Lectures on Viterbo’s theorem for the symplectic cohomology of the cotangent bundle

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In these lectures we will present the proof of Viterbo’s theorem due to Abbondandolo and Schwarz. For a good introduction see [Abouzaid, 2015], and for a good story of Viterbo’s theorem see [Abbondandolo and Schwarz, 2006, Abbondandolo and Schwarz, 2014]. For details on the construction of Floer homology see [Audin and Damian, 2014] and [Santos, 2018]; most of these details can be readily adapted to the construction of symplectic homology. Details on compactness are present in [Abbondandolo and Schwarz, 2006], while details on orientations are present in [Floer and Hofer, 1993] and [Abbondandolo and Schwarz, 2006, Abbondandolo and Schwarz, 2014] as well as [Abouzaid, 2015].

Any mistakes in adaptation are my own responsibility.

Lectures 1 and 2 correspond to section 1, lectures 3 and 4 to section 2, lecture 5 to section 3 and lectures 6 and 7 to section 4.

Sign conventions

The canonical symplectic structure on the cotangent bundle $T^*Q$ is $dp \wedge dq$, where $q \in Q$ and $p \in T_q^*Q$.

The Hamiltonian vector field is $\omega(X_H, \cdot) = -dH$.

Compatibility between symplectic and almost-complex structures is given by $\omega(\cdot, J \cdot)$ being a Riemannian metric.

The Hamiltonian action functional is $\mathcal{A}_H(x) = \mathcal{A}(x) - \int_0^1 H(t, x(t)) \, dt$, where $\mathcal{A}$ denotes the symplectic action functional. This is the same as [Abbondandolo and Schwarz, 2006], while the functional in [Audin and Damian, 2014, Abouzaid, 2015, Santos, 2018] is $-\mathcal{A}_H(x)$.

Floer’s equation is the negative $L^2$-gradient flow of the Hamiltonian action functional, hence $\partial_s u - J(t, u)(\partial_t u - X_t(u)) = 0$. Consequently, this is the same as [Abbondandolo and Schwarz, 2006], while Floer’s equation in [Audin and Damian, 2014, Abouzaid, 2015, Santos, 2018] is $\partial_s u + J(t, u)(\partial_t u - X_t(u)) = 0$.

The Conley-Zehnder index of a symplectic path satisfies $\mu_{CZ}(e^{\pi i t}) = 1$. This is the same as [Abbondandolo and Schwarz, 2006, Abouzaid, 2015, Santos, 2018], but reverse to [Audin and Damian, 2014]. In any case, what matters is how one grades Hamiltonian orbits using $\mu_{CZ}$: [Abouzaid, 2015] and [Santos, 2018] shift and reverse.
1 Construction of symplectic cohomology of the cotangent bundle

1.1 Motivation and Viterbo’s theorem

In symplectic topology the following are very important dynamics problems.

**Conjecture** (Arnol’d). *A Hamiltonian symplectomorphism on a compact symplectic manifold \((M,\omega)\), with non-degenerate fixed points has at least \(\sum b_k(M,\mathbb{F})\) fixed points.*

**Conjecture** (Weinstein). *A compact contact manifold has closed Reeb orbits.*

The Arnol’d conjecture has been tackled with Floer homology, defined as the homology of a chain complex generated by 1-periodic solutions of Hamilton’s equation. For a detailed history of the conjecture see [Salamon, 1999]; the seminal references are [Floer, 1988a] and [Floer, 1988b], the general proof is the main goal of [Fukaya and Ono, 1999].

The Weinstein conjecture is still open, and some attempts have been made using symplectic (co)homology: it is defined in the same way as Floer homology, but for manifolds with contact type boundary, and relates the Hamiltonian dynamics with the Reeb dynamics on the boundary.

A particular case is the unit ball on \(\mathbb{C}^n\), which has contact type boundary, and null symplectic homology; this can be used to show that the Weinstein conjecture holds for spheres (see [Oancea, 2004, section 4.3]).

We are interested in the cotangent bundle of a compact manifold \(Q\) of dimension \(n\). We denote a general point in \(T^*Q\) by \((q,p)\), where \(q \in Q\) and \(p \in T^*_qQ\).

The Liouville 1-form is \(\lambda_{p,q}(v) = p(\pi_*(v)) = pdq\), and the canonical symplectic form is \(\omega = d\lambda = dp \wedge dq\).

Endowing \(Q\) with a Riemannian metric, we can identify \(TQ \cong T^*Q\) canonically and induce a norm on \(T^*Q\), which we denote \(|p|\). We also denote \(\rho(q, p) = |p|\). Defining \(D^*Q = \{\rho \leq 1\}\) and \(S^*Q = \{\rho = 1\}\), we can decompose

\[T^*Q = D^*Q \cup_{S^*Q} S^*Q \times [1, +\infty[,\]

where \(D^*Q\) is compact with contact-type boundary \(S^*Q\), and \(S^*Q \times [1, +\infty[\) is non-compact; sometimes called cylindrical/conical/convex end.

If we consider the kinetic energy \(H(q,p) = |p|^2/2\), then solutions of Hamilton’s equation

\[\dot{x}(t) = X_H(x(t))\]

for \(x : S^1 \to T^*Q\) correspond to 1-periodic geodesics of \(Q\) (with speed \(|p|\)) (see [Abouzaid, 2015, Lemma 1.2.9]).

Since closed geodesics generate \(H_*(\mathcal{L}Q)\), there should be a relation between the symplectic (co)homology of \(T^*Q\) and the singular homology of \(\mathcal{L}Q\).

**Theorem 1.1** (Viterbo). *There is a local system of coefficients \(\eta\) such that

\[SH_*(T^*Q, \eta) \cong H_*(\mathcal{L}Q).\]

Moreover, if \(Q\) is orientable then \(\eta\) is trivial if the second Stiefel-Whitney class vanishes on tori in \(T^*Q\), in which case

\[SH_*(T^*Q) \cong H_*(\mathcal{L}Q).\]
In all cases, 

\[ \text{SH}_* (T^* Q; \mathbb{Z}/2) \cong H_* (LQ; \mathbb{Z}/2). \]

This relation was first unveiled in [Viterbo, 1995], with detailed proofs being presented in [Salamon and Weber, 2006] and [Abbondandolo and Schwarz, 2006]. In the meantime, a sign-incoherence was found out when \( Q \) is not spin, with corrections being provided in [Abouzaid, 2015] and [Abbondandolo and Schwarz, 2014].

The main references for these lectures are [Abouzaid, 2015] and [Abbondandolo and Schwarz, 2006, Abbondandolo and Schwarz, 2014]. The two offer different approaches to the construction of symplectic cohomology and to the proof of the theorem. Before describing these differences, we need to describe the construction of symplectic homology. Our approach is to recall the construction of Floer homology, and analyze the obstructions we might encounter when passing to non-compact symplectic manifolds.

1.2 Construction of Floer Homology and obstructions

The fastest “definition” of Floer homology is the following.

**Definition 1.2.** Consider a symplectic manifold \((W, \omega)\), an Hamiltonian \(H \in C^\infty (W \times S^1)\) with non-degenerate 1-periodic orbits, and a time-dependent a.c.s. \(J_t\) compatible with \(\omega\). The Floer complex \(SC(W, \omega, H, J)\) is generated by \(O(H)\), with differential defined on generators by

\[ \partial_{H,J} x = \sum_{|y|=|x|-1} n(x, y) y, \]

where \(n(x, y)\) is the number of solutions \(u \in C^\infty (\mathbb{R} \times S^1; W)\) of Floer’s equation

\[ \partial_s u - J_t(u) (\partial_t u - X_H(u)) = 0, \]

with asymptotic conditions

\[ \lim_{s \to -\infty} u(s, t) = x(t), \quad \lim_{s \to +\infty} u(s, t) = y(t), \]

modulo the \(\mathbb{R}\)-action \((\sigma \cdot u)(s, t) = u(\sigma + s, t)\).

The homology of this complex, called **symplectic homology**, is denoted

\[ \text{SH}_* (W, \omega, H, J) = H_*(SC(W, \omega, H, \partial_{H,J})). \]

What do we need in order for this definition to make sense? Let us denote by \(M(x, y)\) the set of solutions of (1) and (2), and by \(L(x, y) = M(x, y)/\mathbb{R}\).

- Grading of \(O(H)\);
- Finiteness of \(L(x, y)\):
  - Each \(M(x, y)\) is a manifold of dimension \(|y| - |x|\): Fredholm theory + Transversality;
  - The \(\mathbb{R}\)-action is free and proper: Continuation;
  - \(L(x, y)\) is compact if \(|y| - |x| = 1\): Compactness;
\[ \partial^2 = 0, \text{ shown by seeing that } \bigcup_{|y|=|x|-1=|z|+1} \mathcal{L}(x, y) \times \mathcal{L}(y, z) \text{ is the boundary of the compact 1-dimensional manifold } \mathcal{L}(x, z): \]

- Fredholm theory + Transversality;
- Gluing.

We have already seen that for closed aspherical manifolds with first Chern class vanishing over spheres these conditions are met when we restrict our attention to contractible orbits (and thus contractible solutions of Floer’s equation).

**Remark 1.3.** This is true when we restrict our attention to contractible orbits; in general we would require the manifolds to be atoroidal, and that the first Chern class vanish over torii.

**Remark 1.4.** The above definition only works for coefficients on a ring of characteristic 2. In general, one needs to define \( n(x, y) \) as a signed count of elements of \( \mathcal{L}(x, y) \); we will come back to this in Section 3.

**Remark 1.5.** Floer’s equation is defined as the negative \( L^2 \)-gradient flow of the Hamiltonian action functional, which is only well-defined if \( W \) is atoroidal (aspherical suffices when looking at contractible orbits). This condition is not a priori needed in order for the above construction to work, but we will need it for the compactness property.

The continuation property is local, the Fredholm theory analyzes neighborhoods of compact cylinders, and transversality also does this, over a countable set of pairs of orbits. In these theories the gluing property follows from the Fredholm theory and transversality. For further details, see [Audin and Damian, 2014], [Santos, 2018].

**Remark 1.6.** About transversality, bear in mind that we can either perturb \( H \) or perturb \( J \), both are fine (so long as \( J \) is allowed to be time-dependent). For the proof of Viterbo’s theorem that we will follow, we require the latter.

The difficult part is compactness, and we also need to revisit grading because we are no longer considering only contractible orbits.

### 1.3 Grading Hamiltonian orbits in the cotangent bundle

Given \( x \in \mathcal{O}(H) \), the unitary bundle \( x^*TW \) is trivializable — a \( G \)-bundle over \( S^1 \) is trivializable whenever \( G \) is connected, e.g. [Hatcher, 2017]—, and a choice of trivialization \( \Psi: x^*TW \to S^1 \times \mathbb{R}^{2n} \) gives us a path of symplectic matrices

\[
R_\Psi(t) = \Psi_{x(t)} \left( d\phi_{X_H} \right)_{x(0)} \Psi_{x(0)}^{-1}
\]

such that \( R_\Psi(1) - I \) is non-singular. We can thus assign to it the Conley-Zehnder index \( \mu_{CZ}(R_\Psi) \). This index depends only on the homotopy class of \( \Psi \), which is an element of \([S^1, U(2n)] \cong \mathbb{Z}\).

