Instituto Superior Técnico / University of Lisbon

Departament of Bioengineering

Master on Biomedical Engineering

Signals and Systems in Bioengineering

1st Semester de 2016/2017

João Miguel Sanches

Test 1

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Name : Number:

The duration of the test is 1h30m. The score of each item is 2 when right and -0.5 if wrong. Only one option can be selected in each question.

1. Consider a continuous signal, $x(t)$, sampled with a sampling frequency of $f_s = 10Hz$. How many samples correspond to the first 1.45 seconds of the signal ($t \in [0, 1.45]$)?

   - ☐ a) 14.
   - ■ b) 15.
   - ☐ c) 16.
   - ☐ d) None

   Solution: $t_k = kT_s = k/f_s$ where $k = 0, 1, ...N - 1$.
   $t_{k_{max}} = k_{max}/f_s \leq 1.45 \Rightarrow k_{max} = floor(1.45 * f_s) = floor(14.5) = 14 = N - 1$
   $\Rightarrow N = 15$

2. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ two real unit norm vectors, $\|\mathbf{p}_i\| = 1$, from a non-orthogonal basis. What is the value of the following inner product: $< \mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2 >$?

   - ■ a) 0.
   - ☐ b) 1.
   - ☐ c) 2.
   - ☐ d) None

   Solution:

$$
\begin{aligned}
< \mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2 >=< \mathbf{p}_1, \mathbf{p}_1 > - < \mathbf{p}_1, \mathbf{p}_2 > + < \mathbf{p}_2, \mathbf{p}_1 > - < \mathbf{p}_2, \mathbf{p}_2 >= \\
< \mathbf{p}_1, \mathbf{p}_1 > + < \mathbf{p}_1, \mathbf{p}_2 > (1-1) - < \mathbf{p}_2, \mathbf{p}_2 >= 1 + 0 - 1 = 0 \qquad (1)
\end{aligned}
$$

3. Let $\mathbf{x}$ be a vector form S, a subspace of R^2 spanned by the (**column**) vectors $\{\mathbf{p}_1, \mathbf{p}_2\}$. Consider the following representation of $\mathbf{x}$ as function of the vectors of the basis,

$$\mathbf{x} = a\mathbf{p}_1 + b\mathbf{p}_2 + \mathbf{e}(\theta) \tag{2}$$

where $\mathbf{e}(\theta)$ is an error vector. The optimal set of coefficient, $\theta^* = [a^*, b^*]^T$, that minimize the norm of the error, $\theta^* = \arg\min_\theta \|\mathbf{e}\|$, can be obtained using the **orthogonality principle** as follows

$$\theta^* = R^{-1}P \tag{3}$$

where $R = \Phi^T\Phi$, $P = [< \mathbf{x}, \mathbf{p}_1 >, < \mathbf{x}, \mathbf{p}_2 >]^T$ and $<,>$ denotes the inner product operator.

In this formulation what is Φ?

- ■ a) $\Phi = [\mathbf{p}_1, \mathbf{p}_2]$.
- □ b) $\Phi = [\mathbf{p}_1, \mathbf{p}_2]^T$.
- □ c) $\Phi = \mathbf{p}_1^T\mathbf{p}_2$.
- □ d) None

4. Consider an unknown continuous scalar function $f(t)$ defined in the interval $t \in [0, 1]$ and a set of M observations $F = [f_1, f_2, ..., f_M]^T$ taken at random time points, t_k, from the interval where $f(t)$ is defined. Let also consider a continuous function $g(t) = \sum_{k=0}^{N-1} c_k\phi_k(t)$, a linear combination of N known basis functions, $\phi_k(t)$, evenly distributed in the same time interval, $\phi_k(t) = \phi(t/\Delta - k), k = 0...N-1$ where $\phi(t)$ is a mother interpolation function and Δ is the distance between interpolation functions.

The optimal set of coefficients, $\mathbf{c} = [c_0, c_1, ..., c_{N-1}]^T$ are computed by minimizing the norm of the error $E(\theta) = \|F - G\|$ where $G = \{g(t_k)\}$ with t_i is the time point of the i^{th} observation.

Using the appropriated formulation (2nd laboratory work) the solution is

$$\mathbf{c}^* = (\Theta^T\Theta)^{-1}\Theta^T F$$

where Θ is a function of the observations and interpolation functions.
Under this formulation what are the dimensions of matrix Θ?

