Instituto Superior Técnico / Universidade de Lisboa

Departament of Bioengineering

Master on Biomedical Engineering

Signals and Systems in Bioengineering

1st Semester 2016/2017

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Test 2 / Exam 1

January 7, 2017

Name : Number:

The duration of the Test is 1h30m and the Exam is 3h. The score of each item is 2 when right and -0.5 if wrong in the Test and 1 when right and -0.25 if wrong in the Exam. Only one option can be selected in each question.

Test 1

1. Consider a space with inner-product, $< \mathbf{x}, \mathbf{y} >$ and induced norm $\|x\| = \sqrt{< \mathbf{x}, \mathbf{x} >}$. Which condition is true?

- ☐ a) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\|$.
- ■ b) $| < \mathbf{x}, \mathbf{y} > | \leq \|\mathbf{x}\|.\|\mathbf{y}\|$
- ☐ c) $\|\mathbf{x}+\mathbf{y}\| \geq \|\mathbf{x}\| + \|\mathbf{y}\|$.
- ☐ d) None

2. Let $d(\mathbf{x}, \mathbf{y})$ be a metric function in a space S. The function $g(\mathbf{x}, \mathbf{y}) = \log(1 + d(\mathbf{x}, \mathbf{y}))$

- ☐ a) is not a metric function because it could be negative.
- ☐ b) is not a metric function because $\exists_{\mathbf{x},\mathbf{y}\in\mathbf{S}} : g(\mathbf{x}, \mathbf{y}) \neq g(\mathbf{y}, \mathbf{x})$
- ■ c) is a metric function.
- ☐ d) None

3. Consider the canonical adaptive filter displayed in Fig. 1 where $x(n) = -d(n+1)$ and $h(n)$ is a 4 length FIR filter. In these conditions what is the optimal impulse response of the FIR that minimizes the norm of the error, $\|\mathbf{e}\|$?

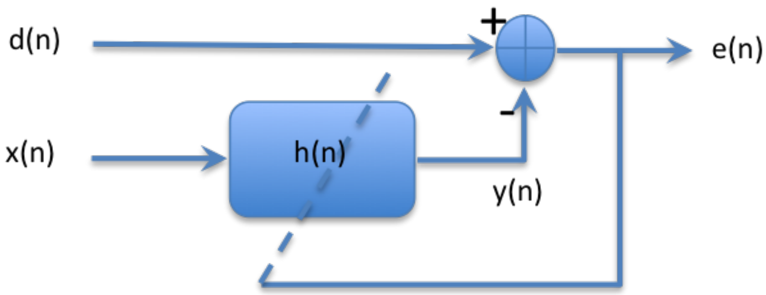

Figura 1: Adaptive filter.

- ☐ a) $h(n) = [1, 0, 0, 0]$.
- ☐ b) $h(n) = [-1, 0, 0, 0]$.
- ■ c) $h(n) = [0, -1, 0, 0]$.
- ☐ d) None

Take into account that $f(n-n_0) = f(n)*\delta(n-n_0)$ and assuming that $h(n) = -\delta(n-n_0)$ then $y(n) = x(n) * h(n) = x(n) * [-\delta(n-n_0)] = -x(n-n_0) = -[-d(m+1)]|_{m=n-n_0} = d(n-n_0+1) \Rightarrow y(n) = d(n)$ if $n_0 = 1 \Rightarrow h(n) = -\delta(n-1)$.

4. Consider the canonical adaptive filter displayed in Fig. 1 where $d(n) = g(n)*x(n)$, $x(n)$ is white Gaussian noise, $x(n) \sim \mathcal{N}(0, \sigma^2)$ and $h(n)$ and $g(n)$ are P length FIR filters (where $g(n)$ is known). In these conditions what is the optimal impulse response of the FIR filter $h(n)$ that minimizes the norm of the error, $\|\mathbf{e}\|$?

   - ■ a) $h(n) = g(n)$.
   - ☐ b) $h(n) = g(-n)$.
   - ☐ c) $h(n) = g^*(n)$.
   - ☐ d) None

   where * denotes conjugation.

   If $h(n) = g(n) \Rightarrow y(n) = g(n)*x(n) \Rightarrow e(n) = d(n) - y(n) = g(n)*x(n) - g(n)*x(n) = 0$

5. What is the period of the signal $x(n) = exp(2\pi(j-1)n/10)$?

   - ☐ a) $10/(j-1)$ samples
   - ☐ b) $10/\sqrt{2}$ sample
   - ☐ c) 10 sample
   - ■ d) None

The signal $x(n) = exp(2\pi(j-1)n/10) = exp(j2\pi n/10)exp(-\pi n/5)$ is not periodic because the second term, $exp(-\pi n/5)$, is a real decaying exponential.

