Instituto Superior Técnico / Universidade de Lisboa

Departament of Bioengineering

Master on Biomedical Engineering

Signals and Systems in Bioengineering

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Exam 2

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Name : Number:

The duration of the Exam is 3h. The score of each item 1 when right and −0.25 if wrong. Only one option can be selected in each question.

Part 1

1. Consider the sequences $x(n)$ and $y(n)$ with lengths 5 and 8 respectively. The sequences $w(n) = x(n) * y(n)$ is the linear convolution of $x(n)$ with $y(n)$ while $z(n) = x(n)(*_{11})y(n)$ is the 11-length circular convolution of $x(n)$ with $y(n)$.

   What equality is true?

   - ☐ a) $w(0) = z(0)$
   - ■ b) $w(1) = z(1)$
   - ■ c) $w(9) = z(9)$
   - ☐ d) None

   $$\begin{aligned} w_{12}(n) &= [w(0), w(1), w(2), w(3), w(4), w(5), w(6), w(7), w(8), w(9), w(10), w(11), 0, 0, ...] \\ z_{11}(n) &= [z(0), w(1), w(2), w(3), w(4), w(5), w(6), w(7), w(8), w(9), w(10), |z(0), w(1), ...] \end{aligned}$$

   where $z(0) = w(0) + w(11) \neq w(0)$

2. Consider the finite length sequence $x(n) = \{1, 2, 3\}$ and the sequence $y(n) = x((4-n)_6)$. What is the value of $y(4)$?

   - ■ a) 1
   - ☐ b) 2
   - ☐ c) 3

- ☐ d) None

$x((4-n)_6) = x(-(n-4))_6$

$$\begin{aligned} x_6(n) &= [...|1|, 2, 3, 0, 0, 0, |\mathbf{1}|, 2, 3, 0, 0, 0, |1|, 2, 3, 0, 0, 0, ... \\ x_6(-n) &= [...|1|, 0, 0, 0, 3, 2, |\mathbf{1}|, 0, 0, 0, 3, 2, |1|, 0, 0, 0, 3, 2, ... \\ x_6(-(n-4)) &= [...|0|, 0, 3, 2, 1, 0, |\mathbf{0}|, 0, 3, 2, 1, 0, |0|, 0, 3, 2, 1, 0, ... \end{aligned}$$

3. Let $p = [p_1, p_2, ..., p_N]$ be a complete, non orthogonal basis of a vector space S. In this case the Graminian is

   - ■ a) Non-diagonal
   - ☐ b) Non-invertible
   - ☐ c) Non-square
   - ☐ d) None

   If the basis of the space is non orthogonal basis some of the off-diagonal elements of the Graminian matrix, $\mathbf{R}$, are not null, $\exists_{i \neq j} : R_{i,j} =< \mathbf{p}_i, \mathbf{p}_j > \neq 0$

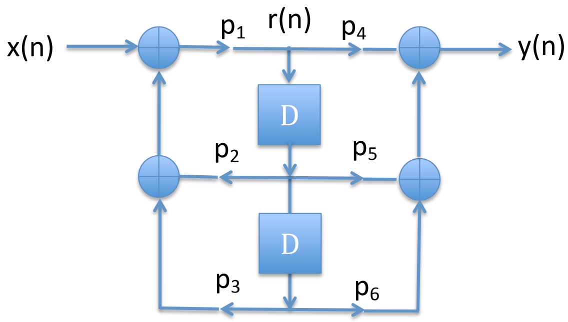

Figura 1: Direct Form II IIR filter

4. Consider the following IIR filter:

$$H(z) = \frac{2 + 0.5z^{-2}}{1 - 1.25z^{-1} + 0.25z^{-2}} \tag{1}$$

   graphically represented in Fig.1 in the direct form II.
   What is the vector of coefficients, $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5, p_6]$?

