Introduction to Stochastic Processes
MMA, LMAC

DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

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Duration: 90 minutes **Test 1**

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Renewal Processes 2.0 points

Consider a renewal process, $\{N_D(t): t \geq 0\}$, consisting of all ODD arrivals of a Poisson process with rate λ. (2.0)

Derive the associated renewal function, $m_D(t)$, by using the Laplace-Stieltjes transform method and capitalizing on the table of important Laplace transforms in the formulae.

- **Renewal process**
  $\{N_D(t): t \geq 0\}$
  $N_D(t) =$ number of ODD arrivals until time t
- **Inter-renewal times**
  $X_1 \sim \text{Exponential}(\lambda)$ because the time until the first (and odd) arrival is exponentially distributed r.v. with mean λ^{-1}.
  $X_i \stackrel{i.i.d.}{\sim} X$, $i = 2, 3, \ldots$
  $X \sim \text{Gamma}(2, \lambda)$ because the time between consecutive odd arrivals is a sum of two independent exponentially distributed r.v. with mean λ^{-1}.
- **Deriving the renewal function**
  The LST of the d.f. of the1st. inter-renewal of X is given by

$$\begin{aligned}\tilde{G}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dG(x) \\ &= E\left(e^{-sX_1}\right) \\ &= M_{X_1}(-s) \\ &\stackrel{form.}{=} \frac{\lambda}{\lambda+s}\end{aligned}$$

The LST of the common d.f. of the remaining inter-renewal times is given by

$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &= E\left(e^{-sX}\right) \\ &= M_X(-s) \\ &\stackrel{form.}{=} \left(\frac{\lambda}{\lambda+s}\right)^2\end{aligned}$$

Moreover, the LST of the renewal function can be obtained in terms of the one of F:

$$\begin{aligned}\tilde{m}_D(s) &\stackrel{form.}{=} \frac{\tilde{G}(s)}{1-\tilde{F}(s)} \\ &= \frac{\frac{\lambda}{\lambda+s}}{1-\left(\frac{\lambda}{\lambda+s}\right)^2} \\ &= \frac{\lambda(\lambda+s)}{2\lambda s+s^2} \\ &= \frac{\lambda^2}{(s+0)(s+2\lambda)} + \frac{\lambda}{s+2\lambda}.\end{aligned}$$

Taking advantage of the LT in the formulae, we successively get:

$$\begin{aligned}\frac{dm_D(t)}{dt} &= LT^{-1}\left[\tilde{m}_D(s), t\right] \\ &= LT^{-1}\left[\frac{\lambda^2}{(s+0)(s+2\lambda)} + \frac{\lambda}{s+2\lambda}, t\right] \\ &= \lambda^2 \times \frac{e^{-0\times t} - e^{-2\lambda\times t}}{2\lambda - 0} + \lambda\, e^{-2\lambda t} \\ &= \frac{\lambda}{2}\left(1+e^{-2\lambda t}\right) \\ m_D(t) &= \int_0^t \frac{\lambda}{2}\left(1+e^{-2\lambda x}\right) dx \\ &= \frac{\lambda t}{2} + \frac{\lambda}{2}\,\frac{1-e^{-2\lambda t}}{2\lambda} \\ &= \frac{\lambda t}{2} + \frac{1-e^{-2\lambda t}}{4}, \quad t \geq 0.\end{aligned}$$

Group 2 — Discrete time Markov chains 9.0 points

1. A communication network consisting of the sequence of stages of binary communication channels is a DTMC $\{X_n : n \in \mathbb{N}_0\}$, where X_n denotes the digit leaving the n^{th} stage of the channel and X_0 denotes the digit entering the first stage. This DTMC has state space $\mathcal{S} = \{0, 1\}$ and TPM

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where $a, b \in (0, 1)$.

(a) Show that the $n-$step TPM is given by (1.5)

$$\mathbf{P}^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

(Do not resort to mathematical induction. Check the note below!)

Note: The eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 1-a-b$. Then, using the spectral decomposition method, we get $\mathbf{P}^n = \lambda_1^n \mathbf{E}_1 + \lambda_2^n \mathbf{E}_2$, where $\mathbf{E}_1 = (\lambda_1 - \lambda_2)^{-1}(\mathbf{P} - \lambda_2 \mathbf{I})$ and $\mathbf{E}_2 = (\lambda_2 - \lambda_1)^{-1}(\mathbf{P} - \lambda_1 \mathbf{I})$ are the constituent matrices of $\mathbf{P}$.

