DM
DEPARTAMENTO DE MATEMÁTICA
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Introduction to Stochastic Processes

MMA, LMAC, MECD

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Duration: 90 minutes **Test 1**

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 0 — Introduction to Stochastic Processes 2.5 points

Let $S(t)$ denote the price of a security[1] at time t. According to a popular model for the stochastic process $\{S(t): t \geq 0\}$, the price remains unchanged until a *shock* occurs, at which time the price is multiplied by a random factor. This model supposes that $S(t) = S(0)\prod_{i=1}^{N(t)} X_i$, where: $S(0)$ is the initial and constant price; $N(t)$ denotes the number of *shocks* up to time t; $\{N(t): t \geq 0\}$ is a Poisson process with rate λ; X_i are i.i.d. r.v. with common expected value μ $(0 < \mu \leq 1)$; $\{N(t): t \geq 0\}$ is independent of the $X_1, X_2, \ldots$; $\prod_{i=1}^{N(t)} X_i = 1$ when $N(t) = 0$. (2.5)

Derive the mean function of $\{S(t): t \geq 0\}$.[2] Is this a second order weakly stationary process?[3]

- **Random shocks**

  $X_i \overset{i.i.d.}{\sim} X, \quad i \in \mathbb{N}$

  $E(X) = \frac{1}{\mu}$

- **Number of random shocks**

  $N(t) =$ number of random shocks up to time t

  $\{N(t): t \geq 0\} \sim PP(\lambda)$

  $N(t) \sim \text{Poisson}(\lambda\, t)$

  $\{N(t): t \geq 0\} \sim PP(\lambda) \quad \perp\!\!\!\perp \quad \{X_i : i \in \mathbb{N}\}$

- **Requested expectation**

  Due to the fact that $X_i \overset{i.i.d.}{\sim} X$ and $\{N(t),\, t \geq 0\} \perp\!\!\!\perp \{X_i : i \in \mathbb{N}\}$, we can successively write:

$$
\begin{aligned}
E[S(t)] &= E\{E[S(t)|N(t)]\} \\
&= S(0) \times E\left\{E\left[\prod_{i=1}^{N(t)} X_i \middle| N(t)\right]\right\} \\
&= S(0) \times E\left[\prod_{i=1}^{N(t)} E(X_i) \middle| N(t)\right] \\
&= S(0) \times E\{[E(X)]^{N(t)}\} \\
&= S(0) \times E\left[\mu^{N(t)}\right] \\
&= S(0) \times P_{N(t)}(\mu),
\end{aligned}
$$

  where $P_{N(t)}(z) = E\left[z^{N(t)}\right] = e^{\lambda t (z-1)}$ $(|z| \leq 1)$ represents the p.g.f. of $N(t)$. Since we admitted that $0 < \mu \leq 1$, we have:

$$E[S(t)] = S(0) \times e^{\lambda t (\mu - 1)}.$$

- **Checking whether the process is second order weakly stationary**

  $E[S(t)]$ depends on t, hence $\{S(t): t \geq 0\}$ is not a [first order weakly stationary process therefore it cannot be a] second order weakly stationary process.

[1] A certificate attesting credit, the ownership of stocks or bonds, or the right to ownership connected with tradable derivatives.

[2] **Hint**: It might be useful to recall that the r.v. $Y \sim \text{Poisson}(\xi)$ has p.g.f. given by $P_Y(z) = E(z^Y) = e^{\xi(z-1)}$ $(|z| \leq 1)$.

[3] **Note**: Do not calculate the autocovariance function to reply to this last question.