Given \( u \in \mathcal{M}(x_-, x_+) \), the local dimension at \( u \) is given by \( \mu_{CZ}(R_-) - \mu_{CZ}(R_+) \), where \( R^\pm \) are computed using a trivialization \( \Psi: u^*TW \to (\mathbb{R} \times S^1) \times \mathbb{R}^{2n} \). Defining a global grading thus corresponds to choosing a homotopy class of a trivialization for a representative of each class in \( \pi_1(W) \); so long as
c_1(TW,J) vanishes over 2-tori, these choices are consistent. Moreover, the chain complex $SC(W,\omega,H,J)$ decomposes into a complex for each class in $[S^1,W]$:

$$SC(W,\omega,H,J) = \bigoplus_{[\gamma] \in [S^1,W]} SC^\gamma(W,\omega,H,J),$$

so the choices in grading do not affect whether this defines an homology. In general, there are no canonical choices, but for $T^*_Q$ there are somewhat canonical choices since it has a Lagrangian distribution, as we will now describe. In any case, the choice of grading should be suitable for Viterbo’s theorem to hold without shifts in grading.

**Version 1:** The vertical distribution $T_vT^*_Q$ is Lagrangian in $TT^*_Q$, thus $TT^*_Q \cong T_vT^*_Q \otimes \mathbb{C}$. In general, given a real (Riemannian) bundle $E$ of rank $n$ over $S^1$, a unitary trivialization $\Psi: E \otimes \mathbb{C} \to S^1 \times \mathbb{R}^{2n}$ induces a map

$$\Gamma_\Psi: S^1 \to \Lambda(n), \quad t \mapsto \Psi_t(E).$$

Fixing the “degree” of this map determines the homotopy class of $\Psi$. This degree is well-defined since $\Lambda(n) \cong U(n)/O(n)$, and the long exact sequence for the fibration $p: U(n) \to U(n)/O(n)$ ends in

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{p_*} \pi_1(U(n)/O(n)) \to \mathbb{Z}/2 \to 1,$$

hence $\pi_1(U(n)/O(n)) \cong \mathbb{Z}$ and $p_* \cong 2 \times$. It follows that changing the trivialization changes the degree by an even number, and so we can always choose a trivialization such that this map has degree 1 or 0; the latter case corresponds to $E$ being orientable, and then we can actually choose $\Psi$ such that $\Gamma_\Psi(t) = \mathbb{R}^n$ for all $t \in S^1$.

**Version 2:** As above, but pass to the determinant line bundle: then $\det(E \otimes \mathbb{C}) = \det(E) \otimes \mathbb{C}$, the map is called “Gauss map”, and $\Lambda(1) \cong \mathbb{RP}^1$. It is clear that the natural map $S^1 \to \mathbb{RP}^1$ shifts degrees by 2.

**Remark 1.7.** The complexification of a real vector bundle has trivial Chern class; in particular the first Chern class of $T^*_Q$ vanishes over 2-tori.

**Definition 1.8.** We define

$$|x| = \begin{cases} 
\mu_{CZ}(R_\Psi), & \text{if } x^*(T_vT^*_Q) \text{ is orientable,} \\
\mu_{CZ}(R_\Psi) - 1, & \text{otherwise,}
\end{cases}$$

where $\Psi$ is a trivialization with $\deg(\Gamma_\Psi) \in \{0,1\}$. We can abbreviate

$$|x| = \mu_{CZ}(R_\Psi) - \deg(\Gamma_\Psi)$$

for any trivialization $\Psi$.

### 1.4 Compactness

Finally, let us discuss compactness. The proof of compactness for closed aspherical symplectic manifolds follows the following lines:
1. Define an action functional in order to uniformly bound the energy of finite energy solutions;

2. Show that the gradient of solutions is uniformly bounded via bubbling off-analysis (using the above bound) - requires $W$ aspherical;

3. Use the Arzelà-Ascoli theorem together with elliptic regularity.

Defining an action functional requires atoroidal $W$; the cotangent bundle is exact hence atoroidal, and the functional takes the form

$$\mathcal{A}_H(x) = \int_{S^1} \lambda_{x(t)}(\dot{x}(t)) - H_t(x(t)) \, dt.$$ 

The energy is bounded by showing that a solution $u$ of Floer’s equation with $E(u) = \int_{\mathbb{R} \times S^1} \|\partial_s u\|^2 \, ds \, dt < \infty$, where we denote $\langle \cdot, \cdot \rangle_t = \omega(\cdot, J_t \cdot)$, has asymptotics $x_{\pm}$, and moreover

$$E(u) = \mathcal{A}_H(x_-) - \mathcal{A}_H(x_+).$$

In the closed case the uniform bound follows from $O(H)$ being finite, but this is no longer automatically the case. Moreover, in order to apply the Arzelà-Ascoli theorem (which we also need for bubbling-off analysis) we need to be in a relatively compact subspace. In general, we have the following Theorem.

**Theorem 1.9** (Gromov compactness). If $W$ is aspherical, a set $U$ of solutions of Floer’s equation such that there are $c > 0$ and $K \subset W$ compact with $E(u) \leq c$ and $u(\mathbb{R} \times S^1) \subset K$ for all $u \in U$ is relatively compact in $C^0_\text{loc}(\mathbb{R} \times S^1; W)$, and by elliptic regularity also in any $C^k_\text{loc}(\mathbb{R} \times S^1; W)$.

The condition $E(u) \leq c$ may be omitted if $W$ is atoroidal. However, restricting arbitrarily to a compact subspace we run into problems: our manifolds of solutions of Floer’s equation may cease being so, and gluing may go wrong.

Let us examine two different solutions to this problem.

**Linear approach:** Consider linear Hamiltonians $H = b \cdot \rho$ for $\rho \geq 1$, for some slope $b > 0$ and a.c.s. which is convex in a neighborhood of $S^*Q$ i.e. $J_t(X_\rho)$ proportional to $-\partial_\rho$ (positive proportionality constant) near $S^*Q$.

**Proposition 1.10.** In these conditions, if $u \in \mathcal{M}(x,y)$ is such that $x(S^1), y(S^1) \subset D^*Q$, then $u(\mathbb{R} \times S^1) \subset D^*Q$.

So we can restrict to the compact set $D^*Q$ without falling into the previous problems. The proof of Proposition 1.10, and the details in this case are in [Abouzaid, 2015].

**Quadratic approach:** Filtration by action: restrict attention to solutions $u \in \mathcal{M}(x,y)$ with $a < \mathcal{A}_H(y) < \mathcal{A}_H(x) < b$; energy automatically bounded. One then needs to show that the image of such solutions is uniformly bounded; this work for “quadratic Hamiltonians” and “uniformly continuous” a.c.s. i.e. satisfying

(H1) $(dH)_{t,q,p}(0,p) - H(t,q,p) \geq h_0 |p|^2 - h_1$ for some $h_0 > 0$;

(H2) $|X_H(t,q,p)| \leq h_3(1 + |p|^2)$.
and \( J \) is uniformly continuous with respect to the standard compatible a.c.s. \( J_0 \) induced by \( TQ \cong T^*Q \).

**Example 1.11.** Hamiltonians of the form
\[
H(t,q,p) = \frac{1}{2}|T(t,q)p - A(t,q)|^2 + V(t,q),
\]
where \( T^*T \geq 0 \), satisfy (H1) and (H2).

**Theorem 1.12.** Let \( H \in C^\infty(S^1 \times T^*Q) \) be an Hamiltonian with non-degenerate 1-periodic orbits satisfying (H1) and (H2) and let \( J \) be uniformly continuous w.r.t \( J_0 \). For any \( a < b \) there is \( \rho_0 \) such that the set of solutions \( u \) of Floer’s equation with \( a < A_H(u(s)) < b \) for all \( s \in \mathbb{R} \) satisfies \( \rho(u(\mathbb{R} \times S^1)) \leq \rho_0 \).

The construction then works normally: by (H1) the Hamiltonian action functional \( A_H \) is uniformly bounded from below; fixing \( x \in O(H) \), any trajectory \( u \in M(x,y) \) then satisfies \( A_H(u(s)) \leq A_H(x) \), and restricting our attention to the space of solutions \( A_H(u(s)) \leq A_H(x) \), we can show the needed compactness properties to define \( \partial x \) and to show that \( \partial^2 x = 0 \). For details, see [Abbondandolo and Schwarz, 2006].

**Remark 1.13.** The condition on \( J \) present in [Abbondandolo and Schwarz, 2006] is not that \( J \) is uniformly continuous but that \( J \) is sufficiently \( L^\infty \)-close to \( J_0 \). The latter condition is used in the proof of [Abbondandolo and Schwarz, 2006, Theorem 1.14], but the former (weaker) condition is seen to be enough by a traditional “freezing coefficients” argument.

### 1.5 Invariance

For closed manifolds we have discussed invariance:

\[
FH(M,\omega;H^1,J^1) \cong FH(M,\omega;H^2,J^2).
\]

The chain map — called **continuation map** in the literature — is constructed by counting solutions of the parametrized Floer equation
\[
\partial_s u - J_{s,t}(u) (\partial_t u - X_{H_{s,t}}(u)) = 0,
\]
where \( J_{s,t} = J_1 \) for \( s \ll -1 \) and \( J_{s,t} = J_2 \) for \( s \gg 1 \), and similarly for \( H \). The construction of this map faces the same problem as before: compactness. In general, if \( H_1 - H_2 \) is compactly supported and \((H_1,J_1),(H_2,J_2)\) satisfy the aforementioned conditions (e.g. convexity near \( S^*Q \) in the linear approach) we still have invariance.

For linear Hamiltonians, it turns out that we can only construct this map if \( \text{slope}(H_1) \geq \text{slope}(H_2) \); but this map is unique in cohomology. We may thus define \( SH_*(T^*Q; b) \cong SH_*(T^*Q; H,J) \), where \( \text{slope}(H) = b \). Symplectic cohomology is defined as the inverse limit
\[
SH_*(T^*Q) = \lim_{b \to +\infty} SH_*(T^*Q; b),
\]
with respect to the aforementioned continuation maps.

For quadratic Hamiltonians, we can always construct continuation maps, and we can simply define \( SH_*(T^*Q) \cong SH_*(T^*Q; H,J) \).
Remark 1.14. In [Abouzaid, 2015] symplectic cohomology is defined, and the continuation maps exist in the reverse direction and symplectic cohomology is then a direct limit.

Remark 1.15. As with constructing the differential, in order to construct the continuation maps, compactness is the main issue. In order to show that these maps are actually chain maps a gluing argument is used, and in order to show that two such maps induce the same map in homology we need to construct a chain homotopy in a similar way, and prove it is in fact a chain homotopy again using a gluing argument. This mimics the proof of (3) presented in [Audin and Damian, 2014]. These gluing properties tend to be an after-thought, so to say, but some extra care needs to be taken when considering orientations, as we will see with Viterbo’s theorem in particular.

Remark 1.16. In the quadratic approach, the continuation maps are only constructed in [Abbondandolo and Schwarz, 2006] for $L^\infty$-close Hamiltonians. This condition can be avoided by considering “slower” homotopies.

Remark 1.17. It is an a posteriori consequence of Viterbo’s theorem that these constructions are equivalent, but this can be proven a priori; for a sketch of such a proof see [Seidel, 2007, Section 3].

2 Viterbo’s theorem for $\mathbb{Z}/2$-coefficients

For details in this section see [Abbondandolo and Schwarz, 2006].

