Instituto Superior Técnico / University of Lisbon

Departament of Bioengineering

Master on Biomedical Engineering

Signal and Systems in Bioengineering

1st Semester de 2014/2015

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Test 2 / Exame 1

January 17, 2015

Name : Number:

The duration of the test is 1h30 and the exam is 3h. The score of each item is 1 when right and -0.25 if wrong. Only one option can be selected in each question.

Part 1

1. Consider the following set of vectors that span a space S, $p_1 = [1; 0; 1]$, $p_2 = [0; 1; 1]$, $p_3 = [0; 1; 0]$. What is the Graminian of this basis?

   - ☐ a) $[\{0, 0, 0\}; \{1, 1, 1\}; \{0, 1, 0\}]$
   - ■ b) $[\{2, 1, 0\}; \{1, 2, 1\}; \{0, 1, 1\}]$
   - ☐ c) I (Identity matrix)
   - ☐ d) None of the above

   where $\{.\}$ represents a line of a matrix.

2. Consider the following optimization problem,

$$\mathbf{c}^* = \arg\min_{\mathbf{c}} \|A\mathbf{c} - \mathbf{x}\|_1 \tag{1}$$

   where A is a $N \times L$ matrix, $\mathbf{x}$ is a N length column vector of observations and $\mathbf{c}$ is a L length vector of coefficients to be estimated. $\|e\|_1 = \sum_{k=1}^{L} |e_k|$ is called $\mathcal{L}_1$ norm where $|a|$ is the absolute value of the scalar a. This optimization problem can solved using the *Iteratve Reweighted Least Squares* (IRLS) algorithm as follows

$$\mathbf{c}^{t+1} = \arg\min_{\mathbf{c}} \mathbf{e}^T W(\mathbf{c}^t)\mathbf{e} \tag{2}$$

   where $\mathbf{e} = A\mathbf{c} - \mathbf{x}$ and $W(\mathbf{c}^t)$ is a diagonal matrix with elements

- ■ a) $w_{i,i} = 1/|e_i|$
- □ b) $w_{i,i} = |e_i|$
- □ c) $w_{i,i} = 1/e_i^2$
- □ d) None of the above.

3. The coefficients of the *Discrete Cosine Transform* (DCT), $X(k)$, are

- ■ a) Real
- □ b) Complex
- □ c) Within the interval $]-1, 1[$
- □ d) None of the above

4. Let $g(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^T(\mathbf{x} + \mathbf{y})$ where $\mathbf{x}$ and $\mathbf{y}$ are N length column vectors. $g(\mathbf{x}, \mathbf{y})$ is

- □ a) a metric function because is strictly positive.
- □ b) a metric because involves the difference $(\mathbf{x} - \mathbf{y})$.
- □ c) a metric because is null if and only if $\mathbf{x} = \mathbf{y}$.
- ■ d) None of the above

5. What is the period of the signal $x(n) = e^{j\pi n}$?

- □ a) 0.
- □ b) 1.
- ■ c) 2.
- □ d) None of the above

6. Consider a discrete signal, $x_d(n), n \in N$, that is obtained by uniformly sampling a continuous signal, $x_c(t), t \in R$, where $x_d(n) = x_c(nT_s)$. $T_s = 1/f_s$ is the sampling period and $f_s = 1000Hz$ is the sampling frequency. What is the minimum frequency separation in the discrete domain (digital frequency axis) if a 1024 length *Fast Fourier Transform* (FFT_{1024}) is used in the analysis?

- ■ a) $\pi/512$ rad/sample.
- □ b) 1000/1024 Hz.
- □ c) 2π rad/sample.
- □ d) None of the above

7. Let us consider an infinite signal, to be filtered by a 25 length impulse response FIR filter . To implement the filtering process by blocks with a 1024 length FFT algorithm, what should be the maximum length of the input blocks to not have overlap of these blocks?

- ■ a) 1000.
- □ b) 1024.
- □ c) 1048.
- □ d) None

8. What is the value of the $\mathcal{L}_0$ norm of the vector $x = [2, 1, 0, 1]$?

- ■ a) 3
- □ b) 2
- □ c) $\sqrt{6}$
- □ d) None of the above

Problem (4) Let $x(n)$ be a N length strictly positive sequence and consider the following non linear auto-regressive (AR) model

$$x(n) = c_1 x(n-1) + c_2 \log(x(n-2)) \tag{3}$$

.

Formulate the estimation problem of the vector of coefficients $\theta = [c_1, c_2]^T$ using matrix notation by minimizing the energy function

$$E(\theta) = \sum \left[c_1 x(n-1) + c_2 \log(x(n-2)) - x(n)\right]^2 \tag{4}$$

with respect to θ,

$$\theta^* = \arg\min_{\theta} E(\theta) \tag{5}$$

Solution

Let $\mathbf{x}$ be a N length column vector. The objective function defined in (4) can be written as follows

$$E(\theta) = (\Psi\theta - \mathbf{x})^T(\Psi\theta - \mathbf{x}) \tag{6}$$

where Ψ is the following $N \times 2$ matrix

$$\Psi = \begin{bmatrix} 0 & 0 \\ x(1) & 0 \\ x(2) & \log(x(1)) \\ x(2) & \log(x(1)) \\ \cdots & \cdots \\ x(N-1) & \log(x(N-2)) \end{bmatrix} \tag{7}$$

$$\theta^* = (\Psi^T\Psi)^{-1}\Psi^T\mathbf{x} \tag{8}$$

Part 2

1. What is the value of the following integral?

$$\int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{4}} dx \tag{9}$$

- ■ a) $2\sqrt{\pi}$.
- □ b) 1.
- □ c) ∞.
- □ d) None of the above

2. Consider the signal $x(n) = \eta(n) + k$ where $\eta(n)$ is a zero mean white Gaussian noise component, $\eta \sim \mathcal{N}(0, \sigma^2)$, and k is a constant. What is the Power Spectrum of $x(n)$?