- **DTMC**
  $\{X_n : n \in \mathbb{N}\}$
  $X_n = n^{th}$ the digit leaving the n^{th} stage of the channel
- **State space**
  $\mathcal{S} = \{0, 1\}$
- **TPM**

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

- **$n-$step TPM**
  Let $\lambda_1 = 1$ and $\lambda_2 = 1-a-b$. Then

$$\begin{aligned}\mathbf{E}_1 &= (\lambda_1 - \lambda_2)^{-1}(\mathbf{P} - \lambda_2 \mathbf{I}) \\ &= \frac{1}{1-(1-a-b)}\left(\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} - (1-a-b)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} \\
\mathbf{E}_2 &= (\lambda_2-\lambda_1)^{-1}(\mathbf{P}-\lambda_1\mathbf{I}) \\
&= \frac{1}{(1-a-b)-1}\left(\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \frac{1}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix} \\
\mathbf{P}^n &= \lambda_1^n\mathbf{E}_1+\lambda_2^n\mathbf{E}_2 \\
&= \mathbf{E}_1+(1-a-b)^n\mathbf{E}_2 \\
&= \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}+\frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.
\end{aligned}
$$

(b) Assume that $a = 0.1$, $b = 0.2$, and $P(X_0 = 0) = P(X_0 = 1) = 0.5$. Find the p.f. of X_n. (1.5)

- **Initial distribution**
  $P(X_0 = 0) = P(X_0 = 1) = 0.5$, $a = 0.1$, and $b = 0.2$, thus
  $$
  \begin{aligned}
  \underline{\alpha} &= [P(X_0=i)]_{i\in\mathscr{S}} \\
  &= [0.5 \quad 0.5]
  \end{aligned}
  $$
- **$n-$step TPM**
  $$
  \begin{aligned}
  \mathbf{P}^n &= \frac{1}{0.1+0.2}\begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}+\frac{(1-0.1-0.2)^n}{0.1+0.2}\begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix} \\
  &= \frac{1}{0.3}\begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}+\frac{0.7^n}{0.3}\begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix} \\
  &= \begin{bmatrix} \frac{2+0.7^n}{3} & \frac{1-0.7^n}{3} \\ \frac{2-2\times0.7^n}{3} & \frac{1+2\times0.7^n}{3} \end{bmatrix}
  \end{aligned}
  $$
- **Requested p.f.**
  $$
  \begin{aligned}
  \underline{\alpha}^n &= [P(X_n=i)]_{i\in\mathscr{S}} \\
  &\overset{[(3.8)]}{=} \underline{\alpha}\times\mathbf{P}^n \\
  &= [0.5 \quad 0.5]\times\begin{bmatrix} \frac{2+0.7^n}{3} & \frac{1-0.7^n}{3} \\ \frac{2-2\times0.7^n}{3} & \frac{1+2\times0.7^n}{3} \end{bmatrix} \\
  &= \begin{bmatrix} \frac{2}{3}-\frac{0.7^n}{6} & \frac{1}{3}+\frac{0.7^n}{6} \end{bmatrix}.
  \end{aligned}
  $$

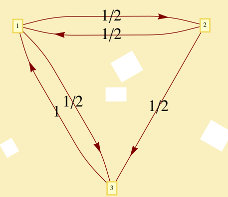
(c) After having classified the two states of this DTMC, obtain the long-run fraction of time that the communication network is in state 0. (1.0)