   - ☐ a) $\mathbf{p} = [1, -1.25, 0.25, 2, 0, 0.5]$

- ■ b) $\mathbf{p} = [1, 1.25, -0.25, 2, 0, 0.5]$
- □ c) $\mathbf{p} = [2, 0, 0.5, 1, 1.25, -0.25]$
- □ d) None

$$\begin{cases} Y(z) = (2 + 0.5z^{-2})R(z) \\ R(z) = X(z)/\left[1 - 1.25z^{-1} + 0.25z^{-2}\right] \end{cases} \Rightarrow$$

$$\begin{cases} y(n) = 2r(n) + 0r(n-1) + 0.5r(n-2) \\ r(n) = x(n) + 1.25r(n-1) - 0.25r(n-2) \end{cases} \Rightarrow \begin{cases} y(n) = p_4 r(n) + p_5 r(n-1) + p_6 r(n-2) \\ r(n) = p_1 \left[x(n) + p_2 r(n-1) + p_3 r(n-2)\right] \end{cases}$$

5. What is the response of the filter $H(z) = 1 - z^{-1}$ to the step signal, $u(n)$?

- □ a) 0
- □ b) $u(n)$
- ■ c) $\delta(n)$
- □ d) None

$Y(z) = (1 - z^{-1})X(z) = X(z) - z^{-1}X(z) \Rightarrow y(n) = u(n) - u(n-1) = \delta(n)$

6. Let us consider an infinite signal, to be filtered by a FIR filter with impulse response length 100. To implement the filtering process by blocks with a 2048 length FFT_{2048} algorithm, what should be the length of the input blocks to not have overlapping of these blocks?

- □ a) 2048.
- □ b) 2000.
- ■ c) 1948.
- □ d) None

$N + P - 1 \leq 2048 \Rightarrow N \leq 2048 - P + 1 = 2048 - 100 + 1 = 1949$

7. Consider the canonical adaptive filter displayed in Fig. 2 and two, independent white (with autocorrelation $\phi(n) = \delta(n)$), Gaussian noise signals, $\eta(n), \epsilon(n) \sim \mathcal{N}(0, 1)$. If $d(n) = \eta(n)$ and $x(n) = \epsilon(n)$ what is the optimal impulse response of the 4-length FIR filter $h(n)$ that minimizes the norm of the error, $\|\mathbf{e}\|$?

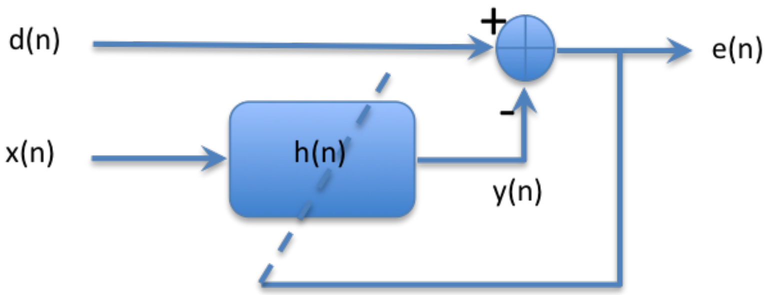

Figura 2: Adaptive filter.

- ■ a) $h(n) = [0, 0, 0, 0]$.
- ■ b) $h(n) = [1, 0, 0, 0]$.
- □ c) $h(n) = [1, 1, 1, 1]$.
- ■ d) None

$J(\mathbf{h}) =$
$E([\eta(n) - h(n) * \epsilon(n))]^2 = E([\eta(n) - \sum_k h_k \epsilon(n-k)]^2) =$
$E(\eta^2(n) - 2\sum_k h_k \eta(n)\epsilon(n-k) + \sum_{k,r} h_k h_r \epsilon(n-k)\epsilon(n-r)) =$
$\sigma_\eta^2 - 2\sum_k h_k E[\eta(n)\epsilon(n-k)] + \sum_{k,r} h_k h_r E(\epsilon(n-k)\epsilon(n-r)) =$
$\sigma_\eta^2 - 2\sum_k h_k \phi_{\eta\epsilon}(-k) + \sum_{k,r} h_k h_r \phi_{\epsilon,\epsilon}(k-r) =$
$\sigma_\eta^2 - 2\sum_k h_k \phi_{\eta\epsilon}(-k) + \sum_{k,r} h_k h_r \delta(k-r)$
$\sigma_\eta^2 - 2\sum_k h_k \phi_{\eta\epsilon}(-k) + \sum_m h_m^2$
$dJ(\mathbf{h})/h_i = 0 \Rightarrow -2\phi_{\eta\epsilon}(-i) + 2h_i = 0 \Rightarrow h_i = \phi_{\eta\epsilon}(-k)$(Option b) or d))

if η and ϵ are independent, then $\phi_{\eta,\epsilon}(-k) = E(\eta(n))E(\epsilon(n-k) = 0$ and then $h_i = 0$ (option a)).