2.1 Legendre transform

Let $L \in C^\infty(S^1 \times TQ)$ be a Lagrangian function satisfying

(L1) There is $\ell_0 > 0$ such that $\nabla_{vv} L(t, q, v) \geq \ell_0 I$,

(L2) There exists $\ell_1 \geq 0$ such that

$$|\nabla_{vv} L(t, q, v)| \leq \ell_1, \quad |\nabla_{qq} L(t, q, v)| \leq \ell_1 (1 + |v|), \quad |\nabla_{qv} L(t, q, v)| \leq \ell_1 (1 + |v|^2).$$

Example 2.1. Lagrangians of the form

$$L(t, q, v) = \frac{1}{2} |T(t, q)v - A(t, q)|^2 - V(t, q),$$

where $T^*T$ is everywhere positive, satisfy (L1) and (L2).

Condition (L1) implies that $L$ is hyper-regular: the fiber-preserving map

$$\mathcal{L}_L : S^1 \times TQ \to S^1 \times T^*Q$$

$$(t, q, v) \mapsto (t, q, dL(t, q, v)|_{T_{v(t,q)}Q})$$

is a diffeomorphism. Denoting $\mathcal{L}_L^{-1}(t, q, p) = (t, q, v(t, q, p))$, we define the Legendre transform of $L$ as the Hamiltonian

$$H : S^1 \times T^*Q \to \mathbb{R}$$

$$(t, q, p) \mapsto p(v(t, q, p)) - L(t, q, v(t, q, p)) = \max_{v \in T^*Q} \langle p[v], L(t, q, v) \rangle.$$
Define a time-dependent vector field \( Y_L = L^{-1}_L X_H \) on \( TQ \). Integral curves of \( Y_L \) are of the form \( y(t) = (q(t), \dot{q}(t)) \), where \( q \) satisfies the Euler-Lagrange equation

\[
D_t \nabla_q L (t, q(t), \dot{q}(t)) = \nabla_q L (t, q(t), \dot{q}(t)).
\]

The Legendre transform gives a bijection between 1-periodic orbits of \( X_H \) and \( Y_L \). In fact, Viterbo’s theorem is not only concerned with counting trajectories but with the fact that this bijection is related to a chain map, and that this bijection works for any quadratic Hamiltonians, even if they are not Legendre transforms of quadratic Lagrangians.

### 2.2 Morse homology for Hilbert manifolds

We begin by describing a specific model of \( H_q(LM) \): Morse homology. For details, see [Abbondandolo and Majer, 2006] or [Abbondandolo and Schwarz, 2006, Section 2.3].

In general, let \( M \) be a smooth Hilbert manifold with a \( C^1 \)-Riemannian metric \( G \), let \( f \in C^2(M) \). Assume that

(i) \( f \) is a Morse function and has only critical points of finite Morse index;

(ii) \( f \) is bounded from below;

(iii) \((M, G)\) is complete;

(iv) \( f \) satisfies the Palais-Smale condition at every level \( c \): a sequence \((p_n) \subset M\) such that \( f(p_n) \to c \) and \( |\nabla f(p_n)| \to 0 \) has a convergent subsequence;

(v) \( -\nabla f \) satisfies the Morse-Smale condition up to order 1 on \( \{a < f < b\} \): if \( m(x) - m(y) \leq 1 \) then \( W^u(x) \) intersects \( W^s(y) \) transversely.

**Theorem 2.2.** Given any metric \( G \) such that (iii) and (iv) hold, there is a dense set of uniformly equivalent metrics to \( G \) such that (v) holds.

Morse homology is defined as usual: \( MC^k(M, f) \) is the free abelian group generated by the critical points with Morse index \( k \), the differential is defined on generators by

\[
\partial f, G x = \sum_{m(y)=k-1} \left| \frac{W^u(x) \cap W^s(y)}{\mathbb{R}} \right| y.
\]

It is known that

\[
MH_k(M, f, G) := H_k (MC^\bullet(M, f), \partial_f, G) \cong H_k(M).
\]

Moreover, for different choices of metrics \( G_1 \) and \( G_2 \) there is a chain isomorphism

\[
(MC^\bullet(M, f), \partial_f, G_1) \cong (MC^\bullet(M, f), \partial_f, G_2).
\]

The following property is important to note: for any \( x \in \text{Crit } f \), the unstable manifold \( W^u(x) \) is relatively compact, with closure contained in

\[
W^u(x) \bigcup_{y \in \text{Crit } f} \bigcup_{f(y) < f(x)} W^u(y).
\]
2.3 Morse homology for $W^{1,2}(S^1; Q)$

For the loop space $LQ$ define

$$W^{1,2}(S^1; Q) = \left\{ x \in C^0(S^1; Q) : \int_{S^1} g_{x(t)}(\dot{x}(t), \dot{x}(t)) \, dt < +\infty \right\}.$$ 

This is a Hilbert manifold locally modelled on $W^{1,2}(x^*TQ)$, well-defined because there is a Sobolev embedding $W^{1,2}(S^1) \hookrightarrow C^0(S^1)$. It is clear that $W^{1,2}(S^1; Q)$ is homotopy equivalent to $LQ$. The Riemannian metric is the obvious one.

Define

$$E_L(q) = \int_{S^1} L(t, q(t), \dot{q}(t)) \, dt.$$

This functional is smooth. A loop $q$ is a critical point of $E_L$ iff $(q, \dot{q})$ is a 1-periodic integral curve of $Y_L$.

By (F1) the Lagrangian $L$ is bounded from below and thus so is $E_L$. If $\phi^1_{Y_L}$ has non-degenerate fixed points, then $E_L$ is Morse. Also from (F1) one can show that the critical points of $E_L$ have finite morse index. That $W^{1,2}(S^1; Q)$ is complete is easy to see.

Proposition 2.3. $E_L$ satisfies the Palais-Smale condition at every level.

Hence conditions (i)-(iv) are satisfied; perturbing the metric if necessary, condition (v) is satisfied, and Morse homology $MH_*(W^{1,2}(S^1; Q), E_L, G)$ is well-defined.

The following is important to note: $E_L(\pi \circ x) \geq A_H(x)$, with equality if and only if $x \in \mathcal{O}(H)$.

2.4 Construction of the chain map

Denote by $\pi: T^*Q \to Q$ the standard projection. Consider a Lagrangian $L \in C^\infty(S^1 \times Q)$ satisfying (L1) and (L2) with Legendrian transform $H \in C^\infty(S^1 \times T^*Q)$ satisfying (H1) and (H2) and with non-degenerate orbits. The chain map

$$\mathcal{V}: (MC(W^{1,2}(S^1; Q), \mathcal{E}_L), \partial_{\mathcal{E}_L, G}) \to (SC(T^*Q, \omega, H), \partial_{H,J})$$

is constructed as follows: given $q \in \text{Crit}(\mathcal{E}_L)$ define

$$\mathcal{V}(q) = \sum_{\substack{x \in \mathcal{O}(x) \mid |x| = m(q)}} n^+(x, q),$$

where $n^+(x, q)$ is the number of solutions $u \in C^\infty(\mathbb{R}^+ \times S^1)$ of

$$\partial_s u - J(t, u)(\partial_t u - X_t(u)) = 0$$

satisfying the asymptotic conditions

$$\lim_{s \to +\infty} u(s) = x, \quad \lim_{s \to 0^+} \pi \circ u(s) \in W^u(q).$$

Denote by $\mathcal{M}^+(q, x)$ the space of such solutions.

In order to show that $\mathcal{M}^+(q, x)$ is finite, we need to show that $\mathcal{M}^+(q, x)$ is a manifold of dimension $m(q) - |x|$, and that it is compact if this number is 0.
2.4.1 Fredholm theory: particular to abstract

Given \( u \in \mathcal{M}^+(q,x) \), denote by \( \bar{u} \colon [0, +\infty) \times S^1 \to T^*Q \) its extension, and let \( \Psi \colon \bar{u}^*T^*Q \to (\mathbb{R}^+ \times S^1) \times \mathbb{R}^{2n} \) be a trivialization such that \( \lim_{s \to +\infty} D_s \Psi = 0 \) uniformly, and such that \( D_t \Psi \) is bounded (which exists since \( \bar{u} \) has exponential decay). Given \( r > 2 \), identify \( W^{1,r}(\bar{u}^*TM) \cong W^{1,r}(\mathbb{R}^+ \times S^1; \mathbb{R}^{2n}) \) via \( \Psi \).

Define a Banach space

\[
V^r = \left\{ Y \in W^{1,r}(\bar{u}^*TT^*Q) : \pi_* Y(0) \in T_{\pi u(0)}W^u(q) \right\},
\]

with norm

\[
\|Y\| = \max \left\{ \|Y\|_{1,r}, \left\| (d\pi)_{u(0)} Y(0) \right\|_2 \right\}.
\]

This space is well-defined since there are Sobolev embeddings

\[
W^{1,r}(\bar{u}^*TT^*Q) \hookrightarrow \to C_0(\bar{u}^*TT^*Q),
\]

\[
W^{1,2}(u(0)^*TT^*Q) \hookrightarrow \to C_0(\bar{u}(0)^*TT^*Q);
\]

see for example [Adams and Fournier, 2003] for the embeddings in trivial bundles.

We may then model maps \( u \colon \mathbb{R}^+ \times S^1 \to T^*Q \) with asymptotic conditions (4) by \( Y \mapsto \exp_u Y \) for \( Y \) in some ball in \( V^r \). These parametrizations have smooth transition functions, and thus define a Banach manifold. On each of these charts we may define

\[
F_{H,J}^u : V^r \to L^r (\bar{u}^*TT^*Q)
\]

\[
y \mapsto \Phi_u(Y)^{-1} F_{H,J}^u(y),
\]

where \( \Phi_u(Y) \) denotes parallel transport along \( \tau \mapsto \exp_u(\tau Y) \) and \( F_{H,J}^u(u) = \partial_s u - J(t,u)(\partial_t u - X_t(u)) \). The map \( F_{H,J}^u \) is well-defined and smooth, with derivative \( D_{\bar{u}}^+ : V^r \to L^r (\bar{u}^*TT^*Q) \)

\[
y \mapsto D_{\bar{u}} Y = JD_t Y - S_{H,J}^u Y.
\]

Locally, \( \mathcal{M}^+(q,x) \) is the zero set of \( F_{H,J}^u \), and we can show it is a manifold of dimension \( m(q) - |x| \) with the implicit function theorem for Banach manifolds, so long as \( D_{\bar{u}}^+ \) is surjective and its kernel has dimension \( m(q) - |x| \). In other words,

Fredholm property: \( D_{\bar{u}}^+ \) is a Fredholm operator with index \( m(q) - |x| \);

Transversality: perturbing \( J \) if necessary, \( D_{\bar{u}}^+ \) is surjective.

Denoting

\[
W_{\psi}^{1,\varphi}(\bar{u}^*TT^*Q) = \left\{ Y \in W^{1,\varphi}(\bar{u}^*TT^*Q) : (d\pi)_{u(0)} Y = 0 \right\},
\]

the map

\[
V^r \to T_{\pi u(0)}W^u(q),
\]

\[
y \mapsto (d\pi)_{u(0)} Y
\]
induces a linear homeomorphism

\[ V^r/W_0^1,r \cong T_{\tau_{\sigma(0)}}W^u(q), \]

which is finite-dimensional of dimension \( m(q) \), hence \( D_u^+ \) is Fredholm if and only if \( D_u^+|_{W^1,r} \) is Fredholm, in which case

\[
\text{ind} \left( D_u^+ \right) = \text{ind} \left( D_u^+|_{W^1,r} \right) + \dim T_{\tau_{\sigma(0)}}W^u(q); = m(q)
\]

it is thus enough to show that \( \text{ind}(D_u^+|_{W^1,r}) = -|x| \).

