

RESOLUÇÃO  
 TESTE 1, V3

MODSIM

(P1) - Análise de Sistemas Não Lineares

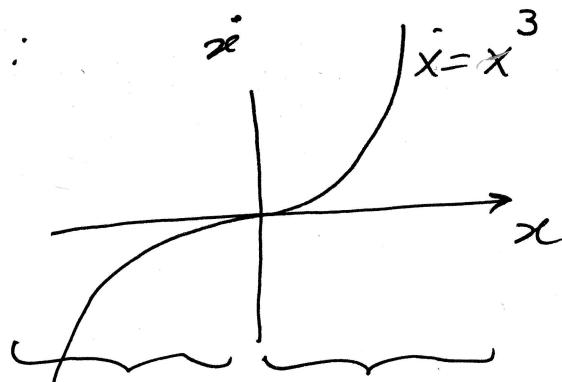
$$1.1 - \dot{x} = x^3 + u; u=0$$

$$\Rightarrow \dot{x} = x^3$$

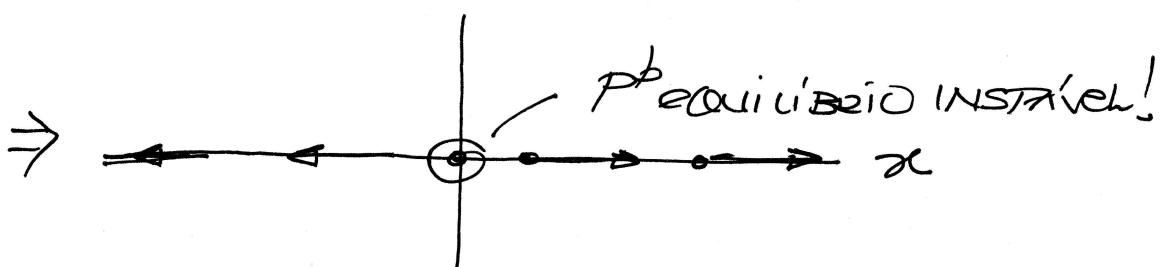
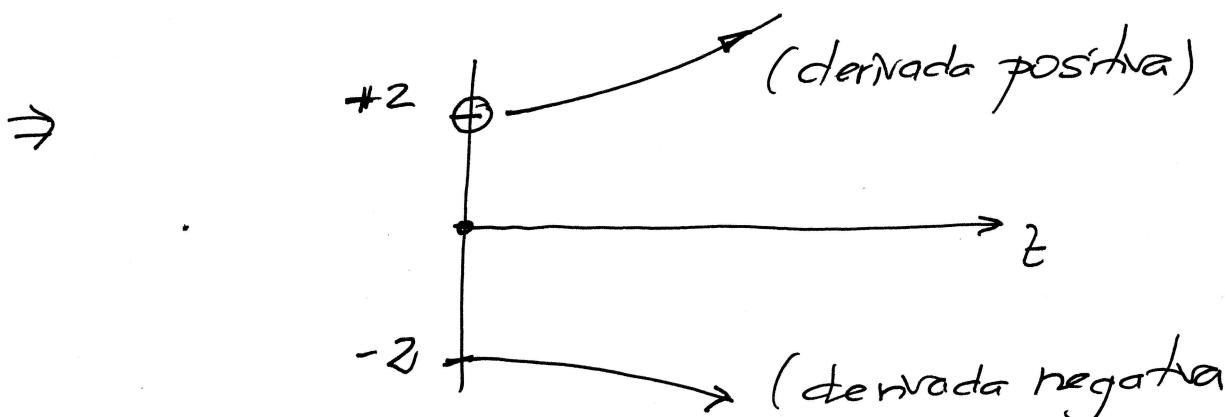
$x=0 \Rightarrow \dot{x}=0$ , ou seja,  $x=0$  é o único ponto de equilíbrio

