\pi/12)$
- ■ c) $y(n) = sin(\pi n/2)$
- ☐ d) None

- After up-sampling the resulting signal is $x(n) = sin(\pi n/6) + sin(5\pi/12)$
- After filtered the resulting signal is $x(n) = sin(\pi n/6)$
- After down-sampled the resulting signal is $x(n) = sin(\pi n/2)$

2. Consider the discrete signal $x(n)$ where **two** (2) additional zero samples are introduced between the original ones, $[...x(i), 0, 0, x(i+1), ...]$. The ideal interpolation of these new samples can be implemented by filtering the augmented signal with an ideal low-pass filtering with a cut-off frequency ω_c. What should be that frequency?

- ☐ a) $\omega_c = \pi/2$
- ■ b) $\omega_c = \pi/3$
- ☐ c) $\omega_c = \pi/4$
- ☐ d) None

Two additional zeros for each sample correspond to an upsampling ratio $D = 3$. In this case, the ideal low-pass interpolation filter has a cut-off frequency of $\omega_c = \pi/D = \pi/3$.

3. Consider a first order unknown system $G(s) = A/(1 + \tau s)$ and its step, $x(t) = u(t)$, response displayed in Fig. 2. What is the time constant, τ, in seconds?

- ☐ a) 25.
- ☐ b) 20.
- ☐ c) 1/4.
- ■ d) None

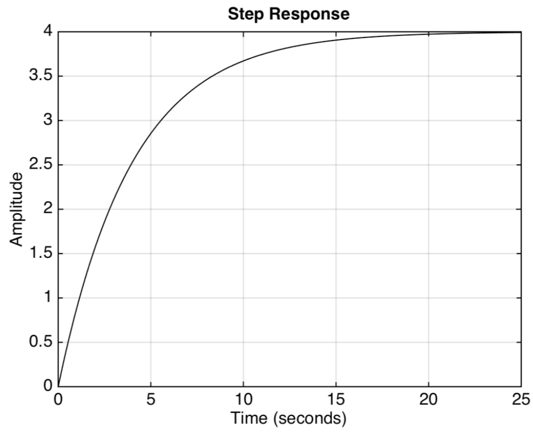

Figura 2: First order step response of an unknown system, $G(s) = A/(1+\tau s)$.

4. Consider two independent random variables, x and y, with variances σ_x^2 and σ_y^2 respectively. What is the variance of $z = xy$?

- ■ a) $\sigma_x^2 * \sigma_y^2$.
- □ b) σ_x^2/σ_y^2.
- □ c) $\sigma_x^2 + \sigma_y^2$.
- ■ d) None of the above

Using $E(x^2) = \sigma_x^2 + \mu_x^2$ and $\mu_z = \mu_x\mu_y$, because x and y are independent, then

$\sigma_z^2 = E((z-\mu_z)^2) = E((xy-\mu_x\mu_y)^2) = E(x^2y^2 - 2xy\mu_x\mu_y + \mu_x^2\mu_y^2) =$
$E(x^2)E(y^2) - \mu_x^2\mu_y^2 = (\sigma_x^2+\mu_x^2)(\sigma_y^2+\mu_y^2) - \mu_x^2\mu_y^2 = \sigma_x^2\sigma_y^2 + \sigma_x^2\mu_y^2 + \sigma_y^2\mu_x^2 + \mu_x^2\mu_y^2 - \mu_x^2\mu_y^2 \Rightarrow$

$\sigma_z^2 = \sigma_x^2\sigma_y^2 + \sigma_x^2\mu_y^2 + \sigma_y^2\mu_x^2$
if $\mu_x = \mu_y = 0 \Rightarrow \sigma_z^2 = \sigma_x^2\sigma_y^2$

5. Consider the system

$$H(z) = \frac{1}{1 - 0.25z^{-1}}. \tag{7}$$

If the input is a zero mean ($\mu_x = 0$) white noise with variance $\sigma_x^2 = 2$, $x \sim N(0,2)$. What is the *power spectral density* (PSD) of the output?

- ■ a) $\frac{2}{1.0625 - 0.5cos(\omega)}$.
- □ b) $\frac{\sqrt{2}}{1.0625 - 0.5cos(\omega)}$.

- ☐ c) $\frac{2}{1.25-0.25cos(\omega)}$.
- ☐ d) None of the above

6. Consider a LTI system with impulse response $h(n) = \delta(n) - \frac{1}{3}\delta(n-1)$ and an input signal with autocorrelation $\phi_{xx}(m) = \delta(m) + 4$. What is the mean of the output signal?

