

FICHA 10 - SOLUÇÕES

AULA PRÁTICA

1. a) $P(\cos^2 x) = P\left(\frac{\cos(2x) + 1}{2}\right) = \frac{1}{4}\text{sen}(2x) + \frac{x}{2}$; OU por partes, com $u' = v = \cos x$.

b) $P(\text{sen}^3 x) = P(\text{sen} x(1 - \cos^2 x)) = -\cos x + \frac{1}{3}\cos^3 x$; OU por partes, com $u' = \text{sen} x \Rightarrow u = -\cos x$ e $v = \text{sen}^2 x \Rightarrow v' = 2\text{sen} x \cos x$.

c) $\frac{1}{3}\text{sen}^3 x - \frac{1}{5}\text{sen}^5 x$, escrevendo $\cos^2 x = 1 - \text{sen}^2 x$; OU por partes, com $u' = \cos x \text{sen}^2 x$, $v = \cos^2 x$.

d) $\frac{1}{3}\text{ch}^3 x$,

e) $3\text{sh} x + \text{sh}^3 x$, escrevendo $\text{ch}^2 x = \text{sh}^2 x + 1$; OU por partes, com $u' = \text{ch} x$, $v = \text{ch}^2 x$.

f) $\frac{1}{3}\text{tg}^3 x$,

2.

a) $-e^{-x}(x^3 + 3x^2 + 6x + 6)$, b) $x \arcsen x + \sqrt{1 - x^2}$, c) $\frac{2}{3}x^{\frac{3}{2}}(\ln x - \frac{2}{3})$

d) $\frac{1}{2}(-x + (x^2 + 1)\text{arctg} x)$, e) $x \ln^2 x - 2x \ln x + 2x$, f) $\frac{1}{2}\text{sen}(2x) \ln(\text{tg} x) - x$,

g) $\frac{1}{2}(\text{sh} x \cos x + \text{ch} x \text{sen} x)$, h) $\frac{2}{3}x^{3/2} \text{arctg} \sqrt{x} - \frac{1}{3}(x - \ln(1 + x))$,

i) $\frac{x}{2}(\cos(\ln x) + \text{sen}(\ln x))$,

3. Sendo $P\left(\frac{1}{1+x}\right) = \ln(x+1)$, para todo o $x \in]-1, +\infty[$, temos $\psi'(x) = \ln(x+1) + C_1$.

A condição $\psi'(0) = 1$ resulta em $C_1 = 1$. Usando primitivação por partes (verifique) temos $P(\ln(x+1) + 1) = (x+1)\ln(x+1)$, ou seja $\psi(x) = (x+1)\ln(x+1) + C_2$. Dado que $\psi(0) = 1$, obtém-se o resultado $\psi(x) = (x+1)\ln(x+1) + 1$.

4.

a) $P\left(\frac{x}{1+(x-1)^2}\right) = P\left(\frac{(x-1)+1}{1+(x-1)^2}\right) = \frac{1}{2}\ln(1+(x-1)^2) + \text{arctg}(x-1)$,

b) $P\left(\frac{1}{x^2+2x+2}\right) = P\left(\frac{1}{(x+1)^2+1}\right) = \text{arctg}(x+1)$,

c) $P\left(\frac{x+1}{(x+2)^3}\right) = P\left(\frac{(x+2)-1}{(x+2)^3}\right) = -\frac{1}{x+2} + \frac{1}{2(x+2)^2}$.

5.

a) $\ln\left|\frac{x+2}{x+1}\right| - \frac{2}{x+2}$, b) $x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\text{arctg} x$,

c) $\ln(x^2+2x+2) + \ln|x| - \text{arctg}(x+1)$, d) $-\frac{2}{x-1} - \frac{1}{2(x-1)^2}$,

e) $-\frac{1}{5}\ln|x-2| + \frac{1}{10}\ln(x^2+1) + \frac{2}{5}\text{arctg}(x)$, f) $\ln|x| - \frac{2}{x-1}$,

6. a) O domínio de $\frac{x}{(x+1)(x+2)^2}$ é $\mathbb{R} \setminus \{-2, -1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln\left(\frac{x+2}{x+1}\right) - \frac{2}{x+2} + C_1, & \text{se } x > -1, \\ \ln\left(-\frac{x+2}{x+1}\right) - \frac{2}{x+2} + C_2, & \text{se } -2 < x < -1, \\ \ln\left(\frac{x+2}{x+1}\right) - \frac{2}{x+2} + C_3, & \text{se } x < -2, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