8. What is the number of complex multiplications involved in the computation of a 1024-length DFT using the *Fast Fourier Transform* (FFT) algorithm?

- □ a) 1048576.
- ■ b) 10240.
- □ c) 1024.
- □ d) None

$N \log_2(N) = 1024 * 10 = 10240$

Problem (2)

Consider the FIR filter $h(n)$ represented in Fig.2 where

$$y(n) = \sum_{k=0}^{p} h(k)x(n-k)$$

where p is the order of the filter and $\mathbf{h} = \{h(0), h(1), ...h(p)\}^T$ are the $p+1$ coefficients of the filter to be estimated in each discrete time point, according the following criterion

$$\mathbf{h}(n) = \arg\min_{\mathbf{h}} J(\mathbf{h}) \tag{2}$$

where

$$J(\mathbf{h}) = \sum_{i=0}^{N-1} e^2(n-i)$$

with $e(n) = d(n) - y(n)$ where N is the length of a sliding window where the minimization of $J(\mathbf{h})$ is performed.

1. **(0.5)** Let $\mathbf{y} = A\mathbf{h}$ where $\mathbf{y} = [y(0), y(1), ..., y(N-1)]^T$.
   Derive A explicitly indicating its dimensions.

$$\begin{bmatrix} y_0 \\ y_1 \\ \cdots\cdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} x_0 & 0 & 0 & \cdots & 0 \\ x_1 & x_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{N-1} & x_{N-2} & x_{N-3} & \cdots & x_{N-p-1} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \cdot\cdot \\ h_p \end{bmatrix} \Rightarrow \mathbf{y} = A\mathbf{h} \Rightarrow \mathbf{e} = A\mathbf{hy} \tag{3}$$

2. **(0.5)** Express $J(\mathbf{h})$ by using matrix notation and **derive** the closed form solution of (2).
   $J(\mathbf{h}) = \mathbf{e}^T\mathbf{e} = (A\mathbf{h} - \mathbf{y})^T(A\mathbf{h} - \mathbf{y}) \Rightarrow \nabla_h J(\mathbf{h}) = A^T(A\mathbf{h} - \mathbf{y}) = 0 \Rightarrow$

   $\mathbf{h}^* = (A^TA)^{-1}A^T\mathbf{y}$

3. **(0.5)** What is the optimal filter $h(n)$ if $p = 10$ and $d(n) = x(n-2)$?
   $y(n) = \sum_{k=0}^{p} h_k x(n-k) = h_0x(0) + h_1x(n-1) + h_2x(n-2) + h_3x(n-3) + ... + h_px(n-p)$.
   If $\mathbf{h} = [0, 0, 1, 0, ..., 0] \Rightarrow y(n) = x(n-2) \Rightarrow e(n) = d(n) - y(n) = 0 \Rightarrow \|e(n)\| = 0$.

4. **(0.5)** Consider the following objective function to be minimized (in theory) in each discrete time point n:

$$J(n, \mathbf{h}) = E[\epsilon^2] = \int_{-\infty}^{+\infty} \epsilon^2 p(\epsilon) d\epsilon, \quad (4)$$

where $\epsilon = d(n) - y(n)$.

Derive the optimal impulsive response of $\mathbf{h}^*$ that minimizes $J(n, \mathbf{h})$ when $d(n)$ and $x(n)$ are white zero mean and unit variance Gaussian noise **independent** signals, $d(n), x(n) \sim \mathcal{N}(0, 1)$.