Tracado de trajectórias:

$$\frac{dx}{dt} = x^3 \Rightarrow$$



derivada < 0      derivada > 0

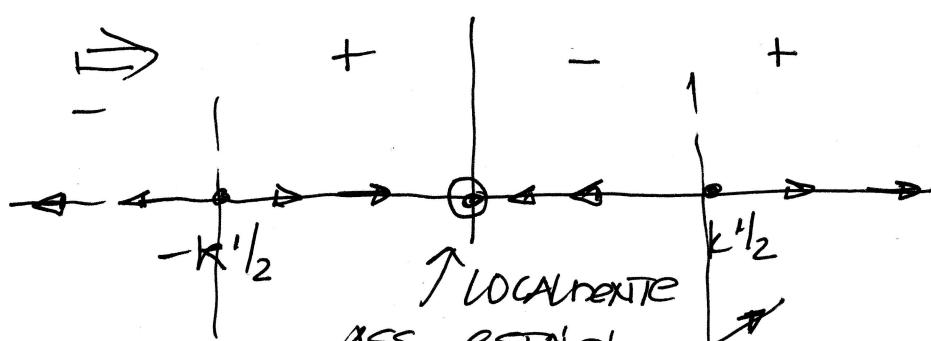
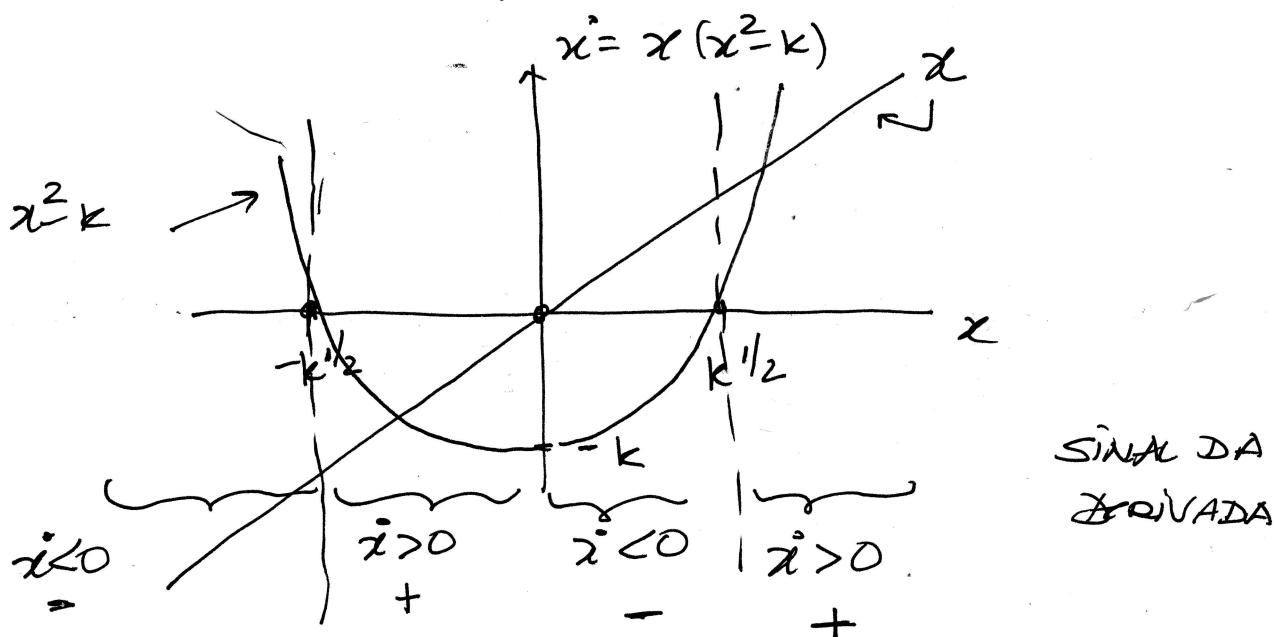