- b) O domínio de $\frac{x^4}{x^4 - 1}$ é $\mathbb{R} \setminus \{-1, 1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} x + \frac{1}{4} \ln\left(\frac{x-1}{x+1}\right) - \frac{1}{2} \operatorname{arctg} x + C_1, & \text{se } x > 1, \\ x + \frac{1}{4} \ln\left(-\frac{x-1}{x+1}\right) - \frac{1}{2} \operatorname{arctg} x + C_2, & \text{se } -1 < x < 1, \\ x + \frac{1}{4} \ln\left(\frac{x-1}{x+1}\right) - \frac{1}{2} \operatorname{arctg} x + C_3, & \text{se } x < -1, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

- c) O domínio de $\frac{3x^2 + 2}{x(x^2 + 2x + 2)}$ é $\mathbb{R} \setminus \{0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln(x^2 + 2x + 2) + \ln(x) - \operatorname{arctg}(x + 1) + C_1, & \text{se } x > 0, \\ \ln(x^2 + 2x + 2) + \ln(-x) - \operatorname{arctg}(x + 1) + C_2, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias.

As restantes alíneas fazem-se de forma semelhante.

7. a) $P\left(\frac{1 + \sqrt{x}}{x(4 - \sqrt{x})}\right) = 2P\left(\frac{1 + t}{t(4 - t)}\right) = \frac{1}{2} \ln \left| \frac{\sqrt{x}}{(4 - \sqrt{x})^5} \right|$, fazendo $\sqrt{x} = t \Leftrightarrow x = t^2$, com $x > 0, x \neq 16$, e $t > 0, t \neq 4$;
- b) $P\left(\frac{e^{4x}}{e^{2x} + 1}\right) = \frac{1}{2}P\left(\frac{t}{t + 1}\right) = \frac{1}{2}e^{2x} - \frac{1}{2} \ln(e^{2x} + 1)$, fazendo $e^{2x} = t \Leftrightarrow x = \frac{1}{2} \ln t$, com $x \in \mathbb{R}$ e $t > 0$;
- c) $P\left(\frac{2 \ln x - 1}{x \ln x (\ln x - 1)^2}\right) = P\left(\frac{2t - 1}{t(t - 1)^2}\right) = \ln \left| \frac{\ln x - 1}{\ln x} \right| - \frac{1}{\ln x - 1}$, fazendo $\ln x = t \Leftrightarrow x = e^t$, com $x \in \mathbb{R}^+ \setminus \{1, e\}$ e $t \in \mathbb{R} \setminus \{0, 1\}$;
- d) $P\left(\frac{1 - \operatorname{tg} x}{1 + \operatorname{tg} x}\right) = P\left(\frac{1 - t}{(1 + t)(1 + t^2)}\right) = \ln |\cos x| + \ln |\operatorname{tg} x + 1|$, fazendo $\operatorname{tg} x = t \Leftrightarrow x = \operatorname{arctg} t$, com $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $x \neq -\frac{\pi}{4}$ e $t \neq -1$.

NOTA: Esta substituição é invertível para $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, mas como a função $\operatorname{tg} x$ é periódica de período π , o resultado é válido no domínio de $\operatorname{tg} x$.

8. a) Fazendo a substituição $\sqrt{1 + 2x} = t \Leftrightarrow x = \frac{1}{2}(t^2 - 1)$, com $x > -\frac{1}{2}$ e $t > 0$, temos (verifique)

$$P\left(\frac{1}{x\sqrt{1 + 2x}}\right) = P\left(\frac{2}{t^2 - 1}\right) = \ln \left| \frac{\sqrt{1 + 2x} - 1}{\sqrt{1 + 2x} + 1} \right|.$$

b) Fazendo a substituição $x = t^6 \Leftrightarrow t = \sqrt[6]{x}$, para $x > 0$, $t > 0$, temos

$$P\left(\frac{1}{\sqrt{x} + \sqrt[3]{x}}\right) = P\left(\frac{6t^5}{t^3 + t^2}\right) = P\left(\frac{6t^3}{t+1}\right) = P\left(t^2 - t + 1 - \frac{1}{t+1}\right).$$

