Advanced Quantum Field Theory
(Version of September 2016)

Jorge Crispim Romão

Physics Department
2016
## Contents

1 Free Field Quantization ................................................. 13
   1.1 General formalism .................................................. 13
      1.1.1 Canonical quantization for particles ...................... 13
      1.1.2 Canonical quantization for fields ......................... 17
      1.1.3 Symmetries and conservation laws ......................... 19
   1.2 Quantization of scalar fields ..................................... 23
      1.2.1 Real scalar field ............................................ 23
      1.2.2 Microscopic causality ...................................... 27
      1.2.3 Vacuum fluctuations ........................................ 28
      1.2.4 Charged scalar field ...................................... 28
      1.2.5 Time ordered product and the Feynman propagator ........ 30
   1.3 Second quantization of the Dirac field .......................... 32
      1.3.1 Canonical formalism for the Dirac field ................. 32
      1.3.2 Microscopic causality ...................................... 35
      1.3.3 Feynman propagator ........................................ 36
   1.4 Electromagnetic field quantization ................................ 37
      1.4.1 Introduction .................................................. 37
      1.4.2 Undefined metric formalism ................................. 39
      1.4.3 Feynman propagator ........................................ 46
   1.5 Discrete Symmetries .................................................. 47
      1.5.1 Parity ......................................................... 47
      1.5.2 Charge conjugation ......................................... 49
      1.5.3 Time reversal ............................................... 50
      1.5.4 The $TCP$ theorem ......................................... 51
   Problems for Chapter 1 ............................................ 52

2 Physical States. S Matrix. LSZ Reduction. .......................... 55
   2.1 Physical states .................................................... 55
   2.2 In states .......................................................... 56
## CONTENTS

2.3 Spectral representation for scalar fields ........................................ 58  
2.4 *Out* states ..................................................................................... 60  
2.5 *S* matrix ......................................................................................... 61  
2.6 Reduction formula for scalar fields ................................................... 62  
2.7 Reduction formula for fermions .......................................................... 65  
2.7.1 States *in* and *out* ..................................................................... 65  
2.7.2 Spectral representation fermions .................................................... 66  
2.7.3 Reduction formula fermions ............................................................ 68  
2.8 Reduction formula for photons .......................................................... 70  
2.9 Cross sections ..................................................................................... 72  
Problems for Chapter 2 .......................................................................... 75

3 Covariant Perturbation Theory ......................................................... 77  
3.1 *U* matrix ......................................................................................... 77  
3.2 Perturbative expansion of Green functions ......................................... 80  
3.3 Wick’s theorem .................................................................................. 82  
3.4 Vacuum–Vacuum amplitudes ............................................................... 87  
3.5 Feynman rules for $\lambda \varphi^4$ ............................................................. 88  
3.6 Feynman rules for QED ....................................................................... 93  
3.6.1 Compton scattering ...................................................................... 94  
3.6.2 Electron–positron elastic scattering (Bhabha scattering) .......... 98  
3.6.3 Fermion Loops .............................................................................. 100  
3.6.4 Feynman rules for QED ................................................................. 102  
3.7 General formalism for getting the Feynman rules ............................. 103  
3.7.1 Example: scalar electrodynamics ............................................... 105  
Problems for Chapter 3 .......................................................................... 108

4 Radiative Corrections ......................................................................... 111  
4.1 QED Renormalization at one-loop ................................................... 111  
4.1.1 Vacuum Polarization .................................................................. 112  
4.1.2 Self-energy of the electron ............................................................ 121  
4.1.3 The Vertex .................................................................................... 125  
4.2 Ward-Takahashi identities in QED ..................................................... 130  
4.2.1 Transversality of the photon propagator $n = 0, p = 1$ ............ 131  
4.2.2 Identity for the vertex $n = 1, p = 0$ ............................................ 133  
4.3 Counterterms and power counting .................................................... 136  
4.4 Finite contributions from RC to physical processes ........................ 140  
4.4.1 Anomalous electron magnetic moment ...................................... 140
CONTENTS

4.4.2 Cancellation of IR divergences in Coulomb scattering . . . . . . . . 142

5 Functional Methods 147

5.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 147
5.2 Generating functional for Green’s functions . . . . . . . . . . . . . . . . . 148
  5.2.1 Green’s functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 148
  5.2.2 Connected Green’s functions . . . . . . . . . . . . . . . . . . . . . . 148
  5.2.3 Truncated Green’s functions . . . . . . . . . . . . . . . . . . . . . . . 150
  5.2.4 Irreducible diagrams . . . . . . . . . . . . . . . . . . . . . . . . . . . 150
5.3 Generating functionals for Green’s functions . . . . . . . . . . . . . . . . . 151
5.4 Feynman rules . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 157
5.5 Path integral for generating functionals . . . . . . . . . . . . . . . . . . . 159
  5.5.1 Quantum mechanics of \( n \) degrees of freedom . . . . . . . . . . . 159
  5.5.2 Field theory . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 161
  5.5.3 Applications . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 162
  5.5.4 Example: perturbation theory for \( \lambda \phi^4 \) . . . . . . . . . . . . 163
  5.5.5 Symmetry factors . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 168
  5.5.6 A comment on the normal ordering . . . . . . . . . . . . . . . . . . . 169
  5.5.7 Generating functionals for fermions . . . . . . . . . . . . . . . . . . . 172
5.6 Change of variables in path integrals. Applications . . . . . . . . . . . . . 172
  5.6.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 172
  5.6.2 Dyson-Schwinger equations . . . . . . . . . . . . . . . . . . . . . . . . 173
  5.6.3 Ward identities . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 177
Problems for Chapter 5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 178

6 Non-Abelian Gauge Theories 183

6.1 Classical theory . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 183
  6.1.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 183
  6.1.2 Covariant derivative . . . . . . . . . . . . . . . . . . . . . . . . . . . . 184
  6.1.3 Tensor \( F_{\mu\nu} \) . . . . . . . . . . . . . . . . . . . . . . . . . . . 185
  6.1.4 Choice of gauge . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 186
  6.1.5 The action and the equations of motion . . . . . . . . . . . . . . . . 186
  6.1.6 Energy–momentum tensor . . . . . . . . . . . . . . . . . . . . . . . . . 187
  6.1.7 Hamiltonian formalism . . . . . . . . . . . . . . . . . . . . . . . . . . 187
6.2 Quantization . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 189
  6.2.1 Systems with \( n \) degrees of freedom . . . . . . . . . . . . . . . . . 189
  6.2.2 QED as a simple example . . . . . . . . . . . . . . . . . . . . . . . . . 192
  6.2.3 Non abelian gauge theories. Non covariant gauges . . . . . . . . . . 194
6.2.4 Non abelian gauge theories in covariant gauges .................................. 196
6.2.5 Gauge invariance of the $S$ matrix .................................................. 201
6.2.6 Faddeev-Popov ghosts ................................................................. 205
6.2.7 Feynman rules in the Lorenz gauge ................................................. 206
6.2.8 Feynman rules for the interaction with matter ................................. 209

6.3 Ward Identities .................................................................................. 211
6.3.1 BRS transformation ................................................................. 211
6.3.2 Ward-Takahashi-Slavnov-Taylor identities .................................... 215
6.3.3 Example: Transversality of vacuum polarization ............................ 219
6.3.4 Gauge invariance of the $S$ matrix ................................................. 223

6.4 Ward Takahashi Identities in QED ...................................................... 225
6.4.1 Ward-Takahashi identities for the functional $Z[J]$ .......................... 225
6.4.2 Ward-Takahashi identities for the functionals $W$ and $\Gamma$ ............ 226
6.4.3 Example: Ward identity for the QED vertex .................................. 227
6.4.4 Ghosts in QED ............................................................................. 227

6.5 Unitarity and Ward Identities .............................................................. 230
6.5.1 Optical Theorem .......................................................................... 230
6.5.2 Cutkosky rules ............................................................................ 231
6.5.3 Example of Unitarity: scalars and fermions .................................. 234
6.5.4 Unitarity and gauge fields ............................................................ 237

Problems for Chapter 6 .......................................................................... 241

7 Renormalization Group ........................................................................... 247
7.1 Callan-Symanzik equation ................................................................. 247
7.1.1 Renormalization scheme with momentum subtraction .................... 247
7.1.2 Renormalization group ............................................................... 249
7.1.3 Callan-Symanzik equation ........................................................... 250
7.1.4 Weinberg’s theorem and the solution of the RG equations ............ 253
7.1.5 Asymptotic solution of the RG equations ..................................... 253

7.2 Minimal subtraction (MS) scheme ..................................................... 256
7.2.1 Renormalization group equations for MS ...................................... 256
7.2.2 Minimal subtraction scheme ........................................................ 257
7.2.3 Physical parameters .................................................................... 259
7.2.4 Renomalization group functions in minimal subtraction ................. 260
7.2.5 $\beta$ and $\gamma$ properties ............................................................ 264
7.2.6 Gauge independence of $\beta$ and $\gamma_m$ in MS ............................... 266

7.3 Effective gauge couplings ................................................................. 270
7.3.1 Fixed points ................................................................................. 270
7.3.2 \( \beta \) function for theories with scalars, fermions and gauge fields . . . . 271
7.3.3 The vacuum of a NAGT as a paramagnetic medium (\( \mu > 1 \)) . . . . 277
7.4 Renormalization group applications . . . . . . . . . . . . . . . . . . . . . . 278
  7.4.1 Scale \( M_X \) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 278
  7.4.2 Scale \( M_Z \) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 280
Problems for Chapter 7 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 291

A Path Integral in Quantum Mechanics 293
A.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 293
A.2 Configuration space . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 293
  A.2.1 Matrix elements of operators . . . . . . . . . . . . . . . . . . . . . . . . . . 294
  A.2.2 Time ordered product of operators . . . . . . . . . . . . . . . . . . . . . . 295
  A.2.3 Exact results I: harmonic oscillator . . . . . . . . . . . . . . . . . . . . . 295
  A.2.4 Exact results II: external force . . . . . . . . . . . . . . . . . . . . . . . . 295
  A.2.5 Perturbation theory . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 296
A.3 Phase space formulation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 296
A.4 Bargmann-Fock space (coherent states) . . . . . . . . . . . . . . . . . . . . . 297
  A.4.1 Normal form for an operator . . . . . . . . . . . . . . . . . . . . . . . . . . 298
  A.4.2 Evolution operator . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 299
  A.4.3 Exact results I: harmonic oscillator . . . . . . . . . . . . . . . . . . . . . 300
  A.4.4 Exact results II: external force . . . . . . . . . . . . . . . . . . . . . . . . 303
A.5 Fermion systems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 304
  A.5.1 Derivatives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 304
  A.5.2 Dot product . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 304
  A.5.3 Integration . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 305
  A.5.4 Representation of operators . . . . . . . . . . . . . . . . . . . . . . . . . . . 306
  A.5.5 Normal form for operators . . . . . . . . . . . . . . . . . . . . . . . . . . . 307
Problems for Appendix A . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 309