The trivialization \( \Psi \) maps \( W^1,r \) to

\[ W^1,r := \{ Y \in W^1, (\mathbb{R}^+ \times S^1; \mathbb{R}^{2n}) : Y(0,t) \in \Gamma_{\Psi}(t), t \in S^1 \}, \]

and conjugates \( D_u^+|_{W^1,r} \) to the operator

\[
L_{S,\lambda}^+: W^1,r (\mathbb{R}^+ \times S^1; \mathbb{R}^{2n}) \rightarrow L^r (\mathbb{R}^+ \times S^1; \mathbb{R}^{2n})
\]

\[
Y \mapsto \partial_s Y - J_0 \partial_t Y - SY,
\]

where \( S(+\infty,t) \) generates \( R_{\Psi}(t) \); i.e.

\[
\frac{dR_{\Psi}(t)}{dt} = J_0 S(+\infty,t) R_{\Psi}(t), \quad t \in [0,1[.
\]

### 2.5 Fredholm theory: abstract setting

Let \( 1 < r < \infty \), let \( S \in C^0(\mathbb{R}^+ \times S^1; \mathbb{R}^{2n \times 2n}) \) be non-degenerate — i.e. \( S(\pm \infty,t) \) generates symplectic paths \( R^{\pm} \) such that \( R^{\pm}(1) - I \) is non-singular — and let \( \lambda \in C^1(S^1; \Lambda(n)) \).

**Theorem 2.4** (Fredholm property). The operator

\[
L_{S,\lambda}^+: W^1_\lambda (\mathbb{R}^+ \times S^1; \mathbb{R}^{2n}) \rightarrow L^r (\mathbb{R}^+ \times S^1; \mathbb{R}^{2n})
\]

\[
Y \mapsto \partial_s Y - J_0 \partial_t Y - SY
\]

is Fredholm.

**Theorem 2.5** (Index formula). The Fredholm index of \( L_{S,\lambda}^+ \) is \( \text{deg}(\lambda) - \mu_{CZ}(R). \)

**Sketch of proof.** First one shows that \( \text{ind}(L_{S,\lambda}^+) \) depends only on \( \mu_{CZ}(R) \) and \( \text{deg} \lambda \); the argument to see that fixing \( \lambda \) and changing \( S \) such that \( \mu_{CZ}(R) \) does not change the Fredholm index is standard; see for example [Santos, 2018, Proposition 7.9.4]. Now if \( \text{deg}(\lambda_1) = \text{deg}(\lambda_2) \) then there is \( \Psi \in C^1(S^1; U(n)) \) such that \( \lambda_2(t) = \Psi(t)\lambda_1(t) \), with \( \text{deg} \Psi = 0 \). Thus we can extend \( \Psi \) to \( \Psi \in C^1(\mathbb{R}^+ \times S^1; U(n)) \) such that \( \lambda_2(t) = \Psi(0,t)\lambda_1(t) \) and \( \Psi(s,t) = I \) for \( s \geq 1 \). Then \( \Phi \) is a fiberwise linear isomorphism on \( (\mathbb{R}^+ \times S^1) \times \mathbb{R}^{2n} \), and

\[
L_{S,\lambda} \Phi = \Phi L_{\tilde{S},\lambda_2},
\]

where \( \tilde{R} = R \).
This means that showing the index formula given \( \mu_{CZ}(R) \) and \( \deg(\lambda) \) is the same as showing it for a given representative of those classes. To ease notation, let us say that in that case the index formula holds for the pair \( (\mu_{CZ}(R), \deg(\lambda))_n \). It is clear that the index formula is additive, since 
\[
L(s_1, s_2, (\lambda_1, \lambda_2)) = L(s_1, \lambda_1) \oplus L(s_2, \lambda_2).
\]
In other words, if the index formula holds for \((a_1, b_1)_n, (a_2, b_2)_n\) then it holds for \((a_1 + a_2, b_1 + b_2)_{n_1, n_2}\). Conversely, if the index formula holds for \((2a, 2b)_{2n}\) then it holds for \((a, b)_n\).

One shows explicitly that the index formula holds in the following specific case for \( n = 1 \): \( S(s, t) = aI \) with \( a \not\in 2\pi\mathbb{Z} \) and \( \lambda(t) = e^{(x+\mathbb{Z})} \). Hence the index formula holds for \((2k + 1, \ell)_1\).

Hence the index formula holds for \((2k, 2\ell)_2 = (2k - 1, \ell)_1 + (1, \ell)_1\), and thus for \((k, \ell)_1\), for all integers \( k, \ell \).

Finally, the index formula holds in all cases for \( n \geq 2 \) since \((k, \ell)_n = (k, \ell)_1 + (0, 0)_1 + \cdots + (0, 0)_1\).

### 2.5.1 Transversality

The proof of transversality is standard, but requires checking the following detail: if \( x \in \mathcal{O}(H) \) then \( \pi \circ x \in \text{Crit} \mathcal{E}_L \) and the constant solution \( x \in \mathcal{M}^+(\pi \circ x, x) \); the linearization \( D^+_u \) is invertible (i.e. surjective and with Fredholm index 0). For details, see [Abbondandolo and Schwarz, 2006, Proposition 3.7]; we note that a fundamental fact in the proof is that \( |x| = m(\pi \circ x) \) (see [Weber, 2002]).

After doing so, one shows that the set of uniformly continuous almost complex structures such that \( D^+_u \) is surjective for all \( u \in \mathcal{M}(q, x) \), for all \( q \in \text{Crit} \mathcal{E}_L \) and \( x \in \mathcal{O}(H) \), is a countable intersection of open dense sets. Hence the set of such \( J \) such that \( \partial_{H, J} \) and \( \mathcal{V} \) is well-defined is itself a countable intersection of open dense sets, hence dense and in particular non-empty.

**Remark 2.6.** In particular, this can be done for any choice of Lagrangian \( L \) in the aforementioned conditions. In [Weber, 2002] it is shown that Lagrangians of the form \( L(t, q, v) = |v|^2 - V(t, q) \) are non-degenerate for generic choice of \( V \).

### 2.5.2 Compactness

As in the definition of \( \partial_{H, J} \), compactness is shown for bounded action. The precise statement is the following.

**Theorem 2.7.** If \( L \) and \( H \) satisfy (L1), (L2), (H1) and (H2), and \( J \) is uniformly continuous w.r.t. \( J_0 \), there is \( \rho_0 \) such that \( \rho(u(\mathbb{R}^+ \times S^1)) \leq \rho_0 \) for all \( u \in \mathcal{M}^+(q, x) \).

**Proof.** Direct consequence of [Abbondandolo and Schwarz, 2006, Theorem 1.14]; the conditions are satisfied:

\[
\mathcal{E}_L(q) \geq \mathcal{E}_L(\pi \circ \bar{u}(0)) \geq \mathcal{A}_H(\bar{u}(0)) \geq \mathcal{A}_H(u(s)) \geq \mathcal{A}_H(u(+\infty)) = \mathcal{A}_H(x)
\]

and \( \|\pi \circ \bar{u}(0)\|_2 \) is uniformly bounded since \( W^u(q) \) is pre-compact in \( W^{1, \frac{3}{2}}(S^1; Q) \) (subsection 2.2).

By Gromov compactness, \( \mathcal{M}^+(q, x) \) is relatively compact.
As in the construction of $\partial_{H,J}$, this compactness expresses itself in terms of broken trajectories: given any sequence $(u_m)$ in $M^+(q,x)$, there is a subsequence converging to a broken trajectory $([c^1], \ldots, [c^n], u, [v^1], \ldots, [v^n])$, for $[c^i] \in \mathcal{L}(q_{i-1}, q_i)$, and $u \in M^+(q_k, x_0)$, and $v^j \in \mathcal{L}(x_{j-1}, x_j)$, with $q_0 = q$ and $x_n = x$. This convergence means that $u_m \to u$ in $C^0_{\text{loc}}([0, +\infty[\times S^1; T^*Q)$ and $s_m^j \cdot u_m \to v_j$, $\phi_t^{L_m} \pi_m u_m(0) \to c^i(0)$ for some sequences of positive $s_m^j$ and negative $t_m^j$.

2.5.3 Gluing

In particular, if $m(q) - |x| = 1$ then $M^+(q,x)$ is a 1-dimensional manifold. It has a compactification $\overline{M^+}(q,x)$, which is a compact 1-manifold with boundary

$$
\bigcup_{m(q_1) = m(q) - 1} \mathcal{L}(q, q_1) \times M^+(q_1, x) \bigcup \bigcup_{|y| = |x| - 1} M^+(q, y) \times \mathcal{L}(y, x).
$$

Thus, with coefficients in a ring of characteristic 2 we have

$$0 = \mathcal{V} \partial_{E, G} + \partial_{H,J} \mathcal{V} = \mathcal{V} \partial_{E, G} - \partial_{H,J} \mathcal{V},$$

as needed.

2.6 The surprisingly short proof of Viterbo’s theorem

**Theorem 2.8** (Viterbo). Let $R$ be a ring with characteristic 2 and let $Q$ be a closed manifold. Let $H \in C^\infty(S^1 \times T^*Q)$ be a 1-periodic Hamiltonian with all 1-periodic orbits non-degenerate, let $L \in C^\infty(S^1 \times TQ)$ be a 1-periodic Lagrangian such that $H$ is the Legendre transform of $L$, and let $J$ be an almost complex structure compatible with $\omega$.

Suppose that the conditions (L1), (L2), (H1), (H2) hold, that $J$ is uniformly continuous w.r.t. the metric induced by $J_0$, that $G$ is a metric on $W^{1,2}(S^1; Q)$ and that $(E_L, G)$ satisfies the Morse-Smale condition up to order 1, and that $J$ is such that all operators $D_u$ and $D_v^+$ with $u \in \mathcal{M}(x,y)$, $v \in M^+(q,x)$ for some $x,y \in \mathcal{O}(H)$ and $q \in \text{Crit } E_L$ are surjective.

Then the map

$$\mathcal{V}: (SC(T^*Q, \omega, H; R), \partial_{H,J}) \to (MC(W^{1,2}(S^1; Q), E_L; R), \partial_{E,L,G})$$

is a chain isomorphism. In particular, $\mathcal{V}$ induces an isomorphism on the level of homology:

$$SH_*(T^*Q, \omega; R) \cong H_*(LQ; R).$$

**Proof.** We have seen that $n^+(q, x) = 0$ if $E_L(q) < A_H(x)$ or $E_L(q) \leq A_H(x)$ and $q \neq \pi \circ x$ (subsection 2.3). Moreover, if $x \in \mathcal{O}(H)$ then $M^+(\pi \circ x, x) = \{x\}$ hence $n^+(\pi \circ x, x) = 1$. Since $A_H$ is bounded from below by $H1$, and since the number of orbits with action bounded from above is finite, we can order $\mathcal{O}_k(H)$ by order of increasing $A_H$, and correspondingly $\text{Crit}_k(E_L)$ by order of increasing $E_L$. In terms of these orderings, the map $\mathcal{V}$ is then represented by an upper-triangular matrix with ones in the diagonal, and is thus an isomorphism. □
3 Orientations in Floer homology

In this section we give a brief introduction to orientations in Floer homology, including a proof that defining the coefficients of $\partial_{H,J}$ as signed counts yields $\partial^2_{H,J} = 0$.

3.1 Revisiting the differential, orientation coherence

The definitions of $\partial$ and $V$ are the same; the numbers $n(x,y)$ and $n^+(q,x)$ are redefined as follows:

$$n(x,y) = \sum_{[u] \in \mathcal{L}(x,y)} \varepsilon([u]),$$

where $\varepsilon: \mathcal{L}(x,y) \to \pm 1$, and

$$n^+(q,x) = \sum_{[u] \in \mathcal{M}^+(q,x)} \varepsilon^+(u),$$

where $\varepsilon^+: \mathcal{M}^+(q,x) \to \pm 1$.