Group 1 — Poisson Processes 8.5 points

1. Passengers arrive to a train station according to a Poisson process with rate λ passengers per hour.

(a) Find the probability that exactly 3 passengers arrive in the first hour, given that at least one passenger arrived in the first 30 minutes, when $\lambda = 8$. (1.5)

- **Stochastic process**

  $\{N(t): t \geq 0\} \sim PP(\lambda)$

  $N(t) =$ number of arrivals of passengers to the train station by time t (time in hours)

  λ

- **Relevant distributions**

  $N(t) \sim \text{Poisson}(\lambda\, t)$

  $(N(s) \mid N(t) = n) \overset{form.}{\sim} \text{Binomial}(n, s/t),\ 0 < s < t$

- **Requested probability**

  With $s = 0.5$ and $t = 1$, we obtain

$$
\begin{aligned}
P[N(1) = 3 \mid N(0.5) > 0] &\overset{Bayes' theo.}{=} \frac{P[N(0.5) > 0 \mid N(1) = 3] \times P[N(1) = 3]}{P[N(0.5) > 0]} \\
&= \frac{\{1 - P[N(0.5) = 0 \mid N(1) = 3]\} \times P[N(1) = 3]}{1 - P[N(0.5) = 0]} \\
&\overset{\lambda=8}{=} \frac{\left[1 - \binom{3}{0} 0.5^0 (1-0.5)^{3-0}\right] \times \frac{e^{-8\times 1}(8\times 1)^3}{3!}}{1 - \frac{e^{-8\times 0.5}(8\times 0.5)^0}{0!}} \\
&= \frac{\left(1 - 0.5^3\right) \times \frac{e^{-8} 8^3}{3!}}{1 - e^{-4}} \\
&\simeq 0.025515.
\end{aligned}
$$

  [Alternatively,

$$
\begin{aligned}
P[N(1) = 3 \mid N(0.5) > 0] &= \frac{P[N(0.5) > 0,\, N(1) = 3]}{P[N(0.5) > 0]} \\
&= \frac{P[N(1) = 3] - P[N(0.5) = 0,\, N(1) = 3]}{1 - P[N(0.5) = 0]} \\
P[N(1) = 3 \mid N(0.5) > 0] &\overset{indep.incr.}{=} \frac{P[N(1) = 3] - P[N(0.5) = 0] \times P[N(1) - N(0.5) = 3]}{1 - P[N(0.5) = 0]} \\
&\overset{station.incr.}{=} \frac{P[N(1) = 3] - P[N(0.5) = 0] \times P[N(1 - 0.5) = 3]}{1 - P[N(0.5) = 0]} \\
&\overset{N(t)\sim Poi(8t)}{=} \frac{\frac{e^{-8\times 1}(8\times 1)^3}{3!} - \frac{e^{-8\times 0.5}(8\times 0.5)^0}{0!} \times \frac{e^{-6\times 0.5}(8\times 0.5)^3}{3!}}{1 - \frac{e^{-8\times 0.5}(8\times 0.5)^0}{0!}} \\
&= \frac{\frac{e^{-8}}{3!} \times \left(8^3 - 4^3\right)}{1 - e^{-4}} \\
&\simeq 0.025515.
\end{aligned}
$$

  Or $P[N(1) = 3 \mid N(0.5) > 0] = \frac{P[N(0.5)>0, N(1)=3]}{P[N(0.5)>0]} = \cdots = \frac{\sum_{i=1}^{3} P[N(0.5)=i, N(1)-N(0.5)=3-i]}{1-P[N(0.5)=0]} = \ldots]$

(b) The number of hours T between successive train arrivals at the station is uniformly distributed on $(0, 1)$. Suppose a train has just left the station. Let X denote the number of people who get on the next train. Find $E(X)$ and $V(X)$.[4] (1.0)

- **R.v.**

  $T =$ time until the next train arrives

  $T \sim \text{uniform}(0, 1)$

  $E(T) \overset{form.}{=} \frac{1}{2}, \quad V(T) \overset{form.}{=} \frac{1}{12}$

  $X =$ number of people who get on the next train

[4] **Hint**: It might be useful to condition X on $T = t$.