6. The *Fast Fourier Transform* (FFT) optimizes the computation of the DFT by removing completely redundant computations. The core of the FFT algorithm, called butterfly, is a structure that computes a 2-length DFT vector of Fourier coefficients $DFT_2(\mathbf{x}) = \mathbf{X} = [X(0), X(1)]^T$ from 2 length sequences, $\mathbf{x} = [x(0), x(1)]^T$. Using matrix notation

$$\mathbf{X} = W\mathbf{x}$$

where W is one of the following 2×2 square matrices. What is that matrix?

- ☐ a) $W = [1, 0; 0, 1]$.
- ■ b) $W = [1, 1; 1, -1]$.
- ☐ c) $W = [1, 0; 1, 0]$.
- ☐ d) None

$X(k) = \sum_{k=0}^{N-1} x(n)e^{-\frac{2\pi}{N}kn}$.
For $N = 2 \Rightarrow$
$X(0) = x(0)e^{-pi*0*0} + x(1)e^{-pi*0*1} = x(0) + x(1)$
$X(1) = x(0)e^{-pi*1*0} + x(1)e^{-pi*1*1} = x(0) - x(1) \Rightarrow$
$\mathbf{X} = W\mathbf{x}$ where $\mathbf{X} = [X(0), X(1)]^T, \mathbf{x} = [x(0), x(1)]^T$ and $W = [1, 1; 1 - 1]$

7. Consider the chirp signal $x(t) = sin(2\pi f(t)t)$ with $0 \leq t \leq 1$ seconds and $f(t) = 100 + 900t^2$ Hz. What is the maximum frequency of the spectrum of $x(t)$?

- ☐ a) 2900 Hz
- ☐ b) 1800 Hz
- ☐ c) 1000 Hz
- ■ d) None

$x(t) = sin(\phi(t))$ where $\phi(t) = 2\pi f(t)t$
$\omega(t) = d\phi(t)/dt = 2\pi f_{inst}(t) = 2\pi[f(t) + t\dot{f}(t)]$ where $\dot{f}(t) = 1800t \Rightarrow$
$f_{inst}(t) = 100 + 900t^2 + 1800t^2 = 100 + 2700t^2$, a monotonic increasing function with time, $t \Rightarrow f_{max} = f_{inst}(1) = 100 + 2700 = 2800$ Hz

8. Consider the analog signal $x(t) = sin(2\pi f_1 t) + sin(2\pi f_2 t)$ with $f_1 = 1000$ Hz and $f_2 = 1010$ Hz. What are the appropriated sampling rate, f_s, and FFT length, N, that should be used to discriminate both peaks in the spectrum of the discrete (sampled) version of $x(t)$, $x_d(n) = x(nT_s)$?

- ☐ a) $f_s = 1000$ Hz, $N = 100$.
- ■ b) $f_s = 2020$ Hz, $N = 256$.
- ☐ c) $f_s = 4000$ Hz, $N = 256$.
- ☐ d) None

$f_s \geq 2f_{max} = 2 * 1010Hz = 2020Hz$ and $\Delta f = f_s/N \Rightarrow$
$N_{FFT} \geq f_s/\Delta f$ where $\Delta f = f_2 - f_1 = 10 \Rightarrow$
$N_{FFT} = 2^n \geq f_s/\Delta f = 2020/10 = 202 \Rightarrow N_{FFT} = 256$

Problem (T=4/Ex=2)

1. Derive the expression of $y(n)$ as function of $x(n)$.

   Consider two N length signals $x(n)$ and $y(n) = DFT_N^{-1}[DFT_M[x(n)]]$ where $M < N$ (for sake of simplicity assume that $N/2 < M < N$).