In order for $\partial^2_{H,J} = 0$, in addition to the gluing property we require orientation coherence: if $x, z \in \mathcal{O}(H)$ are such that $|x| - |z| = 2$ and $([u_1], [v_1])$ and $([u_2], [v_2])$ are endpoints of an arc in $\mathcal{L}(x, z)$, then $\varepsilon([u_1])\varepsilon([v_1]) + \varepsilon([u_2])\varepsilon([v_2]) = 0$.

Similarly, in order for $\partial_{\mathcal{E}_L, G} V - V \partial_{H,J} = 0$ we require the following to hold: if $x \in \mathcal{O}(H)$ and $q \in \mathcal{C} \mathcal{E}_L$ are such that $m(q) - |x| = 1$, we have the following options for endpoints of an arc in $\mathcal{M}^+(q,x)$:

- For $([c_1], u_1)$ and $([c_2], u_2)$ we have $\varepsilon([c_1])\varepsilon^+(u_1) + \varepsilon([c_2])\varepsilon^+(u_2) = 0$;
- For $([c_1], u_1)$ and $([u_2], [v_1])$ we have $\varepsilon([c_1])\varepsilon^+(u_1) = \varepsilon^+(u_2)\varepsilon([v_1])$;
- For $([u_1], [v_1])$ and $([u_2], [v_2])$ we have $\varepsilon^+(u_1)\varepsilon([v_1]) + \varepsilon^+(u_2)\varepsilon([v_2]) = 0$.

3.2 Systems of local coefficients

**Definition 3.1.** By a system of local coefficients we mean a map $\nu$ mapping homotopy classes of cylinders with fixed asymptotics to $\pm 1$, compatible with concatenation.

If $([u_1], [v_1])$ and $([u_2], [v_2])$ are endpoints of an arc in $\mathcal{L}(x, z)$ then $\nu(u_1)\nu(v_1) = \nu(u_2)\nu(v_2)$.

Suppose that we have orientations $\varepsilon$ as in section 3.1. Then we can define a new differential

$$\partial_{H,J,\nu}(x) = \sum_{y \in \mathcal{O}(H)} \left( \sum_{[u] \in \mathcal{L}(x,y)} \varepsilon([u])\nu([u]) \right) y,$$

and define

$$SH_\bullet(T^*Q, \omega; \nu) \cong SH_\bullet(T^*Q, \omega, H, J; \nu) = H_\bullet(SC_\bullet(T^*Q, \omega, H), \partial_{H,J,\nu}).$$

The usual symplectic homology corresponds to taking $\nu \equiv 1$. In general, $SH_\bullet(T^*Q, \omega; \nu)$ depends on the choice of $\nu$.  

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3.3 Coherent orientations for the symplectic homology of the cotangent bundle

For general symplectic manifolds the “canonical” reference is [Floer and Hofer, 1993], in which you can find the proofs of the results presented in this section. We made some adaptations for the cotangent bundle, following [Abbondandolo and Schwarz, 2014].

We begin by describing how to interpret orientation. Recalling that \( M(x,y) \) is defined as the zero-level set of an operator with linearization \( Du \), by the implicit function theorem we have \( TuM(x,y) \cong \ker Du \), and an orientation of \( M(x,y) \) corresponds to an orientation of \( \bigwedge^{\max} \ker Du \), the latter being a particular case of the determinant of a Fredholm operator.

**Definition 3.2.** Let \( S \) be a topological space and let \( X,Y \) be Banach spaces. Given a map \( f: S \to \text{Fred}(X,Y) \), the **determinant line bundle** \( \det(f) \to S \) has fiber at \( s \) given by

\[
\det(f(s)) := \bigwedge^{\max} \ker(f(s)) \otimes \left( \bigwedge^{\max} \coker(f(s)) \right)^*. 
\]

If the map \( f \) is an embedding we denote \( \det(f) \) by \( \det(S) \).

Hence an orientation of \( M(x,y) \) corresponds to an orientation of \( \det([Du: u \in M(x,y)]) \).

### 3.3.1 Coherent orientations in \( \mathbb{C}^n \)

Let us temporarily pass to operators \( W^{1,r}((\mathbb{R} \times S^1; \mathbb{R}^{2n}) \to L^r((\mathbb{R} \times S^1; \mathbb{R}^{2n}) \) via a unitary trivialization the operators \( Du \) correspond to operators of the form \( L_T := \partial_s + J_0 \partial_t + S \), where \( S \in C^0(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \) is non-degenerate, i.e. \( S^\pm(t) := S(\pm \infty,t) \) generate symplectic paths \( R^\pm \) such that \( R^\pm(1) = I \) is non-singular. Denote by \( \Sigma \) the space of all such operators. The subspace \( \Sigma(S^-, S^+) = \{ L_T: T^\pm = S^\pm \} \) is clearly contractible, and thus \( \det(\Sigma(S^-, S^+)) \) is orientable.

Let \( S, T \) be admissible, asymptotically constant and such that \( S^+ = T^- \). Then, for large enough \( \rho > 0 \) we can define

\[
(S^\#_\rho T)(s,t) = \begin{cases} 
S(s + \rho, t), & s \in [-\infty,0] \times S^1 \\
T(s - \rho, t), & s \in [0, +\infty] \times S^1,
\end{cases}
\]

for \( (s,t) \in \mathbb{R} \times S^1 \). Define also \( L_S^\#_\rho L_T = L_{S^\#_\rho T} \in \Sigma(S^-, T^+) \).

**Proposition 3.3.** In the above conditions there is a natural, homotopy-invariant isomorphism

\[
\det(L_S) \otimes \det(L_T) \cong \det(L_{S^\#_\rho T})
\]

for sufficiently large \( \rho > 0 \).

**Remark 3.4.** For the gluing procedure one needs to show that \( Du^\#_\rho v \) has a two-dimensional kernel for large enough \( \rho \); this is essentially done by showing that \( \ker(Du^\#_\rho v) \cong \ker(Du) \#_\rho \ker(Dv) \), which is equivalent to the above since \( Du, Du^\#_\rho v \) are all surjective in this case.
Theorem 3.5. Given $S_0, S_1, S_2$, orientations $o_{S_0, S_1}$ and $o_{S_1, S_2}$ of $\Sigma(S_0, S_1)$ and $\Sigma(S_1, S_2)$ respectively induce an orientation $o_{S_0, S_1} \# o_{S_1, S_2}$ of $\Sigma(S_0, S_2)$. Moreover we have associativity:

$$(o_{S_0, S_1} \# o_{S_1, S_2}) \# o_{S_2, S_3} = o_{S_0, S_3} \# (o_{S_1, S_2} \# o_{S_2, S_3}).$$

Proof. It is enough to define the gluing orientation for specific operators; this is induced for asymptotically constant operators by the canonical isomorphism of Proposition 3.3. The gluing orientation does not depend on the choice of sufficiently large $\rho$ or on the choice of operators since said isomorphism is homotopy-invariant. To see that associativity holds it is also enough to do so for asymptotically constant operators, which also follows from the homotopy invariance of the canonical isomorphism of Proposition 3.3.

Definition 3.6. A coherent orientation for $\Sigma$ is a map $\sigma$ which assigns to each $(S^-, S^\circ)$ an orientation of $\det(\Sigma(S^-, S^\circ))$ such that

$$\sigma(S_0, S_1) \# \sigma(S_1, S_2) = \sigma(S_0, S_2).$$

Theorem 3.7. Coherent orientations exist.

Proof. Choose $S_0$. For $S_1, S_2 \neq S_0$ we sequentially orient $\Sigma(S_0, S_0), \Sigma(S_1, S_0), \Sigma(S_0, S_2)$ and $\Sigma(S_1, S_2)$.

Firstly, $L_{S_0}$ is invertible and thus $\det L_{S_0} = \mathbb{R} \otimes \mathbb{R}^*$, which we orient by choosing $1 \otimes 1^* \in \det L_{S_0}$ as a positive direction. This sets $\sigma(S_0, S_0)$.

Secondly, for any $S_1 \neq S_0$ choose $\sigma(S_1, S_0)$ by orienting $\Sigma(S_1, S_0)$ arbitrarily. Thirdly, for any $S_2 \neq S_0$ set $\sigma(S_2, S_0)$ such that

$$\sigma(S_0, S_2) \# \sigma(S_2, S_0) = \sigma(S_0, S_0).$$

Finally, for any $S_1, S_2 \neq S_0$ set

$$\sigma(S_1, S_2) = \sigma(S_1, S_0) \# \sigma(S_0, S_2).$$

These satisfy the coherent orientation condition by associativity of $\#$. We henceforth fix a coherent orientation $\sigma$ of $\Sigma$.

3.3.2 Coherent orientations in the cotangent bundle

The idea now is to orient $\det(D_u) = \det(\Psi^{-1} L_S \Psi) = \Psi^{-1} \det(L_S)$ for a trivialization $\Psi$: $u^* T^* Q \to (\mathbb{R} \times S^1) \times \mathbb{R}^{2n}$. In order to do so, we need this orientation to not depend on the choice of trivialization, which we do by fixing the asymptotic trivializations.

Lemma 3.8. If $\Psi$ is a unitary vector bundle automorphism of $(\mathbb{R} \times S^1) \times \mathbb{R}^{2n}$ which restricts to the identity on $\{ \pm \infty \} \times S^1$, then any orientation $o$ of $L_S$ is compatible with the orientation $\Psi(o)$ of $\Psi L_S \Psi^{-1}$ in $\det(\Sigma(S^-, S^+))$.

Fix for each $x \in O(H)$ a trivialization $\psi_x: x^* T^* Q \to S^1 \times \mathbb{R}^{2n}$ such that $\deg \Gamma_{\psi_x} \in \{0, 1\}$. Given $x_\pm \in LT^* Q$ denote by $P^\infty(x_-, x_+)$ the set of $u \in C^\infty(\mathbb{R} \times S^1; T^* Q)$ such that $\lim_{s \to \pm \infty} u(s) = x_\pm$ in $C^\infty(S^1; T^* Q)$, satisfying an exponential decay condition

$$|\partial_s u(s, t)|_{L_S} \lesssim e^{-\delta |s|},$$

for some $\delta > 0$. 

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Remark 3.9. The set $P^\infty(x_-, x_+)$ is non-empty if and only if $x_-$ and $x_+$ represent the same conjugacy class in $\pi_1(T^*Q)$. Whenever these sets come up we always assume that this is the case.

Lemma 3.10. For any $u \in P^\infty(x_-, x_+)$, there is a trivialization $\psi_u: u^*TT^*Q \to (R \times S^1) \times R^{2n}$ extending $\psi_{x_\pm}$.

For simplicity we henceforth assume that all trivializations satisfy the above condition.

In terms of orientations, taking trivializations is thus functorial, and the concepts of gluing and coherent orientation can be transported to the more general context we are in now.

The choices of trivializations $\psi_{x_\pm}$ fix generators $S_{x_\pm} \in C^0(S^1; \text{Sym}(2n; R))$ of $R_{x_\pm}$, such that $\psi_uD_u\psi_u^{-1} \in \Sigma(S_{x_-}, S_{x_+})$ for any $u \in P^\infty(x_-, x_+)$. 