B Path Integral in Quantum Field Theory 311
B.1 Path integral quantization . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 311
B.2 Path integral for generating functionals . . . . . . . . . . . . . . . . . . . . . 314
B.3 Fermion systems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 320
Problems Appendix B . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 322

C Useful techniques for renormalization 325
C.1 \( \mu \) parameter . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 325
C.2 Feynman parametrization . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 326
C.3 Wick Rotation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 328
C.4 Scalar integrals in dimensional regularization 329
C.5 Tensor integrals in dimensional regularization 331
C.6 \Gamma function and useful relations 331
C.7 Explicit formulæ for the 1-loop integrals 333
C.7.1 Tadpole integrals 333
C.7.2 Self-Energy integrals 333
C.7.3 Triangle integrals 334
C.7.4 Box integrals 334
C.8 Divergent part of 1-loop integrals 335
C.8.1 Tadpole integrals 335
C.8.2 Self-Energy integrals 335
C.8.3 Triangle integrals 336
C.8.4 Box integrals 336
C.9 Passarino-Veltman Integrals 336
C.9.1 The general definition 336
C.9.2 The divergences 339
C.9.3 Useful results for PV integrals 339
C.10 Examples of 1-loop calculations with PV functions 350
C.10.1 Vacuum Polarization in QED 350
C.10.2 Electron Self-Energy in QED 352
C.10.3 QED Vertex 355
C.11 Modern techniques in a real problem: \( \mu \rightarrow e\gamma \) 360
C.11.1 Neutral scalar charged fermion loop 360
C.11.2 Charged scalar neutral fermion loop 371

D Feynman Rules for the Standard Model 377
D.1 Introduction 377
D.2 The Standard Model 377
D.2.1 Gauge Group \( SU(3)_c \) 377
D.2.2 Gauge Group \( SU(2)_L \) 378
D.2.3 Gauge Group \( U(1)_Y \) 378
D.2.4 The Gauge Field Lagrangian 379
D.2.5 The Fermion Fields Lagrangian 380
D.2.6 The Higgs Lagrangian 380
D.2.7 The Yukawa Lagrangian 381
D.2.8 The Gauge Fixing 381
D.2.9 The Ghost Lagrangian 381
D.2.10 The Complete SM Lagrangian 382
D.3 The Feynman Rules for QCD ............................................... 383
  D.3.1 Propagators ...................................................... 383
  D.3.2 Triple Gauge Interactions ...................................... 383
  D.3.3 Quartic Gauge Interactions .................................... 383
  D.3.4 Fermion Gauge Interactions ................................... 384
  D.3.5 Ghost Interactions ............................................. 384
D.4 The Feynman Rules for the Electroweak Theory ....................... 384
  D.4.1 Propagators ...................................................... 384
  D.4.2 Triple Gauge Interactions ...................................... 385
  D.4.3 Quartic Gauge Interactions .................................... 385
  D.4.4 Charged Current Interaction .................................. 386
  D.4.5 Neutral Current Interaction .................................. 386
  D.4.6 Fermion-Higgs and Fermion-Goldstone Interactions ...... 386
  D.4.7 Triple Higgs-Gauge and Goldstone-Gauge Interactions .. 387
  D.4.8 Quartic Higgs-Gauge and Goldstone-Gauge Interactions .. 388
  D.4.9 Triple Higgs and Goldstone Interactions ................... 390
  D.4.10 Quartic Higgs and Goldstone Interactions ................ 391
  D.4.11 Ghost Propagators ............................................ 392
  D.4.12 Ghost Gauge Interactions ................................... 392
  D.4.13 Ghost Higgs and Ghost Goldstone Interactions .......... 393
Preface

This is a text for an *Advanced Quantum Field Theory* course that I have been teaching for many years at Instituto Superior Técnico. This course was first written in Portuguese. Then, at a latter stage, I added some text in one-loop techniques in English. Then, I realized that this text could be more useful if it was all in English. As the process of revising the whole text shall take a long time, I decided to make available a *mixed language* text. At the moment, around 60% are in English, the rest in Portuguese. My goal is to change this gradually.

This last semester an effort was made to correct known misprints in the equations before the full translation gets done. However, I am certain that many more still remain. If you find errors or misprints, please send me an email.

IST, May 2012

Jorge C. Romão

jorge.romao@ist.utl.pt
Chapter 1

Free Field Quantization

1.1 General formalism

1.1.1 Canonical quantization for particles

Before we study the canonical quantization of systems with an infinite number of degrees of freedom, as it is the case with fields, we will review briefly the quantization of systems with a finite number of degrees of freedom, like a system of particles.

Let us start with a system that consists of one particle with just one degree of freedom, like a particle moving in one space dimension. The classical equations of motion are obtained from the action,

$$ S = \int_{t_1}^{t_2} dt L(q, \dot{q}) . $$

(1.1)

The condition for the minimization of the action, $\delta S = 0$, gives the Euler-Lagrange equations,

$$ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 $$

(1.2)

which are the equations of motion.

Before proceeding to the quantization, it is convenient to change to the Hamiltonian formulation. We start by defining the conjugate momentum $p$, to the coordinate $q$, by

$$ p = \frac{\partial L}{\partial \dot{q}} $$

(1.3)

Then we introduce the Hamiltonian using the Legendre transform

$$ H(p, q) = p \dot{q} - L(q, \dot{q}) $$

(1.4)

In terms of $H$ the equations of motion are,
CHAPTER 1. FREE FIELD QUANTIZATION

\[ \{H, q\}_{PB} = \frac{\partial H}{\partial p} = \dot{q} \]  
(1.5)

\[ \{H, p\}_{PB} = -\frac{\partial H}{\partial q} = \dot{p} \]  
(1.6)

where the Poisson Bracket (PB) is defined by

\[ \{f(p, q), g(p, q)\}_{PB} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} \]  
(1.7)

obviously satisfying

\[ \{p, q\}_{PB} = 1 . \]  
(1.8)

The quantization is done by promoting \( p \) and \( q \) to hermitian operators that instead of Eq. (1.8) will satisfy the commutation relation (\( \hbar = 1 \)),

\[ [p, q] = -i \]  
(1.9)

which is trivially satisfied in the coordinate representation where \( p = -i \frac{\partial}{\partial q} \). The dynamics is given by the Schrödinger equation

\[ H(p, q) \left| \Psi_S(t) \right> = i \frac{\partial}{\partial t} \left| \Psi_S(t) \right> \]  
(1.10)

If we know the state of the system in \( t = 0 \), \( \left| \Psi_S(0) \right> \), then Eq. (1.10) completely determines the state \( \left| \Psi_s(t) \right> \) and therefore the value of any physical observable. This description, where the states are time dependent and the operators, on the contrary, do not depend on time, is known as the Schrödinger representation. There exits and alternative description, where the time dependence goes to the operators and the states are time independent. This is called the Heisenberg representation. To define this representation, we formally integrate Eq. (1.10) to obtain

\[ \left| \Psi_S(t) \right> = e^{-iHt} \left| \Psi_S(0) \right> = e^{-iHt} \left| \Psi_H \right> . \]  
(1.11)

The state in the Heisenberg representation, \( \left| \Psi_H \right> \), is defined as the state in the Schrödinger representation for \( t = 0 \). The unitary operator \( e^{-iHt} \) allows us to go from one representation to the other. If we define the operators in the Heisenberg representation as

\[ O_H(t) = e^{iHt} O_S e^{-iHt} \]  
(1.12)

then the matrix elements are representation independent. In fact,

\[ \left< \Psi_S(t) | O_S | \Psi_S(t) \right> = \left< \Psi_S(0) | e^{iHt} O_S e^{-iHt} | \Psi_S(0) \right> \]  
(1.13)
1.1. GENERAL FORMALISM

\[ \langle \Psi_H | O_H(t) | \Psi_H \rangle \]  

(1.14)

The time evolution of the operator \( O_H(t) \) is then given by the equation

\[ \frac{dO_H(t)}{dt} = i[H, O_H(t)] + \frac{\partial O_H}{\partial t} \]  

(1.15)

which can easily be obtained from Eq. (1.12). The last term in Eq. (1.15) is only present if \( O_S \) explicitly depends on time.

In the non-relativistic theory the difference between the two representations is very small if we work with energy eigenfunctions. If \( \psi_n(q,t) = e^{-i\omega_n t}u_n(q) \) is a Schrödinger wave function, then the Heisenberg wave function is simply \( u_n(q) \). For the relativistic theory, the Heisenberg representation is more convenient, because it is easier to describe the time evolution of operators than that of states. Also, Lorentz covariance is more easily handled in the Heisenberg representation, because time and spatial coordinates are together in the field operators.

In the Heisenberg representation the fundamental commutation relation is now

\[ [p(t), q(t)] = -i \]  

(1.16)

The dynamics is now given by

\[ \frac{dp(t)}{dt} = i[H, p(t)] ; \frac{dq(t)}{dt} = i[H, q(t)] \]  

(1.17)

Notice that in this representation the fundamental equations are similar to the classical equations with the substitution,

\[ \{ , \}_{PB} \implies i[ , ] \]  

(1.18)

In the case of a system with \( n \) degrees of freedom Eqs. (1.16) and (1.17) are generalized to

\[ [p_i(t), q_j(t)] = -i\delta_{ij} \]  

(1.19)

\[ [p_i(t), p_j(t)] = 0 \]  

(1.20)

\[ [q_i(t), q_j(t)] = 0 \]  

(1.21)

and

\[ \dot{p}_i(t) = i[H, p_i(t)] ; \dot{q}_i(t) = i[H, q_i(t)] \]  

(1.22)

Because it is an important example let us look at the harmonic oscillator. The Hamiltonian is

\[ H = \frac{1}{2}(p^2 + \omega_0^2 q^2) \]  

(1.23)

The equations of motion are

\[ \dot{p} = i[H, p] = -\omega_0^2 q \]  

(1.24)
\[ \dot{q} = i[H, q] = p \implies \ddot{q} + \omega_0^2 q = 0. \] (1.25)

It is convenient to introduce the operators
\[ a = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q + ip) ; \quad a^\dagger = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q - ip) \] (1.26)

The equations of motion for \( a \) and \( a^\dagger \) are very simple:
\[ \dot{a}(t) = -i\omega_0 a(t) \quad \text{and} \quad \dot{a}^\dagger(t) = i\omega_0 a^\dagger(t). \] (1.27)