   Let us consider the following signal, that is result of using a M-length DFT smaller than the dimension of the signals, N,
   $\tilde{y}(n) = DFT_M^{-1}[DFT_M[x(n)]] = \begin{cases} x(n) + x(n+M), & 0 \leq n \leq N-M-1 \\ x(n) & N-M \leq n \leq M-1 \end{cases}$
   If in the the previous expression we replace the first M by N,
   $y(n) = DFT_N^{-1}[DFT_M[x(n)]] = \begin{cases} x(n) + x(n+M), & 0 \leq n \leq N-M-1 \\ x(n) & N-M \leq n \leq M-1 \\ 0 & M \leq n \leq N-1 \end{cases}$
   where the difference for the previous case is just a set of padding zeros at the end of $\tilde{y}(n)$

2. Apply the previous result for the signal $x(n) = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]$ for $M = 6$.
   $y(n) = DFT_N^{-1}[DFT_M[x(n)]] = \begin{cases} x(n) + x(n+6), & 0 \leq n \leq 3 \\ x(n) & 4 \leq n \leq 5 \\ 0 & 6 \leq n \leq 9 \end{cases}$
   $= [6, 8, 10, 12, 4, 5, 0, 0, 0, 0]$

Test 2

1. Consider the following procedure to change the sampling rate of a discrete signal $x(n)$ by a factor of $R = 2/3$,

$$y(n) = T_{\downarrow 3}[h(n) * T_{\uparrow 2}[x(n)]]$$

where $h(n)$ is an ideal low-pass filter with a cut-off frequency $\omega_c = \pi/3$. If $x(n) = sin(\pi n/3) + sin(5\pi/6)$ what is $y(n)$?

- ☐ a) $y(n) = sin(\pi n/2) + sin(15\pi/12)$
- ☐ b) $y(n) = sin(15\pi/12)$
- ■ c) $y(n) = sin(\pi n/2)$
- ☐ d) None

- After up-sampling the resulting signal is $x(n) = sin(\pi n/6) + sin(5\pi/12)$
- After filtered the resulting signal is $x(n) = sin(\pi n/6)$
- After down-sampled the resulting signal is $x(n) = sin(\pi n/2)$

2. Consider the filter

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

with an input signal $x(n) = \eta(n) + 1$ where $\eta \sim \mathcal{N}(1, 1)$ is a unit mean, $\mu = 1$, and unit variance, $\sigma^2 = 1$, Gaussian random variable. What is the mean of the output signal, $\mu_y =< y(n) >$?

- ■ a) 4.
- ☐ b) 2.
- ☐ c) 0.
- ☐ d) None

$< y(n) >= H(z)|_{z=1} < x(n) >= H(1) < \eta(n) + 1 >= 2[< \eta(n) > +1] = 2[1 + 1] = 4$

3. Consider

$$H(z) = \frac{1 - 0.25z^{-1}}{1 - 0.5z^{-1}}.$$

What type of filter $G(z) = 1/H(z)$?

- ☐ a) Low pass.
- ■ b) High pass.
- ☐ c) Band pass.
- ☐ d) None

$H(z)$ is a low-pass filter (the pole, located at $p = 0.5$, is closer to 1 than the zero, located at $z = 0.25$). The pole of $G(z)$ is the zero of $H(z)$ and the same for the zero of $G(z)$. Thus, $G(z)$ is a high-pass filter.

4. Let x and y two independent zero mean, $\mu = 0$, and unit variance, $\sigma^2 = 1$ random variables. What is the variance of the variable $z = x - y$

- ■ a) 2.
- ☐ b) 1.
- ☐ c) 0.
- ☐ d) None

$\sigma_z^2 = E[(z - \mu_z)^2] = E[(x - y - \mu_x + \mu_y)^2] = E[((x - \mu_x) - (y - \mu_y))^2] =$
$E[(x - \mu_x)^2 + (y - \mu_y)^2 - 2(x - \mu_x)(y - \mu_y)] =$
$E[(x - \mu_x)^2] + E[(y - \mu_y)^2] - 2E[(x - \mu_x)(y - \mu_y)] =$
$\sigma_x^2 + \sigma_y^2 - 2\int\int(x - \mu_x)(y - \mu_y)p(x, y)dxdy$
$\sigma_x^2 + \sigma_y^2 - 2\int\int(x - \mu_x)(y - \mu_y)p(x)p(y)dxdy$
$\sigma_x^2 + \sigma_y^2 - 2\int(x - \mu_x)dx\int(y - \mu_y)(y)dy$
$\sigma_x^2 + \sigma_y^2 = 2$

5. Consider the discrete stochastic process $x(n) = sin(2\pi n/N) + \eta(n)$ where $\eta(n) \sim \mathcal{N}(0, 1)$ is white Gaussian noise. This process is

