TÉCNICO LISBOA

# SSB
# Signals and Systems in Bioengineering

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# Chapter 1

# Spaces

**Exercise 1.0.1.** *Demonstration of the Cauchy-Schwartz inequality*

*If* $\mathbf{x} = \mathbf{y} = 0$ *the demonstration is trivial. Let us consider the following inequality*

$$0 \leq \|x - \alpha y\|^2 = \|x\|^2 - 2Re\langle \mathbf{x}, \alpha \mathbf{y} \rangle + |\alpha|^2 \|y\|^2 \tag{1.1}$$

*e and let us compute the value of the scalar* $\alpha$ *that minimizes the norm* $\|\mathbf{x} - \alpha \mathbf{y}\|^2$ *by deriving with it with respect* $\alpha$*,*

$$\frac{d\|\mathbf{x} - \alpha \mathbf{y}\|^2}{d\alpha} = -2Re\langle \mathbf{x}, \mathbf{y} \rangle + 2\alpha \|y\|^2 = 0 \Rightarrow \tag{1.2}$$

$$\alpha = \frac{Re\langle \mathbf{x}, \mathbf{y} \rangle}{\|y\|^2} \tag{1.3}$$

*and by replacing*

$$0 \leq \|x - \alpha y\|^2 = \|x\|^2 - 2\frac{Re\langle \mathbf{x}, \alpha \mathbf{y} \rangle}{\|y\|^2} + \frac{Re\langle \mathbf{x}, \alpha \mathbf{y} \rangle}{\|y\|^2} \tag{1.4}$$

$$= \|x\|^2 - \frac{Re\langle \mathbf{x}, \alpha \mathbf{y} \rangle}{\|y\|^2} \Rightarrow \tag{1.5}$$

$$|Re\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|, \tag{1.6}$$

*If* $\mathbf{y} = \alpha \mathbf{x}$ *then*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \alpha \mathbf{x} \rangle = \alpha \langle \mathbf{x}, \mathbf{x} \rangle = \alpha \|x\|^2. \tag{1.7}$$

*Since* $\|\mathbf{y}\| = |\alpha|\|x\| \Rightarrow \|\mathbf{x}\| = \|\mathbf{y}\|/\alpha$ *he have*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \alpha \|x\| \frac{\|\mathbf{y}\|}{\alpha} = \|x\|\|y\|, \tag{1.8}$$

*q.e.d*

*Remark:*

$$\begin{aligned} \|x-y\|^2 &= \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \|x\|^2 - 2Re\langle \mathbf{x}, \mathbf{y} \rangle + \|y\|^2 \end{aligned}$$

*For a real space*

$$\|x-y\|^2 = \|x\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle + \|y\|^2 \tag{1.9}$$

*From the Cauchy-Schwartz inequality it is possible to prove the triangular inequality,*

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &\leq \langle \mathbf{x}, \mathbf{x} \rangle + 2\|\mathbf{x}\|\|\mathbf{y}\| + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \Rightarrow \qquad (1.10) \\ \|\mathbf{x}+\mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\| \qquad (1.11) \end{aligned}$$

*For a space of real functions defined in the interval $[a, b]$ the Cauchy-Schwartz inequality is the following:*

$$\left[\int_a^b f(t)g(t)dt\right]^2 \leq \int_a^b f^2(t)dt \int_a^b g^2(t)dt \tag{1.12}$$

**Exercise 1.0.2.** *(Ex.2.1-1, pag. 121)*
*Considere o vector $x = [1, 2, 3, 4, 5, 6]^T$. Calcule a mtrica $l_p$, $d_p(\mathbf{x}, \mathbf{0})$ para $p = 1, 2, 4, 10, 100, \infty$. Comente o facto de $d_p(\mathbf{x}, \mathbf{0}) \to max(x_i)$ quando $p \to \infty$.*

**Exercise 1.0.3.** *(Ex.2.1-2, pag. 121)(**Solved**)*
*Let $\mathbf{X}$ be an arbitrary set. Show that the following function*

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \tag{1.13}$$

*is a metric function.*

*$d(x, y)$ is a metric function if and only if*
*i)$d(x, y) \geq 0, \forall x, y \in \mathbf{X}$, which is truth ,*
*ii)$d(x, y) = 0$ if and only if $x = y$, which is also truth and finaly*
*iii) the triangular inequality must hold. If $x = y = z$ the prove is trivial. If $(x = y) \wedge (y \neq z)$ or $(x \neq y) \wedge (y = z)$ then $d(x, y) + d(y, z) = 1 + 0 = d(x, z)$. If $x \neq y \neq z$, $d(x, y) + d(y, z) = 1 + 1 = 2 > 1 = d(x, z)$ q.e.d.*

**Exercise 1.0.4.** *(Ex.2.1-4, pag. 121)(**Solved**)*
*Triangular Inequality*

- *For $x, y \in R$ the following inequality holds*

$$|x+y| \leq |x| + |y|. \tag{1.14}$$

  *What is the condition for the equality?*
  ***Demonstration:***
  $|x+y|^2 = x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2 = (|x|+|y|)^2 \Rightarrow |x+y| \leq |x| + |y|$ *q.e.d.*

- *Prove the triangular inequality for $\mathbf{x}, \mathbf{y} \in R^n$.*

$$\|x+y\| \leq \|x\| + \|y\|. \tag{1.15}$$

  *where $\|.\|$ is the Euclidean norm.*
  *Suggestion: Use the fact of $\sum_{i=1}^{n} x_i y_j \leq \|x\|\|y\|$ (Cauchy-Schwartz inequality.)*
  ***Demonstration:***
  $(\|x+y\|)^2 =< x+y, x+y >=< x, x > +2 < x, y > + < y, y > \leq (\|x\|)^2 + 2(\|x\|)(\|y\|) + (\|y\|)^2 = ((\|x\|) + (\|y\|))^2 \Rightarrow (\|x+y\|) \leq (\|x\|)(\|y\|)$ *q.e.d.*

**Exercise 1.0.5.** *(Ex.2.1-5, pag. 122)(**Solved**)*
*Let $(\mathbf{X}, d)$ be a metric space. Show that*

$$g(x,y) = \frac{d(x,y)}{1+d(x,y)} \tag{1.16}$$

*is a metric function $\mathbf{X}$.*
***Prove:***
*First lust us check if the properties of the metric function hold.*
***i)*** $g(x,y) = g(y,x)$*: is true;*
***ii)*** $\dfrac{d}{1+d} \geq 0 \Rightarrow d \geq 1 + d \Rightarrow 0 \geq 1$*, is true;*
***iii)*** $\dfrac{d(x,y)}{1+d(x,y)} = 0 \Rightarrow \{d(x,y) = 0ou1 + d(x,y) = \infty\} \Rightarrow x = y$ *because $d(x,y)$ is a metric function.*
***iv)*** $g(x,z) \leq g(x,y) + g(y,z)$ :
*Let us apply the fuction $g(x)$ to both sides of $d(x,z) \leq d(x,y) + d(y,z)$, and the result is $g(x,z) \leq g(d(x,y) + d(y,z))$. Now, it is enough to prove that $g(a+b) \leq g(a) + g(b)$ :*