Logo,

$$P\left(\frac{1}{\sqrt{x} + \sqrt[3]{x}}\right) = \frac{\sqrt{x}}{3} - \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} - \ln(\sqrt[6]{x} + 1).$$

c) $P\left(\frac{e^{2x}}{(e^{2x} - 1)(1 + e^x)}\right) = \frac{1}{4} \ln \left| \frac{e^x - 1}{e^x + 1} \right| - \frac{1}{2(1 + e^x)}.$

d) Fazendo a substituição $t = \sqrt{1 + e^x} \Leftrightarrow x = \ln(t^2 - 1)$, para $x \in \mathbb{R}$, $t > 1$, temos

$$P\left(\frac{1}{\sqrt{1 + e^x}}\right) = P\left(\frac{1}{t} \frac{2t}{t^2 - 1}\right) = P\left(\frac{2}{(t-1)(t+1)}\right).$$

e assim (verifique)

$$P\left(\frac{1}{\sqrt{1 + e^x}}\right) = \ln \left| \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right|.$$

e) Fazendo a substituição $x = \cos t \Leftrightarrow t = \arccos x$, com $t \in]0, \pi[$, $t \neq \pi/2$, temos

$$P\left(\frac{\sqrt{1 - x^2}}{x^4}\right) = P\left(-\frac{\operatorname{sen}^2 t}{\cos^4 t}\right) = P\left(-\operatorname{tg}^2 t \frac{1}{\cos^2 t}\right) = -\frac{1}{3} \operatorname{tg}^3 t.$$

Logo, notando que se $t = \arccos x$ então de $1 + \operatorname{tg}^2 t = \frac{1}{\cos^2 t}$ vem $\operatorname{tg} t = \sqrt{\frac{1}{\cos^2 t} - 1} = \sqrt{\frac{1}{x^2} - 1}$ temos

$$P\left(\frac{\sqrt{1 - x^2}}{x^4}\right) = -\frac{1}{3} \left(\frac{1}{x^2} - 1\right)^{3/2}.$$

f) $P\left(\frac{1}{\sqrt{x(1-x)}}\right) = 2 \operatorname{arcsen} \sqrt{x}.$

g) $P\left(\frac{1}{2 + \operatorname{tg} x}\right) = P\left(\frac{1}{(2+t)(1+t^2)}\right) = \frac{1}{5} P\left(\frac{1}{2+t} + \frac{-t+2}{1+t^2}\right) = \frac{1}{5} \ln |2 + \operatorname{tg} x| - \frac{1}{10} \ln(1 + \operatorname{tg}^2 x) + \frac{2}{5} x.$

h) Fazendo a substituição $\operatorname{sen} x = t \Leftrightarrow x = \operatorname{arcsen} t$, para $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $x \neq 0$, obtém-se (verifique)

$$P\left(\frac{1}{\operatorname{sen}^2 x \cos x}\right) = P\left(\frac{\cos x}{\operatorname{sen}^2 x(1 - \operatorname{sen}^2 x)}\right) = P\left(\frac{1}{t^2(1-t^2)}\right).$$

Logo (verifique)

$$P\left(\frac{1}{\operatorname{sen}^2 x \cos x}\right) = -\frac{1}{\operatorname{sen} x} + \frac{1}{2} \ln \left| \frac{1 + \operatorname{sen} x}{1 - \operatorname{sen} x} \right|.$$