P1.2 -

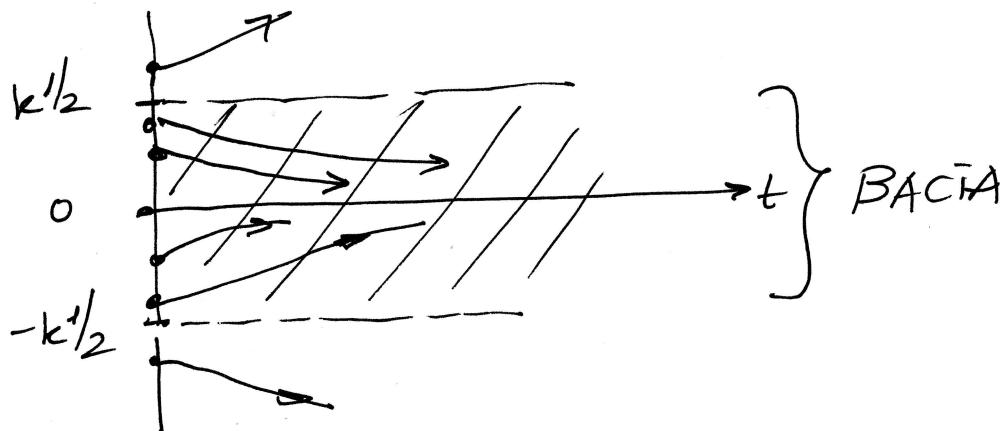
$$\dot{x} = x(x^2 - k)$$

Pontos de equilíbrio:

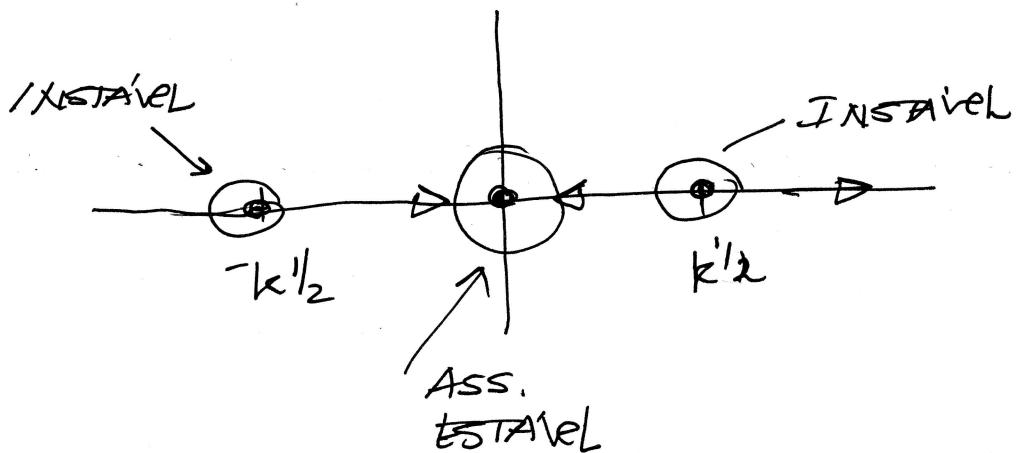
$$x(x^2 - k) = 0 \Leftrightarrow x = 0, +k^{1/2}, -k^{1/2}$$

Tracado de trajectórias  $\Rightarrow$  investigar o sinal de  $\dot{x}$ 

TRAJECTÓRIAS  
no  
TEMPO


## Bacia de Atracção de 0



ESTABILIDADE LOCAL DOS PONTOS DE EQUILÍBRIO



PI.3 - ANÁLISE DE ESTABILIDADE recorrendo a LINEARIZAÇÕES

$$\ddot{x} = x^3 - kx \quad ; \quad \text{seja } x = x_{\text{eq}} + \delta x$$

$$\Rightarrow (\ddot{x}_{\text{eq}} + \ddot{\delta x}) = (x_{\text{eq}} + \delta x)^3 - k(x_{\text{eq}} + \delta x)$$

$$\approx \underbrace{(x_{\text{eq}}^3)}_{\sim} + 3x_{\text{eq}}^2 \delta x - k \delta x - \underbrace{k x_{\text{eq}}}_{\text{equilíbrio!}}$$

$$\boxed{x_{\text{eq}}^3 - k x_{\text{eq}} = 0} \quad (\text{equilíbrio!})$$

$$\Rightarrow \boxed{\dot{\delta x} = 3x_{eq}^2 \delta x - k \delta x} = \boxed{(3x_{eq}^2 - k) \delta x}$$

da forma  $\dot{\delta x} = a \delta x$ ;  $a = 3x_{eq}^2 - k$

Análise das pontas de equilíbrio

$$x_{eq}=0 \Rightarrow \dot{\delta x} = -k \delta x \Rightarrow \begin{array}{l} \text{POUQUIL. LOGARÍTMIC} \\ \text{ASSIMPT. ESTÁVEL} \end{array}$$