They have the solution
\[ a(t) = a_0 e^{-i\omega_0 t} \quad \text{and} \quad a^\dagger(t) = a^\dagger_0 e^{i\omega_0 t} \] (1.28)

and obey the commutation relations
\[ [a, a^\dagger] = [a_0, a^\dagger_0] = 1 \] (1.29)
\[ [a, a] = [a_0, a_0] = 0 \] (1.30)
\[ [a^\dagger, a^\dagger] = [a^\dagger_0, a^\dagger_0] = 0 \] (1.31)

In terms of \( a, a^\dagger \) the Hamiltonian reads
\[ H = \frac{1}{2} \omega_0 (a^\dagger a + aa^\dagger) = \frac{1}{2} \omega_0 (a^\dagger_0 a_0 + a_0 a^\dagger_0) \] (1.32)
\[ = \omega_0 a^\dagger_0 a_0 + \frac{1}{2} \omega_0 \] (1.33)

where we have used
\[ [H, a_0] = -\omega_0 a_0, \quad [H, a^\dagger_0] = \omega_0 a^\dagger_0 \] (1.34)

We see that \( a_0 \) decreases the energy of a state by the quantity \( \omega_0 \) while \( a^\dagger_0 \) increases the energy by the same amount. As the Hamiltonian is a sum of squares the eigenvalues must be positive. Then it should exist a ground state (state with the lowest energy), \( |0\rangle \), defined by the condition
\[ a_0 |0\rangle = 0 \] (1.35)

The state \( |n\rangle \) is obtained by the application of \( (a^\dagger_0)^n \). If we define
\[ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger_0)^n |0\rangle \] (1.36)
then
\[ \langle m|n\rangle = \delta_{mn} \] (1.37)
and
\[ H |n\rangle = (n + \frac{1}{2}) \omega_0 |n\rangle \] (1.38)

We will see that, in the quantum field theory, the equivalent of \( a_0 \) and \( a^\dagger_0 \) are the creation and annihilation operators.
1.1. GENERAL FORMALISM

1.1.2 Canonical quantization for fields

Let us move now to field theory, that is, systems with an infinite number of degrees of freedom. To specify the state of the system, we must give for all space-time points one number (or more if we are not dealing with a scalar field). The equivalent of the coordinates \( q_i(t) \) and velocities, \( \dot{q}_i \), are here the fields \( \varphi(\vec{x}, t) \) and their derivatives, \( \partial^\mu \varphi(\vec{x}, t) \). The action is now

\[
S = \int d^4x L(\varphi, \partial^\mu \varphi) \tag{1.39}
\]

where the Lagrangian density \( L \), is a functional of the fields \( \varphi \) and their derivatives \( \partial^\mu \varphi \).

Let us consider closed systems for which \( L \) does not depend explicitly on the coordinates \( x^\mu \) (energy and linear momentum are therefore conserved). For simplicity let us consider systems described by \( n \) scalar fields \( \varphi_r(x), r = 1, 2, \cdots n \). The stationarity of the action, \( \delta S = 0 \), implies the equations of motion, the so-called Euler-Lagrange equations,

\[
\partial^\mu \frac{\partial L}{\partial (\partial^\mu \varphi_r)} - \frac{\partial L}{\partial \varphi_r} = 0 \quad r = 1, \cdots n \tag{1.40}
\]

For the case of real scalar fields with no interactions that we are considering, we can easily see that the Lagrangian density should be,

\[
L = \sum_{r=1}^{n} \left[ \frac{1}{2} \partial^\mu \varphi_r \partial_\mu \varphi_r - \frac{1}{2} m^2 \varphi_r \varphi_r \right] \tag{1.41}
\]

in order to obtain the Klein-Gordon equations as the equations of motion,

\[
(\Box + m^2) \varphi_r = 0 \quad ; \quad r = 1, \cdots n \tag{1.42}
\]

To define the canonical quantization rules we have to change to the Hamiltonian formalism, in particular we need to define the conjugate momentum \( \pi(x) \) for the field \( \varphi(x) \). To make an analogy with systems with \( n \) degrees of freedom, we divide the 3-dimensional space in cells with elementary volume \( \Delta V_i \). Then we introduce the coordinate \( \varphi_i(t) \) as the average of \( \varphi(\vec{x}, t) \) in the volume element \( \Delta V_i \), that is,

\[
\varphi_i(t) \equiv \frac{1}{\Delta V_i} \int_{(\Delta V_i)} d^3x \varphi(\vec{x}, t) \tag{1.43}
\]

and also

\[
\dot{\varphi}_i(t) \equiv \frac{1}{\Delta V_i} \int_{(\Delta V_i)} d^3x \dot{\varphi}(\vec{x}, t) . \tag{1.44}
\]

Then

\[
L = \int d^3x L \rightarrow \sum_i \Delta V_i \pi_i . \tag{1.45}
\]

Therefore the canonical momentum is now

\[
p_i(t) = \frac{\partial L}{\partial \dot{\varphi}_i(t)} = \Delta V_i \frac{\partial \pi_i}{\partial \dot{\varphi}_i(t)} = \Delta V_i \pi_i(t) \tag{1.46}
\]
and the Hamiltonian

\[ H = \sum_i p_i \dot{\varphi}_i - L = \sum_i \Delta V_i (\pi_i \dot{\varphi}_i - \mathbf{Z}_i) \]  

(1.47)

Going now into the limit of the continuum, we define the conjugate momentum,

\[ \pi(\vec{x},t) = \frac{\partial \mathcal{L}(\varphi, \dot{\varphi})}{\partial \dot{\varphi}(\vec{x},t)} \]  

(1.48)

in such a way that its average value in \( \Delta V_i \) is \( \pi_i(t) \) defined in Eq. (1.46). Eq. (1.47) suggests the introduction of an Hamiltonian density such that

\[ H = \int d^3 x \mathcal{H} \]  

(1.49)

\[ \mathcal{H} = \pi \dot{\varphi} - \mathcal{L} . \]  

(1.50)

To define the rules of the canonical quantization we start with the coordinates \( \varphi_i(t) \) and conjugate momenta \( p_i(t) \). We have

\[ [p_i(t), \varphi_j(t)] = -i \delta_{ij} \]  

\[ [\varphi_i(t), \varphi_j(t)] = 0 \]  

\[ [p_i(t), p_j(t)] = 0 \]  

(1.51)

In terms of momentum \( \pi_i(t) \) we have

\[ [\pi_i(t), \varphi_j(t)] = -i \pi_i(t) \delta_{ij} \]  

(1.52)

Going into the continuum limit, \( \Delta V_i \to 0 \), we obtain

\[ [\varphi(\vec{x}, t), \varphi(\vec{x}', t)] = 0 \]  

(1.53)

\[ [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \]  

(1.54)

\[ [\pi(\vec{x}, t), \varphi(\vec{x}', t)] = -i \delta(\vec{x} - \vec{x}') \]  

(1.55)

These relations are the basis of the canonical quantization. For the case of \( n \) scalar fields, the generalization is:

\[ [\varphi_r(\vec{x}, t), \varphi_s(\vec{x}', t)] = 0 \]  

(1.56)

\[ [\pi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = 0 \]  

(1.57)

\[ [\pi_r(\vec{x}, t), \varphi_s(\vec{x}', t)] = -i \delta_{rs} \delta(\vec{x} - \vec{x}') \]  

(1.58)

where

\[ \pi_r(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_r(\vec{x}, t)} \]  

(1.59)

and the Hamiltonian is

\[ H = \int d^3 x \mathcal{H} \]  

(1.60)

with

\[ \mathcal{H} = \sum_{r=1}^n \pi_r \dot{\varphi}_r - \mathcal{L} . \]  

(1.61)
1.1.3 Symmetries and conservation laws

The Lagrangian formalism gives us a powerful method to relate symmetries and conservation laws. At the classical level the fundamental result is the following theorem.

**Noether’s Theorem**

To each continuous symmetry transformation that leaves $L$ and the equations of motion invariant, corresponds one conservation law.

**Proof:**

We will make a generic proof for the most general case, and then consider particular cases. We take a general change of inertial frame, including Lorentz transformations and translations. For infinitesimal transformations we have

$$x'^\mu = x^\mu + \varepsilon^\mu + \omega^\mu_{\nu}x'^\nu \equiv x^\mu + \delta x^\mu, \quad \delta x^\mu = \varepsilon^\mu + \omega^\mu_{\nu}x'^\nu \quad (1.62)$$

where $\varepsilon^\mu$ and $\omega^\mu_{\nu}$ are infinitesimal constant parameters. For the fields, under such a transformation we have two types of variations.

$$\delta \varphi_r(x) \equiv \varphi_r'(x) - \varphi_r(x) \quad (1.63)$$

$$\delta_T \varphi_r(x) \equiv \varphi_r'(x') - \varphi_r(x) \quad (1.64)$$

They are related because (we neglect second order terms)

$$\delta_T \varphi_r(x) = [\varphi_r'(x') - \varphi_r(x')] + [\varphi_r(x') - \varphi_r(x)]$$

$$= \delta \varphi_r(x') + \frac{\partial \varphi_r}{\partial x_\beta} \delta x_\beta = \delta \varphi_r(x) + \frac{\partial \varphi_r}{\partial x_\beta} \delta x_\beta \quad (1.65)$$

Now the invariance of the Lagrangian

$$L(\varphi'_r(x'), \partial_\alpha \varphi'(x')) = L(\varphi_r(x), \partial_\alpha \varphi(x)) \quad (1.66)$$

can be written as

$$0 = L(\varphi'_r(x'), \partial_\alpha \varphi'(x')) - L(\varphi_r(x), \partial_\alpha \varphi(x))$$

$$= \delta L + \frac{\partial L}{\partial x_\beta} \delta x_\beta \quad (1.67)$$

Now we calculate $\delta L$ (using the equations of motion)

$$\delta L = \frac{\partial L}{\partial \varphi_r} \delta \varphi_r + \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \delta (\partial_\alpha \varphi_r)$$

$$= \frac{\partial}{\partial x_\alpha} \left[ \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \delta \varphi_r \right]$$

$$= \frac{\partial}{\partial x_\alpha} \left[ \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \delta_T \varphi_r - \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \frac{\partial L}{\partial x_\beta} \delta x_\beta \right] \quad (1.68)$$
CHAPTER 1. FREE FIELD QUANTIZATION

where we have used Eq. (1.65). Introducing now Eq. (1.68) in Eq. (1.67) we obtain

\[
0 = \frac{\partial}{\partial x_\alpha} \left[ \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \delta T \varphi_r - \left( \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \frac{\partial L}{\partial x^\beta} - L g^{\alpha\beta} \right) \delta x^\beta \right]
\]

\[
= \frac{\partial}{\partial x_\alpha} \left[ \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \delta T \varphi_r - T^{\alpha\beta} \delta x^\beta \right]
\]

\[
= \partial_\alpha J^\alpha
\]  

(1.69)

We have defined the conserved current \( J^\alpha \) and the tensor \( T^{\alpha\beta} \) by

\[
T^{\alpha\beta} = \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \frac{\partial L}{\partial x^\beta} - L g^{\alpha\beta}
\]  

(1.70)

\[
J^\alpha = \frac{\partial L}{\partial (\partial_\alpha \varphi_r)} \delta T \varphi_r - T^{\alpha\beta} \delta x^\beta
\]  

(1.71)

This ends the proof of Noether’s theorem.