SUPLEMENTARES

1. a) $P(\operatorname{sen}^3 x \cos^4 x) = P(\operatorname{sen} x(1 - \cos^2 x) \cos^4 x) = P(\operatorname{sen} x(\cos^4 x - \cos^6 x)) =$
 $= -\frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x$; OU por partes, com $u' = \operatorname{sen} x \cos^4 x \Rightarrow u = -\frac{1}{5} \cos^5 x$, $v = \operatorname{sen}^2 x$.
- b) $\frac{1}{4} \operatorname{sen}^4 x - \frac{1}{6} \operatorname{sen}^6 x$ (como c) e d));
- c) $P(4 \cos^2 x \operatorname{sen}^2 x) = P(\operatorname{sen}^2(2x)) = \frac{x}{2} - \frac{1}{8} \operatorname{sen}(4x)$;
- d) $\frac{1}{2}(\operatorname{sh} x \operatorname{ch} x + x)$, por partes, com $u' = v = \operatorname{ch} x$ e usar $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$.
- e) $P(\operatorname{tg}^3 x + \operatorname{tg}^4 x) = P((\sec^2 x - 1) \operatorname{tg} x) + P((\sec^2 x - 1) \operatorname{tg}^2 x) = \frac{1}{2} \operatorname{tg}^2 x + \ln |\cos x| +$
 $\frac{1}{3} \operatorname{tg}^3 x - \operatorname{tg} x + x$;
- f) $\operatorname{tg} x + \frac{1}{3} \operatorname{tg}^3 x$, escrevendo $\sec^2 x = 1 + \operatorname{tg}^2 x$, OU por partes, $u' = \sec^2 x \Rightarrow u = \operatorname{tg} x$,
 $v = \sec^2 x \Rightarrow v' = 2 \sec^2 x \operatorname{tg} x$.

2.

- a) $e^x(e^x + x - 1) - e^{2x}/2$, b) $-x \cos x + \operatorname{sen} x$, c) $e^x(\operatorname{sen} x - \cos x)/2$,
- d) $P(x^3 e^{x^2}) = P(x^2 \cdot x e^{x^2}) = (x^2 - 1) \frac{e^{x^2}}{2}$, e) $(x^2 + 2) \operatorname{ch} x - 2x \operatorname{sh} x$,
- f) $x \operatorname{arctg} x - \frac{1}{2} \ln(1 + x^2)$, g) $\frac{1}{4}(1 + x^2)^2 \operatorname{arctg} x - x/4 - x^3/12$,
- h) $x \ln |1/x + 1| + \ln |x + 1|$, i) $\frac{x^3}{3} \ln^2 x - \frac{2}{9} x^2 \ln x + \frac{2}{27} x^3$,
- j) $x(\ln^3 x - 3 \ln^2 x + 6 \ln x - 6)$, k) $-\frac{1}{x} \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x}$,
- l) $-\cos x \ln(1 + \operatorname{sen} x) + x + \cos x$, m) $-(1 - x^2)^{3/2} \operatorname{arcsen} x + x - x^3/3$,
- n) $-\frac{\ln x}{1+x} + \ln \left| \frac{x}{1+x} \right|$, o) $\frac{1}{1 + \ln^2 3} 3^x (\operatorname{sen} x + \ln 3 \cos x)$,
- p) $\frac{2}{3} x^{3/2} \operatorname{arctg}(1/\sqrt{x}) + \frac{1}{3}(x - \ln(x + 1))$,
- q) $P(x^n \ln x) = \frac{1}{n+1} x^{n+1} \ln x - P\left(\frac{1}{n+1} x^{n+1} \frac{1}{x}\right) = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1}$.

3. Use primitivação por partes, com $u' = 1$, $v = \ln^n |x|$. Temos $P(\ln^2 |x|) = x \ln^2 |x| - 2x \ln |x| + 2x$ em $\mathbb{R} \setminus \{0\}$. Logo

$$F(x) = \begin{cases} x \ln^2 |x| - 2x \ln |x| + 2x + C, & \text{se } x > 0 \\ x \ln^2 |x| - 2x \ln |x| + 2x - C, & \text{se } x < 0 \end{cases}$$

com $C \in \mathbb{R}$, já que somando constantes C em $x > 0$ e C' em $x < 0$ temos $F(1) = 2 + C = -F(-1) = -(2 + C')$ logo $C' = -C$.

4.

$$\text{a) } P\left(\frac{1}{1-x}\right) = -\ln|1-x|, \quad \text{b) } P\left(\frac{1}{(x-3)^3}\right) = -\frac{1}{2(x-3)^2},$$

$$\text{c) } P\left(\frac{x+1}{x^2+1}\right) = \frac{1}{2}\ln(x^2+1) + \operatorname{arctg} x, \quad \text{d) } P\left(\frac{2x+1}{x^2+4}\right) = \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right),$$

$$\text{e) } P\left(\frac{x+1}{a^2+x^2}\right) = \frac{1}{2}\ln(a^2+x^2) + \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right),$$

$$\text{f) } P\left(\frac{1}{x^2+x+1}\right) = P\left(\frac{1}{\left(x+\frac{1}{2}\right)^2+3/4}\right) = \frac{4}{3}P\left(\frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\right) = \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right).$$