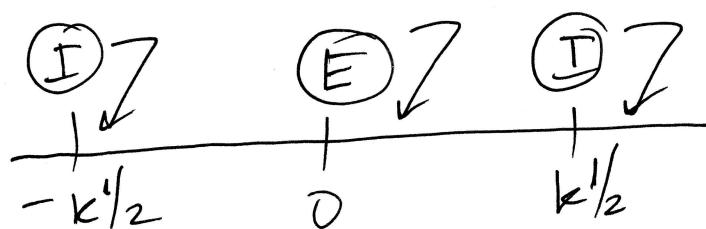
$< 0$

$$x_{eq}=k^{1/2} \Rightarrow \dot{\delta x} = 3x_{eq}^2 - k = \Rightarrow \text{INSTAVEL}$$

$$= 3k - k = 2k > 0$$

$$x_{eq}=-k^{1/2} \Rightarrow \dot{\delta x} = 2k > 0 \Rightarrow \text{INSTAVEL}$$

/



P2)

Modelação em Espaço de Estados

Q2.1 -

$$x_1 = z_1$$

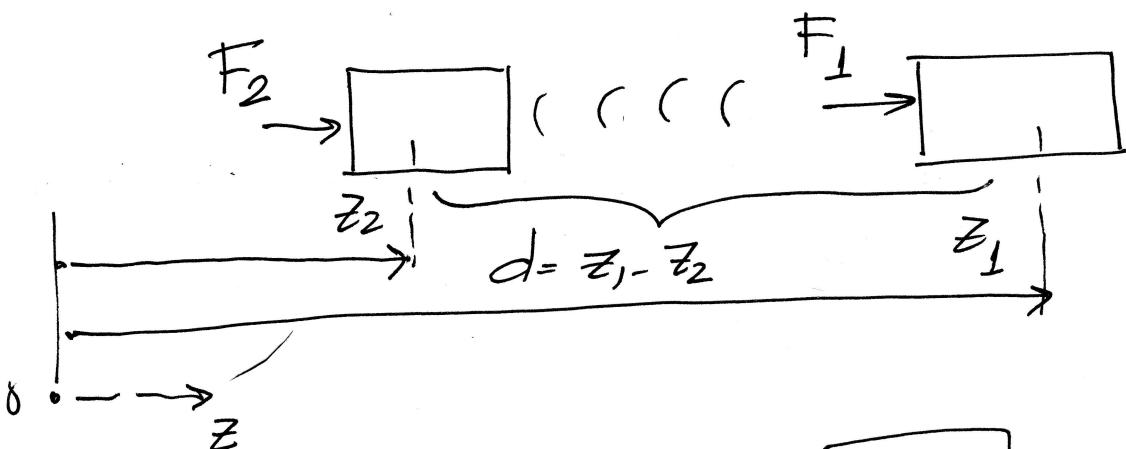
$$x_3 = \ell = z_1 - z_2 \quad \text{cte!}$$

$$\dot{x}_2 = \dot{z}_1$$

$$\dot{x}_4 = \dot{\ell} = \dot{z}_1 - \dot{z}_2$$

$$u = F_1$$

$$y = e$$



Objectivo: conduzir  $e = d - r \rightarrow 0$

ou seja,

$$d - r \rightarrow 0,$$

$$d \rightarrow r \Rightarrow z_1 - z_r \rightarrow r$$

(manter as plataformas a distância constante  $f(x_3) = r$ )

$$z_1 - z_2 - r$$

Estratégia: fazer  $F_2 = F_1 + k_1 e + k_2 \dot{e}$ ;  $k_1, k_2 > 0$

Resolução

Determinar a evolução das variáveis de estado

$$x_1, x_2, x_3, x_4$$

em função da entrada  $u = F_1$

$$(x_1) \quad \dot{x}_1 = \ddot{z}_1 = x_2 \quad \checkmark$$

$$(x_2) \quad \dot{x}_2 = \ddot{x}_1 = F_1 \quad (F = m \cdot \text{aceleração}) \\ = u$$

$$(x_3) \quad x_3 = e = z_1 - z_2 - r$$

$$\Rightarrow \dot{x}_3 = \dot{e} = x_4 \quad \checkmark$$

$$(x_4) \quad \dot{x}_4 = \ddot{e} = \ddot{z}_1 - \ddot{z}_2$$

notar:  $\ddot{z}_1 = \dot{x}_2 = \ddot{x}_1 = F_1$  (vee acima)

$$\ddot{z}_2 = F_2 \quad (F = m \cdot \text{aceleração})$$

$$= F_1 + k_1 e + k_2 \dot{e} \Rightarrow$$

$\dot{x}_4 = F_1 - F_2 - k_1 e - k_2 \dot{e}$   
 $= -k_1 x_3 - k_2 x_4$

$$\Rightarrow \dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_1 & -k_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = (0 \ 0 \ 1 \ 0) x = e$$

Q.22

Subsistema  $\overset{\sim}{A}$ 

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Estabilidade à origem?