- **Requested mean and variance**
The arrivals of passengers are governed by a $PP(\lambda)$ [and the arrivals of trains by a RP with IRT uniformly distributed in $(0,1)$], hence $](X \mid T=t) \sim N(t) \sim \text{Poisson}(\lambda\, t)$ and $E(X \mid T=t) = V(X \mid T=t) = \lambda\, t$. Hence

$$\begin{aligned} E(X) &= E[E(X \mid T)] \\ &= E(\lambda\, T) \\ &= \lambda\, E(T) \\ &= \frac{\lambda}{2} \overset{\lambda=8}{=} 4 \\ V(X) &= V[E(X \mid T)] + E[V(X \mid T)] \\ &= V(\lambda\, T) + E(\lambda\, T) \\ &= \lambda^2\, V(T) + \lambda\, E(T) \\ &= \frac{\lambda^2}{12} + \frac{\lambda}{2} \\ &= \frac{\lambda^2 + 6\lambda}{12} \overset{\lambda=8}{=} \frac{28}{3}. \end{aligned}$$

2. Assume signals are emitted according to a Poisson process with rate λ. The signals are registered by a counter that locks up for a fixed period of time $\tau > 0$ every time it registers a signal. While it is locked, it cannot register any other emitted signal.

(a) What is the probability that the counter is not locked at time t ($t > \tau$), assuming that it is not locked at time 0? (1.0)

- **Stochastic process**
$\{N(t) : t \geq 0\} \sim PP(\lambda)$
$N(t)$ = number of signals emitted until t

- **Non-homogenous Bernoulli splitting**
The counter is not locked at time t if the last event before t occurred more than τ time units ago.

$$\begin{aligned} p(t) &= P[0 \text{ events in } (t-\tau, t]] \\ &= P[N(t) - N(t-\tau) = 0] \\ &\overset{stat.incr.}{=} P[N(\tau) = 0] \\ &\overset{N(\tau)\sim Poisson(\lambda\tau)}{=} e^{-\lambda\tau}, \quad t > \tau. \end{aligned}$$

In case $t \leq \tau$, the counter is not locked at time t iff no events occurred in $(0, t]$; the associated probability equals $p(t) = P[N(t) = 0] = e^{-\lambda t}$.

(b) Find the probability that the counter registers at least one signal in the interval $(0, t]$, when $\lambda = \tau = 1$ and $t = 2$. (1.0)

- **R.v.**
Then the number of register signals by time t, $N_R(t)$, results from a non-homogenous Bernoulli splitting of $\{N(t) : t \geq 0\}$ and

$$N_R(t) \overset{form.}{\sim} \text{Poisson}\left(\lambda \int_0^t p(s)\, ds\right),$$

where

$$\begin{aligned} \lambda \int_0^t p(s)\, ds &= \lambda \left(\int_0^\tau e^{-\lambda t}\, ds + \int_\tau^t e^{-\lambda\tau}\, ds \right) \\ &= \lambda \left(\frac{1 - e^{-\lambda\tau}}{\lambda} + (t-\tau) e^{-\lambda\tau} \right) \end{aligned}$$

$$\lambda \int_0^t p(s)\, ds \overset{\lambda=\tau=1,\, t=2}{=} 1.$$

- **Requested probability** For $\lambda = \tau = 1$ and $t = 2$, we obtain

$$\begin{aligned} P[N_R(t) > 0] &= 1 - P[N_R(t) = 0] \\ &= 1 - P_{Poisson(1)}(0) \\ &= 1 - e^{-1} \\ &\simeq 0.632121. \end{aligned}$$

3. Admit that orders arrive to a depot according to a non-homogeneous Poisson process with mean function $m(t) = \ln(1+t)$, $t \geq 0$. (2.0)

Compute the probability that the time between the arrivals of the first and second orders belongs to the interval $[1,2]$.[5]

- **Stochastic process**
$\{N(t) : t > 0\} \sim NHPP$
$N(t)$ = number of orders arrived to the depot until time t