5.

$$\text{a) } \ln|x| - \ln|x+1| = \ln\left|\frac{x}{x+1}\right|, \quad \text{b) } \ln|x| - \ln|x-1| = \frac{2}{x-1},$$

$$\text{c) } -\ln|x| + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right), \quad \text{d) } 2\ln|x-1| - \ln|x| + \frac{1}{x},$$

$$\text{e) } \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1|, \quad \text{f) } \frac{x^2}{2} + \ln|x+1| + \frac{1}{x+1},$$

$$\text{g) } x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\operatorname{arctg} x, \quad \text{h) } \frac{1}{2}\ln(x^2+4) + \operatorname{arctg}\left(\frac{x}{2}\right) + \frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|,$$

$$\text{i) } \ln|x-3| + \ln|x+1| - \frac{1}{x+1}, \quad \text{j) } \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{x+1}, \quad \text{k) } \ln|x-3| - \frac{1}{x+1},$$

$$\text{l) } \ln|1+x| - \frac{1}{2}\ln(x^2+2x+2), \quad \text{m) } 7\ln|x+1| - 2\ln(x^2+x+1) - \frac{8}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right).$$

6. a) O domínio de $\frac{1}{x^2+x}$ é $\mathbb{R} \setminus \{-1, 0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln x - \ln(x+1) + C_1, & \text{se } x > 0, \\ \ln(-x) - \ln(x+1) + C_2, & \text{se } -1 < x < 0, \\ \ln(-x) - \ln(-x-1) + C_3, & \text{se } x < -1, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.b) O domínio de $\frac{x+1}{x(x-1)^2}$ é $\mathbb{R} \setminus \{0, 1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln x - \ln(x-1) - \frac{2}{x-1} + C_1, & \text{se } x > 1, \\ \ln x - \ln(-x+1) - \frac{2}{x-1} + C_2, & \text{se } 0 < x < 1, \\ \ln(-x) - \ln(-x+1) - \frac{2}{x-1} + C_3, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.c) O domínio de $\frac{x^2+x-4}{x(x^2+4)}$ é $\mathbb{R} \setminus \{0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} -\ln x + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_1, & \text{se } x > 0, \\ -\ln(-x) + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_2, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias.

d) O domínio de $\frac{x^2+1}{x^2(x-1)}$ é $\mathbb{R} \setminus \{0, 1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} 2 \ln(x-1) - \ln x + 1/x + C_1, & \text{se } x > 1, \\ 2 \ln(1-x) - \ln x + 1/x + C_2, & \text{se } 0 < x < 1, \\ 2 \ln(1-x) - \ln(-x) + 1/x + C_3, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

As restantes alíneas fazem-se de forma semelhante.

7. a) $\frac{1}{2}e^{x^2+2x} + C$, com $C \in \mathbb{R}$.

b)

$$P\left(\frac{x+3}{x^4-x^2}\right) = -\ln|x| + \frac{3}{x} + 2 \ln|x-1| - \ln|x+1| = \frac{3}{x} + \ln \frac{(x-1)^2}{|x(x+1)|}.$$

A forma geral da primitiva em $]1, +\infty[$ é $G(x) = \frac{3}{x} + \ln \frac{(x-1)^2}{x(x+1)} + K$, com $K \in \mathbb{R}$. Tem-se

$$\lim_{x \rightarrow +\infty} G(x) = \lim_{x \rightarrow +\infty} \frac{3}{x} + \ln \frac{(x-1)^2}{x(x+1)} + K = \ln(1) + K = K,$$

logo $\lim_{x \rightarrow +\infty} G(x) = 3 \Leftrightarrow K = 3$.