Calcular os valores próprios de  $\overset{\sim}{A}$ 

$$\det(\lambda I - \overset{\sim}{A}) = \begin{vmatrix} 1 & -1 \\ k_1 & \lambda + k_2 \end{vmatrix} = \underbrace{\lambda^2 + k_2 \lambda + k_1}_{> 0}$$

$$k_1, k_2 > 0$$

 $\Rightarrow$  raízes com parte real negativa

 $\Rightarrow$  ESTABILIDADE ASSIMPTÓTICA ✓

P3

$$G_1(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3(s+1)}$$

P3.1

$$\Rightarrow s^3(s+1)y(s) = u(s)$$

$$\Rightarrow s^4y(s) + s^3y(s) = u(s)$$

↓ TEMPO

$$\overset{\text{IV}}{y}(t) + \overset{\text{III}}{y}(t) = u(t) \quad (@)$$

Atribuição de variáveis de estado:

$$x_1 = y ; \quad x_2 = \dot{y} ; \quad x_3 = \ddot{y} ; \quad x_4 = \overset{\dots}{\underset{(y^{\text{II}})}{\dddot{y}}} ; \quad x_5 = \overset{\dots}{\underset{(y^{\text{III}})}{\ddot{y}}}$$

$$\Rightarrow \begin{array}{l} \overset{\circ}{x}_1 = x_2 \\ \overset{\circ}{x}_2 = x_3 \\ \overset{\circ}{x}_3 = x_4 \\ \overset{\circ}{x}_4 = -x_4 + u \end{array} ; \quad y = x_1$$

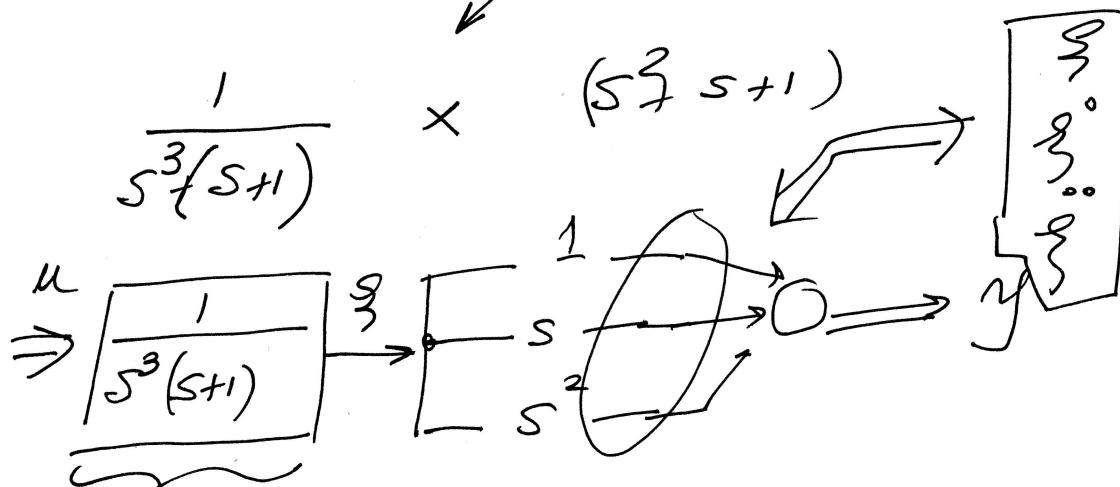
(de @)

$$\Rightarrow \dot{x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_A x + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_B u ; \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}}_C x$$