- **Mean value and intensity functions**
For $t \geq 0$,

$$\begin{aligned} m(t) &= \int_0^t \lambda(s)\, ds \\ &= \ln(1+t) \\ \lambda(t) &= \frac{d\, m(t)}{dt} \\ &= \frac{1}{1+t} \end{aligned}$$

- **R.v.**
X_2 = time between the arrivals of the first and second order

- **Requested probability**
Please note that, for $n = 1$ and our particular NHPP,

$$\begin{aligned} P(X_{n+1} > t) &\overset{form.}{=} \int_0^{+\infty} \lambda(s)\, e^{-m(t+s)} \frac{[m(s)]^{n-1}}{(n-1)!}\, ds \\ &\overset{n=1}{=} \int_0^{+\infty} \frac{1}{1+s} e^{-\ln(1+t+s)}\, ds \\ &= \int_0^{+\infty} \frac{1}{(1+s)(1+t+s)}\, ds \\ &\overset{note}{=} \left. \frac{\ln(1+s)}{t} - \frac{\ln(1+t+s)}{t} \right|_{s=0}^{+\infty} \\ &= \left. \frac{1}{t} \times \ln\left(\frac{1+s}{1+t+s} \right) \right|_{s=0}^{+\infty} \\ &= \frac{\ln(1+t)}{t}. \end{aligned}$$

Hence, the requested probability can be written as

$$\begin{aligned} P(X_2 \in [1,2]) &= P(X_{1+1} > 1) - P(X_{1+1} > 2) \\ &= \frac{\ln(1+1)}{1} - \frac{\ln(2+1)}{2} \\ &= \ln(2) - \frac{\ln(3)}{2} \simeq 0.143841. \end{aligned}$$

[5]**Note:** $\int \frac{1}{(1+s)(1+t+s)}\, ds = \frac{\ln(1+s)}{t} - \frac{\ln(1+t+s)}{t}$.

4. The water of a certain reservoir is depleted at a constant rate of 1 000 units daily. The reservoir is refilled by randomly occurring rainfalls; they occur according to a Poisson process with rate 0.2 per day; and the amount of water added to the reservoir by a rainfall is 5 000 units with probability 0.8 or 8 000 units with probability 0.2. The present water level is at 5 000 units.

(a) Define in some detail the r.v. $W(t)$ representing the water level of the reservoir at time t. (1.0)

- **Auxiliary r.v. and stochastic process**
  X_i = water refill by the i^{th} rainfall
  $X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$

$$P(X = x) = \begin{cases} 0.8, & x = 5000 \\ 0.2, & x = 8000 \\ 0, & x \neq 5000, 8000 \end{cases}$$

  $N(t)$ = number of rainfalls up to time (in days) t
  $N(t) \sim \text{Poisson}(\lambda\, t = 0.2t)$

  We admit that $\{N(t),\ t \geq 0\} \sim PP(\lambda = 0.2) \perp\!\!\!\perp \{X_i : i \in \mathbb{N}\}$.

- **Defining** $W(t)$
  Since the water of the reservoir is depleted at a constant rate of 1 000 units daily, *etc.*, the water level of the reservoir at time t is given by

$$W(t) = \max\left\{0, 5000 - 1000\,t + \sum_{i=1}^{N(t)} X_i\right\}$$

  (current level − water depleted + water from the rainfalls).

- **Obs.**
  $\left\{\sum_{i=1}^{N(t)} X_i : t \geq 0\right\}$ is a compound PP.

(b) Find the probability that the reservoir will be empty within the next five days? (1.0)

- **Event**
  Capitalizing on the fact that the r.v. X_i are positive, the reservoir will be empty within the next five days iff

$$\begin{aligned} W(5) &= 0 \\ 5000 - 1000 \times 5 + \sum_{i=1}^{N(5)} X_i &\leq 0 \\ \sum_{i=1}^{N(5)} X_i &\leq 0 \\ N(5) &= 0, \end{aligned}$$

  that is, iff it does not rain in those 5 days.