8. c) $P\left(\frac{1}{(1+x^2)^2}\right) = \frac{x}{2(1+x^2)} + \frac{1}{2} \operatorname{arctg} x$.

$$P\left(\frac{1}{(1+x^2)^3}\right) = \frac{x}{4(1+x^2)^2} + \frac{3x}{8(1+x^2)} + \frac{3}{8} \operatorname{arctg} x.$$

9. a) $P\left(\frac{1}{\sqrt[3]{x}(1+\sqrt[3]{x^4})}\right) = P\left(\frac{3t^2}{1+t^4}\right) = \frac{3}{2} \operatorname{arctg} \sqrt[3]{x^2}$, fazendo $\sqrt[3]{x} = t \Leftrightarrow x = t^3$;

b) $P\left(\frac{\sqrt{x-1}}{x}\right) = P\left(\frac{2t^2}{t^2+1}\right) = 2\sqrt{x-1} - 2 \operatorname{arctg} \sqrt{x-1}$, fazendo $\sqrt{x-1} = t \Leftrightarrow x = t^2 + 1$, com $x > 1$ e $t > 0$;

c) $P\left(\frac{1}{1+e^{2x}}\right) = P\left(\frac{1}{1+t} \cdot \frac{1}{2t}\right) = \frac{1}{2} \ln \left| \frac{t}{1+t} \right| = \frac{1}{2} \ln \left| \frac{e^{2x}}{1+e^{2x}} \right|$, fazendo $e^{2x} = t \Leftrightarrow x = \frac{1}{2} \ln t$, com $x \in \mathbb{R}$ e $t > 0$;

d) $P\left(\frac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right) = P\left(\frac{t^2}{(1+t^2)(t-1)^2}\right) = -\frac{1}{4} \ln(1+e^{2x}) + \frac{1}{2} \ln|e^x-1| - \frac{1}{2} \frac{1}{e^x-1}$, fazendo $e^x = t \Leftrightarrow x = \ln t$, com $x \in \mathbb{R} \setminus \{0\}$ e $t > 0, t \neq 1$;

e) $P\left(\frac{\ln x}{x(\ln x-1)^2}\right) = P\left(\frac{t}{(t-1)^2}\right) = \ln|\ln x-1| - \frac{1}{\ln x-1}$, fazendo $\ln x = t \Leftrightarrow x = e^t$, com $x \in \mathbb{R}^+ \setminus \{e\}$ e $t \in \mathbb{R} \setminus \{1\}$.

10. a) Fazendo a substituição $\sqrt{x} = t \Leftrightarrow x = t^2$, com $x > 0$ e $t > 0$, temos (verifique)

$$P\left(\frac{5}{2(x+1)(\sqrt{x}+2)}\right) = P\left(\frac{5t}{(t^2+1)(t+2)}\right) = \ln(x+1) + \operatorname{arctg} \sqrt{x} - 2 \ln(\sqrt{x}+1).$$

b) $P\left(\frac{1}{(2-x)\sqrt{1-x}}\right) = -2 \operatorname{arctg} \sqrt{1-x}$

c) Fazendo a substituição $\sqrt[4]{1+x} = t \Leftrightarrow x = t^4 - 1$, com $x > -1$ e $t > 0$, temos

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = P\left(\frac{1}{(t^4-1)t} 4t^3\right) = P\left(\frac{4t^2}{t^4-1}\right) = P\left(\frac{4t^2}{(t-1)(t+1)(t^2+1)}\right)$$

e assim (verifique),

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = \ln \left| \frac{\sqrt[4]{1+x}-1}{\sqrt[4]{1+x}+1} \right| + 2 \operatorname{arctg} \sqrt[4]{1+x}.$$

d) $P\left(\frac{e^{x/2}}{\sqrt{1-e^x}}\right) = -2 \operatorname{arcsen} \sqrt{1-e^x}$ (pode fazer também $t = e^{x/2}$ - mais simples).

e) Fazendo a substituição $\ln x = t \Leftrightarrow x = e^t$, para $x \in \mathbb{R}^+ \setminus \{e^{-2}, e^2\}$, $t \neq \pm 2$, temos

$$P\left(\frac{1}{x(4-\ln^2(x))}\right) = P\left(\frac{1}{e^t(4-t^2)} e^t\right) = P\left(\frac{1}{(2-t)(2+t)}\right).$$

e assim (verifique)

$$P\left(\frac{1}{x(4-\ln^2(x))}\right) = \frac{1}{4} \ln \left| \frac{2+\ln x}{2-\ln x} \right|.$$

f) Fazendo a substituição $\ln x = t \Leftrightarrow x = e^t$, para $x \in \mathbb{R}^+ \setminus \{1, e\}$, $t \in \mathbb{R} \setminus \{0, 1\}$, temos

$$P\left(\frac{1}{x \ln x (1-\ln x)}\right) = P\left(\frac{1}{t(1-t)}\right) = \ln \left| \frac{\ln x}{1-\ln x} \right|.$$