- **Classification of the states of the DTMC**
  Since $a \neq 0$ and $b \neq 0$, we are dealing with a finite single closed communicating class, i.e., with an irreducible DTMC with finite state space. Hence, all states are positive recurrent[, by Prop. 3.35].
  Furthermore, the DTMC is aperiodic, namely because $a \neq 1$ and $b \neq 1$.
- **Requested long-run fraction of time**
  Since the DTMC is irreducible, positive recurrent and aperiodic we can add that long-run fraction of time that the communication network is in state 0 equals
  $$
  \begin{aligned}
  \lim_{n\to+\infty} P_{i0}^n &\overset{(a)}{=} \lim_{n\to+\infty}\left(\frac{2}{3}-\frac{0.7^n}{6}\right) \\
  &= \frac{2}{3}, \quad i\in\mathscr{S}.
  \end{aligned}
  $$
  [Alternatively, the DTMC is irreducible, positive recurrent and aperiodic then
  $$\lim_{n\to+\infty} P_{ij}^n = \pi_j > 0,\ i,j\in\mathscr{S},$$
  where the row vector $\underline{\pi} = [\pi_j]_{j\in\mathscr{S}}$ is the unique stationary distribution satisfying
  $$\begin{cases} \pi_j = \sum_{i\in\mathscr{S}}\pi_i P_{ij},\ j\in\mathscr{S} \\ \sum_{j\in\mathscr{S}}\pi_j = 1, \end{cases}$$
  that is,
  $$\begin{cases} \pi_0 = \pi_0 P_{00}+\pi_1 P_{10} \\ \pi_0+\pi_1 = 1 \end{cases}$$
  $$\begin{cases} \pi_0 = \pi_0\times(1-a)+(1-\pi_0)\times b \\ \pi_1 = 1-\pi_0 \end{cases}$$
  $$\begin{cases} \pi_0 = \pi_0\times0.9+(1-\pi_0)\times0.2 \\ - \end{cases}$$
  $$\begin{cases} \pi_0\times(1-0.9+0.2) = 0.2 \\ - \end{cases}$$
  $$\begin{cases} \pi_0 = \frac{2}{3} \\ \pi_1 = 1-\frac{2}{3} = \frac{1}{3}.] \end{cases}$$

2. Consider a brand-switching model such that a customer keeps switching between brands 1, 2 and 3 according to the TPM
$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}.$$

(a) Draw the associated transition diagram and classify the states of this DTMC. Are the states periodic? (1.5)

- **DTMC**
  $\{X_n : n\in\mathbb{N}\}$
  X_n = brand bought in the n^{th} acquisition
- **State space**
  $\mathscr{S} = \{1,2,3\}$
- **TPM**
  $$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$
- **Transition diagram**



- **Classification/periodicity of the states of the DTMC**
  - Judging by the transition diagram, all states communicate with one another, thus $\mathscr{S} = \{1,2,3\}$ is a finite single closed communicating class [hence the DTMC has a finite state space and is irreducible]. With that being said, [by Prop. 3.35,] all states are positive recurrent.

– The transition diagram leads us to conclude that $P_{22}^n > 0$, $n = 2,3,4,\ldots$, therefore we can return to state 2 after $2,3,4,\ldots$ transitions, $d(2) = gcd\{n \in \mathbb{N} : P_{22}^n > 0\} = 1$ and this state is aperiodic.
The same holds for the remaining states of this closed (and positive recurrent) communicating class, $\mathscr{S} = \{1,2,3\}$.
[After all (a)periodicity is a class property.]

(b) Suppose brands 1, 2 and 3 cost 100, 150 and 200 euro (respectively). Assume a discount factor $\alpha = \frac{2}{3}$ and compute the expected total discounted expenditure of a customer starting with brand 1. (1.5)

Note: Check the footnote on the next page![1]

- **Vector of costs**

$$\begin{aligned}\underline{c} &= [c(j)]_{j\in\mathscr{S}} \\ &= \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix}\end{aligned}$$

- **Discount factor**
$\alpha = \frac{2}{3}$

- **Vector of the expected total discounted expenditures**
[According to Prop. 3.86,] The expected total discounted cost incurred over the infinite horizon, starting at state i, is equal to

$$\phi(i) = E\left[\sum_{n=0}^{+\infty} \alpha^n c(X_n) \mid X_0 = i\right].$$

Equivalently, $\underline{\phi} = [\phi(i)]_{i\in\mathscr{S}} = (\mathbf{I} - \alpha\mathbf{P})^{-1} \times \underline{c}$. [This result is valid for any TPM. Indeed, no assumptions are made about irreducibility, periodicity, transience or recurrence of the DTMC in Prop. 3.86.]
In this particular case, we get