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b} = \frac{a+b+2ab}{(1+a)(1+b)} = \frac{a+b+2ab}{1+a+b+ab}. \tag{1.17}$$

*which is equivalent to,*

$\frac{(a+b)}{(1+a+b)} \leq \frac{(a+b)+2ab}{(1+a+b)+ab} \Rightarrow$ *para* $n = a+b$, $c = 2ab$ *e* $1+a+b$, $\frac{n}{d} \leq \frac{n+2c}{d+c} \Rightarrow$

$\Rightarrow nd+nc \leq nd+2cd \Rightarrow n \leq 2d \Rightarrow n \leq 2(n+1) \Rightarrow n \leq 2n+2 \Rightarrow n+2 \geq 0$, *which is always true. Therefore, ,*

$$\frac{d(x,z}{1+d(x,y)} \leq g(d(x,y)+d(y,z)) \leq g(d(x,y)+g(d(y,z)) \tag{1.18}$$

$$g(d(x,y)+g(d(y,z)) = \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \tag{1.19}$$

$\Rightarrow \frac{d(x,y)}{1+d(x,y)}$ *is a metric function (q.e.d).*

**Exercise 1.0.6.** *(Ex.2.1-8, pag. 122)*
*Show that in a the metric space* $(R^n, d_p)$ *the function* $d_p(\mathbf{x},\mathbf{y})$ *decreases with* $p$*, that is,* $d_p(\mathbf{x},\mathbf{y}) \geq d_q(\mathbf{x},\mathbf{y})$ *if* $p \leq q$*. Suggestion: Compute the derivative with respect to p and show it is less or equal to zero. Use the inequality of the a logarithmic sum. This inequality states that for non negative sequences* $a_1, a_2, ..., a_n$ *and* $b_1, b_2, ..., b_n$*,*

$$\sum_{i=1}^{n} a_i \log(a_i/b_i) \geq \left[\sum_{i=1}^{n} a_i\right] \log\left[\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}\right] \tag{1.20}$$

*Make* $b_i = 1$ *and* $a_i = |x_i - y_i|^p$*. Use also the fact of, for non negative sequences* $\{\alpha_i\}$ *such that* $\sum_{i=1}^{n} \alpha_i = 1$ *the maximum of*

$$\sum_{i=1}^{n} \alpha_i \log \alpha_i \tag{1.21}$$

*is zero.*

**Exercise 1.0.7.** *(Ex.2.1-20, pag. 123)(**Solved**)*
*Show that, if* $\{x_n\}$ *is a sequence where*

$$d(x_{n+1}, x_n) < Cr^n \tag{1.22}$$

*for $0 \leq r < 1$, than $\{x_n\}$ is a Cauchy sequence.*

*A sequence is a Cauchy sequence if for any $\epsilon > 0$, there is a $N > 0$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$.*

*Let $n > m$, the distance*

$$\begin{aligned} d(x_n, x_m) &< d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \\ &< d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ &= Cr^{n-1} + Cr^{n-2} + ... + Cr^{m+1} \\ &= C(r^{n-1} + r^{n-2} + ... + r^{m+1}) \\ &= Cr^m \frac{1 - r^{n-m}}{1 - r} \\ &< Cr^m/(1 - r) = \epsilon \Rightarrow \\ N &= ceil(\log((1 - r)\epsilon/C)/\log(r)) \end{aligned} \tag{1.23}$$

*which means, when $n, m > N$, $d(x_n, x_m) < \epsilon$*

**Exercise 1.0.8.** *(Ex.2.1-21, pag. 123)(**Solved**)*
*Let $p_n = (x_n, y_n, z_n) \in R^3$. Show that if $\{p_n\}$ is a Cauchy sequence with the following metric*

$$d(p_j, p_k) = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2}, \tag{1.24}$$

*then $\{x_n\}$, $\{y_n\}$ e $\{z_n\}$ are also Cauchy sequences with the following metric $d(x_j, x_k) = |x_j - x_k|$.*

***Demonstration:***
*A sequence is called Cauchy sequence if for every $\varepsilon > 0$, there is a $N > 0$ such that $d(x_n, x_m) < \varepsilon$ for any $m, n > N$.*
*$p_n = (x_n, y_n, z_n) \in R^3$ is a Cauchy sequence with the metric:*

$$d(p_j, p_k) = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2} \tag{1.25}$$

*Therefore, for $j, k > N$,*
*$d(p_j, p_k) = \sqrt{\delta_x^2 + \delta_y^2 + \delta_z^2} \leq \varepsilon \Rightarrow \delta_x^2 + \delta_y^2 + \delta_z^2 \leq \varepsilon^2 \Rightarrow 0 \leq \delta_x^2 \leq \varepsilon^2 - \delta_y^2 - \delta_z^2 \leq \varepsilon^2 \Rightarrow |\delta_x| = |x_j - x_k| \leq \varepsilon, \bigvee j, k > N$, which means, each sequence $x_n, y_n$ and $z_n$ are Cauchy sequences under the norm $d(x, y) = |x - y|$, q.e.d.*

**Exercise 1.0.9.** *(Ex.2.3-30, pag. 124)*
*Let $S$ the set of all solutions of the following differential equation defined in $C^3[0, \infty]$*

$$\frac{d^3x}{dt^3} + b\frac{d^2x}{dt^3} + c\frac{dx}{dt} + dx = 0. \tag{1.26}$$

*Show that $S$ is a linear subspace of $C^3[0,\infty]$.*

**Exercise 1.0.10.** *(Ex.2.3-33, pag. 124)* ***(Solved)***
*Show that in a normed linear space*

$$|\|x\| - \|y\|| \leq \|x - y\|. \tag{1.27}$$

*If $x = y$ the demonstration is trivial. Let $\|x\| > \|y\|$. The goal is to prove that*

$$\begin{aligned} \|x\| - \|y\| < \|x - y\| \Rightarrow \\ \|x\| < \|y\| + \|x - y\| \end{aligned} \tag{1.28}$$

*and making $e = x - y \Rightarrow x = y + e$ we obtain*

$$\|e + y\| < \|y\| + \|e\| \tag{1.29}$$

*which is true because results directly from the triangular inequality. If $\|x\| < \|y\|$ the demonstration is similar.*

**Exercise 1.0.11.** *(Ex.2.3-34, pag. 124)* ***(Solved)***
*Prove the convexity of the norm function.*
***Demonstration:***
*By definition, a scalar function $f : X \times X \longrightarrow R$ is a convex function if and only if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$ with $0 \leq \alpha \leq 1$.*
*The norm is $\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|(1-\alpha)y\| = |\alpha|\|x\| + |1-\alpha|\|y\| = \alpha\|x\| + (1-\alpha)\|y\|$ q.e.d.*

**Exercise 1.0.12.** *(Ex.2.4-40, pag. 124)*
*Compute the inner product $\langle \mathbf{f}, \mathbf{g} \rangle$ using the definition*

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(t)g(t)dt. \tag{1.30}$$

1. $f(t) = t^2 + 2t$, $g(t) = t + 1$.
2. $f(t) = e^{-t}$, $g(t) = t + 1$.
3. $f(t) = cos(2\pi t)$, $g(t) = sin(2\pi t)$.

