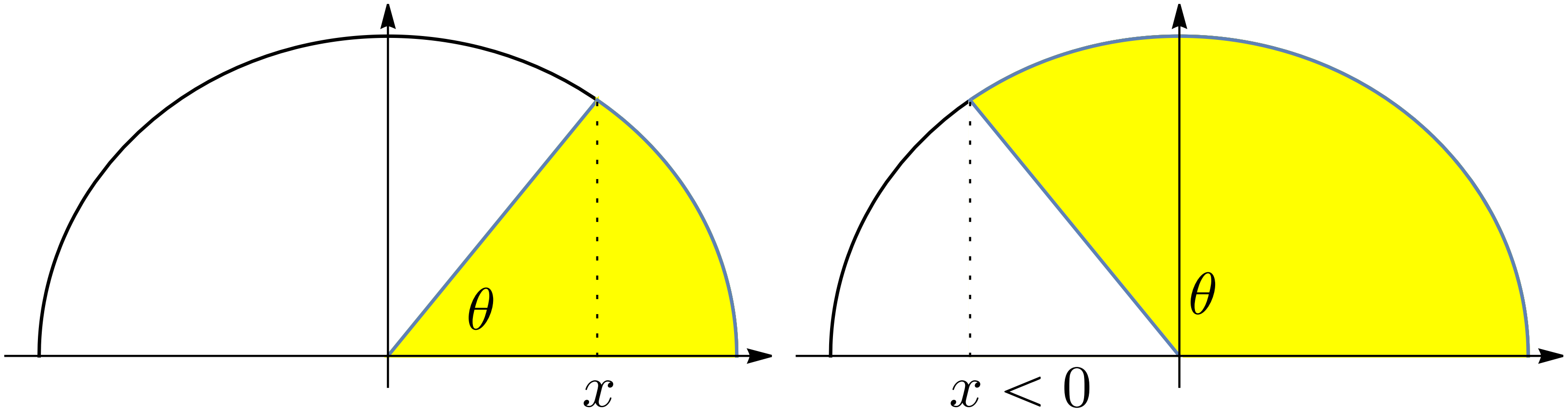
Aula de Hoje: Funções Trigonométricas

Última Aula

$$\arccos x = \theta \iff \cos\theta = x \quad \text{e} \quad \theta \in [0,\pi]$$



$$\arccos x = \theta = 2 \times \text{Área} = 2\left(\frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,\mathrm{d}t\right)$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

A Função Seno

Definição. $\operatorname{sen}\theta = -\cos'\theta$.

Para $\theta \in]0,\pi[$: $x = \cos\theta \Leftrightarrow \theta = \arccos x$.

$$\cos'\theta = \frac{1}{\arccos' x} = \frac{1}{-1/\sqrt{1-x^2}} = -\sqrt{1-x^2} = -\sqrt{1-\cos^2\theta}$$

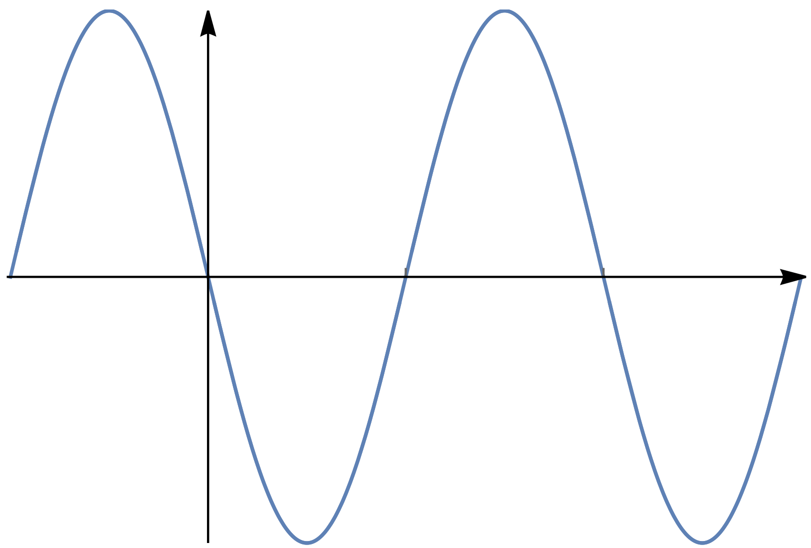
- $\operatorname{sen}\theta = \sqrt{1-\cos^2\theta}$ logo $\operatorname{sen}^2\theta = 1-\cos^2\theta$
- $\operatorname{sen}'\theta = \dfrac{1}{2\sqrt{1-\cos^2\theta}}(-2\cos\theta)(-\operatorname{sen}\theta) = \cos\theta$

$$\lim_{\theta\to 0^+}\cos'\theta = 0 \quad \text{e} \quad \lim_{\theta\to\pi^-}\cos'\theta = 0$$