Before we apply it to particular cases let us also introduce some useful notation. We define for infinitesimal transformations

\[
\varphi_r'(x') = S_{rs}(a) \varphi_s(x), \quad S_{rs}(\omega) = \delta_{rs} + \frac{1}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta}_{rs} \rightarrow \delta T \varphi_r(x) = \frac{1}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta}_{rs}
\]  

(1.72)

1) Translations

First we consider the case of translations. For this case we have

\[
\delta T \varphi_r = 0, \quad \delta x^\mu = \varepsilon^\mu
\]  

(1.73)

From the above and using the fact that \( \varepsilon^\mu \) is arbitrary and constant we get from Eq. (1.71)

\[
\partial_\mu J^\mu = 0 \quad \rightarrow \quad \partial_\mu T^{\mu\nu} = 0
\]  

(1.74)

where \( T^{\mu\nu} \) is the energy-momentum tensor defined above in Eq. (1.70),

\[
T^{\mu\nu} = -g^{\mu\nu} L + \sum_r \frac{\partial L}{\partial (\partial_\mu \varphi_r)} \partial^\nu \varphi_r
\]  

(1.75)

Using these relations we can define the conserved quantities

\[
P^\mu = \int d^3 x T^{0\mu} \quad \Rightarrow \quad \frac{d P^\mu}{dt} = 0
\]  

(1.76)

Noticing that \( T^{00} = H \), it is easy to realize that \( P^\mu \) should be the 4-momentum vector. Therefore we conclude that invariance for translations leads to the conservation of energy and momentum.
2) Lorentz transformations

Consider the infinitesimal Lorentz transformations
\[ x'^\mu = x^\mu + \omega^\mu_\nu x^\nu, \quad \delta T\varphi_r(x) = \frac{1}{2} \omega_{\alpha\beta} \Sigma_{rs}^{\alpha\beta} \] (1.77)

where we have indicated the total variation of the fields using the conventions of Eq. (1.72). For instance for scalar fields we have \( \delta T\varphi_r(x) = 0 \) and for spinors
\[ \delta_T\psi_r(x) = \frac{1}{8} [\gamma_\mu, \gamma_\nu]_{rs} \omega^\mu\nu \psi_s(x) \] (1.78)

Inserting these variations in the conserved current, Eq. (1.71), and factoring out the constant parameters \( \omega_{\alpha\beta} \) we obtain
\[ \partial_\mu M^{\mu\alpha\beta} = 0 \quad \text{with} \quad M^{\mu\alpha\beta} = x^\alpha T^\mu\beta - x^\beta T^\mu\alpha + \frac{\partial L}{\partial (\partial_\mu\varphi_r)} \Sigma_{rs}^{\alpha\beta} \varphi_s \] (1.79)

The conserved angular momentum is then
\[ M^{\alpha\beta} = \int d^3x M^{0\alpha\beta} = \int d^3x \left[ x^\alpha T^0\beta - x^\beta T^0\alpha + \sum_{r,s} \pi_r \Sigma_{rs}^{\alpha\beta} \varphi_s \right] \] (1.80)

with
\[ \frac{dM^{\alpha\beta}}{dt} = 0 \] (1.81)

3) Internal Symmetries

Let us consider that the Lagrangian is invariant for an infinitesimal internal symmetry transformation
\[ \delta_T\varphi_r(x) = -i\varepsilon \lambda_{rs} \varphi_s(x), \quad \delta x^\mu = 0 \] (1.82)

where we explicitly indicate that there are no change in the coordinates, only in the fields. Then substituting in the current we easily obtain
\[ \partial_\mu J^\mu = 0 \quad \text{where} \quad J^\mu = -i \frac{\partial L}{\partial (\partial_\mu\varphi_r)} \lambda_{rs} \varphi_s \] (1.83)

This leads to the conserved charge
\[ Q(\lambda) = -i \int d^3x \pi_r \lambda_{rs} \varphi_s \quad : \quad \frac{dQ(\lambda)}{dt} = 0 \] (1.84)

These relations between symmetries and conservation laws were derived for the classical theory. Let us see now what happens when we quantize the theory. In the quantum theory the fields \( \varphi_r(x) \) become operators acting on the Hilbert space of the states. The physical observables are related with the matrix elements of these operators. We have therefore to require Lorentz covariance for those matrix elements. This in turn requires that the operators have to fulfill certain conditions.
This means that the classical fields relation
\[ \varphi'_r(x') = S_{rs}(a)\varphi_s(x) \] (1.85)
should be in the quantum theory
\[ \langle \Phi'_\alpha | \varphi_r(x') | \Phi'\beta \rangle = S_{rs}(a) \langle \Phi_\alpha | \varphi_s(x) | \Phi_\beta \rangle \] (1.86)
It should exist an unitary transformation \( U(a, b) \) that should relate the two inertial frames
\[ \left| \Phi' \right\rangle = U(a, b) \left| \Phi \right\rangle \] (1.87)
where \( a_{\mu\nu} \) e \( b^\mu \) are defined by
\[ x'^\mu = a_{\mu\nu} x^\nu + b^\mu \] (1.88)
Using Eq. (1.87) in Eq. (1.86) we get that the field operators should transform as
\[ U(b)\varphi_r(x)U^{-1}(a, b) = S_{rs}^{(-1)}(a)\varphi_s(ax + b) \] (1.89)
Let us look at the consequences of this relation for translations and Lorentz transformations. We consider first the translations. Eq. (1.89) is then
\[ U(b)\varphi_r(x)U^{-1}(b) = \varphi_r(x + b) \] (1.90)
For infinitesimal translations we can write
\[ U(\varepsilon) \equiv e^{i\varepsilon_{\mu}P^\mu} \simeq 1 + i\varepsilon_{\mu}P^\mu \] (1.91)
where \( P^\mu \) is an hermitian operator. Then Eq. (1.90) gives
\[ i[\mathcal{P}^\mu, \varphi_r(x)] = \partial^\mu \varphi_r(x) \] (1.92)
The correspondence with classical mechanics and non relativistic quantum theory suggests that we identify \( \mathcal{P}^\mu \) with the 4-momentum, that is, \( \mathcal{P}^\mu \equiv P^\mu \) where \( P^\mu \) has been defined in Eq. (1.76).
As we have an explicit expression for \( P^\mu \) and we know the commutation relations of the quantum theory, the Eq. (1.92) becomes an additional requirement that the theory has to verify in order to be invariant under translations. We will see explicitly that this is indeed the case for the theories in which we are interested.
For Lorentz transformations \( x'^\mu = a_{\mu\nu} x^\nu \), we write for an infinitesimal transformation
\[ a^\mu_{\nu} = g^\mu_{\nu} + \omega^\mu_{\nu} + O(\omega^2) \] (1.93)
and therefore
\[ U(\omega) \equiv 1 - \frac{i}{2} \omega_{\mu\nu}M^\mu_{\nu} \] (1.94)
We then obtain from Eq. (1.89) the requirement
\[ i[M^\mu_{\nu}, \varphi_r(x)] = x^\mu \partial^\nu \varphi_r - x^\nu \partial^\mu \varphi_r + \Sigma_{r,s}^{\mu\nu} \varphi_s(x) \] (1.95)
Once more the classical correspondence lead us to identify \( M^\mu_{\nu} = M^\mu_{\nu} \) where the angular momentum \( M^\mu_{\nu} \) is defined in Eq. (1.80). For each theory we will have to verify Eq. (1.95) for the theory to be invariant under Lorentz transformations. We will see that this is true for the cases of interest.
1.2 Quantization of scalar fields

1.2.1 Real scalar field

The real scalar field described by the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial_{\mu} \varphi - \frac{1}{2} m^2 \varphi^2 \]  

(1.96)

to which corresponds the Klein-Gordon equation

\[(\Box + m^2)\varphi = 0\]  

(1.97)

is the simplest example, and in fact was already used to introduce the general formalism. As we have seen the conjugate momentum is

\[ \pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} \]  

(1.98)

and the commutation relations are

\[
[\varphi(\vec{x}, t), \varphi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0
\]  

\[
[\pi(\vec{x}, t), \varphi(\vec{x}', t)] = -i\delta^3(\vec{x} - \vec{x}')
\]  

(1.99)

The Hamiltonian is given by,

\[ H = P^0 = \int d^3x \mathcal{H} \]  

\[ = \int d^3x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} m^2 \varphi^2 \right] \]  

(1.100)

and the linear momentum is

\[ \vec{P} = -\int d^3x \pi \nabla \varphi \]  

(1.101)

Using Eqs. (1.100) and (1.101) it is easy to verify that

\[ i[P^\mu, \varphi] = \partial^\mu \varphi \]  

(1.102)

showing the invariance of the theory for the translations. In the same way we can verify the invariance under Lorentz transformations, Eq. (1.95), with \( \Sigma_{\mu}^{\nu} = 0 \) (spin zero).