**Exercise 1.0.13.** *(Ex.2.4-41, pag. 125)*
*Compute the inner product based on the Eucliden norm using a vector operation,*

1. $x = [1, 2, -3, 4]^T$, $y = [2, 3, 4, 1]^T$.
2. $x = [2, 3]^T$, $y = [1, -2]^T$.

**Exercise 1.0.14.** *(Ex.2.5-43, pag. 125)(**Solved**)*
*Show that for an induced norm $\|.\|$ over a real vector space:*

1. *The parallelogram law is true:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.31}$$

*Illustrate graphically the parallelogram law.*
***Demonstration:***
$\|x+y\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = \|x\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + \|y\|^2$ *and*
$\|x-y\|^2 = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle - 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = \|x\|^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle + \|y\|^2$.
*Summing both previous equations, leads to:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1.32}$$

*q.e.d.*

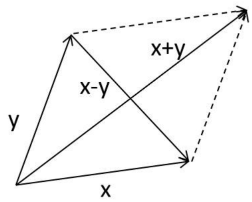


Figure 1.1: Parallelogram law.

2. *Show that*

$$\langle \mathbf{x}, \mathbf{y}\rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4}, \tag{1.33}$$

*the polarization identity.*
***Demonstration:***

$$\|x + y\|^2 - \|x - y\|\ 2 = 4\langle \mathbf{x}, \mathbf{y}\rangle \Rightarrow \langle \mathbf{x}, \mathbf{y}\rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4} \tag{1.34}$$

*q.e.d.*

**Exercise 1.0.15.** *(Ex.2.6-44, pag. 125)*
*For the following inner product operation*

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(t)g(t)dt \tag{1.35}$$

*show that the Cauchy-Schwartz inequality is true for*

1. *(Solved) $f(t) = e^{-t}$, $g(t) = t + 1$.*
   ***Demonstration:***
   $\|f\| = \int_0^1 \varepsilon^{-2t}dt = -\frac{1}{2}(\varepsilon^{-2t})|_0^1 = -\frac{1}{2}(\varepsilon^{-2} - 1) = \frac{1}{2}(1 - \varepsilon^{-2})$
   $\|g\| = \int_0^1 (t+1)^2 dt = \int_0^1 (t^2 + 2t + 1)dt = (\frac{t^3}{3} + t^2 + t)|_0^1 = \frac{1}{3} + 1 + 1 = \frac{7}{3}$
   $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 (t+1)\varepsilon^{-t}dt = \int_0^1 (t\varepsilon^{-t} + \varepsilon^{-t})dt = -t\varepsilon^{-t}|_0^1 + \int_0^1 \varepsilon^{-t} + \int_0^1 \varepsilon^{-t} = -(t+2)\varepsilon^{-t}|_0^1 = 2 - 3\varepsilon^{-1}$

   $\|f\|\|g\| = \frac{7}{6}(2 - 3\varepsilon^{-2}) = 1.0088$

   $|\langle \mathbf{f}, \mathbf{g} \rangle| = |2 - 3\varepsilon^{-1}| = 0.8964 \Rightarrow \|f\|\|g\| > |\langle \mathbf{f}, \mathbf{g} \rangle|$
   *q.e.d*

2. $f(t) = e^{-t}$, $g(t) = -5e^{-t}$

**Exercise 1.0.16.** *(Ex.2.6-45, pag. 125)(**Solved**)*
*Show that the following inequality is true:*

$$-1 \leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|x\|_2 \|y\|_2} \leq 1 \tag{1.36}$$

**Demonstration:**
Cauchy-Schwartz inequality: $\|x\|\|y\| \geqslant |\langle \mathbf{x}, \mathbf{y} \rangle|$

Se $\langle \mathbf{x}, \mathbf{y} \rangle > 0 \Rightarrow \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|x\|\|y\|} \leqslant 1$

Se $\langle \mathbf{x}, \mathbf{y} \rangle < 0 \Rightarrow -\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|x\|\|y\|} \leqslant 1 \Rightarrow \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|x\|\|y\|} \geqslant -1 \Rightarrow -1 \leqslant \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|x\|\|y\|} \leqslant 1$

**Exercise 1.0.17.** *(Ex.2.7-47, pag. 125)*
*Let $x_1(t) = 3t^2 - 1$, $x_2(t) = 5t^3 - 3t$, $x_3(t) = 2t^2 - t$ and $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^1 f(t)g(t)dt$. Compute the angle of each pairwise combination of these functions and identify the pairs that are orthogonal.*

**Exercise 1.0.18.** *(Ex.2.7-48, pag. 125)(**Solved**)*
*Let*

$$\begin{aligned} x_1 &= [1, 2, 4, -2]^T \\ x_2 &= [5, -2, -3, 1]^T \\ x_3 &= [1, 2, 1, 2]^T. \end{aligned}$$

*Compute the angle between these vectors using the Euclidean inner product and identify the pairs that are orthogonal.*

$$\|x_1\| = \sqrt{< x_1, x_1 >} = \sqrt{1+4+16+4} = \sqrt{25} = 5$$
$$\|x_2\| = \sqrt{< x_2, x_2 >} = \sqrt{25+4+9+1} = \sqrt{39}$$
$$\|x_3\| = \sqrt{< x_3, x_3 >} = \sqrt{1+4+1+4} = \sqrt{10}$$

$$cos(\theta_{1,2}) = \frac{< x_1, x_2 >}{\|x_1\|\,\|x_2\|} = \frac{5-4-12-2}{5\sqrt{39}} \longrightarrow \theta_{1,2} = acos(\frac{-23}{5\sqrt{39}}) = 137,4417$$
$$cos(\theta_{1,3}) = \frac{< x_1, x_3 >}{\|x_1\|\,\|x_3\|} = \frac{1+4+4-4}{5\sqrt{10}} = \frac{1}{\sqrt{10}} \longrightarrow \theta_{1,3} = 71,5651$$
$$cos(\theta_{2,3}) = \frac{< x_2, x_3 >}{\|x_2\|\,\|x_3\|} = \frac{5-4-3+2}{\sqrt{390}} = \frac{0}{\sqrt{390}} \longrightarrow \theta_{2,3} = \pi/2$$

The vectors $x_2$ e $x_3$ are orthogonal.