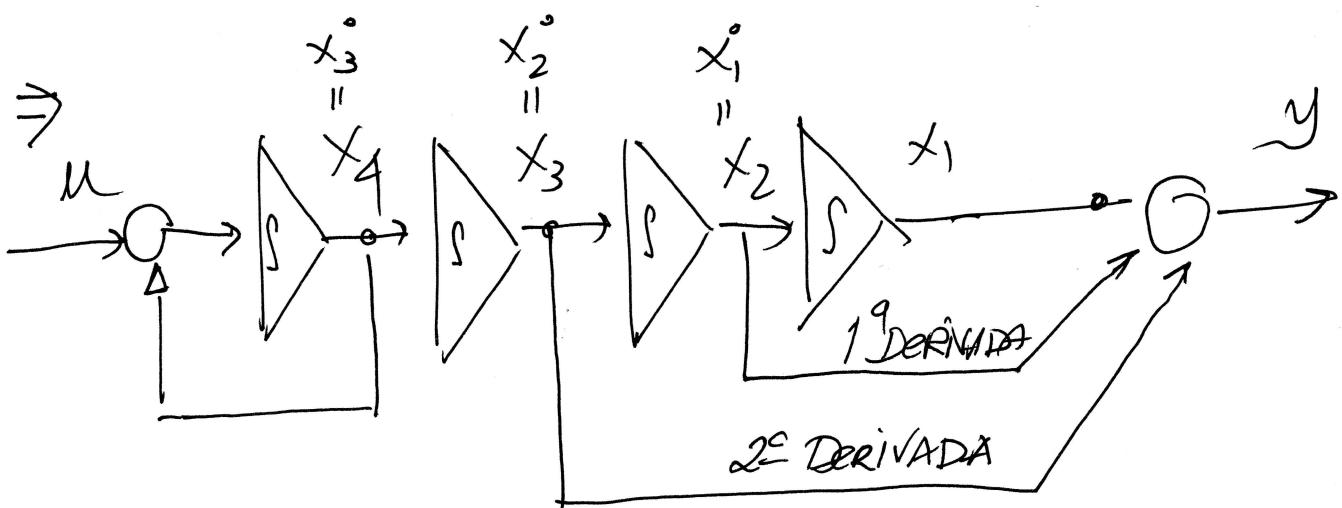
P3.2

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{s^2 + s + 1}{s^3(s+1)}$$



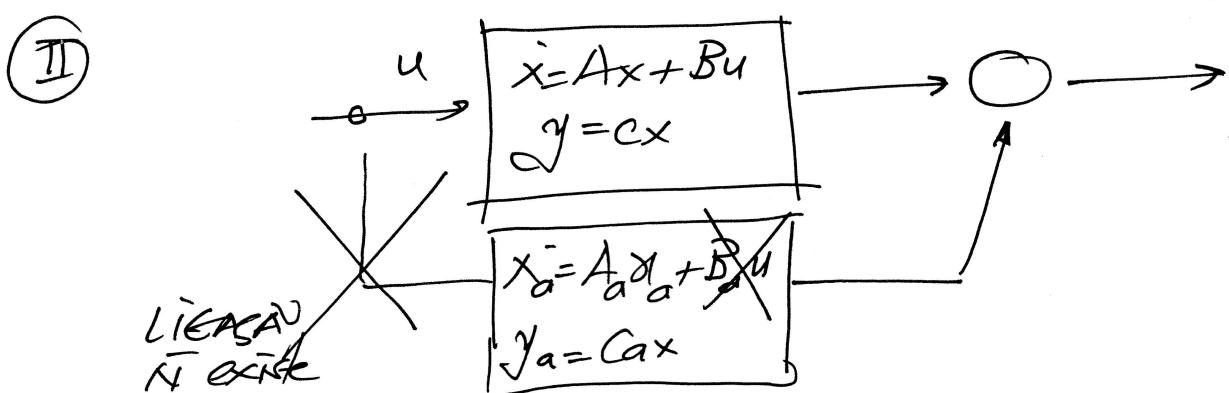
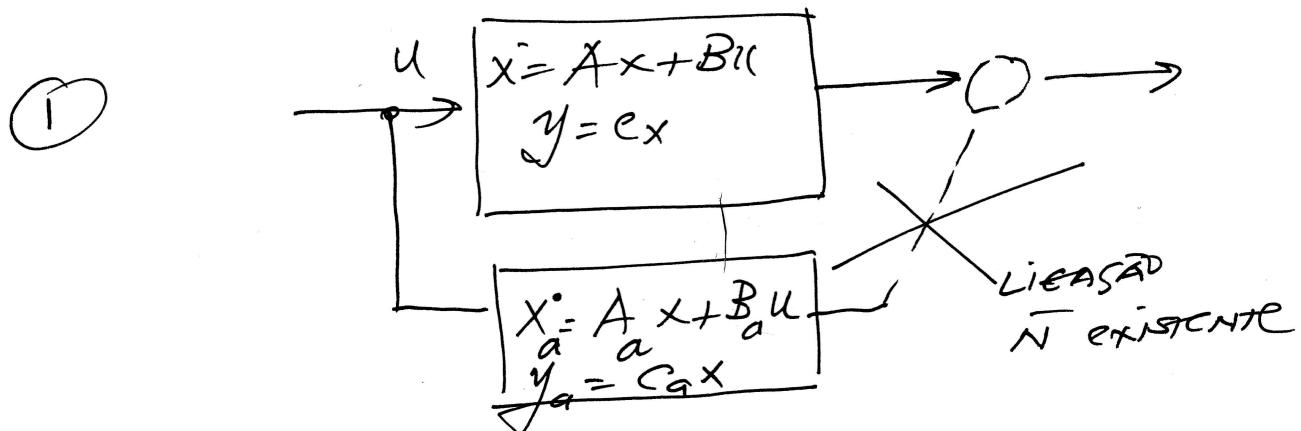
Antigo sistema :