- **Requested probability**

$$\begin{aligned} P[W(5) = 0] &= P[N(5) = 0] \\ &\overset{N(5)\sim Poisson(0.2\times 5)}{=} F_{Poisson(1)}(0) \\ &\overset{tables}{=} 0.3679. \end{aligned}$$

Group 2 — Renewal Processes	9.0 points

1. Consider a renewal process $\{N(t) : t \geq 0\}$, with renewal function $m(t) = \frac{\lambda t}{2} - \frac{1-e^{-2\lambda t}}{4}$, $t \geq 0$.

(a) Derive the inter-renewal distribution of $\{N(t) : t \geq 0\}$. Comment. (2.5)

- **Deriving the inter-renewal distribution**
  Since $m(t) = \frac{\lambda t}{2} - \frac{1-e^{-2\lambda t}}{4}$, $t \geq 0$, the LST of the renewal function is given by

$$\begin{aligned} \tilde{m}(s) &= \int_{0^-}^{+\infty} e^{-st}\, dm(t) \\ &= \int_{0^-}^{+\infty} e^{-st} \times \frac{\lambda}{2}\left(1 - e^{-2\lambda t}\right) dt \qquad \left[= \frac{\lambda}{2}\left(LT[1,s] - LT[e^{-2\lambda t}, s]\right) \quad \text{(see formulae)}\right] \\ &= \frac{\lambda}{2s}\int_{0^-}^{+\infty} s\, e^{-st}\, dt - \frac{\lambda}{2(2\lambda+s)}\int_{0^-}^{+\infty} (2\lambda+s)\, e^{-(2\lambda+s)t}\, dt \\ &= \frac{\lambda}{2s}\int_{0^-}^{+\infty} f_{exp(s)}(t)\, dt - \frac{\lambda}{2(2\lambda+s)}\int_{0^-}^{+\infty} f_{exp(2\lambda+s)}(t)\, dt \qquad (s > 0) \\ &= \frac{\lambda}{2s} - \frac{\lambda}{2(2\lambda+s)} \\ &= \frac{2\lambda^2 + \lambda s - \lambda s}{2s(2\lambda+s)} \\ &= \frac{\lambda^2}{s(2\lambda+s)}. \end{aligned}$$

  Moreover, since $\tilde{m}(s) \overset{form.}{=} \frac{\tilde{F}(s)}{1-\tilde{F}(s)}$ the LST of the inter-renewal distribution can be obtained in terms of the one of m:

$$\begin{aligned} \tilde{F}(s) &\overset{form.}{=} \frac{\tilde{m}(s)}{1+\tilde{m}(s)} \\ &= \frac{\frac{\lambda^2}{s(2\lambda+s)}}{1 + \frac{\lambda^2}{s(2\lambda+s)}} \\ &= \frac{\frac{\lambda^2}{s(2\lambda+s)}}{\frac{s^2+2\lambda s+\lambda^2}{s(2\lambda+s)}} \\ &= \left(\frac{\lambda}{\lambda+s}\right)^2. \end{aligned}$$

  Taking advantage of the m.g.f. in the formulae, we get:

$$\begin{aligned} \tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &= E\left(e^{-sX}\right) \\ &= M_X(-s) \\ \left(\frac{\lambda}{\lambda+s}\right)^2 &\equiv M_{gamma(2,\lambda)}(-s), \end{aligned}$$

  that is, the inter-renewal times are $X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$, where $X \sim \text{gamma}(2, \lambda)$.

- **Comment**
  $\{N(t) : t \geq 0\}$ consists of all even arrivals of a $PP(\lambda)$.