Definition 3.11. By orientation $o$ of the set $D(x, y) = \{D_u : u \in P^\infty(x, y)\}$ we mean a map assigning to each $u \in P^\infty(x, y)$ an orientation $o(u)$ of $D_u$, such that

$$o(u) = \psi_u (o_{S_x, S_y} (\psi_u D_u \psi_u^{-1}))$$

for some fixed orientation $o_{S_x, S_y}$ of $\det(\Sigma(S_x, S_y))$. We abbreviate $o = \psi o_{S_x, S_y}$ and $o_{S_x, S_y} = \psi^{-1} o$.

The above definition makes sense by Lemma 3.8.

Given orientations $o_{x, y}$ and $o_{y, z}$ of $D(x, y)$ and $D(y, z)$, respectively, we can define an orientation $o_{x, y} \# o_{y, z}$ of $D(x, z)$ by gluing:

$$o_{x, y} \# o_{y, z} = \psi (\psi^{-1} o_{x, y} \# \psi^{-1} o_{y, z}).$$

Gluing is associative by Theorem 3.5, in the sense that $(o_{x, y} \# o_{y, z}) \# o_{z, w} = o_{x, y} \# (o_{y, z} \# o_{z, w}).$

For the chosen coherent orientation $\sigma$ of $\Sigma$, denote also by $\sigma$ the corresponding orientation of $D$: i.e. a map assigning to pairs $(x, y) \in \mathcal{O}(H)$ an orientation of $D(x, y)$ defined by $\sigma(x, y) = \psi \sigma(S_x, S_y)$. This new orientation is also coherent:

$$\sigma(x, y) \# \sigma(y, z) = \sigma(x, z).$$

3.4 Floer/symplectic homology with integer coefficients

If $|x| - |y| = 1$ then $M(x, y)$ is a 1-manifold endowed with a free and proper $R$-action. For $u \in M(x, y)$ we have $\ker D_u = R \cdot \partial_u u$; consider the orientation $\sigma_s$ for which $\partial_u u$ is positively oriented. Define $\varepsilon(u) \in \{\pm 1\}$ by

$$\sigma(u) = \varepsilon(u) \sigma_s(u),$$

for $u \in M(x, y)$. It is clear that $\varepsilon(u) = \varepsilon([u])$ is well-defined.

Now suppose that $([u_1], [v_1]) \in L(x, y_1) \times L(y_1, z)$ and $([u_2], [v_2]) \in L(x, y_2) \times L(y_2, z)$ are endpoints of an arc in $L(x, z)$. Then

$$\varepsilon(u_1) \varepsilon(v_1) \sigma_s(u_1) \# \sigma_s(v_1) = \varepsilon(u_2) \varepsilon(v_2) \sigma_s(u_2) \# \sigma_s(v_2),$$

hence

$$\varepsilon(u_1) \varepsilon(v_1) \sigma_s(u_1) \# \sigma_s(v_1) = \sigma(x, z) = \varepsilon(u_2) \varepsilon(v_2) \sigma_s(u_2) \# \sigma_s(v_2).$$
Lemma 3.12. Suppose we are in the above conditions. Let $o$ be an orientation of $D(x, z)$ and let $V(w) \in \ker D_w$ be such that $(\partial_s w, V(w))$ is a positively oriented basis of $\ker D_w$, for all $w \in M(x, z)$. The orientation $\sigma_s(u)\#\sigma_s(v)$ agrees with the orientation $o$ of $D$ if and only if $V(w)$ projects into a vector field on $L(x, z)$ which is inner-pointing near $([u], [v])$.

Proof. Recall the gluing procedure (e.g. [Audin and Damian, 2014, Chapter 9], [Santos, 2018, Chapter 10]), which yields a curve $\rho \mapsto \varphi(\rho) \in M(x, z)$ such that $\lim_{\rho \to \pi} [\varphi(\rho)] = ([u], [v])$ in $L(x, z)$. By gluing $\psi_u$ and $\psi_v$ to get a trivialization $\psi_{u\#_p v}$, the gluing procedure for the trivialized operators gives an isomorphism $\ker(D_{u\#_p v}) \cong \ker(D_u) \oplus \det(D_v)$ via

$$
\partial_s u \mapsto \partial_s u \#_p 0,
\partial_s v \mapsto 0 \#_p \partial_s v.
$$

The orientations $\sigma_s(u)$ and $\sigma_s(v)$ correspond to taking $\partial_s u$ and $\partial_s v$ to be positively oriented, and thus $\partial_s(u)\#\partial_s(v)$ corresponds to taking $(\partial_s u \#_p 0, 0 \#_p \partial_s v)$ to be positively oriented.

On the other hand, the basis $(\partial_s \varphi(\rho), \partial_v \varphi(\rho))$ of $\ker(D_{\varphi(\rho)})$ is approximated by the basis $(\partial_s(u \#_p v), \partial_p(u \#_p v))$ of $\ker(D_u)\#_p \ker(D_v)$, and we have

$$
\partial_s (u \#_p v) / \partial_p (u \#_p v) = (\partial_s u \#_p \partial_s v) / (\partial_s u \#_p (-\partial_s v)) = -2 (\partial_s u \#_p 0) / (0 \#_p \partial_s v);
$$

since $\partial_p \varphi(\rho)$ projects to an outward-pointing vector near $([u], [v])$, we arrive at the desired conclusion.

On an arc, a positively oriented vector field is inner pointing at one of the end points and outward pointing at the other, thus exactly one of $\sigma_s(u_1)\#\sigma_s(v_1)$ and $\sigma_s(u_2)\#\sigma_s(v_2)$ agrees with $\sigma(x, z)$. In conclusion,

$$
\varepsilon(u_1) \varepsilon(v_1) = -\varepsilon(u_2) \varepsilon(v_2),
$$

as needed to define Floer/symplectic homology with integer coefficients.

Remark 3.13. The data (or choices) made in order to define the signs in Floer homology are a coherent orientation of $\Sigma$ and a trivialization with prescribed degree of the Gauss map for each one-periodic orbit of $H$. The coherent orientation itself is defined by the choices of orientations of $\Sigma(S_1, S_0)$ for fixed $S_0$ and variable $S_1 \neq S_0$.

Choosing differently does not affect the chain isomorphism type of the symplectic chain complex.

4 The general form of Viterbo’s theorem

As we have mentioned, there is no general isomorphism $SH_*(T^*Q, \omega) \to H_*(LQ)$. In [Abbondandolo and Schwarz, 2014], it is mentioned that $SH_*(T^*\mathbb{CP}^2, \omega; k) = 0$ if $k \neq 2$ but clearly $H_*(L\mathbb{CP}^2; k) \neq 0$ (due to P. Seidel, following [Kragh, 2018]).

In this section we describe the proof that Viterbo’s theorem holds for twisted Floer homology, defined with a local system of coefficients which is trivial if the second Stiefel-Whitney class of $Q$ vanishes over 2-tori.
4.1 Orienting half-cylinders in $\mathbb{C}^n$

In order to define $\varepsilon^+(u)$ we need to orient $M^+(q, x)$. In order to do so we begin by orienting the operators $L^+_{S, \lambda}$.

Denote by $\lambda: S^1 \to \Lambda(n)$ either $\lambda(t) = \mathbb{R}^n$ or $\lambda(t) = e^{\pi it} \mathbb{R}^n$, and omit $\lambda$ from the above notation, considering $L^+_{S, \lambda}$ to be one of the operators, with $\lambda$ fixed. The space $\Sigma^+$ of operators $L^+_{S, \lambda}$ such that $S$ is non-degenerate decomposes into contractible spaces $\Sigma^+(S^+) = \{ L_T \in \Sigma^+ : T^+ = S^+ \}$. Hence $\det(\Sigma^+(S^+))$ is orientable.

Given orientations $o_{S_0}$ and $o_{S_0, S_1}$ of $\Sigma^+(S_0)$ and $\Sigma(S_0, S_1)$, we can define a glued orientation $o_{S_0} \# o_{S_0, S_1}$ of $\Sigma^+(S_1)$. This gluing operation is also associative

$$(o_{S_0} \# o_{S_0, S_1}) \# o_{S_1, S_2} = o_{S_0} \# (o_{S_0, S_1} \# o_{S_1, S_2}).$$

A coherent orientation $\sigma$ of $\Sigma^+$ maps each non-degenerate $S^+$ to an orientation of $\Sigma^+(S^+)$, in such a way that $\sigma(S_1) = \sigma(S_0) \# \sigma(S_0, S_1)$.

**Theorem 4.1.** Coherent orientations of $\Sigma^+$ exist.

**Proof.** Choose non-degenerate $S_0$ and an orientation $\sigma(S_0)$ of $\Sigma^+(S_0)$. Set $\sigma(S_1) = \sigma(S_0) \# \sigma(S_0, S_1)$. This is well-defined by associativity and since $\sigma$ is a coherent orientation of $\Sigma$. $\square$

**Remark 4.2.** There are only two coherent orientations of $\Sigma^+$.

To orient the operators $D^+_a |_{W^+_{2, r}}$ we need to choose more specific trivializations. For each $x \in O(H)$ choose $\psi_x: x^* TT^* Q \to S^1 \times \mathbb{R}^{2n}$ such that $(\psi_x)_t(T_v T^* Q) = \lambda(t)$ for $t \in S^1$ (i.e. $\psi_x$ is vertical-preserving). We need to see that there are vertical-preserving extensions of $\psi_x$ to half-cylinders.

**Lemma 4.3.** Given $u \in M^+(q, x)$, there is a unitary trivialization

$$\psi_u: \bar{u}^* TT^* Q \to \left( \mathbb{R}^n \times S^1 \right) \times \mathbb{R}^{2n}$$

extending $\psi_x$ and such that $(\psi_u)_{(s, t)}(T_v T^* Q) = \lambda(t)$ for $(s, t) \in \mathbb{R}^n$.

**Proof.** Via any trivialization $\psi_u$ extending $\psi_x$, the question becomes: is there a unitary vector bundle automorphism $U$ of $\left( \mathbb{R}^n \times S^1 \right) \times \mathbb{R}^{2n}$ such that $U(+\infty, t) = I$ and $U(s, t) = (\psi_u)_{(s, t)}(T_v T^* Q) = \lambda(t)$ for $(s, t) \in \mathbb{R}^n \times S^1$. This is true since the maps $(s, t) \mapsto (\psi_u)_{(s, t)}(T_v T^* Q)$ and $(s, t) \mapsto \lambda(t)$ are homotopic in $\Lambda(n)$. $\square$

We also need to check that the choice of vertical-preserving trivialization does not affect orientation. The composition of vertical-preserving trivializations maps, at each point $(s, t)$, the subspace $\lambda(t) = e^{\pi t} \mathbb{R}^n$ to itself, and thus maps $\mathbb{R}^n$ to itself. We can identify such maps with the subgroup $O(n) \subset U(n)$. The following lemma is a consequence of Lemma 4.9; it should be clear from the discussion of section 4.4.