   - ☐ a) 4/9.
   - ☐ b) 2/3.
   - ■ c) 4/3.
   - ☐ d) None of the above

7. A closed-loop real system with complex conjugated poles is always

   - ☐ a) unstable.
   - ☐ b) stable.
   - ■ c) overshot
   - ☐ d) None of the above

8. Consider the discrete stochastic process $x(n) = sin(2\pi n/N) + \eta(n)$ where $\eta(n) \sim \mathcal{N}(0,1)$ is white Gaussian noise. This process is

   - ☐ a) Stationary.
   - ☐ b) Ergodic.
   - ☐ c) White.
   - ■ d) None

   - It is not stationary because it is time varying.
   - It is not Ergodic because it is not stationary
   - It is not white because
     $E[x(n)x(n+m)] = sin(2\pi n/N)sin(2\pi(n+m)/N) + E[sin(2\pi sin(2\pi n/N)sin(2\pi(n+m)/N) + \psi_\eta(m) =$
     $sin(2\pi n/N)sin(2\pi(n+m)/N) + \delta(m) \neq \delta(m)$

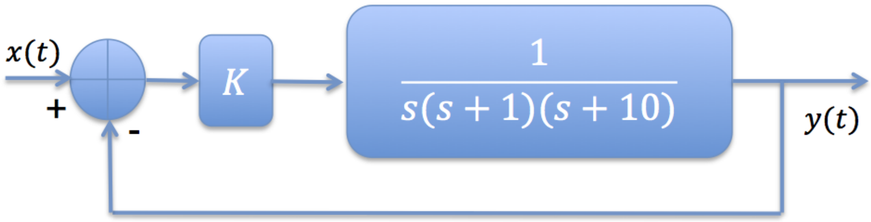

Figura 3: Unit feedback control system.

Problem (T=4/Ex=2)
Consider the feedback system represented in Fig.3.

1. **(2)** Draw the root-locus (RL) for $K > 0$ and $K < 0$. Compute explicitly the **break in** and **break out** points at the real axis, **asymptotic center and angle** of the asymptotes and the **sections of the real axis** of RL.

   (a) break in and break out points: Characteristic equation: $1 + \frac{k}{s(s+1)(s+10)} = 0 \Rightarrow k = -s(s+1)(s+10) \Rightarrow$
   $\frac{dk}{ds} = 3s^2 + 22s + 10 = 0 \Rightarrow s_{1,2} = -0,4869; -6,8464$

   (b) Imaginary axis crossing point: $\theta_1 + \theta_2 + \theta_3 = (2r+1)\pi \Rightarrow$
   $\pi/2 + \tan^{-1}(\omega) + \tan^{-1}(\omega/10) = \pi \Rightarrow$
   $tan\left(\tan^{-1}(\omega) + \tan^{-1}(\omega/10)\right) = tan\left(\pi/2\right) = \frac{\omega+\omega/10}{1-\omega^2/10} = \infty \Rightarrow$
   $1 - \omega^2/10 = 0 \Rightarrow \omega = \pm\sqrt{10}$

   (c) $\sigma_{ass} = \left(\sum_k p_k - \sum_r z_r\right)/(N-M) = -11/3 = -3.33$

   (d) $\phi(k > 0) = \{\pm\pi/3; \pi\}$

   (e) $\phi(k < 0) = \{0; \pm 2\pi/3;\}$

2. **(0,5)** Compute the values of K for which the closed-loop poles are equal (double-poles) and indicate their positions in the RL.

   The double-poles occur at the break-in and break-out points of the real axis: $k = -s(s+1)(s+10)|_{s=[-0,4869;-6,8464]} = [2,38; -126,23]$

3. **(0,5)** What is the minimum value of K for which the system is stable without overshooting?
   The system is stable without overshoot for $k \in [0; 2,38]$, that correspond to the part of the RL where the closed loop poles are real and negative (stable).

4. **(1)** For which value of K the system becomes critically stable? Indicate over the RL the location of the poles for this value of K. What is the frequency of the oscillations?

See item 1b.
For this position: $s = j\omega = j\sqrt{10} \Rightarrow$
$k(j\sqrt{10}) = -s(s+1)(s+10)|_{s=j\omega=j\sqrt{10}} = 110$

The frequency of oscillations is $\omega = \sqrt{10} rad/sec$.