**Exercise 1.0.19.** *Show that $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are orthogonal in $L^2(-\pi, \pi)$.*

**Exercise 1.0.20.** *(**solved**)*
*Let $V$ the space spanned by the following functions $p_1(x) = \cos(x)/\sqrt{\pi}$ and $p_2(x) = \sin(x)/\sqrt{\pi}$ in $L^2(-\pi, \pi)$.*

1. *Show that $\{p_1, p_2\}$ is an orthonormal bais .*
   *To be an orthonormal basis its elements need to be pairwise orthogonal with unit norm, that is, $< e_i, e_j >= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$*
   *$< e_1, e_2 >=< e_2, e_1 >= 0$, como mostrado no exerccio anterior.*

   $< e_1, e_1 >= \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 + cos(2t))dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1dt + \frac{1}{4\pi}\int_{-\pi}^{\pi} 2cos(2t)dt = \frac{2\pi}{2\pi} + \frac{1}{4\pi}sin(2t)|_{-\pi}^{\pi} = 1 + 0 = 1$

   $< e_2, e_2 >= \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - cos(2t))dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1dt - \frac{1}{4\pi}\int_{-\pi}^{\pi} 2cos(2t)dt = \frac{2\pi}{4\pi} - \frac{1}{4\pi}sin(2t)|_{-\pi}^{\pi} = 1 - 0 = 1$

2. *Compute the projection of $f(x) = x$ into $V$.*
   $f_0 =< f, e_1 > e_1+ < f, e_2 > e_2$

$$
\begin{aligned}
< f, e_1 > &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) cos(x) dx = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x cos(x) dx \\
&= \frac{1}{\sqrt{\pi}} \left[ x sin(x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} sin(x) dx \right] = 0 - \frac{1}{\sqrt{\pi}} cos(x)|_{-\pi}^{\pi} = 0 \\
< f, e_2 > &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) sin(x) dx = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x sin(x) dx \\
&= \frac{1}{\sqrt{\pi}} \left[ -x cos(x)|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} cos(x) dx \right] = \\
&= \frac{1}{\sqrt{\pi}} \left[ 2\pi + sin(x)|_{-\pi}^{\pi} \right] = \frac{2\pi}{\sqrt{\pi}} = 2\sqrt{\pi} \\
f_0 &= < f, e_2 > e_2 = 2\sqrt{\pi} \frac{sin(x)}{\sqrt{\pi}} = 2 sin(x)
\end{aligned}
$$

3. *Consider $W$ the space spanned by $\phi(x) = 1$ and $\psi = \begin{cases} 1, & 0 \le x \le 1/2; \\ -1, & 1/2 \le x \le 1. \end{cases}$ in the interval $[0, 1]$. The functions $\phi(x)$ and $\psi(x)$ are called scale and wavelet functions respectively.*

   (a) *Show that these functions are orthogonal in $L^2(0, 1)$.*
   $< \phi, \psi >= \int_0^1 \phi(x)\psi(x)dx = \int_0^{1/2} 1dx - \int_{1/2}^1 1dx = x|_0^{1/2} - x|_{1/2}^1 = 1/2 - (1 - 1/2) = 0$

   (b) *Compute the projection of $f(x) = x$ in $W$.*
   $< f, \phi >= \int_0^1 f(x)dx = \int_0^1 xdx = \frac{x^2}{2}|_0^1 = 1/2$
   $< f, \psi >= \int_0^{1/2} xdx - \int_{1/2}^1 xdx = \frac{x^2}{2}|_0^{1/2} - \frac{x^2}{2}|_{1/2}^1 = \frac{1}{8} - \left[\frac{1}{2} - \frac{1}{8}\right] = -\frac{1}{4}$
   $f_1(x) =< f, \phi > \phi(x)+ < f, \psi > \psi(x) = 1/2\phi(x) - 1/4\psi(x) = \begin{cases} 1/4 & 0 \le x < 1/2 \\ 3/4 & 1/2 \le x < 1 \end{cases}$

**Exercise 1.0.21.** *(Ex.2.7-49, pag. 126)*
*Show that a set of nonzero vectors $\{p_1, p_2, ..., p_m\}$ that are mutually orthogonal, so that, $\langle \mathbf{p_i}, \mathbf{p_j} \rangle = 0, i \neq j$, is linearly independent. (Orthogonality implies linear independence.)*

**Exercise 1.0.22.** *(Ex.2.12-57, pag. 126)*
*Show that if $V$ and $W$ are subspaces of a vector space $S$ then their intersection, $V \bigcap W$, is also a subspace.*

**Exercise 1.0.23.** *(Ex.2.12-64, pag. 127)(**Solved**)*
*If $v$ is a vector, show that the matrix which projects onto span($v$) is*

$$P_v = \frac{vv^H}{v^H v}. \tag{1.37}$$

**Demonstration:** By the orthogonality principle

$$\begin{cases} \langle \mathbf{x} - \alpha \mathbf{v}, \mathbf{v} \rangle = 0 \\ \mathbf{y} = \alpha v \end{cases} \quad \begin{cases} v^T x = \alpha v^T v \Rightarrow \alpha = \frac{v^T x}{v^T v} \\ \mathbf{y} = v \frac{v^T x}{v^T v} = \frac{v^T x}{v^T v} x = P_v x \end{cases}$$

**Exercise 1.0.24.** *(Ex.2.13-68, pag. 127)(**Solved**)*
*Let $A$ be a matrix which can be factored as*

$$A = U \Sigma V^H$$

*where $U^H U = I$, $V^H V = I$. $\Sigma$ is a real diagonal matrix. This factorization is called single value decomposition (SVD).*

*Let $A = [p_1, p_2, ..., p_n]$ and $V = span(p_1, p_2, ..., p_n)$ and the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = y^T x$, between $v = P_A x$ where $P_A = A(A^H A)^{-1} A^H$. Show that $P_A = P_U$.*

**Demonstration:**

$$\begin{aligned} P_A &= (U\Sigma V^H)((U\Sigma V^H)^H (U\Sigma V^H))^{-1} (U\Sigma V^H)^H = \\ &= (U\Sigma V^H)(V\Sigma^H U^H U \Sigma V^H)^{-1} (V\Sigma^H U^H) = \\ &= (U\Sigma V^H)(V\Sigma^H \Sigma V^H)^{-1} (V\Sigma^H U^H) = \\ &= (U\Sigma V^H)(V(\Sigma^H \Sigma)^{-1} V^H)(V\Sigma^H U^H) = \\ &= U\Sigma(\Sigma^H \Sigma)^{-1} \Sigma^H U^H = U\Sigma\Sigma^{-1}\Sigma^{-H}\Sigma^H U^H = I \\ P_U &= U(U^H U)^{-1} U^H = I \end{aligned}$$

q.e.d.