$$\begin{aligned}\underline{\phi} &= (\mathbf{I} - \alpha\mathbf{P})^{-1} \times \underline{c} \\ &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3} \times \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}\right)^{-1} \times \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & 0 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix} \\ [&= \begin{bmatrix} \frac{27}{16} & \frac{9}{16} & \frac{3}{4} \\ \frac{15}{16} & \frac{21}{16} & \frac{3}{4} \\ \frac{9}{8} & \frac{3}{8} & \frac{3}{2} \end{bmatrix} \times \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix} = \begin{bmatrix} \frac{3225}{8} \\ \frac{3525}{8} \\ \frac{1875}{4} \end{bmatrix}]\end{aligned}$$

- **Requested expected total discounted expenditure (customer starting with brand 1)**

$$\begin{aligned}\phi(1) &= \text{1st. row of } (\mathbf{I} - \alpha\mathbf{P})^{-1} \times \underline{c} \\ &= \begin{bmatrix} \frac{27}{16} & \frac{9}{16} & \frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix} \\ &= \frac{3225}{8} \\ &= 403.125.\end{aligned}$$

(c) Determine $f_{ij}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$, for $i, n = 1,2,3$ and $j = 3$. (2.0)

[1] The following result may come handy: $\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{27}{16} & \frac{9}{16} & \frac{3}{4} \\ \frac{15}{16} & \frac{21}{16} & \frac{3}{4} \\ \frac{9}{8} & \frac{3}{8} & \frac{3}{2} \end{bmatrix}$.

- **Requested probabilities**
Let:
  i) $f_{ij}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$ be the probability of reaching state j for the first time starting from state i, for $i, j \in \mathscr{S}$ and $n \in \mathbb{N}$;
  ii) $\underline{f}_j^n = [f_{ij}^n]_{i\in\mathscr{S}}$ be the associated vector for fixed $j \in \mathscr{S}$ and $n \in \mathbb{N}$.

According to the formulae,

$$\underline{f}_j^n = \begin{cases} \underline{f}_j^1 = [P_{ij}]_{i\in\mathscr{S}}, & n = 1 \\ {}^{(j)}\mathbf{P} \times \underline{f}_j^{n-1} = \left[{}^{(j)}\mathbf{P}\right]^{n-1} \times \underline{f}_j^1, & n = 2,3,\ldots, \end{cases}$$

where ${}^{(j)}\mathbf{P}$ is obtained by setting all the entries of the j^{th} column of $\mathbf{P}$ equal to 0. When $j = 3$, we successively get

$$\begin{aligned}{}^{(3)}\mathbf{P} &= \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ \underline{f}_3^1 &= [P_{i3}]_{i\in\mathscr{S}} \\ &= \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \\ \underline{f}_3^2 &= {}^{(3)}\mathbf{P} \times \underline{f}_3^1 \\ &= \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} \\ \underline{f}_3^3 &= {}^{(3)}\mathbf{P} \times \underline{f}_3^2 \\ &= \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.125 \\ 0.125 \\ 0.25 \end{bmatrix}.\end{aligned}$$

Group 3 — Continuous time Markov chains	9.0 points

1. A gas station has one diesel fuel pump for trucks only and has capacity for three trucks (including the one at the pump). Trucks arrive according to a Poisson process with rate λ and each truck takes an exponentially distributed time with expected value μ^{-1} to fill its fuel tank. Let $X(t)$ be the number of trucks at the gas station at time t.

(a) Draw the rate diagram and derive the infinitesimal generator $\mathbf{R}$ of the CTMC $\{X(t) : t \geq 0\}$. (1.0)

- **CTMC**
$\{X(t) : t \geq 0\}$
$X(t)$ = number of trucks at the gas station at time t

- **State space**
$\mathscr{S} = \{0,1,2,3\}$

- **Birth/death rates**
  We can interpret an arrival of a truck as a *birth* and note that a filled tank means the departure of the truck or *death*. Moreover, there is only one server and the system has a finite capacity. Thus,
  $$\begin{aligned}\lambda_i &= \lambda, \quad i=0,1,2\\ \mu_i &= \mu, \quad i=1,2,3.\end{aligned}$$
  [By the way, we are dealing with an $M/M/1/3$ system...]