- □ a) $N \times M$.
- ■ b) $M \times N$.
- □ c) $N \times N$.
- □ d) None

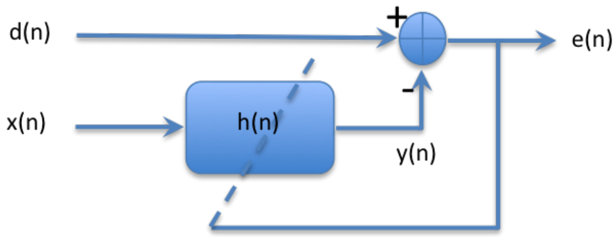

Figura 1: Adaptive filter.

5. Consider the canonical adaptive filter displayed in Fig. 1 where $d(n) = x(n)$ and $h(n)$ is a 4 length FIR filter. In this conditions what is the optimal impulse response of the FIR that minimizes the norm of the error, $\|\mathbf{e}\|$?

   - ☐ a) $h(n) = \mathbf{x} * \mathbf{x}$.
   - ☐ b) $h(n) = [0, 0, 0, 0]$.
   - ■ c) $h(n) = [1, 0, 0, 0]$.
   - ☐ d) None

   Solution: If $d(n) = x(n)$ and $\mathbf{h} = [1, 0, 0, 0, 0]$ then $y(n) = x(n) \Rightarrow e(n) = d(n) - y(n) = x(n) - x(n) = 0$. In these conditions the optimality criterion, the minimization of $\|\mathbf{e}\|$, holds and the error norm is the minimum possible $\|\mathbf{e}\| = \|[0, 0, ..., 0]\| = 0$.

6. Consider the complex signal $x(n) = [1; j; -1; -j]$. What is the 4 length DFT value for $k = 4$, $X_4(4)$?

   - ■ a) 0.
   - ☐ b) 4.
   - ☐ c) $2(1 - j)$.
   - ☐ d) None

7. Consider a $N = 1000$ length complex signal $x(n)$ to be filtered by a $P = 24$ length FIR filter with impulse response $h(n)$.

   What is the minimum length of the *Fast Fourier Transform* (FFT) used to compute the DFTs needed to filter $x(n)$ in the frequency domain without loss of information?

   - ☐ a) 1000.
   - ☐ b) 1023.
   - ■ c) 1024.

- ☐ d) None

Solution: Let us consider the integer r. $N_{FFT} = 2^r \geq length(x_k * h) = N + P - 1 = 1000 + 24 - 1 = 1023 \Rightarrow r \geq \log_2(1023) = \ln(1023)/\ln(2) = 9.9986 \Rightarrow r = 10 \Rightarrow N_{FFT} = 2^r = 1024$.

8. Consider two finite length sequences $x_5(n)$ and $y_6(n)$ with lengths 5 and 6 respectively. $w(n) = x_5(n) * y_6(n)$ and $z(n) = x_5(n)(\star_8)y_6(n)$ are the linear $(*)$ and circular length 8, $(\star_8)$, convolutions respectively. Which condition is true?

   - ☐ a) $w(0) = z(0)$.
   - ■ b) $w(2) = z(2)$.
   - ☐ c) $w(10) = z(10)$.
   - ☐ d) None

Problem (4) Let $x(n)$ be a N length strictly positive sequence and consider the following non linear model

$$x(n) = x^{\alpha}(n-1)x^{\beta}(n-2)e^{\epsilon(n)}$$

Formulate the estimation problem of the vector of coefficients $\theta = [\alpha, \beta]^T$ using matrix notation by minimizing the norm of the error, $\|\epsilon\|$.

Solution:

$$\begin{aligned} x(n) &= x^{\alpha}(n-1)x^{\beta}(n-2)e^{\epsilon(n)} \Rightarrow \\ \log[x(n)] &= \alpha \log[x(n-1)] + \beta \log[x(n-2)] + \epsilon(n) \Rightarrow \\ \tilde{\mathbf{y}} &= \alpha \tilde{\mathbf{x}}_1 + \beta \tilde{\mathbf{x}}_2 + \mathbf{e} \Rightarrow \\ \tilde{\mathbf{y}} &= [\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2]\,\theta + \mathbf{e} \end{aligned} \tag{4}$$

where $\theta = [\alpha, \beta]^T$, $\tilde{\mathbf{y}} = [\log[x(n)]]$, $\tilde{\mathbf{x}}_1 = [\log[x(n-1)]]$, $\tilde{\mathbf{x}}_2 = [\log[x(n-2)]]$ and $\mathbf{e} = [\epsilon(n)]$. Solve (4) for θ by minimizing the norm of the error: $\|\epsilon\|$.