Novo sistema : "semelhante ao antigo", mas a observação vai "pescar" o vector  $(\ddot{y}, \dot{y}, y)^T$ .



$$G(s) = \frac{1}{s^3(s+1)} \Rightarrow \dot{x} = Ax + Bu \\ y = Cx$$

Ideia básica: introduzir variáveis de estado "espúrias" que não contribuem para a relações entrada-saída



Exemplo : (I)  $\dot{x}_a^* = -x_a + u$

(II)  $\dot{x}_a^* = -x_a$   
 $y_a = x_a$

- P4 -

$$\boxed{\begin{aligned} \dot{P} &= V \\ \dot{V} &= -\beta |V|V + F' \end{aligned}}$$

A.1)

$$P \text{ em equilíbrio} \quad V = F = 0$$

Descrição em Espaço de Estados

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} P \\ V \end{pmatrix} ; \quad u = F'$$

$$\Rightarrow \dot{x} = g(x, u), \text{ com}$$

$$g(x, u) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\beta x_2 |x| + u \end{bmatrix} = \begin{cases} g_1(x, u) \\ g_2(x, u) \end{cases}$$

Linearizar

$$x = x_{eq} + \delta x; \quad u = u_{eq} + \delta u$$

$$\Rightarrow (\dot{x}_{eq} + \dot{\delta x}) = \cancel{g}(x_{eq} + \delta x, u_{eq} + \delta u)$$

$$\Rightarrow \dot{\delta x} = \underbrace{g(x_{eq}, u_{eq})}_{0} + \frac{\partial g}{\partial x} \Big|_{eq} \delta x + \frac{\partial g}{\partial u} \Big|_{eq} \delta u$$

Notar:

$$\left. \frac{\partial g}{\partial x} \right|_{eq} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}_{eq} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underbrace{\quad}_{A}$$

$$\left. \frac{\partial g}{\partial u} \right|_{eq} = \begin{bmatrix} \frac{\partial g_1}{\partial u} \\ \frac{\partial g_2}{\partial u} \end{bmatrix}_{eq} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underbrace{\quad}_{B}$$

$$\Rightarrow \delta \ddot{x} = A \delta x + B \delta u$$

(usar, com abuso de notação)

$x$  em vez de  $\delta x$ )

$u$  em vez de  $\delta u$ )