(b) Admit that the time unit is an hour and $\lambda = 1$. Calculate an approximate value to the probability that less than 24 arrivals occur in the first two days. (1.5)

- **Renewal process**
  $\{N(t) : t \geq 0\}$
  $N(t)$ = number of *even* arrivals of a $PP(\lambda = 1)$ until time t

- **Inter-renewal times**
  $X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$
  $X \sim \text{gamma}(2, \lambda = 1)$
  $\mu = E(X) \overset{form.}{=} \frac{2}{\lambda} = 2$ [The ERT leads to the value of μ: $\lim_{t\to+\infty} m(t)/t = 1/\mu$.]
  $\sigma^2 = V(X) \overset{form.}{=} \frac{2}{\lambda^2} = 2$ [In this problem, we do not need to know σ^2 to obtain the prob.]

- **Requested approximate probability**

$$P[N(t) < n] \overset{form.}{\simeq} \Phi\left(\frac{n - t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right) \overset{t=48,n=200}{=} \Phi\left(\frac{24 - 48/2}{\sqrt{48 \times 2/2^3}}\right) = \Phi(0) = 0.5.$$

(c) Show that the renewal function verifies Blackwell's theorem. (1.0)

- **Verification of Blackwell's theorem**

Since the inter-renewal distribution is non-lattice,

$$\begin{aligned}\lim_{t\to+\infty}[m(t+a) - m(t)] &= \lim_{t\to+\infty}\left[\frac{\lambda(t+a)}{2} - \frac{1-e^{-2\lambda(t+a)}}{4}\right] - \left[\frac{\lambda t}{2} - \frac{1-e^{-2\lambda t}}{4}\right] \\ &= \lim_{t\to+\infty}\left[\frac{\lambda a}{2} - \frac{e^{-2\lambda t} - e^{-2\lambda(t+a)}}{4}\right] \\ &= \frac{\lambda a}{2} \\ &\equiv \frac{a}{\mu},\end{aligned}$$

hence verifying Blackwell's theorem.

2. Three machines operate independently and all of them are needed for a system to work. Machine i operates for an exponential time with rate λ_i ($i = 1,2,3$) before it fails. When a machine fails, the system is shut down and repair begins on the failed machine.[6] The expected time to repair machine i is μ_i ($i = 1,2,3$). Once a failed machine is repaired, it is as good as new and all machines are restarted.

(a) Obtain the expected duration of the up and down cycle. (1.5)

- **R.v.**

Y_i = operating time of machine i

$Y_i \overset{indep.}{\sim}$ exponential(λ_i), $\quad i = 1,2,3$

W_i = repair time of machine i

$E(W_i) = \mu_i, \quad i = 1,2,3$

- **Up time**

Since all three machines are needed for a system to work (the system is UP) for a time

$$U = \min_{i=1,2,3} Y_i \overset{form.}{\sim} \text{exponential}\left(\sum_{i=1}^{3}\lambda_i\right)$$

$$E(U) = \frac{1}{\sum_{i=1}^{3}\lambda_i}.$$

- **Down time**

Note that machine i is being repaired if it was the first of the three machines to fail. Thus, the probability that machine i is being repaired equals

$$p_i = P(Y_i = \min_{j=1,2,3} Y_j) \overset{form.}{=} \frac{\lambda_i}{\sum_{j=1}^{3}\lambda_i}.$$

In that case machine i is being repaired for a period time W_i with expected value $E(W_i) = \mu_i$. Let D represent the repair time (the system is DOWN). Then

$$\begin{aligned}E(D) &= \sum_{i=1}^{3} E(W \mid \text{machine } i \text{ is being repaired}) \times P(\text{machine } i \text{ is being repaired}) \\ &= \sum_{i=1}^{3} E(W_i) \times \frac{\lambda_i}{\sum_{j=1}^{3}\lambda_i} \\ &= \sum_{i=1}^{3}\mu_i \times \frac{\lambda_i}{\sum_{j=1}^{3}\lambda_i} \\ &\overset{form.}{=} \frac{\sum_{i=1}^{3}\mu_i \times \lambda_i}{\sum_{i=1}^{3}\lambda_i}\end{aligned}$$