In order to define the states of the theory it is convenient to have eigenstates of energy and momentum. To build these states we start by making a spectral Fourier decomposition of \( \varphi(\vec{x}, t) \) in plane waves:

\[ \varphi(\vec{x}, t) = \int \tilde{d}k \left[ a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x} \right] \]  

(1.103)

where

\[ \tilde{d}k \equiv \frac{d^3k}{(2\pi)^32\omega_k}; \quad \omega_k = +\sqrt{|\vec{k}|^2 + m^2} \]  

(1.104)
is the Lorentz invariant integration measure. As in the quantum theory \( \varphi \) is an operator, also \( a(k) \) and \( a^\dagger(k) \) should be operators. As \( \varphi \) is real, then \( a^\dagger(k) \) should be the hermitian conjugate to \( a(k) \). In order to determine their commutation relations we start by solving Eq. (1.103) in order to \( a(k) \) and \( a^\dagger(k) \). Using the properties of the delta function, we get

\[
a(k) = i \int d^3x e^{ik \cdot x} \partial_0 \varphi(x)
\]

\[
a^\dagger(k) = -i \int d^3x e^{-ik \cdot x} \partial_0 \varphi(x)
\]

where we have introduced the notation

\[
a \partial_0 b = \frac{\partial b}{\partial t} - \frac{\partial a}{\partial t} b
\]

The second member of Eq. (1.105) is time independent as can be checked explicitly (see Problem 1.3). This observation is important in order to be able to choose equal times in the commutation relations. We get

\[
[a(k), a^\dagger(k')] = \int d^3x \int d^3y \left[ e^{ik \cdot x} \partial_0 \varphi(x, t), e^{-ik' \cdot y} \partial_0 \varphi(y, t) \right] = (2\pi)^3 2\omega_k \delta(\vec{k} - \vec{k}')
\]

and

\[
[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0
\]

We then see that, except for a small difference in the normalization, \( a(k) \) and \( a^\dagger(k) \) should be interpreted as annihilation and creation operators of states with momentum \( k^\mu \). To show this, we observe that

\[
H = \frac{1}{2} \int \tilde{d}k \omega_k \left[ a^\dagger(k)a(k) + a(k)a^\dagger(k) \right]
\]

\[
\vec{P} = \frac{1}{2} \int \tilde{d}k \vec{k} \left[ a^\dagger(k)a(k) + a(k)a^\dagger(k) \right]
\]

Using these explicit forms we can then obtain

\[
[P^\mu, a^\dagger(k)] = k^\mu a^\dagger(k)
\]

\[
[P^\mu, a(k)] = -k^\mu a(k)
\]

showing that \( a^\dagger(k) \) adds momentum \( k^\mu \) and that \( a(k) \) destroys momentum \( k^\mu \). That the quantization procedure has produced an infinity number of oscillators should come as no surprise. In fact \( a(k), a^\dagger(k) \) correspond to the quantization of the normal modes of the classical Klein-Gordon field.

By analogy with the harmonic oscillator, we are now in position of finding the eigenstates of \( H \). We start by defining the base state, that in quantum field theory is called the vacuum. We have

\[
a(k)|0\rangle_k = 0 \quad ; \quad \forall_k
\]
Then the vacuum, that we will denote by $|0\rangle$, will be formally given by

$$|0\rangle = \Pi_k |0\rangle_k$$

and we will assume that it is normalized, that is $\langle 0 | 0 \rangle = 1$. If now we calculate the vacuum energy, we find immediately the first problem with infinities in Quantum Field Theory (QFT). In fact

$$\langle 0 | H | 0 \rangle = \frac{1}{2} \int d\mathbf{k} \omega_k \left[ \langle 0 \left| a(k) a(k) + a(k) a^\dagger(k) \right| 0 \rangle \right]$$

$$= \frac{1}{2} \int d\mathbf{k} \omega_k \left[ \langle 0 \left| a(k), a^\dagger(k) \right| 0 \rangle \right]$$

$$= \frac{1}{2} \int d^3 k \omega_k (2\pi)^3 2\omega_k \delta^3(0)$$

$$= \frac{1}{2} \int d^3 k \omega_k \delta^3(0) = \infty$$

(1.115)

This infinity can be understood as the the (infinite) sum of the zero point energy of all quantum oscillators. In the discrete case we would have, $\sum_k \frac{1}{2} \omega_k = \infty$. This infinity can be easily removed. We start by noticing that we only measure energies as differences with respect to the vacuum energy, and those will be finite. We will then define the energy of the vacuum as being zero. Technically this is done as follows. We define a new operator $P_{N.O.}^\mu$ as

$$P_{N.O.}^\mu = \frac{1}{2} \int d\mathbf{k} k^\mu \left[ a^\dagger(k) a(k) + a(k) a^\dagger(k) \right]$$

$$- \frac{1}{2} \int d\mathbf{k} k^\mu \left[ \langle 0 \left| a(k), a^\dagger(k) \right| 0 \rangle \right]$$

$$= \int d\mathbf{k} k^\mu a^\dagger(k) a(k)$$

(1.116)

Now $\langle 0 | P_{N.O.}^\mu | 0 \rangle = 0$. The ordering of operators where the annihilation operators appear on the right of the creation operators is called normal ordering and the usual notation is

$$: \frac{1}{2} (a^\dagger(k) a(k) + a(k) a^\dagger(k)) : \equiv a^\dagger(k) a(k)$$

(1.117)

Therefore to remove the infinity of the energy and momentum corresponds to choose the normal ordering to our operators. We will adopt this convention in the following dropping the subscript “N.O.” to simplify the notation. This should not appear as an ad hoc procedure. In fact, in going from the classical theory where we have products of fields into the quantum theory where the fields are operators, we should have a prescription for the correct ordering of such products. We have just seen that this should be the normal ordering.

Once we have the vacuum we can build the states by applying the the creation operators $a^\dagger(k)$. As in the case of the harmonic oscillator, we can define the number operator,

$$N = \int d\mathbf{k} a^\dagger(k) a(k)$$

(1.118)
CHAPTER 1. FREE FIELD QUANTIZATION

It is easy to see that $N$ commutes with $H$ and therefore the eigenstates of $H$ are also eigenstates of $N$. The state with one particle of momentum $k^\mu$ is obtained as $a^\dagger(k)|0\rangle$. In fact we have

$$P^\mu a^\dagger(k)|0\rangle = \int d k' k'^\mu a^\dagger(k') a(k') a^\dagger(k)|0\rangle$$

$$= \int d^3 k' k'^\mu \delta^3(\vec{k'} - \vec{k}) a^\dagger(k)|0\rangle = k^\mu a^\dagger(k)|0\rangle \quad (1.119)$$

and

$$Na^\dagger(k)|0\rangle = a^\dagger(k)|0\rangle \quad (1.120)$$

In a similar way, the state $a^\dagger(k_1)\ldots a^\dagger(k_n)|0\rangle$ would be a state with $n$ particles. However, the states that we have just defined have a problem. They are not normalizable and therefore they can not form a basis for the Hilbert space of the quantum field theory, the so-called Fock space. The origin of the problem is related to the use of plane waves and states with exact momentum. This can be solved forming states that are superpositions of plane waves

$$|1\rangle = \lambda \int \vec{d}k C(k) a^\dagger(k)|0\rangle \quad (1.121)$$

Then

$$\langle 1|1\rangle = \lambda^2 \int \vec{d}k_1 \vec{d}k_2 C^*(k_1) C(k_2) \langle 0|a(k_1) a^\dagger(k_2)|0\rangle$$

$$= \lambda^2 \int \vec{d}k |C(k)|^2 = 1 \quad (1.122)$$

and therefore

$$\lambda = \left( \int \vec{d}k |C(k)|^2 \right)^{-1/2} \quad (1.123)$$

with the condition that $\int \vec{d}k |C(k)|^2 < \infty$. If $k$ is only different from zero in a neighborhood of a given 4-momentum $k^\mu$, then the state will have a well defined momentum (within some experimental error).

A basis for the Fock space can then be constructed from the $n$–particle normalized states

$$|n\rangle = \left( n! \int \vec{d}k_1 \ldots \vec{d}k_n |C(k_1, \ldots k_n)|^2 \right)^{-1/2}$$

$$\int \vec{d}k_1 \ldots \vec{d}k_n C(k_1, \ldots k_n) a^\dagger(k_1) \ldots a^\dagger(k_n)|0\rangle \quad (1.124)$$

that satisfy

$$\langle n|n\rangle = 1 \quad (1.125)$$

$$N |n\rangle = n |n\rangle \quad (1.126)$$
1.2. QUANTIZATION OF SCALAR FIELDS

Due to the commutation relations of the operators $a^\dagger(k)$ in Eq. (1.124), the functions $C(k_1\cdots k_n)$ are symmetric, that is,

$$C(\cdots k_i, \cdots k_j, \cdots) = C(\cdots k_j, \cdots k_i, \cdots)$$  \hspace{1cm} (1.127)

This shows that the quanta that appear in the canonical quantization of real scalar fields obey the Bose–Einstein statistics. This interpretation in terms of particles, with creation and annihilation operators, that results from the canonical quantization, is usually called second quantization, as opposed to the description in terms of wave functions (the first quantization).

1.2.2 Microscopic causality

Classically, the fields can be measured with an arbitrary precision. In a relativistic quantum theory we have several problems. The first, results from the fact that the fields are now operators. This means that the observables should be connected with the matrix elements of the operators and not with the operators. Besides this question, we can only speak of measuring $\varphi$ in two space-time points $x$ and $y$ if $[\varphi(x), \varphi(y)]$ vanishes. Let us look at the conditions needed for this to occur.

$$[\varphi(x), \varphi(y)] = \int \tilde{dk}_1\tilde{dk}_2 \left\{ \left[ a(k_1), a^\dagger(k_2) \right] e^{-ik_1\cdot x + ik_2\cdot y} + \left[ a^\dagger(k_1), a(k_2) \right] e^{ik_1\cdot x - ik_2\cdot y} \right\}$$

$$= \int \tilde{dk}_1 \left( e^{-ik_1\cdot (x-y)} - e^{ik_1\cdot (x-y)} \right)$$

$$\equiv i\Delta(x - y)$$  \hspace{1cm} (1.128)

The function $\Delta(x - y)$ is Lorentz invariant and satisfies the relations

$$(\Box + m^2)\Delta(x - y) = 0$$  \hspace{1cm} (1.129)

$$\Delta(x - y) = -\Delta(y - x)$$  \hspace{1cm} (1.130)

$$\Delta(\vec{x} - \vec{y}, 0) = 0$$  \hspace{1cm} (1.131)

The last relation ensures that the equal time commutator of two fields vanishes. Lorentz invariance implies then,

$$\Delta(x - y) = 0 \quad \forall (x - y)^2 < 0$$  \hspace{1cm} (1.132)

This means that for two points that can not be physically connected, that is for which $(x - y)^2 < 0$, the fields interpreted as physical observables, can then be independently measured. This result is known as Microscopic Causality. We note that

$$\partial^\mu \Delta(x - y)|_{x^0 = y^0} = -\delta^3(\vec{x} - \vec{y})$$  \hspace{1cm} (1.133)

which ensures the canonical commutation relation, Eq. (1.99).
1.2.3 Vacuum fluctuations

It is well known from Quantum Mechanics that, in an harmonic oscillator, the coordinate is not well defined for the energy eigenstates, that is
\[
\langle n| q^2 |n \rangle > (\langle n| q |n \rangle)^2 = 0
\] (1.134)

In Quantum Field Theory, we deal with an infinite set of oscillators, and therefore we will have the same behavior, that is,
\[
\langle 0| \varphi(x) \varphi(y) |0 \rangle \neq 0
\] (1.135)

although
\[
\langle 0| \varphi(x) |0 \rangle = 0
\] (1.136)

We can calculate Eq. (1.135). We have
\[
\langle 0| \varphi(x) \varphi(y) |0 \rangle = \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} e^{i k_1 \cdot (x-y)} \langle a(k_1) a^\dagger(k_2) |0 \rangle
\]
\[
\equiv \Delta_+(x-y)
\] (1.137)

The function \( \Delta_+(x-y) \) corresponds to the positive frequency part of \( \Delta(x-y) \). When \( y \to x \) this expression diverges quadratically,
\[
\langle 0| \varphi^2(x) |0 \rangle = \Delta_+(0) = \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} e^{i k_1 \cdot (x-y)} \langle a(k_1) a^\dagger(k_2) |0 \rangle
\]
\[
\equiv \Delta_+(x-y)
\] (1.138)

This divergence can not be eliminated in the way we did with the energy of the vacuum. In fact these vacuum fluctuations, as they are known, do have observable consequences like, for instance, the Lamb shift. We will be less worried with the result of Eq. (1.138), if we notice that for measuring the square of the operator \( \varphi \) at \( x \) we need frequencies arbitrarily large, that is, an infinite amount of energy. Physically only averages over a finite space-time region have meaning.