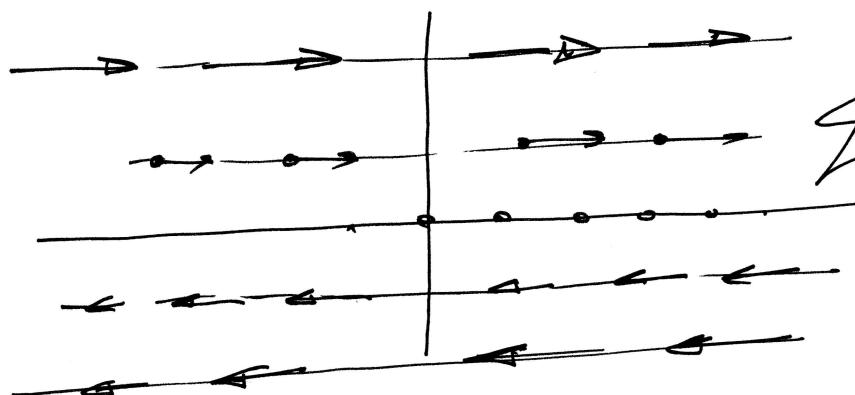
$$\Rightarrow \dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$\Rightarrow \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}}$$

**4.2**

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned} \quad (\text{isoclinas}) \Rightarrow \frac{dx_2}{dx_1} = 0$$

(isoclinas SEMPRE horizontais!)



$\Rightarrow p$  equilíbrio  
( $p = x_1$ ,  
arbitrário!)

**4.3**

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10x_1 - 11x_2 \end{aligned}$$

$$\Rightarrow \dot{x} = \begin{pmatrix} 0 & 1 \\ -10 & -11 \end{pmatrix} x$$

$A$

$$\boxed{\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -10 \end{array}} \Rightarrow \text{ESTAvel!}$$

Calculo de valores próprios

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -10 & \lambda + 11 \end{vmatrix} = \lambda^2 + 11\lambda + 10 = 0$$

↓

P44

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_1$$

$$\Rightarrow \dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$$

$\xrightarrow{A}$

Cálculo de valores próprios

$$\det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

$$\Rightarrow \boxed{\lambda_1 = -1, \lambda_2 = +1} \quad \leftarrow \text{POXO EM SCA, INSTALEL}$$

Cálculo de vetores próprios

$$A v = \lambda v$$

$$\lambda_1 = -1 \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = - \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$\Rightarrow v_{12} = -v_{11} \quad \leftarrow \boxed{v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$\lambda_2 = +1 \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

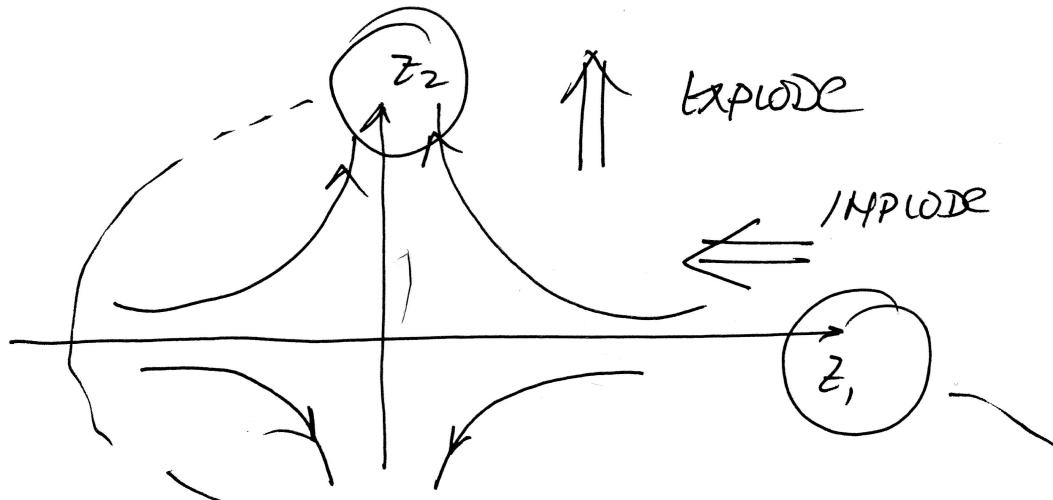
$$\Rightarrow v_{22} = v_{21} \quad \Rightarrow \boxed{v_2 = \begin{bmatrix} 1 \\ +1 \end{bmatrix}}$$

Por mudança de coordenadas adequada,  
o sistema linearizado será' (nas coord. z)

$$\dot{z} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} z$$

$$\downarrow \quad z_1(t) = z_1(0) e^{-t}$$

$$z_2(t) = z_2(0) e^{+t}$$



$\downarrow$  De novo bas coordenadas \*