- □ a) $P_x(\omega) = 1 + 2\pi k$.
- ■ b) $P_x(\omega) = \sigma^2 + 2\pi\delta(\omega)k^2$.
- □ c) $P_x(\omega) = 2\pi\sigma^2\delta(\omega) + k$.
- □ d) None of the above

3. Consider the LTI system described the following difference equations

$$y(n) = x(n) + 0.5y(n-1) \tag{10}$$

What is the mean value of the output signal if the input is $x(n) = \eta(n) + 2$ where $\eta \sim \mathcal{N}(\mu, \sigma^2)$ is white Gaussian noise with $\mu = 2$ and $\sigma = 2$

- ■ a) 8.
- □ b) 4.
- □ c) 2.
- ■d) None of the above

4. Consider the problem in the previous question. What is the *power spectral density* (PSD) of the output?

- □ a) $P_y(\omega) = 8\pi\delta(\omega) + 16$
- ■ b) $P_y(\omega) = 8\pi\delta(\omega) + \frac{4}{1.25 - cos(\omega)}$
- □ c) $P_y(\omega) = 32\pi\delta(\omega) + \frac{4}{1.25 - cos(\omega)}$
- ■ d) None of the above

5. Let $H(z)$ be an ideal lowpass filter with cutoff frequency $\omega_c = \pi/4$ and $H(\omega)|_{\omega=0} = 1$. Consider the input signal $x(n) = 1 + \cos\left(\frac{\pi}{3}n\right) + \eta$ where η is zero mean white noise with variance $\sigma_\eta^2 = 1$. What is the power spectrum of the output, $P_y(\omega)$?

   - ☐ a) $P_y(\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - \pi/3) + \pi\delta(\omega + \pi/3) + 1$ for $\omega \in [\pi/2, \pi/2]$.
   - ■ b) $P_y(\omega) = 2\pi\delta(\omega) + 1$ for $\omega \in [\pi/4, \pi/4]$.
   - ☐ b) $P_y(\omega) = 2\pi\delta(\omega) + 1$.
   - ☐ d) None of the above

6. Consider a system with the following open-loop transfer function

$$G(s) = \frac{1}{s-1} \tag{11}$$

   and an unitary negative feedback output topology where the controller is just a gain, $C(s) = K$. What is the value of K that makes the closed loop system stable?

   - ☐ a) $K = 0$
   - ■ b) $K = 1$
   - ☐ c) $K = \infty$
   - ☐ d) None of the above

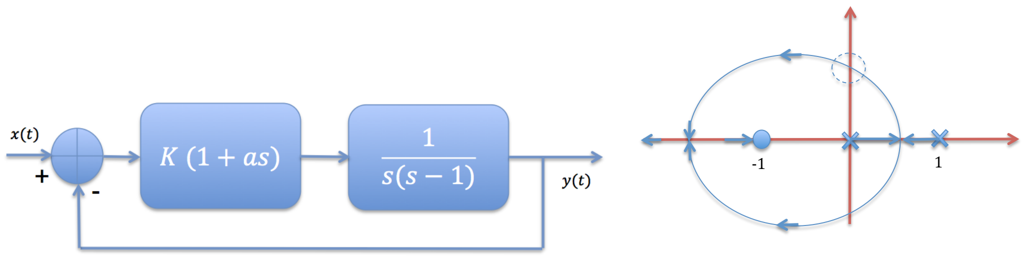
7. Repeat the previous question with

$$G(s) = \frac{1}{s(s-1)} \tag{12}$$

   - ☐ a) $K = 0$
   - ☐ b) $K = 1$
   - ☐ c) $K = \infty$
   - ■ d) None of the above

8. What is the final value of the response to the unity step of the closed loop output unitary feedback topology of the previous system, $G(s) = 1/(s(s-1))$ for, $K = 1$

   - ☐ a) $y(\infty) = 0$
   - ☐ b) $y(\infty) = 1$
   - ■ c) $y(\infty) = \infty$
   - ☐ d) None of the above



(a) Unit feedback (b) Rlocus

Figura 1: Unit feedback control system.

Problem (2) Consider the feedback system represented in Fig.1.

1. Derive the corresponding root-locus for $\alpha = 0$ and $K > 0$ without using the root-locus rules. Do it based on the analytic expression of the closed loop poles.

2. Is the closed-loop system stable for $K > 0$?

3. The root-locus for $\alpha = 1$ and $K > 0$ is displayed in the figure above (right).

   What is the value of K for which the closed loop poles are in the imaginary axis (see dashed circle in the figure)? Derive that value from the expression obtained in 1)

   (this is the value of K that stabilize the system)

4. For the value of K obtained in the previous item what are the characteristics of $y(t)$, namely, concerning its periodicity and oscillating frequency.

5. Derive the interval for K where the closed loop poles are complex.

6. Compute the value of K that leads to double real stable poles. For that K compute the location of the corresponding poles.