1.2.4 Charged scalar field

The description in terms of real fields does not allow the distinction between particles and anti-particles. It applies only the those cases were the particle and anti-particle are identical, like the \( \pi^0 \). For the more usual case where particles and anti-particles are distinct, it is necessary to have some charge (electric or other) that allows us to distinguish them. For this we need complex fields.

The theory for the scalar complex field can be easily obtained from two real scalar fields \( \varphi_1 \) and \( \varphi_2 \) with the same mass. If we denote the complex field \( \varphi \) by,
\[
\varphi = \frac{\varphi_1 + i \varphi_2}{\sqrt{2}}
\] (1.139)

then
\[
\mathcal{L} = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2) =: \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi
\] (1.140)
which leads to the equations of motion
\[(\Box + m^2)\varphi = 0 ; \quad (\Box + m^2)\varphi^\dagger = 0 \quad (1.141)\]
The classical theory given in Eq. (1.140) has, at the classical level, a conserved current,
\[\partial_\mu J^\mu = 0, \quad J^\mu = i\varphi^\dagger \partial^\mu \varphi \quad (1.142)\]
Therefore we expect, at the quantum level, the charge
\[Q = \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \varphi) : \quad (1.143)\]
to be conserved, that is, \([H, Q] = 0\). To show this we need to know the commutation
relations for the field \(\varphi\). The definition Eq. (1.139), and the commutation relations for
\(\varphi_1\) and \(\varphi_2\) allow us to obtain the following relations for \(\varphi\) and \(\varphi^\dagger\):
\[[\varphi(x), \varphi(y)] = [\varphi^\dagger(x), \varphi^\dagger(y)] = 0 \quad (1.144)\]
\[[\varphi(x), \varphi^\dagger(y)] = i\Delta(x - y) \quad (1.145)\]
For equal times we can get from Eq. (1.145)
\[[\pi(\vec{x}, t), \varphi(\vec{y}, t)] = [\pi^\dagger(\vec{x}, t), \varphi^\dagger(\vec{y}, t)] = -i\delta^3(\vec{x} - \vec{y}) \quad (1.146)\]
where
\[\pi = \dot{\varphi}^\dagger \quad ; \quad \pi^\dagger = \dot{\varphi} \quad (1.147)\]
The plane waves expansion is then
\[\varphi(x) = \int d\vec{k} \left[ a_+(k)e^{-ik\cdot x} + a_+^\dagger(k)e^{ik\cdot x} \right] \quad (1.148)\]
where the definition of \(a_\pm(k)\) is
\[a_\pm(k) = a_1(k) \pm ia_2(k) \quad ; \quad a_\pm^\dagger(k) = a_1^\dagger(k) \mp ia_2^\dagger(k) \quad (1.149)\]
The algebra of the operators \(a_\pm\) it is easily obtained from the algebra of the operators
\(a_\gamma s\). We get the following non-vanishing commutators:
\[[a_+(k), a_+^\dagger(k')] = [a_-(k), a_-^\dagger(k')] = (2\pi)^32\omega_k\delta^3(\vec{k} - \vec{k}') \quad (1.150)\]
therefore allowing us to interpret \(a_+\) and \(a_+^\dagger\) as annihilation and creation operators of
quanta of type +, and similarly for the quanta of type −. We can construct the number
operators for those quanta:
\[N_\pm = \int d\vec{k} a_\pm^\dagger(k)a_\pm(k) \quad (1.151)\]
One can easily verify that
\[ N_+ + N_- = N_1 + N_2 \] (1.152)
where
\[ N_i = \int \tilde{d}k \ a_\dagger_i(k) a_i(k) \] (1.153)
The energy-momentum operator can be written in terms of the + and − operators,
\[ P^\mu = \int \tilde{d}k \ k^\mu \left[ a_\dagger_+(k)a_+(k) + a_\dagger_-(k)a_-(k) \right] \] (1.154)
where we have already considered the normal ordering. Using the decomposition in Eq. (1.148), we obtain for the charge \( Q \):
\[ Q = \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \varphi) : \]
\[ = \int \tilde{d}k \left[ a_\dagger_+(k)a_+(k) - a_\dagger_-(k)a_-(k) \right] \]
\[ = N_+ - N_- \] (1.155)
Using the commutation relation in Eq. (1.150) one can easily verify that
\[ [H, Q] = 0 \] (1.156)
showing that the charge \( Q \) is conserved. The Eq. (1.155) allows us to interpret the ± quanta as having charge ±1. However, before introducing interactions, the theory is symmetric, and we can not distinguish between the two types of quanta. From the commutation relations (1.150) we obtain,
\[ [P^\mu, a_\dagger_+(k)] = k^\mu a_\dagger_+(k) \]
\[ [Q, a_\dagger_+(k)] = +a_\dagger_+(k) \] (1.157)
showing that \( a_\dagger_+(k) \) creates a quanta with 4-momentum \( k^\mu \) and charge +1. In a similar way we can show that \( a_- \) creates a quanta with charge −1 and that \( a_{\pm}(k) \) annihilates quanta of charge ±1, respectively.

### 1.2.5 Time ordered product and the Feynman propagator

The operator \( \varphi^\dagger \) creates a particle with charge +1 or annihilates a particle with charge −1. In both cases it adds a total charge +1. In a similar way \( \varphi \) annihilates one unit of charge. Let us construct a state of one particle (not normalized) with charge +1 by application of \( \varphi^\dagger \) in the vacuum:
\[ |\Psi_+ (\vec{x}, t)\rangle \equiv \varphi^\dagger (\vec{x}, t) |0\rangle \] (1.158)
The amplitude to propagate the state \( |\Psi_+\rangle \) into the future to the point \((\vec{x}', t')\) with \( t' > t \) is given by
\[ \theta(t' - t) \langle \Psi_+ (\vec{x}', t') |\Psi_+ (\vec{x}, t)\rangle = \theta(t' - t) \langle 0|\varphi(\vec{x}', t')\varphi^\dagger (\vec{x}, t) |0\rangle \] (1.159)
1.2. QUANTIZATION OF SCALAR FIELDS

In $\varphi^{\dagger}(\vec{x}, t)|0\rangle$ only the operator $a^{\dagger}_{+}(k)$ is active, while in $\langle 0|\varphi(\vec{x}', t')$ the same happens to $a_{+}(k)$. Therefore Eq. (1.159) is the matrix element that creates a quanta of charge +1 in $(\vec{x}, t)$ and annihilates it in $(\vec{x}', t')$ with $t' > t$.

There exists another way of increasing the charge by +1 unit in $(\vec{x}, t)$ and decreasing it by $-1$ in $(\vec{x}', t')$. This is achieved if we create a quanta of charge $-1$ in $\vec{x}'$ at time $t'$ and let it propagate to $\vec{x}$ where it is absorbed at time $t > t'$. The amplitude is then,

$$\theta(t - t') \langle \Psi_{-}(\vec{x}, t)|\Psi_{-}(\vec{x}', t') \rangle = \langle 0|\varphi^{\dagger}(\vec{x}, t)\varphi(\vec{x}', t')|0\rangle \theta(t - t') \quad (1.160)$$

Since we can not distinguish the two paths we must sum of the two amplitudes in Eqs. (1.159) and (1.160). This is the so-called Feynman propagator. It can be written in a more compact way if we introduce the time ordered product. Given two operators $a(x)$ and $b(x')$ we define the time ordered product $T$ by,

$$T a(x)b(x') = \theta(t - t')a(x)b(x') + \theta(t' - t)b(x')a(x) \quad (1.161)$$

In this prescription the older times are always to the right of the more recent times. It can be applied to an arbitrary number of operators. With this definition, the Feynman propagator reads,

$$\Delta F(x' - x) = \langle 0|T\varphi(x')\varphi^{\dagger}(x)|0\rangle \quad (1.162)$$

Using the $\varphi$ and $\varphi^{\dagger}$ decomposition we can calculate $\Delta F$ (for free fields, of course)

$$\Delta F(x' - x) = \int dk \left[ \theta(t' - t)e^{-ik(x'-x)} + \theta(t - t')e^{ik(x'-x)} \right] \quad (1.163)$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x'-x)} \quad (1.164)$$

$$\equiv \int \frac{d^4k}{(2\pi)^4} \Delta F(k)e^{-ik(x'-x)}$$

where

$$\Delta F(k) = \frac{i}{k^2 - m^2 + i\varepsilon} \quad (1.165)$$

$\Delta F(k)$ is the propagator in momenta space (Fourier transform). The equivalence between Eq. (1.164) and Eq. (1.163) is done using integration in the complex plane of the time component $k^0$, with the help of the residue theorem. The contour is defined by the $i\varepsilon$ prescription, as indicated in Fig. (1.1). Applying the operator $(\Box'_{x} + m^2)$ to $\Delta F(x' - x)$ in any of the forms of Eq. (1.163) one can show that

$$(\Box'_{x} + m^2)\Delta F(x' - x) = -i\delta^4(x' - x) \quad (1.166)$$

that is, $\Delta F(x' - x)$ is the Green’s function for the Klein-Gordon equation with Feynman boundary conditions.

In the presence of interactions, Feynman propagator looses the simple form of Eq. (1.165). However, as we will see, the free propagator plays a key role in perturbation theory.
1.3 Second quantization of the Dirac field

Let us now apply the formalism of second quantization to the Dirac field. As we will see, something has to be changed, otherwise we would be led to a theory obeying Bose statistics, while we know that electrons have spin 1/2 and obey Fermi statistics.