A Função Seno (Continuação)

$\cos'(\theta)$ é ímpar e periódica de período 2π porque:

- $f(x) = f(-x) \;\Rightarrow\; f'(x) = -f'(-x)$
- $f(x+2\pi) = f(x) \;\Rightarrow\; f'(x+2\pi) = f'(x)$



- $\cos'\theta = -\sqrt{1-\cos^2\theta}$
- $\lim_{\theta\to 0^+} \cos'\theta = 0$
- $\lim_{\theta\to \pi^-} \cos'\theta = 0$
- $\lim_{\theta\to k\pi} \cos'\theta = 0$
- $\cos'(k\pi) = 0$

Pela paridade, as igualdades

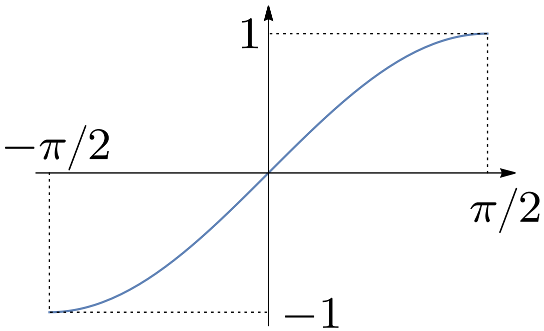
$$\operatorname{sen}^2\theta = 1-\cos^2\theta \quad \text{e} \quad \operatorname{sen}'\theta = \cos\theta$$

são válidas em $[-\pi,\pi]$. Pela periodicidade, são válidas para qualquer $\theta\in\mathbb{R}$.

A Função Arco-Seno

Definição. A função $\operatorname{arcsen} x$ é a inversa da restrição de $\operatorname{sen}\theta$ a $[-\pi/2, , \pi/2]$:

$$\operatorname{arcsen} x = \theta \quad \Leftrightarrow \quad \left(\operatorname{sen}\theta = x \quad \text{e} \quad -\pi/2 \leqslant \theta \leqslant \pi/2\right)$$



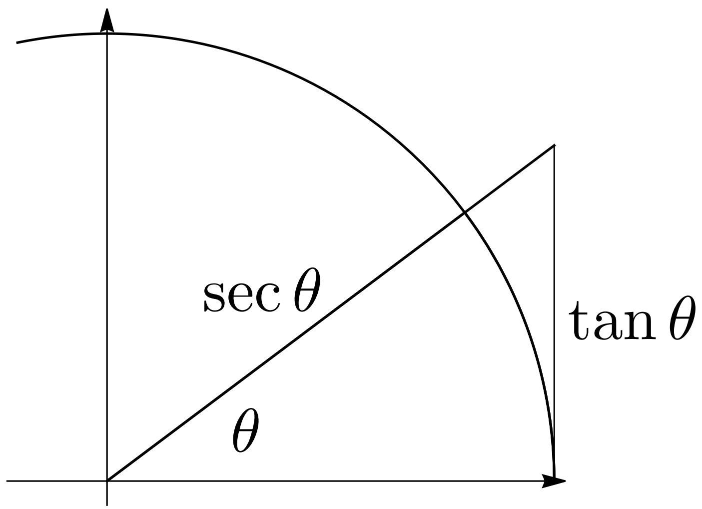
$\operatorname{sen}'(\theta) = \cos(\theta)$ anula-se para $\theta = \pm\pi/2$, correspondendo a $x = \pm 1$.

$$\text{Para } x \neq \pm 1: \quad \operatorname{arcsen}' x = \frac{1}{\operatorname{sen}'\theta} = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1-\operatorname{sen}^2\theta}}$$

No intervalo $]-\pi/2, \pi/2[$, temos $\cos\theta > 0$, portanto:

$$\operatorname{arcsen}' x = \frac{1}{\sqrt{1-\operatorname{sen}^2\theta}} = \frac{1}{\sqrt{1-x^2}}$$

