Instituto Superior Técnico / Tecnical University of Lisbon

Departament of Bioengineering

**Master on Biomedical Engineering**

**Digital Signal Processing in Bioengineering**

2nd Semester de 2012/2013

João Miguel Sanches

# Test 1 - Solution

April 23, 2013

Name : Number:

---

The duration of the test is 1h30m. The score of each item is 2 when right and $-0.5$ if wrong. Only one option can be selected in each question.

1. Let $V$ and $U$ two sub-spaces of a space $S$. The intersection $U \cap V$
   - ■ a) it is necessarily a sub-space of $S$.
   - □ b) it is a sub-space of $U$ or $V$ but not from both simultaneously.
   - □ c) it is not a sub-space of $S$.
   - □ d) None
2. What is the angle between the functions $cos(x)$ and $sin(x)$, defined in the interval $[-\pi, \pi]$?
   - □ a) 0.
   - □ b) $\pi/4$.
   - ■ c) $\pi/2$.
   - □ d) None
3. Consider a discrete signal obtained with a sample rate of 20kHz. What is the corresponding continuous spectral frequency in Hz of the $\text{FFT}_{1024}$ coefficient $X(16)$?
   - ■ a) 312.5 Hz.
   - □ b) 0.0982 rad/sample.
   - □ c) 125 Hz.
   - □ d) None
4. Let $x(n) = [1, -1, 1, 0\ ]$. What is the value of 4 dimension DFT coefficient $X_4(1)$?
   - □ a) 0.

- ☐ b) 1.
- ■ c) $j$.
- ☐ d) None

5. What is the period of the signal $y(n) = sin(n)$?

- ☐ a) 1 sample.
- ☐ b) $2\pi$ rad/sample.
- ☐ c) 1 second.
- ■ d) None

6. Let $g(x, y) = sin(d(x, y))$ be a function of two vectors $x$ and $y$, where $d(x, y)$ is a metric function. The function $g(x, y)$

- ☐ a) is a metric function because it is strictly non negative.
- ☐ b) it is not a metric function because $g(x, x) \neq 0$
- ☐ c) it is a metric function because its value increases when $\|x - y\|$ increases.
- ■ d) None

7. Let $x(n)$ and $y(n)$ two discrete sequences of length 16 with $DFT$ coefficients $X_{16}(k)$ and $Y_{16}(k)$ respectively, where $Y(k) = \begin{cases} X(k) & \text{for } k \text{ even} \\ -X(k) & \text{for } k \text{ odd} \end{cases}$.

What is the right option?

- ■ a) $y(2) = x(10)$.
- ☐ b) $y(1) = x(11)$.
- ☐ c) $y(0) = x(12)$.
- ☐ d) None

8. The goal is to filter, in real time, an audio signal from a microphone with a 50 length impulse response FIR filter. The signal should be processed with a 500 sample length blocks and the convolution is performed by using a 512 length FFT algorithm. What is the number of overlapped samples of the input blocks.

- ☐ a) 12.
- ■ b) 37.
- ☐ c) 50.
- ☐ d) None

**Problem (4)** Let $(x_i, y_i, z_i)$ be $N$ data triplet observations with the underlying model

$$z_i = \alpha x_i^{\beta+\gamma y_i} \tag{1}$$

- a) Derive the expression of the square norm of the error vector, $\mathbf{e} = \{e_i\}$, with $e_i = z_i - \alpha x_i^{\beta+\gamma y_i}$ and the closed form solution of the minimizer vector of parameters,

$$(\alpha, \beta, \gamma) = \arg\min_{\alpha,\beta,\gamma} \|\mathbf{e}\|_2^2 \tag{2}$$

**Solution**

$$z_i = \alpha x_i^{\beta\gamma y_i} = \alpha e^{\log(x_i)(\beta+\gamma z_i)} \Rightarrow \log z_i = \log\alpha + \beta\log(x_i) + \gamma y_i \log(x_i)$$

Solve

$$\theta = \arg\min_{\theta} \|\mathbf{e}\|^2 \tag{3}$$

where

$$e_i = \hat{\alpha} + \beta\hat{x}_i + \gamma\hat{y}_i - \hat{z}_i \tag{4}$$

and

$$\begin{aligned} \hat{z}_i &= \log z_i \\ \hat{x}_i &= \log x_i \\ \hat{y}_i &= y_i \log x_i \end{aligned}$$

Vector $\mathbf{e} = \{e_i\}$ can be written as follows

$$\mathbf{e} = \Phi\theta - \mathbf{z} \tag{5}$$

where $\mathbf{z} = \{\hat{z}_i\}$, $\theta = \{\log(\alpha), \beta, \gamma\}^T$ and

$$\Phi = \begin{pmatrix} 1 & \hat{x}_1 & \hat{y}_1 \\ ... & ... & .... \\ 1 & \hat{x}_i & \hat{y}_i \\ ... & ... & .... \\ 1 & \hat{x}_N & \hat{y}_N \end{pmatrix} \tag{6}$$

Problem (3) can be solved by finding the stationary point of $\mathbf{e}^T(\theta)\mathbf{e}(\theta)$ with respect to $\theta$,

$$\begin{aligned} \nabla_\theta \mathbf{e}^T\mathbf{e} &= \nabla_\theta(\Phi\theta - \mathbf{z})^T(\Phi\theta - \mathbf{z}) = \Phi^T(\Phi\theta - \mathbf{z}) = 0 \Rightarrow \\ \theta^* &= (\Phi^T\Phi)^{-1}\Phi^T\mathbf{z} = pinv(\Phi)\mathbf{z} \end{aligned} \tag{7}$$

- b) Propose an iterative algorithm to compute the optimum vector of parameters $(\alpha, \beta, \gamma)$ that minimizes the $L_1$ norm of the error vector

$$(\alpha, \beta, \gamma) = \arg\min_{\alpha,\beta,\gamma} \|\mathbf{e}\|_1 \tag{8}$$

**Solution**

$$\|\mathbf{e}(\theta)\|_1 = \sum_i |e_i(\theta)| = \sum_i \frac{1}{|e_i(\theta)|} e_i^2(\theta) = \sum_i w_i(\theta) e_i^2(\theta) \tag{9}$$

where

$$w_i(\theta) = \frac{1}{|e_i(\theta)|} \tag{10}$$

By using matrix notation the $L_1$ norm of $\mathbf{e}$ can be written as follows

$$\|\mathbf{e}\|_1 = \mathbf{e}^T(\theta) W(\theta) \mathbf{e}(\theta) \tag{11}$$

where $W(\theta)$ is a diagonal matrix with $w_{ii} = \frac{1}{|e_i(\theta)|}$. The minimizer of (9) can be found as

$$\theta^* = \arg\min_\theta \mathbf{e}^T(\theta) W(\theta^*) \mathbf{e}(\theta) \tag{12}$$

where $\theta^*$ is the minimizer we are looking for. Therefore, the following iterative method is used

$$\theta^t = \arg\min_\theta \mathbf{e}^T(\theta) W(\theta^{t-1}) \mathbf{e}(\theta) \tag{13}$$

where $t$ is the iteration index and $\theta^{t-1}$ is the optimum $\theta$ obtained in the previous iteration. The initialization of $W$ is the identity matrix,$W_0 = I$. The optimization of (13) in each iteration is performed by using the method derived in a).