1.3.1 Canonical formalism for the Dirac field

The Lagrangian density that leads to the Dirac equation is
\[
\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi
\]  
(1.167)

The conjugate momentum to \(\psi_\alpha\) is
\[
\pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = i\psi^\dagger_\alpha
\]  
(1.168)

while the conjugate momentum to \(\psi^\dagger_\alpha\) vanishes. The Hamiltonian density is then
\[
\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi
\]  
(1.169)

The requirement of translational and Lorentz invariance for \(\mathcal{L}\) leads to the tensors \(T^{\mu\nu}\) and \(M^{\mu\nu\lambda}\). Using the obvious generalizations of Eqs. (1.75) and (1.79) we get
\[
T^{\mu\nu} = i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - g^{\mu\nu}\mathcal{L}
\]  
(1.170)

and
\[
M^{\mu\nu\lambda} = i\bar{\psi}\gamma^{\mu}(x^{\nu}\partial^{\lambda} - x^{\lambda}\partial^{\nu} + \sigma^{\mu\lambda})\psi - \left(x^{\nu}g^{\mu\lambda} - x^{\mu}g^{\nu\lambda}\right)\mathcal{L}
\]  
(1.171)

where
\[
\sigma^{\mu\lambda} = \frac{1}{4}[\gamma^{\nu}, \gamma^{\lambda}]
\]  
(1.172)

The 4-momentum \(P^\mu\) and the angular momentum tensor \(M^{\nu\lambda}\) are then given by,
\[
P^\mu = \int d^3x T^{0\mu}
\]
\[ M^{\mu\lambda} = \int d^3 x M^{\mu\nu\lambda} \] 

or 

\[ H = \int d^3 x \psi^\dagger (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi \]

\[ \vec{P} = \int d^3 x \psi^\dagger (-i\vec{\nabla}) \psi \] 

If we define the angular momentum vector \( \vec{J} \equiv (M^{23}, M^{31}, M^{12}) \) we get 

\[ \vec{J} = \int d^3 x \psi^\dagger \left( \vec{r} \times \frac{1}{i} \vec{\nabla} + \frac{1}{2} \vec{\Sigma} \right) \psi \] 

which has the familiar aspect \( \vec{J} = \vec{L} + \vec{S} \). We can also identify a conserved current, \( \partial_\mu j^\mu = 0 \), with 

\[ j^\mu = \psi \gamma^\mu \psi, \] 

which will give the conserved charge 

\[ Q = \int d^3 x \psi^\dagger \psi \] 

All that we have done so far is at the classical level. To apply the canonical formalism, we have to enforce commutation relations and verify the Lorentz invariance of the theory. This will lead us into problems. To see what are the problems and how to solve them, we will introduce the plane wave expansions, 

\[ \psi(x) = \int dp \sum_s \left[ b(p,s)u(p,s)e^{-ip \cdot x} + d^\dagger(p,s)v(p,s)e^{ip \cdot x} \right] \] 

\[ \psi^\dagger(x) = \int dp \sum_s \left[ b^\dagger(p,s)u(p,s)e^{ip \cdot x} + d(p,s)v^\dagger(p,s)e^{-ip \cdot x} \right] \] 

where \( u(p,s) \) and \( v(p,s) \) are the spinors for positive and negative energy, respectively, introduced in the study of the Dirac equation and \( b, b^\dagger, d \) and \( d^\dagger \) are operators. To see what are the problems with the canonical quantization of fermions, let us calculate \( P^\mu \). We get 

\[ P^\mu = \int dk \, k^\mu \sum_s \left[ b^\dagger(k,s)b(k,s) - d(k,s)d^\dagger(k,s) \right] \] 

where we have used the orthogonality and closure relations for the spinors \( u(p,s) \) and \( v(p,s) \). From Eq. (1.179) we realize that if we define the vacuum as \( b(k,s) \left| 0 \right> = d(k,s) \left| 0 \right> = 0 \) and if we quantize with commutators then particles \( b \) and particles \( d \) will contribute with opposite signs to the energy and the theory will not have a stable ground state. In fact, this was the problem already encountered in the study of the negative energy solutions of the Dirac equation, and this is the reason for the negative sign in Eq. (1.179). Dirac’s hole theory required Fermi statistics for the electrons and we will see how spin and statistics are related.

To discover what are the relations that \( b, b^\dagger, d \) and \( d^\dagger \) should obey, we recall that at the quantum level it is always necessary to verify Lorentz invariance. This gives, 

\[ i[P_\mu, \psi(x)] = \partial_\mu \psi; \quad i[P_\mu, \bar{\psi}(x)] = \partial_\mu \bar{\psi} \]

(1.180)
CHAPTER 1. FREE FIELD QUANTIZATION

Instead of imposing canonical quantization commutators and, as a consequence, verifying Eq. (1.180) we will do the other way around. We start with Eq. (1.180) and we will discover the appropriate relations for the operators. Using the expansions Eqs. (1.177) and (1.178) we can show that Eq. (1.180) leads to

\[ [P_\mu, b(k, s)] = -k_\mu b(k, s) ; [P_\mu, b^\dagger(k, s)] = k_\mu b^\dagger(k, s) \]

(1.181)

Using Eq. (1.179) for \( P_\mu \) we get

\[
\sum_{s'} \left[ \left( b^\dagger(p, s')b(p, s') - d(p, s')d^\dagger(p, s') \right) b(k, s) \right] = -(2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{p})b(k, s) \]

(1.183)

and three other similar relations. If we assume that

\[ [d^\dagger(p, s')d(p, s'), b(k, s)] = 0 \]

(1.184)

the condition from Eq. (1.183) reads

\[
\sum_{s'} \left[ b^\dagger(p, s') \{ b(p, s'), b(k, s) \} - \{ b^\dagger(p, s'), b(k, s) \} b(p, s') \right] = -(2\pi)^3 2k^0 \delta^3(\vec{p} - \vec{k})b(k, s)
\]

(1.185)

where the parenthesis \( \{, \} \) denote anti-commutators. It is easy to see that Eq. (1.185) is verified if we impose the canonical commutation relations. We should have

\[
\{ b^\dagger(p, s), b(k, s) \} = (2\pi)^3 2k^0 \delta^3(\vec{p} - \vec{k}) \delta_{ss'}
\]

(1.186)

and all the other anti-commutators vanish. Note that as b anti-commutes with d and \( d^\dagger \), then it commutes with \( d^\dagger d \) and therefore Eq. (1.184) is verified.

With the anti-commutator relations both contributions to \( P_\mu \) in Eq. (1.179) are positive. As in boson case we have to subtract the zero point energy. This is done, as usual, by taking all quantities normal ordered. Therefore we have for \( P_\mu \),

\[
P_\mu = \int \tilde{dk} \ k^\mu \sum_s \left( b^\dagger(k, s)b(k, s) - d(k, s)d^\dagger(k, s) \right) :
\]

(1.187)

and for the charge

\[
Q = \int d^3x : \psi^\dagger(x)\psi(x) :
\]

(1.188)
1.3. SECOND QUANTIZATION OF THE DIRAC FIELD

which means that the quanta of \( b \) type have charge +1 while those of \( d \) type have charge −1. It is interesting to note that was the second quantization of the Dirac field that introduced the − sign in Eq. (1.188), making the charge operator without a definite sign, while in Dirac theory was the probability density that was positive defined. The reverse is true for bosons. We can easily show that

\[
[Q, b^\dagger(k, s)] = b^\dagger(k, s) \quad [Q, d(k, s)] = d(k, s)
\]

and then

\[
[Q, \psi] = -\psi; \quad [Q, \overline{\psi}] = \overline{\psi}
\] (1.189)

In QED the charge is given by \( eQ \) (\( e < 0 \)). Therefore we see that \( \psi \) creates positrons and annihilates electrons and the opposite happens with \( \overline{\psi} \).

We can introduce the number operators

\[
N^+(p, s) = b^\dagger(p, s)b(p, s) \quad N^-(p, s) = d^\dagger(p, s)d(p, s)
\] (1.190)

and we can rewrite

\[
P^\mu = \int \widetilde{dk} k^\mu \sum_s \left( N^+(k, s) + N^-(k, s) \right)
\]

\[
Q = \int \widetilde{dk} \sum_s \left( N^+(k, s) - N^-(k, s) \right)
\] (1.192)

Using the anti-commutator relations in Eq. (1.186) it is now easy to verify that the theory is Lorentz invariant, that is (see Problem 1.4),

\[
i[M^\mu, \psi] = \left( x^\mu \partial^\nu - x^\nu \partial^\mu \right) \psi + \Sigma^\mu \nu \psi.
\] (1.193)

1.3.2 Microscopic causality

The anti-commutation relations in Eq. (1.186) can be used to find the anti-commutation relations at equal times for the fields. We get

\[
\{\psi_\alpha(x, t), \psi_\beta^\dagger(y, t)\} = \delta^3(x - y)\delta_{\alpha\beta}
\] (1.194)

and

\[
\{\psi_\alpha(x, t), \psi_\beta(y, t)\} = \{\psi_\alpha^\dagger(x, t), \psi_\beta^\dagger(y, t)\} = 0
\] (1.195)

These relations can be generalized to unequal times

\[
\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \int \overline{dp} \left[ \left( (\slashed{p} + m)\gamma^0 \right)_{\alpha\beta} e^{-ip(x-y)} - \left( -\slashed{p} + m \right)\gamma^0_{\alpha\beta} e^{ip(x-y)} \right]
\]

\[
= \left[ (i\partial_x + m)\gamma^0 \right]_{\alpha\beta} i\Delta(x - y)
\] (1.196)

where the \( \Delta(x - y) \) function was defined in Eq. (1.128) for the scalar field. The fact that \( \gamma^0 \) appears in Eq. (1.196) is due to the fact that in Eq. (1.196) we took \( \psi^\dagger \) and not \( \overline{\psi} \). In fact, if we multiply on the right by \( \gamma^0 \) we get

\[
\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = (i\partial_x + m)_{\alpha\beta} i\Delta(x - y)
\] (1.197)
CHAPTER 1. FREE FIELD QUANTIZATION

\{\psi_\alpha(x), \psi_\beta(y)\} = \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)\} = 0 \quad (1.198)