**Lemma 4.4.** If $\overline{\Psi}$ is a vertical-preserving, unitary vector bundle automorphism of $\left( \mathbb{R}^n \times S^1 \right) \times \mathbb{R}^{2n}$, which restricts to the identity on $\{ +\infty \} \times S^1$, then any orientation $o$ of $L^+_{S}$ is compatible with the orientation $\overline{\Psi}(o)$ of $\Psi L^+_{S} \Psi^{-1}$ in $\det(\Sigma^+(S^+))$. 20
4.2 Orienting hybrid trajectories

Given an half-cylinder \( u \in \mathcal{M}(q,x) \), since \( V_r/W^1_r \cong T_{\pi_0u(0)}W^u(q) \) we have the following commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
0 & \rightarrow & \ker D^+_u|_{W^1_r} & \rightarrow & \ker D^+_u \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W^1_r & \rightarrow & V_r \\
\downarrow D^+_u|_{W^1_r} & & \downarrow D^+_u & & \downarrow D^+_u \\
L^r(\psi^{-1}TT^*Q) & \rightarrow & L^r(\psi^{-1}TT^*Q) & \rightarrow & 0 \\
\downarrow coker D^+_u|_{W^1_r} & & \downarrow coker D^+_u & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

from which we find the exact sequence

\[
0 \rightarrow \ker D^+_u|_{W^1_r} \rightarrow \ker D^+_u \rightarrow T_{\pi_0u(0)}W^u(q) \rightarrow \coker D^+_u|_{W^1_r} \rightarrow \coker D^+_u \rightarrow 0
\]

which yields a canonical isomorphism ([Floer and Hofer, 1993, Lemma 18])

\[
\bigwedge \ker D^+_u|_{W^1_r} \otimes \bigwedge T_{\pi_0u(0)}W^u(q) \otimes \bigwedge \coker D^+_u \cong \bigwedge \ker D^+_u \otimes \bigwedge \coker D^+_u|_{W^1_r},
\]

or equivalently

\[
det (D^+_u) \cong det \left( D^+_u|_{W^1_r} \right) \otimes \bigwedge T_{\pi_0u(0)}W^u(q). \tag{6}
\]

Since \( W^u(q) \) is the image of an immersion of \( \mathbb{R}^{m(q)} \) ([Abbondandolo and Majer, 2006, Theorem 1.18]), it is contractible. Given an orientation \( o_q \) of \( W^u(q) \) and an orientation \( o_{S^x} \) of \( L^+_S = \psi_u(D^+_u|_{W^1_r}) \psi_u^{-1} \in \Sigma^+(S^x) \), we thus have an induced orientation \( \psi_u(o_{S^x}) \otimes o_q \) of \( det(D^+_u) \).

We define \( \sigma(x) = \psi_u(\sigma(S^x)) \). We give \( \mathcal{M}(q,x) \) the orientation \( \sigma(q,x) \) induced by \( \sigma(x) \otimes \sigma(q) \) via the isomorphism (6), where the second \( \sigma \) attributes to each \( q \in \text{Crit}\ E_u \) an orientation of \( W^u(q) \).

We can define gluing of orientations, and then the orientation \( \sigma(x) \) is coherent in the sense that \( \sigma(x) \# \sigma(x,y) = \sigma(y) \).

In the particular case, \( m(q) = |x| \) then \( \dim \mathcal{M}(q,x) = 0 \) and an orientation of \( \mathcal{M}(q,x) \) is simply a map \( \mathcal{M}(q,x) \rightarrow \{ \pm 1 \} \); define \( \varepsilon^+(u) \) as the image of \( u \) under this map, for \( u \in \mathcal{M}(q,x) \). We can see \( \varepsilon^+(u) \) in a simpler way: under transversality, \( D^+_u \) is surjective hence bijective, thus \( D^+_u|_{W^1_r} \) is injective; the exact sequence (5) yields an isomorphism

\[
T_{\pi_0u(0)}W^u(q) \rightarrow \coker D^+_u|_{W^1_r}, \tag{7}
\]

and \( \varepsilon^+(u) = 1 \) if and only if this isomorphism is orientation preserving.
Lemma 4.8. We have orientation and
Remark 4.5. We can choose a coherent orientation
the pre-gluing solution over \( x \in \mathcal{O}(H) \) such that \( q = \pi \circ x \). This choice forces each \( n^+(\pi \circ x, x) \) to be +1, for \( x \in \mathcal{O}(H) \).

4.3 Obstruction to \( V \) to being a chain map and correction

In order to analyze whether \( V \) is a chain map, we need to analyze orientation coherence (or the lack of it) near broken trajectories. The first type of broken trajectory does not cause any trouble.

Let \( q_0, q_1 \in \mathcal{E}_L \) and \( x \in \mathcal{O}(H) \) be such that \( m(q_0) - 1 = m(q_1) = |x| \). Given \([c] \in \mathcal{L}(q_0, q_1)\) and \( u \in \mathcal{M}^+(q_1, x)\) we have

\[
\sigma(x, q_0) = \sigma(x) \otimes \sigma(q_0) \otimes \sigma(q_1)^* \otimes \sigma(q_1) \cong \epsilon(x) \otimes \sigma(c) \otimes \sigma(q_1) \\
\cong \sigma(x, q_1) \otimes \sigma(c) = \epsilon^+(u) \epsilon([c]) \sigma_s(c).
\]

Lemma 4.6. Let \(([c], u) \in \mathcal{L}(q_0, q_1) \times \mathcal{M}^+(q_1, x)\) be such that \( m(q_0) - 1 = m(q_1) = |x| \). Then \( \epsilon^+(u) \epsilon([c]) = 1 \) if and only if the outward-pointing vector near \([c], u\) is positively oriented with respect to \( \sigma(x, q_0) \).

Now let \( q \in \mathcal{E}_L \) and \( x, y \in \mathcal{O}(H) \) be such that \( m(q) = |x| = |y| + 1 \). Given \( u \in \mathcal{M}^+(q, x) \) and \([v] \in \mathcal{L}(x, y)\) we have

\[
\sigma(q, y) = \sigma(u) \# \sigma(v) \cong \epsilon^+(u) \epsilon([v]) \sigma_s(v).
\]

The gluing procedure near \((u, [v])\) gives a parametrization \( \rho \mapsto \varphi(\rho) \in \mathcal{M}^+(q, y) \) such that \( \lim_{\rho \to \pm \infty} \varphi(\rho) = (u, [v]) \) in \( \mathcal{M}^+(q, y) \), approximated by the pre-gluing

\[
u_{\#, \rho} v(s, t) = \begin{cases} u(s, t), & 0 \leq s \leq \rho - 1, \\ v(s - 2\rho, t), & \rho + 1 \leq s, \end{cases}
\]

which interpolates both functions for \( \rho - 1 \leq s \leq \rho + 1 \). We run into a problem: we cannot generally glue the trivializations \( \psi_u \) and \( \psi_v \) to find a vertical-preserving trivialization \( \psi_{u \#, \rho} \), since \( \psi_{\rho} \) is not necessarily vertical-preserving. If we can choose \( \psi_{\rho} \) such that it is vertical-preserving, then this problem is avoided, and we have the following result.

Lemma 4.7. Let \((u, [v]) \in \mathcal{M}^+(q, x) \times \mathcal{L}(x, y)\) be such that \( m(q) = |x| = |y| + 1 \). Suppose that \( \psi_u \) and \( \psi_v \) are vertical-preserving. Then \( \epsilon^+(u) \epsilon([v]) = 1 \) if and only if the outward-pointing vector near \((u, [v])\) is positively oriented with respect to \( \sigma(x, q_0) \).

Now, as usual, let \( \psi_{\rho} \) be a unitary trivialization extending \( \psi_x \) and \( \psi_y \), and let \( \vartheta_{u \#, \rho} \) be a vertical-preserving unitary trivialization extending \( \psi_{\rho} \). The question is: does the change of trivialization

\[
U := (\psi_{u \#, \rho} \psi_{\rho}) \circ \vartheta_{u \#, \rho}^{-1} : \mathbb{R}^+ \times S^1 \to U(n)
\]

preserve or reverse orientation? This question is well-posed since \( U(+\infty, t) = I \) for \( t \in S^1 \), hence it acts on \( \Sigma^+(S^1) \). Defining \( \eta_{S, \lambda}(U) \) as 1 if \( U \) preserves orientation and -1 otherwise, we can rewrite Lemma 4.7 as follows.

Lemma 4.8. We have \( \epsilon^+(u) \epsilon([v]) \eta_{S, \lambda}(U) = 1 \) if and only if the outward-pointing vector near \((u, [v])\) is positively oriented with respect to \( \sigma(x, q_0) \).
4.4 Viterbo’s theorem with the naive local system of coefficients

In order to understand how Viterbo’s theorem may fail, or how to correct it with a local system of coefficients, we need to understand the map \( \eta_{S_y, \lambda} \).

Firstly, what is its domain? The change of trivialization \( U \) in (8) satisfies the following asymptotic conditions:

- \( U(0, t) \in O(n) \),
- \( U(+\infty, t) = I \),

for \( t \in S^1 \), since both trivializations extend \( \psi_y \), and since both trivializations are vertical-preserving over \( \{0\} \times S^1 \). Denote by \( F_\iota \) (after the homotopy fiber of the inclusion \( \iota: LO(n) \to LU(n) \)) the space of such \( U \).

Secondly, are the indices \( S_y \) and \( \lambda \) necessary? Analogously to [Floer and Hofer, 1993, Lemma 13], the effect of \( U \) on orientation does not depend on \( S_y \); it turns out (Lemma 4.13) that it depends on \( \lambda \). We henceforth ignore the index \( S_y \).

It is clear that \( F_\iota \) is a topological group and that \( \eta_\lambda: F_\iota \to \{\pm 1\} \) is a continuous group homomorphism, hence it is described by a group homomorphism \( \pi_0(F_\iota) \to \{\pm 1\} \), which we also denote \( \eta_\lambda \).

In order to continue, we need to describe \( \pi_0(F_\iota) \), which we do via the fibration sequence of \( \iota \).

**Lemma 4.9.** Let \( \varphi: \mathbb{R}^+ \to [0, 1] \) be an increasing homeomorphism. There is a short exact sequence

\[
0 \to \pi_1(U(n)) \to \pi_0(F_\iota) \xrightarrow{(ev_0)} \pi_0(LO(n)) \to 0. \tag{9}
\]

Denoting by \( \Omega U(n) \) the space of loops starting at \( I \), the homomorphisms are induced by

\[
\Omega U(n) \to F_\iota,
\]

\[
\alpha \mapsto ((s, t) \mapsto \alpha(\varphi(s)))
\]

and \( ev_0: F_\iota \to LO(n) \).

Moreover, \( \pi_0(\Omega) \) is generated by

\[
(s, t) \mapsto \begin{bmatrix}
-e^{\pi i \varphi(s)} & 0 \\
0 & I
\end{bmatrix}
\]

and by an \( U \) extending

\[
U(0, t) = \begin{bmatrix}
\cos(2\pi t) & -\sin(2\pi t) & 0 \\
\sin(2\pi t) & \cos(2\pi t) & 0 \\
0 & 0 & I
\end{bmatrix}
\].

**Proof.** The long exact sequence in homotopy coming from the fibration sequence associated to \( \iota \) ends in

\[
\cdots \to \pi_1(LO(n)) \xrightarrow{\iota_*} \pi_1(LU(n)) \xrightarrow{\partial} \pi_0(F_\iota) \xrightarrow{(ev_0)} \pi_0(LO(n)) \xrightarrow{\iota_*} \pi_0(LU(n)).
\]

The fact that from here we can get (9) follows from the following observations:
- \( \text{ev}_0 \) is surjective on \( \pi_0 \) since \( \pi_1(O(n)) \to \pi_1(U(n)) \) is trivial, due to the factorization \( SO(n) \hookrightarrow SU(n) \hookrightarrow U(n) \) with \( SU(n) \) simply-connected, and since \( U(n) \) is path-connected;

- \( \partial \) is injective since \( \pi_1(LO(n)) \to \pi_1(LU(n)) \) is trivial, due to the factorization \( SO(n) \hookrightarrow SU(n) \hookrightarrow U(n) \) with \( SU(n) \) simply-connected, and since \( U(n) \) is path-connected;

- \( \pi_1(LU(n)) \cong \pi_1(U(n)) \) since \( \pi_2(U(n)) = 0 \).