$J(\mathbf{h}) =$
$E([d(n) - h(n) * x(n))]^2 = E([d(n) - \sum_k h_k x(n-k)]^2) =$
$E(d^2(n) - 2\sum_k h_k d(n)x(n-k) + \sum_{k,r} h_k h_r x(n-k)x(n-r)) =$
$\sigma_d^2 - 2\sum_k h_k E[d(n)x(n-k)] + \sum_{k,r} h_k h_r E(x(n-k)x(n-r)) =$
$\sigma_d^2 - 2\sum_k h_k E[d(n)]E[x(n-k)] + \sum_{k,r} h_k h_r \phi_{\epsilon,\epsilon}(k-r) =$
$\sigma_\eta^2 - 2\sum_k h_k 0 + \sum_{k,r} h_k h_r \delta(k-r)$

$J(\mathbf{h}) = \sigma_\eta^2 + \sum_m h_m^2 \Rightarrow dJ(\mathbf{h})/dh_i = 2h_i = 0 \Rightarrow h_i = 0$

Part 2

1. Consider the discrete signal $x(n)$ where **three** (3) additional zero samples are introduced between the original ones, $[...x(i), 0, 0, 0, x(i+1), ...]$. The ideal interpolation of these new samples can be implemented by filtering the augmented signal with an ideal low-pass filtering with a cut-off frequency ω_c. What should be that frequency?

   - ■ a) $\omega_c = \pi/4$
   - □ b) $\omega_c = \pi/3$
   - □ c) $\omega_c = \pi$
   - □ d) None

   Three additional zeros for each sample correspond to an upsampling ratio $D = 4$. In this case, the ideal low-pass interpolation filter has a cut-off frequency of $\omega_c = \pi/D = \pi/4$.

2. If you want to to change the sampling rate of a discrete signal $x(n)$ by a factor of $R = 0.75$ what is the appropriated sequence of operations?

   - □ a) $T_{\downarrow 4}[T_{\uparrow 3}[h * x(n)]]$.
   - □ b) $T_{\uparrow 3}[h * T_{\downarrow 4}[x(n)]]$.
   - ■ c) $T_{\downarrow 4}[h * T_{\uparrow 3}[x(n)]]$.
   - □ d) None

3. Consider the discrete stochastic process $x(n) = h(n) * \eta(n)$ where $h(n)$ is a *Linear Time Invariant* (LTI) system and $\eta(n) \sim \mathcal{N}(0, 1)$ is stationary white Gaussian noise. The stochastic process $x(n)$ is

   - □ a) Non-stationary.
   - ■ b) Ergodic.
   - □ c) White.
   - □ d) None

4. Consider a first order unknown system $G(s) = A/(1 + \tau s)$ and its step $(x(t) = u(t))$ response displayed in Fig. 3. What is the time constant, τ, in seconds?

   - □ a) 0.25.
   - ■ b) 4.
   - □ c) 20.

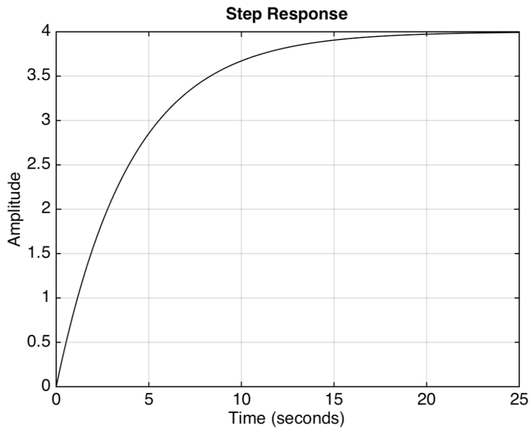

Figura 3: First order step response of an unknown system, $G(s) = A/(1 + \tau s)$.

- ☐ d) None

5. Consider an unknown second order system $G(s)$ and the step ($x(t) = u(t)$) response displayed in Fig. 4. What is the overshooting, S?

- ☐ a) 170%.
- ☐ b) 100%.
- ■ c) 70%.
- ☐ d) None

6. Consider a $N \times N$ image, $\mathbf{X}$, convolved with the following mask,

$$\mathcal{L} = \begin{bmatrix} -1 & -2 & -1 \\ -2 & 12 & -2 \\ -1 & -2 & -1 \end{bmatrix}. \tag{5}$$

What type of filtering is performed by this convolution operation?