- **Expected length of an up and down cycle**

$$\begin{aligned}E(X) &= E(U) + E(D) \\ &= \frac{1}{\sum_{i=1}^{3}\lambda_i} + \frac{\sum_{i=1}^{3}\mu_i \times \lambda_i}{\sum_{i=1}^{3}\lambda_i} \\ &= \frac{1 + \sum_{i=1}^{3}\mu_i \times \lambda_i}{\sum_{i=1}^{3}\lambda_i}\end{aligned}$$

(b) Derive an expression for the long-run proportion of time the system is working. (1.0)

- **State variable**

$$Z(t) = \begin{cases} 1, & \text{if all three machines are operating (i.e., system is up) at time } t \\ 0, & \text{if a machine is being repaired (i.e., system is down) at time } t \end{cases}$$

- **Alternating renewal process**

$\{Z(t) : t \geq 0\}$

- **Long-run proportion of time the system is working**

$E(U + D) < +\infty$, thus

$$\begin{aligned}\lim_{t\to+\infty} P(t) &= \lim_{t\to+\infty} P[Z(t) = 1] \\ &= \lim_{t\to+\infty} P(\text{system is up at time } t) \\ &\overset{form.}{=} \frac{E(U)}{E(U) + E(D)} \\ &= \frac{\frac{1}{\sum_{i=1}^{3}\lambda_i}}{\frac{1+\sum_{i=1}^{3}\mu_i\times\lambda_i}{\sum_{i=1}^{3}\lambda_i}} \\ &= \frac{1}{1 + \sum_{i=1}^{3}\mu_i \times \lambda_i}.\end{aligned}$$

(c) What is the long-run proportion of time machine 1 is being repaired? (1.5)

- **Important**

To calculate the requested quantity we are going to think of a renewal reward process by supposing that we earn 1 monetary unit per time unit that machine 1 is being repaired (during an up and down cycle).

- **Renewal process**

$\{N(t) : t \geq 0\}$

$N(t)$ = number completed up and down cycle by time t

- **Expected inter-renewal time**

$E(X) = E(U) + E(D) = \frac{1+\sum_{i=1}^{3}\mu_i\times\lambda_i}{\sum_{i=1}^{3}\lambda_i}$

- **Reward renewal process**

$\{R(t) = \sum_{n=1}^{N(t)} R_n : t \geq 0\}$

[6] During this repair, the remaining machines are in a state of suspended animation, that is, neither working nor being repaired.

$R(t)$ = total reward gained until time t

$(X_n, R_n) \overset{i.i.d.}{\sim} (X, R)$, $n \in \mathbb{N}$

$$R = \begin{cases} W_1, & \text{if machine 1 is being repaired} \\ 0, & \text{otherwise} \end{cases}$$

- **Expected reward per cycle**

$$\begin{aligned} E(R) &= E(W_1) \times P(\text{machine 1 is being repaired}) + 0 \times P(\text{machine 1 is being repaired}) \\ &= \mu_1 \times \frac{\lambda_1}{\sum_{i=1}^{3} \lambda_i} \end{aligned}$$

- **Long-run expected total reward per time unit**

Since $E(X), E(R) < +\infty$, we can apply the ERT for renewal reward processes and get

$$\begin{aligned} \lim_{t \to +\infty} \frac{E[R(t)]}{t} &= \frac{E(R)}{E(X)} \\ &= \frac{\mu_1 \times \frac{\lambda_1}{\sum_{i=1}^{3} \lambda_i}}{\frac{1+\sum_{i=1}^{3} \mu_i \times \lambda_i}{\sum_{i=1}^{3} \lambda_i}} \\ &= \frac{\mu_1 \times \lambda_1}{1 + \sum_{i=1}^{3} \mu_i \times \lambda_i}. \end{aligned}$$