**Exercise 1.0.25.** *(Ex.3.8-3, pag. 216)*
*Consider the following data sets $x = \{2, 2.5, 3, 5, 9\}$ e $y = \{-4.2, -5, 2, 1, 24.3\}$.*

1. *Display graphically these data sets.*
2. *Compute the linear regression parameters ($y = ax + b$) that better fits the data under the least square (LS) criterion.*

3. *Assuming that the first and last observations are more accurate formulate the estimation process under the weight LS approach, by specifying the weight matrix. Compute the new regression parameters, $a$ and $b$.*

**Exercise 1.0.26.** *(Ex.3.8-(4-7), pag. 216)*
*Formulate the regression problem for the following models:*

$$\begin{aligned} y_i &= a_0 + a_1 x_i + a_2 x_i^2 \\ z_i &= a x_i + b y_i + c \\ y_i &= c e^{a x_i} \\ y_i &= a x_i^b \end{aligned}$$

**Exercise 1.0.27.** *(Ex.3.8-10, pag. 217)*
*Consider the following definition of inner product of matrices*

$$\langle \mathbf{X}, \mathbf{Y} \rangle = trace(XY^H).$$

*The goal is to approximate the matrix $Y$ with the following linear combination,*

$$Y = c_1 X_1 + c_2 X_2 + ... + c_m X_m + E.$$

*Using the orthogonality principle, determine a set of normal equations that can be used to find $\{c_1, c_2, ..., c_m\}$ that minimize the indiced norm of $E$.*

**Exercise 1.0.28.** *(Ex.3.9-12, pag. 217)*
*For the data sequence, $\{1, 1, 2, 3, 5, 8, 13\}$,*

1. *Write the data matrix and the Grammian $A^H A$ using (i) covariance, and (ii) the autocorrelation methods. Assume $m = 2$.*

2. *This sequence will be used to train a linear predictor. The desired signal $d(t)$ is the value of $x(t)$, and the data used are the two prior samples,*

$$x(t) = a_1 x(t-1) + a_2 x(t-2) + e(t),$$

*where $e(t)$ is the prediction error. Determine the least squares coefficients for the predictor using the covariance and autocorrelation methods.*

3. *Determine the minimum least-squares error for both methods.*

**Exercise 1.0.29.** *(Ex.3.15-30, pag. 223)(**Solved**)*
*Using the projection theorem, solve the finite dimensional problem*

$$\begin{aligned} \hat{\mathbf{x}} &= \arg\min_x \mathbf{x}^H Q \mathbf{x} \\ \textit{subject to } A\mathbf{x} &= \mathbf{b} \end{aligned}$$

*where* $\mathbf{x} \in C^n$*,* $Q$ *is a positive-definite symmetric matrix, and* $A$ *is an* $m \times n$ *matrix with* $m < n$*.*

**Demonstration:**

$$\mathrm{A}\mathbf{x} = \mathbf{b} \Rightarrow \begin{bmatrix} \mathbf{y}_1^H \\ \mathbf{y}_2^H \\ \vdots \\ \mathbf{y}_m^H \end{bmatrix} \mathbf{x} = \mathbf{b} \Rightarrow \begin{cases} \langle \mathbf{x}, \mathbf{y_1} \rangle = b_1 \\ \langle \mathbf{x}, \mathbf{y_2} \rangle = b_2 \\ \vdots \\ \langle \mathbf{x}, \mathbf{y_m} \rangle = b_m \end{cases}$$

where $\mathbf{y}_i^H$ are the lines of $A$, $\hat{\mathbf{x}} = [\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_m]\mathbf{c} = A^H \mathbf{c}$ and it is assumed that the solution belongs to the subspace spanned by the vectors $\mathbf{y}_i$,that is, $\hat{\mathbf{x}} \in V = span(\mathbf{y}_1, \cdots, \mathbf{y}_m)$.

Let $\mathbf{x} = \hat{\mathbf{x}} + \varepsilon$ be the system of equations described above:

$$\begin{cases} \langle \hat{\mathbf{x}} + \varepsilon, \mathbf{y_1} \rangle_Q = b_1 \\ \langle \hat{\mathbf{x}} + \varepsilon, \mathbf{y_2} \rangle_Q = b_2 \\ \vdots \\ \langle \hat{\mathbf{x}} + \varepsilon, \mathbf{y_m} \rangle_Q = b_m \end{cases} \Rightarrow \begin{cases} \langle \hat{\mathbf{x}}, \mathbf{y_1} \rangle_Q + \langle \varepsilon, \mathbf{y_1} \rangle_Q = b_1 \\ \langle \hat{\mathbf{x}}, \mathbf{y_2} \rangle_Q + \langle \varepsilon, \mathbf{y_2} \rangle_Q = b_2 \\ \vdots \\ \langle \hat{\mathbf{x}}, \mathbf{y_m} \rangle_Q + \langle \varepsilon, \mathbf{y_m} \rangle_Q = b_m \end{cases} \Rightarrow AQ\hat{\mathbf{x}} = \mathbf{b}$$

where it was assumed that $\langle \varepsilon, \mathbf{y_1} \rangle_Q = \langle \varepsilon, \mathbf{y_2} \rangle_Q = \cdots = \langle \varepsilon, \mathbf{y_m} \rangle_Q = \oslash$ because $\hat{\mathbf{x}} \in V$.

By the dual approximation $\hat{\mathbf{x}} \in V \Rightarrow \hat{\mathbf{x}} = A^T c$.

$$\begin{cases} AQ\hat{\mathbf{x}} = \mathbf{b} \\ \hat{\mathbf{x}} = A^T \mathbf{c} \end{cases} \Rightarrow AQA^T \mathbf{c} = \mathbf{b} \Rightarrow \mathbf{c} = (AQA^T)^{-1}\mathbf{b} \Rightarrow \hat{\mathbf{x}} = A^T(AQA^T)^{-1}\mathbf{b}$$

**Exercise 1.0.30.** *Compute the angle between signals* $t$ *and* $t^2$*, defined in the interval* $[-1, 1]$

**Exercise 1.0.31.** *Let* $g(x, y) = e^{d(x,y)}$ *where* $d(x, y)$ *is a metric function. Is* $g(x, y)$ *a metric function? Why?*