- ☐ a) Low-pass (smoothing).
- ■ b) High-pass (sharpening)
- ☐ c) All-pass (equalization).
- ☐ d) None

7. Let $y(n) = [1 + \alpha x(n)]\, c(n)$ where $c(n) = cos(\omega_c n)$. $X(\omega)$ is the spectrum of $x(n)$. What is the spectrum (Fourier transform) of $y(n)$ in the interval $[0, \pi]$?

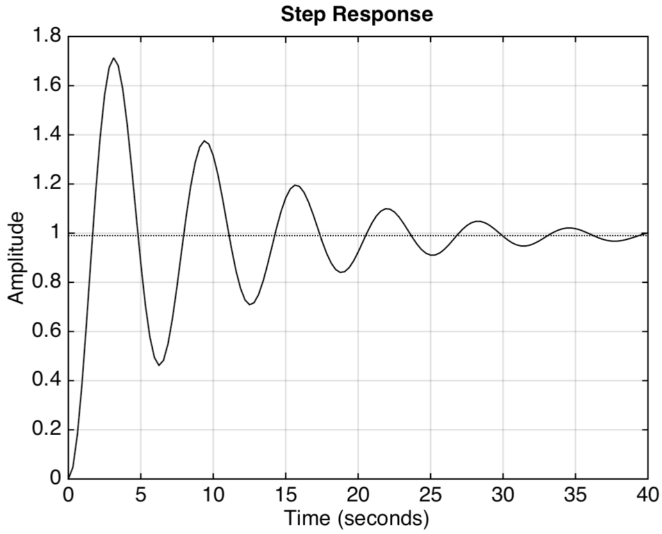

Figura 4: Second order step response of an unknown system.

- ☐ a) $Y(\omega) = \pi\delta(\omega) + \alpha X(\omega - \omega_c)/2$.
- ☐ b) $Y(\omega) = X(\omega) + \alpha X(\omega - \omega_c)/2$
- ☐ c) $Y(\omega) = \alpha X(\omega - \omega_c)$
- ■ d) None

8. Consider an unit feedback loop of the continuous system

$$G(s) = 1/[s(s-1)].$$

Select the controller $C(s)$ that is able to stabilize the close loop system for a given K?

- ☐ a) $C(s) = K$.
- ■ b) $C(s) = K(s + 10)$.
- ☐ c) $C(s) = K(s + 10)/s$.
- ☐ d) None

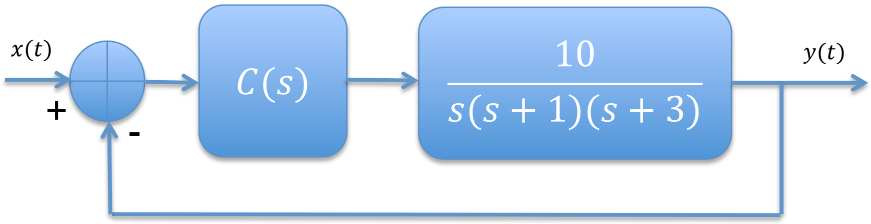

Figura 5: Unit feedback control system.

Problem (2)

Consider the feedback system represented in Fig.5 with a proportional controller $C(s) = K$.

1. **(1)** Draw the root-locus (RL) for $K > 0$ and $K < 0$. Compute explicitly the **break in** and **break out** points at the real axis, **asymptotic center and angle** of the asymptotes and the **sections of the real axis** of the RL.

2. **(0,5)** Stability interval

   - Compute the stability interval of K, $[K_{min}, K_{max}]$, that is, the interval of K where the closed loop system is stable.
   - For the limits of that interval compute the corresponding location of the three closed loop poles.
     (Notice that when the difference between the number of poles and zeros is greater or equal to 2, $n - m \geq 2$, the sum of the open loop and closed loop poles is equal, $\sum_i p_{ol}(i) = \sum_i p_{cl}(i)$).

3. **(0,5)** Consider now the following controller $C(s) = K(s + 2)$. Sketch the new RL and comment about the stability of the new closed loop system (compute again the asymptotic center and angle).