- **Rate diagram**
  Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node i to node j (if $q_{ij}>0$) with q_{ij} written on it. These rates coincide with the birth and death rates...
  $$\begin{tikzcd} \boxed{0} \arrow[r, bend left, "\lambda"] & \boxed{1} \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & \boxed{2} \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & \boxed{3} \arrow[l, bend left, "\mu"] \end{tikzcd}$$

- **Infinitesimal generator**
  This matrix has entries
  $$r_{ij} = \begin{cases} q_{ij}, & i \neq j \\ -\nu_i = -\sum_{m\in\mathscr{S}} q_{im}, & j = i \end{cases}$$
  and in this case it is equal to
  $$\begin{aligned}\mathbf{R} &= [r_{ij}]_{i,j\in\mathscr{S}}\\ &= \begin{bmatrix} -\lambda & \lambda & 0 & 0\\ \mu & -(\lambda+\mu) & \lambda & 0\\ 0 & \mu & -(\lambda+\mu) & \lambda\\ 0 & 0 & \mu & -\mu \end{bmatrix}.\end{aligned}$$

(b) Write the Kolmogorov's forward differential equations in terms of $P_j(t) \equiv P_{0j}(t) = P[X(t)=j \mid X(0)=0]$, for $j\in\mathbb{N}_0$. (Do not try to solve the differential equations!) (1.5)

- **Kolmogorov's forward differential equations**
  These can be written in matrix form:
  $$\frac{d\,\mathbf{P}(t)}{dt} = \left[\frac{d\,P_{ij}(t)}{dt}\right]_{i,j\in\mathscr{S}} \overset{form.}{=} \mathbf{P}(t)\times\mathbf{R}.$$
  Since $i=0$, we are only interested in the first row of the previous matrix
  $$\left[\frac{d\,P_{0j}(t)}{dt}\right]_{j\in\mathscr{S}} = \left[P_{0j}(t)\right]_{j\in\mathscr{S}}\times\mathbf{R}.$$
  Hence the following Kolmogorov's forward differential equations:
  $$\frac{d\,P_j(t)}{dt} \overset{form.}{=} P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j+\mu_j),\ j\in\mathscr{S},$$
  reads as follows:
  $$\begin{aligned}\frac{d\,P_0(t)}{dt} &= P_1(t)\,\mu - P_0(t)\,\lambda\\ \frac{d\,P_j(t)}{dt} &= P_{j-1}(t)\,\lambda + P_{j+1}(t)\,\mu - P_j(t)\,(\lambda+\mu),\ j=1,2\\ \frac{d\,P_3(t)}{dt} &= P_2(t)\,\lambda - P_3(t)\,\mu.\end{aligned}$$

(c) After having admitted that $\rho=\frac{\lambda}{\mu}\in\mathbb{R}^+\backslash\{1\}$, derive the equilibrium probabilities $P_j = \lim_{t\to+\infty}P_j(t)$ for this birth and death process in terms of ρ. (2.0)

- **[Obs.**
  Since this CTMC has a finite state space, we only need to verify that $\rho=\frac{\lambda}{\mu}<+\infty$ to guarantee the existence of equilibrium probabilities $P_j=\lim_{t\to+\infty}P_j(t)$.
  We considered $\rho\neq 1$ to avoid a 2nd. trivial case: $P_j=\frac{1}{4}$, $j=0,1,2,3$.]

- **Equilibrium probabilities** $P_j=\lim_{t\to+\infty}P_j(t)$
  $$\begin{aligned}P_0 &= \left[1+\sum_{n=1}^{+\infty}\left(\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)\right]^{-1}\\ &= \left[1+\sum_{n=1}^{3}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}\\ &= \left[1+\sum_{n=1}^{3}\rho^n\right]^{-1}\\ &\overset{\rho\neq1}{=} \left(1+\rho\,\frac{1-\rho^3}{1-\rho}\right)^{-1}\\ &= \left(\frac{1-\rho+\rho-\rho^4}{1-\rho}\right)^{-1}\\ &= \frac{1-\rho}{1-\rho^4}\\ P_j &= P_0\times\prod_{i=0}^{j-1}\frac{\lambda_i}{\mu_{i+1}}\\ &= P_0\times\left(\frac{\lambda}{\mu}\right)^j\\ &= \frac{1-\rho}{1-\rho^4}\,\rho^j, \quad j=1,2,3.\end{aligned}$$