With regards to the short exact sequence we need to know what \( \partial \) is. It is induced by the inclusion \( \Omega LU(n) \to F_i \): we can see \( U \in F_i \) with \( U(0,t) = I \) for \( t \in S^1 \) as a closed loop in \( LU(n) \), starting at the free loop \( s \mapsto U(s,0) \). Since \( \pi_2(U(n)) = 0 \) we have the following commutative diagram,

\[
\begin{array}{ccc}
\pi_0(\Omega LU(n)) & \to & \pi_0(F_i) \\
\downarrow & & \downarrow \\
\pi_0(LU(n)) & \cong & \pi_0(F_i)
\end{array}
\]

the dashed arrow being the morphism in (9).

From the short exact sequence it follows that \( \pi_0(G) \) is generated by the image of the generator of \( \pi_1(U(n)) \cong \mathbb{Z} \) and lifts of the generators of \( \pi_0(O(n)) \) and \( \pi_1(SO(n)) \); the latter are (10) and \( U \). It is thus enough to notice that the square of (10) is the image of a generator of \( \pi_1(U(n)) \). \( \square \)

Remark 4.10. If \( n = 1 \) the proof still holds with (10) as the sole generator.

From the above lemma it is easy to see that the short exact sequence (9) is simply one of the following:

- If \( n = 1 \):
  \[ 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0; \]

- If \( n = 2 \):
  \[ 0 \to \mathbb{Z} \xrightarrow{(0,2)} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to 0; \]

- If \( n \geq 3 \):
  \[ 0 \to \mathbb{Z} \xrightarrow{(0,2)} \mathbb{Z}/2 \oplus \mathbb{Z} \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0. \]

In particular, the image of \( \pi_1(U(n)) \) is contained in \( \ker \eta_\lambda \), thus the dashed arrow in the following commutative diagram exists:

\[
\begin{array}{ccc}
F_i & \xrightarrow{\eta_\lambda} & \{\pm 1\} \\
\downarrow & & \downarrow \\
LO(n) & \xrightarrow{\text{ev}_0} & \{\pm 1\}
\end{array}
\]

We also denote this morphism by \( \eta_\lambda \), as well as the induced map \( \pi_0(LO(n)) \to \{\pm 1\} \).
Definition 4.11. We define the system of local coefficients $\eta$ as follows: given $x, y \in \mathcal{O}(H)$ and $u \in \mathcal{M}(x, y)$, take $\vartheta_u$ to be a vertical-preserving trivialization of $u^*TT^*Q$ extending $\psi_y$; we define

$$\eta(u) = \eta_{\lambda} \left( \psi_x \circ (\vartheta_u)_{\infty}^{-1} \right).$$

In the circumstances of Lemma 4.8, we have

$$\eta_{\lambda}(U) = \eta_{\lambda}(U(0)) = \eta_{\lambda} \left( (\psi_u \# \rho \psi_v)^0 \circ (\vartheta_u \# \vartheta_v)_{0}^{-1} \right) = \eta(u)$$

since the loops

$$(\psi_u \# \rho \psi_v)^0 \circ (\vartheta_u \# \vartheta_v)_{0}^{-1}, \quad \psi_x \circ (\vartheta_u)_{\infty}^{-1}$$

are homotopic in $O(n)$. As such, the sign in Lemma 4.8 can be written as $\varepsilon^+(u)v([v])\eta(u)$, and the proof of Viterbo’s theorem with coefficients in $\eta$ is immediate.

Theorem 4.12 (Viterbo). Let $Q$ be a closed manifold and let $\eta$ be the above local system of coefficients. Let $H \in C^\infty(S^1 \times T^*Q)$ be a 1-periodic Hamiltonian with all 1-periodic orbits non-degenerate, let $L \in C^\infty(S^1 \times TQ)$ be a 1-periodic Lagrangian such that $H$ is the Legendre transform of $L$, and let $J$ be an almost complex structure compatible with $\omega$.

Suppose that the conditions (L1), (L2), (H1), (H2) hold, that $J$ is uniformly continuous w.r.t. to the metric induced by $J_0$, that $G$ is a metric on $W^{1,2}(S^1; Q)$ such that $(\mathcal{E}_L, G)$ satisfy the Morse-Smale condition up to order 1, and that $J$ is such that all operators $D_u$ and $D_v^+$ with $u \in \mathcal{M}(x, y)$, $v \in \mathcal{M}^+(q, x)$ for some $x, y \in \mathcal{O}(H)$ and $q \in \text{Crit } \mathcal{E}_L$ are surjective.

Then the map

$$V : (SC(T^*Q, \omega, H, J, \partial_{H,I,J}), \partial_{H,I,J}) \rightarrow (MC(W^{1,2}(S^1; Q), \mathcal{E}_L), \partial_{\mathcal{E}_L,G})$$

is a chain isomorphism. In particular, $V$ induces an isomorphism on the level of homology:

$$SH_\bullet(T^*Q, \omega; \eta) \cong H_\bullet(LQ).$$

4.5 Considerations on the local system of coefficients $\eta$

There is an additional symplification of $\eta_\lambda$ which we have not mentioned: it is straightforward to see that $\eta_{\lambda}(U_1, U_2) = \eta_{\lambda}(U_1)\eta_{\lambda}(U_2)$. As such, the computation of $\eta_\lambda$ rests on two essential computations.

Lemma 4.13. Let $n = 1$. Then $(s, t) \mapsto -e^{\pi i \varphi(s)}$ is orientation-reversing for $\lambda(t) = R$ and orientation-preserving for $\lambda(t) = e^{\pi i t}R$.

Lemma 4.14. Let $n = 2$. Then $U : \mathbb{R}^+ \times S^1 \rightarrow U(2)$ satisfying $U(+\infty, t) = I$ and

$$U(0, t) = \begin{bmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{bmatrix}$$

for $t \in S^1$ is orientation-reversing for both $\lambda$. 

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The first computation is immediate for $\lambda(t) = e^{\pi i t} \mathbb{R}$ since we can choose bijective representatives $L^+_S$, for $\lambda(t) = \mathbb{R}$ it is analogous to [Abbondandolo and Schwarz, 2014, Lemma 4.1]. The second computation in the case $\lambda(t) = \mathbb{R}^2$ is [Abbondandolo and Schwarz, 2014, Lemma 4.4], and in the case $\lambda(t) = e^{\pi i t} \mathbb{R}^2$ should be adaptable. In any case the results are correct a posteriori by comparing with the system of local coefficients in [Abouzaid, 2015, Equation (31.2)]; see also Proposition 4.2.8 in the same text.

We summarize the conclusions as follows.

**Theorem 4.15.** A map $U: \mathbb{R}^T \times S^1 \to U(n)$ such that $U(+, t) = I$ and $U(0, t) \in O(n)$ is orientation-preserving for $\lambda(t) = e^{\pi i t} \mathbb{R}^n$ if and only if $U(0) \in \mathcal{LO}(n)$ lifts to $\mathcal{LPin}^\pm(n)$, and orientation-preserving for $\lambda(t) = \mathbb{R}^n$ if and only if $U(0) \in LSO(n)$ lifts to $\mathcal{LSpin}(n)$ or $U(0) \in LSO^-(n)$ does not lift to $\mathcal{LPin}^\pm(n)$.

We can thus re-define $\eta$ as follows. Given $x, y \in \mathcal{O}(H)$ and $u \in \mathcal{P}^\infty(x, y)$, take $\vartheta_u$ to be a vertical-preserving trivialization of $u^*T T^* Q$ extending $\psi_y$; denoting $U = (\vartheta_u)|_{-\infty} \circ \psi_1^\infty \in \mathcal{LO}(n)$, we define $\eta(u)$ to be 1 if $U$ satisfies either of the conditions in Theorem 4.15.

### 4.6 $\eta$ for orientable manifolds

If $Q$ is orientable, then all $x \in \mathcal{O}(H)$ are orientable and we can additionally fix an orientation of $Q$ and require that the trivializations $\psi_x$ preserve this orientation on the vertical spaces; i.e. require that $\psi_x|_{T_x T^* Q} : T_x T^* Q \to S^1 \times \mathbb{R}^n$ is orientation-preserving. Then all mentions of $O(n)$ can be replaced by $SO(n)$, and the lifts in Theorem 4.15 are simply lifts to $\Spin(n)$. Under these assumptions, we have the following result.

**Lemma 4.16.** Let $x, y \in \mathcal{O}(H)$ and $u, v \in \mathcal{P}^\infty(x, y)$. Then $\eta(u) = \eta(v)$ if and only if the Riemannian vector bundle $E := (u \# v)^* T_x T^* Q$ over $T^2$ has a Spin-structure, where $u \# v : T^2 \to T^* Q$ is the torus obtained from gluing $u$ and $v$. In fact, identifying $\{ \pm 1 \}$ with $\mathbb{Z}/2$ we have $w_2(E) = \eta(u) \eta(v) \in H^2(T^2; \mathbb{Z}/2)$.

**Proof.** There are vertical-preserving unitary trivializations $\vartheta_u$ and $\vartheta_v$ of $u^*T T^* Q$ and $v^*T T^* Q$, respectively, extending $\psi_y$ but not necessarily $\psi_x$. Restricting these trivializations to the vertical subbundles identifies $u^*T_x T^* Q$ and $v^*T_x T^* Q$ with the trivial Riemannian bundles over $\mathbb{R}^+ \times S^1$, which have trivial Spin-structures. The Riemannian vector bundle $(u \# v)^* T_x T^* Q$ is then defined by the cocycle

\[
(\vartheta_u)|_{-\infty} \circ (\vartheta_v)|_{-\infty}^{-1} = ((\vartheta_u)|_{-\infty} \circ \psi_u^{-1}) \circ (\psi_x \circ (\vartheta_v)|_{-\infty}^{-1})
\]

Hence $E$ has a Spin-structure if and only if this cocycle lifts to $\mathcal{LSpin}(n)$, which is the case if and only if $\eta(u) \eta(v) = 1$; that is, $\eta(u) = \eta(v)$.

Since $w_2(E)$ is the obstruction to the existence of a Spin-structure (e.g. [Kirby and Taylor, 1990]) and $H^2(T^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$, it follows that $w_2(E) = \eta(u) \eta(v)$.

**Proposition 4.17.** The local system of coefficients $\eta$ is trivial if $w_2(u^*T Q) = 0$ for all $u : T^2 \to Q$. If this is the case, then we can choose trivializations such that $\eta \equiv 1$. 

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Proof. The first statement is immediate from Lemma 4.16. For the second statement, for each $\gamma \in [S^1; T^*Q]$ choose $x_\gamma \in \mathcal{O}(H)$ with $[x_\gamma] = \gamma$ and a vertical-preserving trivialization $\psi_{x_\gamma} : x_\gamma^* T^*Q \to S^1 \times \mathbb{R}^{2n}$. For each $x \in \mathcal{O}(H)$ with $[x] = \gamma$ and $x \neq x_\gamma$, choose any cylinder $u \in \mathcal{P}^\infty(x, x_\gamma)$ and vertical-preserving trivialization $\psi_u : u^* T^*Q \to (\mathbb{R} \times S^1) \times \mathbb{R}^{2n}$ extending $\psi_{x_\gamma}$; set $\psi_x = (\psi_u)^{-\infty}$. It is immediate that $\eta(x, x_\gamma) = \eta(u) = 1$ and thus $\eta(x, y) = \eta(x, x_\gamma) \eta(y, x_\gamma) = 1$ for all $[x], [y] = \gamma$.

References