# Chapter 2

# Transforms

**Exercise 2.0.32.** *Verifique que as funes* $\phi_k(n)$ *e* $\phi_r(n)$ *so ortogonais em qualquer intervalo de dimenso* $N$*, por exemplo,* $[n_0, n_0+N-1]$ *e que formam um conjunto completo, isto , apenas a sequncia nula  ortogonal a todas as funes de base.*

$$\begin{aligned}
\langle \phi_{\mathbf{k}}(\mathbf{n}), \phi_{\mathbf{r}}(\mathbf{n}) \rangle &= \sum_{n=n_0}^{n_0+N-1} \phi_k(n)\phi_r^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}rn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \frac{1}{N}\left\{ \begin{array}{ll} N, & k=r; \\ \frac{1-e^{j2\pi(k-r)}}{1-e^{-j\frac{2\pi}{N}(k-r)}}, & k \neq r. \end{array} \right. \\
&= \left\{ \begin{array}{ll} 1, & k=r; \\ 0, & k \neq r. \end{array} \right. = \delta(k-r) \qquad (2.1)
\end{aligned}$$

Estas funes, de perodo $N, N/2, N/3, ..., 1, 0$, para $k = 1, 2, ..., N, 0$ respectivamente, podem ento ser utilizadas para representar sequncias discretas de perodo $T \leq N$ atravs da seguinte combinao linear

$$\tilde{x}(n) = \sum_{k=0}^{N} X(k)\phi_k(n) = \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn} \qquad (2.2)$$

em que $X(k)$ so os coeficientes da combinao linear.

**Exercise 2.0.33.** *Deduza a equao que permite calcular os coeficientes* $X(k)$ *a partir de* $\tilde{x}(n)$*, equao de anlise. Sugesto: Calcule* $\langle \phi_{\mathbf{r}}(\mathbf{n}), \sum_{\mathbf{k=0}}^{\mathbf{N}} \mathbf{X}(\mathbf{k})\phi_{\mathbf{k}}(\mathbf{n}) \rangle$

**Exercise 2.0.34.** *Verifique a seguinte igualdade,*

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} = \left\{ \begin{array}{ll} 1, & se\ k = mN; \\ 0, & caso\ contrrio. \end{array} \right. \qquad (2.3)$$

**Exercise 2.0.35.** *Considere a seguinte sequncia peridica (onda quadrada) de perodo* $N = 10$, $x(n) = ...1, 1, 1, 1, 1, 0, 0, 0, 0, 0, ....$ *e o sistema descrito pela equao:* $y(n) = 0.8y(n-1) + 0.2x(n)$.

1. *Calcule a DFS do sinal peridico* $x(n)$.
2. *Qual  a sada e a respectiva DFS?*

**Exercise 2.0.36.** *Considere o sistema linear de 1 ordem* $y(n) = ay(n - 1 + x(n)$ *com* $a \in [0, 1[$ *e* $x(n)$ *peridico de perodo* $N$*. Determine a resposta impulsiva do filtro FIR com uma resposta forada igual  do filtro apresentado.*

**Exercise 2.0.37.** *Considere as seguintes sequncias peridicas de perodo* $N = 4$,

$$\tilde{x}(n) = ...0, 2, 0, 1, ...$$
$$\tilde{y}(n) = ... - 1, 0, 1, 0, ...$$

1. *Determine os coeficientes da srie de Fourier discreta de ambas as sequncias.*
2. *Usando os coeficientes que calculou na alnea anterior, calcule a convoluo peridica das sequncias.*
3. *Considere a sequncia* $\tilde{z}(n) = \tilde{x}(n) + \tilde{y}(n)$*. Determine a sua convoluo com* $\tilde{x}(n)$.

**Exercise 2.0.38.** *Considere a sequncia* $\tilde{w}(n) = \tilde{x}(n) + \tilde{y}(n)$*. Sabendo que* $\tilde{x}(n)$ *tem perodo* $N$ *e que* $\tilde{y}(n)$ *tem perodo* $M$*:*

1. *Mostre que* $\tilde{w}(n)$ *tem perodo* $NM$
2. *Determine* $\tilde{W}(K)$ *em funo de* $\tilde{X}(k)$ *e de* $\tilde{Y}(k)$.

**Exercise 2.0.39.** *Calcular a DFT das seguintes sequncias:*

1. $x(n) = \delta(n)$.
2. $x(n) = \delta(n - n_0)$ *para* $0 \leq n_0 \leq N$.
3. $x(n) = a(n)$ *para* $0 \leq n \leq N - 1$.

**Exercise 2.0.40.** *Dada a sequncia de durao finita* $x(n) = \{1, 2, 3\}$ *com* $0 \leq n \leq 3$*, desenhar* $x((-n)_4)$.

**Exercise 2.0.41.** *Pretende-se determinar o espectro de um sinal analgico usando uma DFT de 1024 pontos. Se o sinal for amostrado a* $10kHz$*, qual a resoluo que se obtm em frequncia?*

**Exercise 2.0.42.** *A DFT de uma sequncia de durao finita corresponde a amostras da sua transformada Z no circulo de raio unitrio. Por exemplo, a DFT de ordem 10 da sequncia x(n) de comprimento 10, corresponde a 10 amostras de* $X(z)$ *igualmente espaadas. Pretende-se determinar a sequncia* $x_1(n)$ *que tem como DFT as amostras de X(z) em* $z = 0.5e^{j\frac{2\pi}{10}k+\frac{\pi}{10}}$.

**Exercise 2.0.43.** *Uma sequncia de durao finita tem a seguinte DFT:* $X(k) = \{4, 3, 2, 1, 0, 1, 2, 3,$ *para* $0 \leq k < 8$*. Determine os 16 pontos da DFT da sequncia* $y(n)$ *obtida da seguinte forma*

$$y(n) = \begin{cases} x(n/2), & n \text{ par;} \\ 0, & n \text{ mpar.} \end{cases} \tag{2.4}$$

**Exercise 2.0.44.** *Considere a sequncia de durao finita x(n) com* $x(n) = 0$ *para* $n \in Z\backslash[0, N-1]$*. Pretende-se determinar M amostras de* $X(z)$ *igualmente espaadas sobre o crculo de unitrio com* $M < N$*. Uma das amostras est em* $z = 1$*. Como obter as M amostras de* $X(z)$ *usando uma DFT de M pontos de uma sequncia de M pontos obtida a partir de x(n)?*

**Exercise 2.0.45.** *Considere duas sequncias de durao finita* $x(n)$ *e* $y(n)$*, nulas para* $n < 0$ *e* $\begin{pmatrix} x(n) = 0 & n \geq 8 \\ y(n) = 0 & n \geq 20 \end{pmatrix}$*. As DFTs de 20 pontos de ambas as sequncias so multiplicadas e calculada a transformada inversa do resultado,* $r(n)$*. Indique quais os pontos de* $r(n)$ *que correspondem a pontos que se obteriam pela convoluo linear de* $x(n)$ *com* $y(n)$*.*