g) Fazendo a substituição $t = \operatorname{sen} x \Leftrightarrow x = \operatorname{arcsen} t$, com $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, temos

$$P(\sec x) = P\left(\frac{\cos x}{1-\operatorname{sen}^2 x}\right) = P\left(\frac{1}{1-t^2}\right) = \frac{1}{2} \ln \left| \frac{1+\operatorname{sen} x}{1-\operatorname{sen} x} \right| = \ln |\sec x + \operatorname{tg} x|.$$

h) $P(\sec^3 x) = \frac{1}{4} \ln \left| \frac{1+\operatorname{sen} x}{1-\operatorname{sen} x} \right| + \frac{1}{4(1-\operatorname{sen} x)} - \frac{1}{4(1+\operatorname{sen} x)} = \frac{1}{2} \ln \left| \frac{1+\operatorname{sen} x}{\cos x} \right| + \frac{\operatorname{sen} x}{2 \cos^2 x} = \frac{1}{2} \ln |\sec x + \operatorname{tg} x| + \frac{1}{2} \sec x \operatorname{tg} x.$

i) $P\left(\frac{\cos x}{1+\operatorname{sen} x - \cos^2 x}\right) = P\left(\frac{\cos x}{\operatorname{sen} x + \operatorname{sen}^2 x}\right) = \ln \left| \frac{\operatorname{sen} x}{1+\operatorname{sen} x} \right|.$

j) Temos

$$P\left(\frac{\operatorname{sen} x}{\operatorname{sen}^2(x) + 3(\cos x - 1)}\right) = P\left(\frac{\operatorname{sen} x}{1 - \cos^2(x) + 3(\cos x - 1)}\right) = P\left(\frac{-\operatorname{sen} x}{\cos^2(x) - 3 \cos x + 2}\right)$$

Fazendo a substituição $t = \cos x \Leftrightarrow x = \operatorname{arccos} t$, com $x \in]0, \pi[$, temos

$$P\left(\frac{-\operatorname{sen} x}{\cos^2(x) - 3 \cos x + 2}\right) = P\left(\frac{1}{t^2 - 3t + 2}\right)$$

e assim (verifique)

$$P\left(\frac{\operatorname{sen} x}{\operatorname{sen}^2(x) + 3(\cos x - 1)}\right) = \ln \left| \frac{\cos x - 2}{\cos x - 1} \right|.$$

k) Fazendo a substituição $t = \cos x \Leftrightarrow x = \arccos t$, com $x \in]0, \pi[$, temos

$$P\left(\frac{1}{\operatorname{sen} x(1 + \cos x)}\right) = P\left(\frac{-\operatorname{sen} x}{(\cos x - 1)(1 + \cos x)^2}\right) = P\left(\frac{1}{(t - 1)(1 + t)^2}\right)$$

e obtém-se (verifique)

$$P\left(\frac{1}{\operatorname{sen} x(1 + \cos x)}\right) = \frac{1}{4} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + \frac{1}{2(1 + \cos x)}.$$

l) Como na alínea anterior, fazendo a substituição $t = \operatorname{sen} x \Leftrightarrow x = \operatorname{arcsen} t$, com $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, temos

$$P\left(\frac{1}{\cos x(1 - \operatorname{sen} x)}\right) = P\left(\frac{\cos x}{(1 + \operatorname{sen} x)(1 - \operatorname{sen} x)^2}\right) = P\left(\frac{1}{(1 + t)(1 - t)^2}\right).$$

e assim

$$P\left(\frac{1}{\cos x(1 - \operatorname{sen} x)}\right) = \frac{1}{4} \ln \left| \frac{1 + \operatorname{sen} x}{1 - \operatorname{sen} x} \right| + \frac{1}{2(1 - \operatorname{sen} x)}.$$

$$11. \text{ b) } P\left(\frac{1}{1 + \operatorname{sen} x + \cos x}\right) = \ln \left| \operatorname{tg} \frac{x}{2} \right| - \ln \left| 1 + \operatorname{tg} \frac{x}{2} \right|,$$

$$P\left(\frac{\operatorname{sen} x}{1 - \operatorname{sen} x}\right) = \frac{2}{1 - \operatorname{tg}(x/2)} - x.$$