(d) Obtain the expected number of truck at the gas station in equilibrium in terms of ρ. (0.5)

- **Requested expected value**
  [We are dealing with a $M/M/1/3$ queue and we are supposed to calculate $E(L_s)$...]
  Let L_s be the number of trucks at the gas station in equilibrium. Then
  $$\begin{aligned}E(L_s) &= \sum_{j=0}^{3} j\,P_j\\ &= \frac{1-\rho}{1-\rho^4}\times\sum_{j=1}^{3} j\,\rho^j\\ &= \frac{(1-\rho)(\rho+2\rho^2+3\rho^3)}{1-\rho^4}\\ &= \frac{\rho-\rho^2+2\rho^2-2\rho^3+3\rho^3-3\rho^4)}{1-\rho^4}\\ &= \frac{\rho+\rho^2+\rho^3-3\rho^4}{1-\rho^4}\end{aligned}$$

2. A small car rental company has a fleet of 6 cars. Customers arrive according to a Poisson process with a rate of 5 customers per day. A customer rents a car for an exponential time with a mean of 1.5 days. Arriving customers for which no car is available are lost.

(a) Determine the fraction of arriving customers for which no car is available. (1.5)

- **Birth-death queueing system**
  $M/M/m/m$
  $m=6$

- **State space**
  $\mathscr{S} = \{0, 1, \dots, m\}$

- **Birth/death rates**
  $\lambda_k = \lambda = 5,\ k \in \{0, 1, \dots, m-1\}$
  $\mu_k = k\,\mu = 1.5^{-1}\, k = \frac{2k}{3},\ k \in \{1, 2, \dots, m\}$

- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{m\mu} = \frac{5}{6\times 2/3} = 1.25 < +\infty$

- **Performance measure (in the long-run)**
  L_s = number of rented cars

$$P(L_s = k) = \begin{cases} \frac{\frac{(m\rho)^k}{k!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}}, & k = 0, 1, \dots, m \\ 0, & k = m+1, m+2, \dots \end{cases}$$

- **Requested probability**

$$\begin{aligned} P(L_s = m) &= B(m, m\rho) \\ &= \frac{\frac{(m\rho)^m}{m!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}} \\ &\overset{m=6,\lambda=5,\mu=2/3}{=} \frac{\frac{7.5^6}{6!}}{\sum_{j=0}^{6} \frac{7.5^j}{j!}} \\ &\simeq 0.361541. \end{aligned}$$

(b) Admit that the daily costs (depreciation, insurance, maintenance, etc.) amount to 60 euro per car and the car rental is priced at 110 euro per day. Obtain the mean daily profit. (1.5)

- **R.v.**
  $C^{M/M/m/m}$ = daily profit $= 110 \times L_s - m \times 60$

- **Mean daily profit**

$$\begin{aligned} E\left(C^{M/M/m/m}\right) &= 110 \times E(L_s) - m \times 60 \\ &\overset{form}{=} 110 \times m\rho \times [1 - B(m, m\rho)] - 60\,m \\ &\overset{(a), m\rho=\lambda/\mu=7.5}{\simeq} 110 \times 7.5 \times (1 - 0.361541) - 60 \times 6 \\ &\simeq 166.729. \end{aligned}$$

(c) The company is considering to add 3 extra cars to the fleet. Will this decision increase the mean profit per day? (1.0)

Note: $B(9, 7.5) \simeq 0.147397$.

- **Birth-death queueing system**
  $M/M/m/m$
  $m = 9$

- **Updated mean daily profit**

$$\begin{aligned} E\left(C^{M/M/m/m}\right) &\overset{(b)}{=} 110 \times m\rho \times [1 - B(m, m\rho)] - 60\,m \\ &\overset{m=9, m\rho=\lambda/\mu=7.5}{\simeq} 110 \times 7.5 \times (1 - 0.147397) - 60 \times 9 \\ &\simeq 163.397. \end{aligned}$$

- **Comment**
  $E\left(C^{M/M/9/9}\right) = 163.397 < E\left(C^{M/M/6/6}\right) = 166.729$, thus adding a car to the fleet does not pay off.