**Exercise 2.0.46.** *Pretende-se realizar um filtro FIR de ordem 50 com a tcnica de overlap-save: i) sobreposio de V amostras nas seces de entrada, ii) das sadas de cada seco devem-se retirar M amostras que se devem juntar para obter a sada desejada, filtrada. Assuma que os segmentos de entrada tm 100 amostras de comprimento e que a DFT tem dimenso 128. Alm disso, assuma que a sequncia de sada da convoluo circular indexada do ponto 0 ao ponto 127.*

1. *Determine V.*
2. *Determine M.*
3. *Determine o ndice do incio e do fim dos M pontos extrados, ou seja, quais dos 128 pontos da convoluo circular devero ser extrados para juntar aos resultados da seco anterior.*

**Exercise 2.0.47.** *Usando a definio, determine a DFT das seguintes sequncias (caso no existam diga a razo).*

1. $x(n) = 0.5^n u(n)$
2. $x(n) = 0.5^{|n|}$
3. $x(n) = 2^n u(-n)$
4. $x(n) = 0.5^n u(-n)$
5. $x(n) = 2^{|n|}$
6. $x(n) = 3(8)^{|n|} cos(0.1\pi n)$

**Exercise 2.0.48.** *Calcule a DFT das seguintes sequncias:*

1. $x(n) = [1, 0, -1, 0]$
2. $x(n) = [j, 0, j, 1]$
3. $x(n) = [1, 1, 1, 1, 1, 1, 1, 1]$
4. $x(n) = cos(0.25\pi n)$ *para* $n = 0, ..., 7$.
5. $x(n) = 0.9^n$ *para* $n = 0, ..., 7$.

**Exercise 2.0.49.** *Sejam* $X(k) = [1, j, -1, -j]$ *e* $H(k) = [0, 1, -1, 1]$ *as DFTs de duas sequncias* $x(n)$ *e* $h(n)$ *respectivamente. Usando as propriedades da DFT (sem calcular explicitamente* $x(n)$ *e* $y(n)$*) determine as DFTs das seguintes sequncias:*

1. $x((n-1)_4)$
2. $x((n+3)_4)$
3. $y(n) = h(n) \circledast x(n)$, *em que* $\circledast$ *significa convoluo circular.*
4. $(-1)^n x(n)$
5. $j^n x(n)$
6. $x((-n)_4)$
7. $x((2-n)_4)$

**Exercise 2.0.50.** *Duas sequncias finitas $h(n)$ e $x(n)$ tm as seguintes DFTs, $X(k) = DFT(x) = [1, -2, 1, -2]$ e $H(k) = DFT(h) = [1, j, 1, -j]$. Seja $y(n)$ uma sequncia de dimenso 4, obtida atravs da convoluo circular de $h(n)$ com $x(n)$, i.e., $y(n) = h(n) \circledast x(n)$. Sem calcular explicitamente $h(n)$ e $x(n)$,*

1. *determine $DFT(x((n-1)_4))$ e $DFT(h((n+2)_4))$ onde os deslocamentos so circulares.*
2. *determine os valores $y(0)$ e $y(1)$.*

**Exercise 2.0.51.** *Seja $X(k)$ a DFT de $x(n)$ e sejam as seguintes sequncias:*

$$s(n) = [x(0), ..., x(N-1), x(0), ..., x(N-1)] \tag{2.5}$$

$$y(n) = [x(0), ..., x(N-1), \underbrace{0, ..., 0}_{N zeros}]. \tag{2.6}$$

1. *Mostre que $S(2k) = Y(2k) = X(k)$, para $k = 0, ..., N-1$.*
2. *Mostre que $S(2k+1) = 0$, para $k = 0, ..., N-1$.*
3. *Se $y(n) = s(n)w(n)$, com $w(n) = [\underbrace{1, ..., 1}_{N}, \underbrace{0, ..., 0}_{N}]$, determine uma expresso (frmula de interpolao) para $Y(2k+1)$.*

**Exercise 2.0.52.** *Um problema que existe em muitos pacotes de clculo cientfico (como o MatLab, p.ex.)  o facto de apenas se ter acesso a ndices positivos. Suponha que quer determinar a DFT da seguinte sequncia: $x(n) = [0, 1, 2, 3, \underbrace{4}_{x(0)}, 5, 6, 7]$, e o algoritmo apenas lhe permite calcular a DFT de $[x(0), x(1), ..., x(N-1)]$. Como resolveria o problema?*

**Exercise 2.0.53.** *Seja $X(k) = DFT(x(n))$ com $n, k = 0, ..., N-1$. Determine a relao entre $X(k)$ e as seguintes DFTs,*

1. *$DFT(x^*(n))$*
2. *$DFT(x((-n)_N))$*
3. *$DFT(Re(x(n)))$*
4. *$DFT(Im(x(n)))$*
5. *Aplique os resultados anteriores  sequncia $x(n)$ tal que $DFT(x(n)) = [1, 2, 3, 4, 5, 6, 7, 8]$.*

**Exercise 2.0.54.** *Seja* $X(k) = DFT(x(n))$ *com* $n, k = 0, ..., N-1$. *Determine a relao entre* $X(k)$ *e as seguintes DFTs,*

1. $IDFT(X^*(k))$
2. $IDFT(X((-k)_N))$
3. $IDFT(Re(X(k)))$
4. $IDFT(Im(X(k)))$
5. *Aplique os resultados anteriores sequncia* $X(k)$ *em que* $IDFT(X(k)) = [1, 2j, -2j, 0, 1, -j, j]$.

**Exercise 2.0.55.** *Seja* $x(n)$ *uma sequncia peridica cujo perodo* $[1, -2, \underbrace{3}_{x(0)} -45-6]$. *Esta sequncia a entrada de um sistema linear e invariante no tempo cuja resposta impulsiva* $h(n) = 0.8^{|n|}$. *Determine um perodo da sequncia da sada* $y(n)$.

**Exercise 2.0.56.** *Considere uma DFT de 9 pontos.*

1. *Desenhe o diagrama de fluxo com decimao no tempo usando uma decomposio* $3 \times 3$.
2. *Quantas multiplicaes complexas por potncias de* $e^{j2\pi/N}$ *so necessrias?*
3. *possvel realizar todos os clculos usando apenas um vector (in place)?*

**Exercise 2.0.57.** *O clculo da DFT exige geralmente a realizao de produtos de nmeros complexos. Considere o produto,* $(a + jb)(c + jd) = (ac - bd) + j(bc + ad)$. *Nesta forma a multiplicao complexa requer 4 multiplicaes e 2 somas reais. Verifique que a multiplicao complexa pode tambm ser efectuada com 3 multiplicaes e 5 somas reais:* $[(a-b)d + (c-d)a] + j[(a-b)d + (c+d)b]$.