As Funções Tangente e Secante

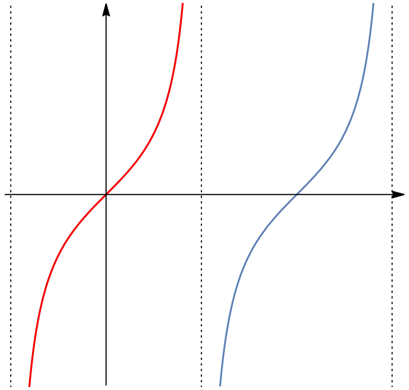


- $\tan\theta = \dfrac{\operatorname{sen}\theta}{\cos\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $1 + \tan^2\theta = \sec^2\theta$

$$\tan'\theta = \frac{\operatorname{sen}'\theta\cos\theta - \operatorname{sen}\theta\cos'\theta}{\cos^2\theta} = \frac{\cos^2\theta + \operatorname{sen}^2\theta}{\cos^2\theta} = \sec^2\theta$$

$$\sec'\theta = -\frac{\cos'\theta}{\cos^2\theta} = \frac{\operatorname{sen}\theta}{\cos^2\theta} = \frac{\operatorname{sen}\theta}{\cos\theta}\,\frac{1}{\cos\theta} = \tan\theta\sec\theta$$

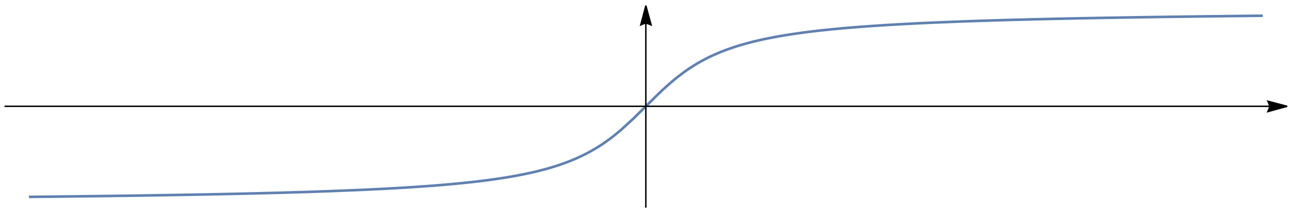
A Função Arco-Tangente



- O domínio da tangente é $D = \mathbb{R} \backslash \{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\}$;
- $\tan' \theta = \sec^2 \theta = 1/\cos^2 \theta > 0$ pelo que a tangente é crescente em intervalos.

$\arctan t$ é a inversa da restrição de $\tan \theta$ a $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$:

$$\arctan x = \theta \quad \Leftrightarrow \quad \left(\tan \theta = x \quad \text{e} \quad -\pi/2 < \theta < \pi/2\right)$$



$$\arctan' x = \frac{1}{\tan' \theta} = \frac{1}{\sec^2 \theta} = \frac{1}{1 + \tan^2 \theta} = \frac{1}{1 + x^2}\ .$$

Tabela de Primitivas

$$\int \cos x \,\mathrm{d}x = \operatorname{sen} x + C \qquad \int \operatorname{sen} x \,\mathrm{d}x = -\cos x + C$$

$$\int \sec^2 x \,\mathrm{d}x = \tan x + C \qquad \int \tan x \ \sec x \,\mathrm{d}x = \sec x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = \operatorname{arcsen} x + C \qquad \int \frac{1}{1+x^2} \,\mathrm{d}x = \arctan x + C$$

Exemplos

- $$\int 2x\cos(x^2)\,\mathrm{d}x \qquad \big(y = x^2, \quad \mathrm{d}y = 2x\,\mathrm{d}x\big)$$
  $$= \int \cos(y)\,\mathrm{d}y = \operatorname{sen}(y) + C = \operatorname{sen}(x^2) + C$$
- $$\int \operatorname{sen}^4(x)\cos(x)\,\mathrm{d}x \qquad \big(y = \operatorname{sen} x, \quad \mathrm{d}y = \cos x\,\mathrm{d}x\big)$$
  $$= \int y^4\,\mathrm{d}y = \frac{y^5}{5} + C = \frac{\operatorname{sen}^5 x}{5} + C$$
- $$\int \frac{x\,\mathrm{d}x}{1+x^4} = \frac{1}{2}\int \frac{2x\,\mathrm{d}x}{1+(x^2)^2} = \frac{1}{2}\arctan(x^2) + C$$

Exemplos

$$\int_0^{\sqrt{\pi/2}} \frac{2x\cos(x^2)}{1+\operatorname{sen}^2(x^2)}\,\mathrm{d}x \qquad (y = x^2, \quad \mathrm{d}y = 2x\,\mathrm{d}x)$$

$$= \int_{y=0}^{\pi/2} \frac{\cos y}{1+\operatorname{sen}^2 y}\,\mathrm{d}y \qquad (z = \operatorname{sen} y, \quad \mathrm{d}z = \cos y\,\mathrm{d}y)$$

$$= \int_{z=0}^{1} \frac{1}{1+z^2}\,\mathrm{d}z$$

$$= \Big[\arctan z\Big]_0^1$$

$$= \arctan 1 - \arctan 0 = \frac{\pi}{4}$$