- ☐ a) Stationary.
- ☐ b) Ergodic.
- ☐ c) White.
- ■ d) None

- It is not stationary because it is time varying.
- It is not Ergodic because it is not stationary
- It is not white because
  $E[x(n)x(n + m)] = sin(2\pi n/N)sin(2\pi(n + m)/N) + E[sin(2\pi n/N)\eta(n + m)] + E[sin(2\pi(n + m)/N)\eta(n)] + E[\eta(n)\eta(n + m)] =$
  $sin(2\pi n/N)sin(2\pi(n + m)/N) + \psi_\eta(m) =$
  $sin(2\pi n/N)sin(2\pi(n + m)/N) + \delta(m) \neq \delta(m)$

6. Consider the feedback control system of blood glucose through the insulin secretion by the pancreas. Under the perspective of the canonical feedback control scheme, displayed in Fig. 2, where is the sensor that monitors the glucose concentration in bloodstream?

   - ☐ a) Liver.
   - ■ b) Pancreas.
   - ☐ c) Muscle.
   - ☐ d) None

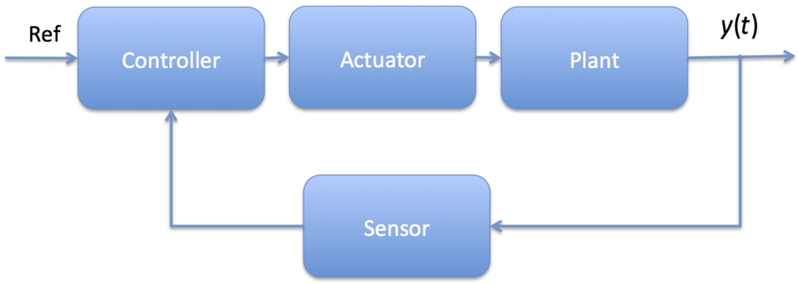

Figura 2: Canonical Feedback Control.

7. Consider an unit feedback loop of the continuous system

$$G(s) = (s+1)/(s(s-1)).$$

   For a given $K > 0$ the output of the closed loop system presents a pure sinusoidal oscillation. What is its frequency?

   - ☐ a) $\omega = 0.5$ rad/sec.
   - ■ b) $\omega = 1$ rad/sec.
   - ☐ c) $\omega = 10$ rad/sec.
   - ☐ d) None

   A pure sinusoidal oscillation occurs when the poles are pure imaginary.
   Characteristic equation $1 + K(s+1)/(s(s-1)) = 0 \Rightarrow K = 1 \Rightarrow p_{1,2} = \pm j \Rightarrow w_d = 1$ rad/sec.

8. Consider an unit feedback loop of the continuous system

$$G(s) = 1/(s+1)$$

   with a controller $C(s)$. Select the appropriated controller to obtain a null (zero) static error (in response to the step function).

- ☐ a) $C(s) = K$.
- ☐ b) $C(s) = K(s + 10)$.
- ■ c) $C(s) = K(s + 10)/s$.
- ☐ d) None

The static error of position (in response to a step function) is zero only for type 1 or higher open loop transfer functions, that is, functions with at least one zero pole.

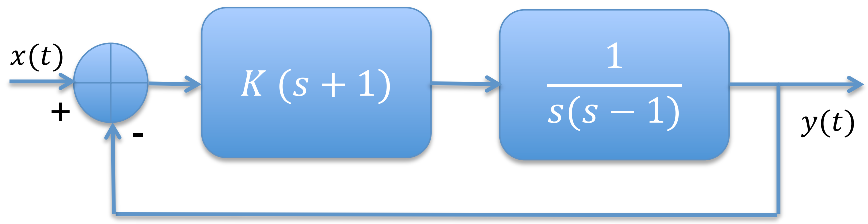

Figura 3: Unit feedback control system.

Problem (T=4/Ex=2)
Consider the feedback system represented in Fig.3.

1. **(2)** Draw the root-locus (RL) for $K > 0$ and $K < 0$. Compute explicitly the **break in** and **break out** points at the real axis, **asymptotic center and angle** of the asymptotes and the **sections of the real axis** of RL.

2. **(0,5)** Compute the values of K for which the closed-loop poles are equal (double-poles) and indicate their positions in the RL.

3. **(0,5)** What is the minimum value of K for which the system is stable without overshooting?

4. **(0,5)** For which value of K the system becomes stable? Indicate over the RL the location of the poles for this value of K.

5. **(0,5)** Compute the discrete version of the system, $G(z)$, using the poles and zero mapping technique, needed if the design of the controller is to be performed in the discrete domain and display their positions in the discrete complex plane together with the unit circle.