**Exercise 2.0.58.** *Considere as seguintes sequncias de comprimento 8:*

$$\begin{aligned} x(n) &= [3, 0, 2, 0, 1, 0, -2, 0] \\ y(n) &= [2, 0, 1, 0, 2, 0, 1, 0]. \end{aligned} \tag{2.7}$$

*Resolva as alneas seguintes usando o menor nmero de operaes possvel.*

1. *Determine a DFT de* $x(n)$ *usando o algoritmo FFT.*
2. *Determine a DFT de* $y(n)$ *usando o algoritmo FFT.*

3. *Determine a sequncia $z(n)$ resultante da convoluo circular de ordem 8 de $x(n)$ com $y(n)$.*

**Exercise 2.0.59.** *Construa o grfico de fluxo do algoritmo FFT de base 2 com 16 pontos e dcimao no tempo. Indique os factores multiplicativos diferentes de 1. Indique as amostras das sequncias de entrada e de sada. Determine o nmero de multiplicaes e somas reais necessrias.*

**Exercise 2.0.60.** *Considere a sequncia complexa $x(n)$ de comprimento $N$ que satisfaz a seguinte condio de simetria: $x(n) = -x(((n * N/2))_N)$ com $0 \leq n \leq N-1$ em que $N$ par.*

1. *Mostre que a DFT de $X(k) = 0$ quando $k$ par.*
2. *Indique como calcular os termos impares da DFT de $x(n)$ usando apenas uma DFT de $N/2$ pontos com alguns clculos adicionais.*

**Exercise 2.0.61.** *Considere uma sequncia $x(n)$ de comprimento $N$ cuja DFT $X(k)$ com $k \in [0, N-1]$. O algoritmo seguinte calcula as amostras pares da DFT, para $N$ par, usando uma nica DFT de $N/2$ pontos.*

- *Crie a sequncia* $y(n) = \begin{cases} x(n) + x(n+N/2), & 0 \leq n \leq N/2-1; \\ 0, & \textit{caso contrrio.} \end{cases}$
- *Calcule a DFT de $N/2$ pontos de $y(n)$, $Y(k)$ com $k \in [0, N/2-1]$.*
- *As amostras pares de $X(k)$ so $X(k) = Y(k/2)$ para $k \in [0, N-1]$.*

1. *Mostre que o algoritmo apresentado produz os resultados pretendidos.*
2. *Considere agora que a sequncia $y(n)$ criada a partir de $x(n)$ por:*

$$y(n) = \begin{cases} \sum_{r=-\infty}^{\infty} x(n+rM), & 0 \leq n \leq M-1; \\ 0, & \textit{caso contrrio.} \end{cases} \tag{2.8}$$

   *Determine a relao entre a DFT de $M$ pontos $Y(k)$ e a transformada de Fourier de $x(n)$. Mostre que a alnea anterior um caso particular desta.*

3. *Desenvolva um algoritmo semelhante ao exposto para calcular as amostras mpares de $X(k)$ usando apenas uma DFT de $N/2$ pontos.*

**Exercise 2.0.62.** *Considere as sequncias $x(n)$ de comprimento $L$ e $h(n)$ de comprimento $P$. Pretende-se calcular a convoluo linear destas duas sequncias, $y(n) = x(n) * h(n)$.*

1. *Qual o comprimento de $y(n)$?*

2. *Quantas multiplicaes reais so necessrias para calcular directamente $y(n)$?*

3. *Defina um procedimento para usar uma DFT na determinao das amostras de $y(n)$. Determine a menor dimenso das transformadas directa e inversa em termos de L e P.*

4. *Assuma que $L = P = N/2$ em que N uma potncia de 2 e a dimenso da DFT. Determine uma expresso para o nmero de multiplicaes reais necessrias para calcular $y(n)$ usando o mtodo da alnea anterior com FFT de base 2. Com base nessa expresso qual o menor valor de N para o qual o mtodo da FFT requer menos multiplicaes reais que o clculo directo da convoluo.*

**Exercise 2.0.63.** *A entrada e a sada de um sistema linear e invariante no tempo satisfazem a seguinte equao s diferenas:*

$$y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \tag{2.9}$$

*Suponha de que dispe de um programa para calcular DFTs de sequncias de comprimento $N = 2^v$. Descreva um procedimento que utilize esse programa e que permita calcular: $H(e^{j\frac{2\pi}{512}k})$ para $k \in [0, 511]$.*

# Chapter 3

# Random Variables

1. Consider the following filter

$$H(z) = -\frac{1}{1 - 0.5z^{-1}} \tag{3.1}$$

   For a zero mean white noise input, $\epsilon(n)$, with variance $\sigma^2 = 1$, what is the mean of the output signal, $\mu_y = E[y(n)]$?

   - □ a) $\mu_y = \sqrt{2}$.
   - □ b) $\mu_y = 1$.
   - □ c) $\mu_y = -\sqrt{2}$.
   - □ d) None

2. Consider the following filter

$$H(z) = -\frac{1}{1 - 0.25z^{-1}} \tag{3.2}$$

   For a zero mean white noise input, $\epsilon(n)$, with variance $\sigma^2 = 1$, what is *power density spectrum* (PDS) of the output signal?

   - □ a) $1/(1.0625 - 0.5cos(\omega))$.
   - □ b) $4/(1.25 - cos(\omega))$.
   - □ c) $4/(1.25 - 0.25cos(\omega))$.
   - □ d) None

3. Let $x(n)$ be a stationary random process with *Power Density Spectrum* (PDS) $P_x(\omega)$ and let $y(n) = x(n) + c$, where $c$ is a constant. What is the PDS of $y$, $P_y(\omega)$?

- ☐ a) $P_y(\omega) = P_x(\omega) + 2\pi c(c + 2\mu_x)\delta(\omega)$.
- ☐ b) $P_y(\omega) = P_x(\omega) + 2\pi c^2\delta(\omega)$.
- ☐ c) $P_y(\omega) = P_x(\omega) + c$.
- ☐ d) None

4. Let us consider the adaptive filter that uses the LMS algorithm where $w_0 = [0, 0, 0, 0]^T$, $x = y = [..., -1, -1, .., -1, 1]$ and

   be a stationary random process with *Power Density Spectrum* (PDS) $P_x(\omega)$ and let $y(n) = x(n) + c$, where $c$ is a constant. What is the PDS of $y$, $P_y(\omega)$?

   - ☐ a) $P_y(\omega) = P_x(\omega) + 2\pi c(c + 2\mu_x)\delta(\omega)$.
   - ☐ b) $P_y(\omega) = P_x(\omega) + 2\pi c^2\delta(\omega)$.
   - ☐ c) $P_y(\omega) = P_x(\omega) + c$.
   - ☐ d) None