We can easily verify the covariance of Eq. (1.197). We use

\begin{align*}
U(a, b)\psi(x)U^{-1}(a, b) &= S^{-1}(a)\psi(ax + b) \\
U(a, b)\bar{\psi}(x)U^{-1}(a, b) &= \bar{\psi}(ax + b)S(a)
\end{align*}

\begin{align*}
S^{-1}\gamma^\mu S &= a^\mu\gamma^\nu \quad (1.199)
\end{align*}

to get

\begin{align*}
U(a, b)\{\psi_\alpha(x), \psi_\beta(y)\}U^{-1}(a, b) &= \\
&= S^{-1}_{\alpha\gamma}(a)\{\psi_\gamma(ax + b), \bar{\psi}_\lambda(ay + b)\}S_{\lambda\beta}(a) \\
&= S^{-1}_{\alpha\gamma}(a)(i\partial_{ax} + m)_{\gamma\lambda}i\Delta(ax - ay)S_{\lambda\beta}(a) \\
&= (i\partial + m)_{\alpha\beta}i\Delta(x - y) \quad (1.200)
\end{align*}

where we have used the invariance of \(\Delta(x - y)\) and the result \(S^{-1}i\partial_{ax}S = i\partial_x\). For \((x - y)^2 < 0\) the anti-commutators vanish, because \(\Delta(x - y)\) also vanishes. This result allows us to show that any two observables built as bilinear products of \(\bar{\psi}\) e \(\psi\) commute for two spacetime points for which \((x - y)^2 < 0\). Therefore

\begin{align*}
\{\bar{\psi}_\alpha(x)\psi_\beta(x), \bar{\psi}_\lambda(y)\psi_\tau(y)\} &= \\
&= \bar{\psi}_\alpha(x)\{\psi_\beta(x), \bar{\psi}_\lambda(y)\}\psi_\tau(y) - \{\bar{\psi}_\alpha(x), \bar{\psi}_\lambda(y)\}\psi_\beta(x)\psi_\tau(y) \\
&\quad + \bar{\psi}_\lambda(y)\{\bar{\psi}_\alpha(x), \psi_\beta(x)\}\psi_\tau(y) - \bar{\psi}_\lambda(y)\{\psi_\beta(x), \bar{\psi}_\alpha(x)\}\psi_\tau(y) \\
&= 0 \quad (1.201)
\end{align*}

for \((x - y)^2 < 0\). In this way the microscopic causality is satisfied for the physical observables, such as the charge density or the momentum density.

1.3.3 Feynman propagator

For the Dirac field, as in the case of the charged scalar field, there are two ways of increasing the charge by one unit in \(x'\) and decrease it by one unit in \(x\) (note that the electron has negative charge). These ways are

\begin{align*}
\theta(t' - t)\left\langle 0|\psi_\beta(x')\psi_\alpha^\dagger(x)|0\right\rangle \quad (1.202) \\
\theta(t - t')\left\langle 0|\psi_\alpha^\dagger(x)\psi_\beta(x')|0\right\rangle \quad (1.203)
\end{align*}

In Eq. (1.202) an electron of positive energy is created at \(\vec{x}'\) in the instant \(t\), propagates until \(\vec{x}'\) where is annihilated at time \(t' > t\). In Eq. (1.203) a positron of positive energy is created in \(x'\) and annihilated at \(x\) with \(t > t'\). The Feynman propagator is obtained
summing the two amplitudes. Due the exchange of $\psi_\beta$ and $\overline{\psi}_\alpha$, there must be a minus sign between these two amplitudes. Multiplying by $\gamma^0$, in order to get $\overline{\psi}$ instead of $\psi^\dagger$, we get for the Feynman propagator,

$$S_F(x' - x)_{\alpha\beta} = \theta(t' - t) \langle 0|\psi_\alpha(x')\overline{\psi}_\beta(x)|0 \rangle - \theta(t - t') \langle 0|\overline{\psi}_\beta(x)\psi_\alpha(x')|0 \rangle \equiv \langle 0|T\psi_\alpha(x')\overline{\psi}_\beta(x)|0 \rangle \quad (1.204)$$

where we have defined the time ordered product for fermion fields,

$$T\eta(x)\chi(y) \equiv \theta(x^0 - y^0)\eta(x)\chi(y) - \theta(y^0 - x^0)\chi(y)\eta(x) \quad (1.205)$$

Inserting in Eq. (1.204) the expansions for $\psi$ and $\overline{\psi}$ we get,

$$S_F(x' - x)_{\alpha\beta} = \int \frac{d^4k}{(2\pi)^4} \left[ \frac{i(k + m)_{\alpha\beta}}{k^2 - m^2 + i\varepsilon} e^{-ik(x' - x)} + \frac{(-k + m)_{\alpha\beta}}{k^2 - m^2 + i\varepsilon} e^{ik(x' - x)} \right]$$

$$\equiv \int \frac{d^4k}{(2\pi)^4} S_F(k)_{\alpha\beta} e^{-ik(x' - x)} \quad (1.206)$$

where $S_F(k)$ is the Feynman propagator in momenta space. We can also verify that Feynman’s propagator is the Green function for the Dirac equation, that is (see Problem 1.7),

$$(i\slashed{D} - m)_{\lambda\alpha} S_F(x' - x)_{\alpha\beta} = i\delta_{\lambda\beta}\delta^4(x' - x) \quad (1.207)$$

### 1.4 Electromagnetic field quantization

#### 1.4.1 Introduction

The free electromagnetic field is described by the classical Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (1.208)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.209)$$

The free field Maxwell equations are

$$\partial_\alpha F^{\alpha\beta} = 0 \quad (1.210)$$

that corresponds to the usual equations in 3-vector notation,

$$\vec{\nabla} \cdot \vec{E} = 0 \quad ; \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} \quad (1.211)$$
The other Maxwell equations are a consequence of Eq. (1.209) and can be written as,

\[ \partial_\alpha \tilde{F}^{\alpha\beta} = 0 \; ; \; \tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \]  

(1.212)

corresponding to

\[ \tilde{\nabla} \cdot \tilde{B} = 0 \; ; \; \tilde{\nabla} \times \tilde{E} = -\frac{\partial \tilde{B}}{\partial t} \]  

(1.213)

Classically, the quantities with physical significance are the fields \( \vec{E} \) and \( \vec{B} \), and the potentials \( A^\mu \) are auxiliary quantities that are not unique due to the gauge invariance of the theory. In quantum theory the potentials \( A^\mu \) are the ones playing the leading role, for instance in the minimal prescription. We have therefore to formulate the quantum fields theory in terms of \( A^\mu \) and not of \( \vec{E} \) and \( \vec{B} \).

When we try to apply the canonical quantization to the potentials \( A^\mu \) we immediately run into difficulties. For instance, if we define the conjugate momentum as,

\[ \pi^\mu = \frac{\partial L}{\partial (\dot{A}^\mu)} \]  

(1.214)

we get

\[ \pi^k = \frac{\partial L}{\partial (\dot{A}^k)} = -\dot{A}^k - \frac{\partial A^0}{\partial x^k} = E^k \]

\[ \pi^0 = \frac{\partial L}{\partial \dot{A}^0} = 0 \]  

(1.215)

Therefore the conjugate momentum to the coordinate \( A^0 \) vanishes and does not allow us to use directly the canonical formalism. The problem has its origin in the fact that the photon, that we want to describe, has only two degrees of freedom (positive or negative helicity) but we are using a field \( A^\mu \) with four degrees of freedom. In fact, we have to impose constraints on \( A^\mu \) in such a way that it describes the photon. This problem can be addressed in three different ways:

i) **Radiation Gauge**

Historically, this was the first method to be used. It is based in the fact that it is always possible to choose a gauge, called the *radiation gauge*, where

\[ A^0 = 0 \; ; \; \nabla \cdot \vec{A} = 0 \]  

(1.216)

that is, the potential \( \vec{A} \) is transverse. The conditions in Eq. (1.216) reduce the number of degrees of freedom to two, the transverse components of \( \vec{A} \). It is then possible to apply the canonical formalism to these transverse components and quantize the electromagnetic field in this way. The problem with this method is that we lose explicit Lorentz covariance. It is then necessary to show that this is recovered in the final result. This method is followed in many text books, for instance in Bjorken and Drell [1].
ii) **Quantization of systems with constraints**

It can be shown that the electromagnetism is an example of an Hamilton generalized system, that is a system where there are constraints among the variables. The way to quantize these systems was developed by Dirac for systems of particles with \( n \) degrees of freedom. The generalization to quantum field theories is done using the formalism of path integrals. We will study this method in Chapter 6, as it will be shown, this is the only method that can be applied to non-abelian gauge theories, like the Standard Model.

iii) **Undefined metric formalism**

There is another method that works for the electromagnetism, called the formalism of the undefined metric, developed by Gupta and Bleuler [2, 3]. In this formalism, that we will study below, Lorentz covariance is kept, that is we will always work with the 4-vector \( A_\mu \), but the price to pay is the appearance of states with negative norm. We have then to define the Hilbert space of the physical states as a sub-space where the norm is positive. We see that in all cases, in order to maintain the explicit Lorentz covariance, we have to complicate the formalism. We will follow the book of Silvan Schweber [4].

### 1.4.2 Undefined metric formalism

To solve the difficulty of the vanishing of \( \pi^0 \), we will start by modifying the Maxwell Lagrangian introducing a new term,

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2
\]  

(1.217)

where \( \xi \) is a dimensionless parameter. The equations of motion are now,

\[
\Box A^\mu - \left( 1 - \frac{1}{\xi} \right) \partial^\mu (\partial \cdot A) = 0
\]  

(1.218)

and the conjugate momenta

\[
\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \frac{1}{\xi} g^{\mu 0} (\partial \cdot A)
\]  

(1.219)

that is

\[
\begin{cases}
\pi^0 = -\frac{1}{\xi} (\partial \cdot A) \\
\pi^k = E^k
\end{cases}
\]

We remark that the Lagrangian of Eq. (1.216) and the equations of motion, Eq. (1.218), reduce to Maxwell theory in the gauge \( \partial \cdot A = 0 \). This why we say that the choice of Eq. (1.216) corresponds to a class of Lorenz gauges with parameter \( \xi \). With this abuse of language (in fact we are not setting \( \partial \cdot A = 0 \), otherwise the problems would come back) the value of \( \xi = 1 \) is known as the *Feynman gauge* and \( \xi = 0 \) as the *Landau gauge*.

From Eq. (1.218) we get

\[
\Box (\partial \cdot A) = 0
\]  

(1.220)
implying that \((\partial \cdot A)\) is a massless scalar field. Although it would be possible to continue with a general \(\xi\), from now on we will take the case of the so-called *Feynman gauge*, where \(\xi = 1\). Then the equation of motion coincide with the Maxwell theory in the Lorenz gauge. As we do not have anymore \(\pi^0 = 0\), we can impose the canonical commutation relations at equal times:

\[
[\pi^\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = -ig^{\mu\nu}\delta^3(\vec{x} - \vec{y}) \\
[A_\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = [\pi_\mu(\vec{x}, t), \pi_\nu(\vec{y}, t)] = 0
\]  

(1.221)

Knowing that \([A_\mu(\vec{x}, t), A_\mu(\vec{y}, t)] = 0\) at equal times, we can conclude that the space derivatives of \(A_\mu\) also commute at equal times. Then, noticing that

\[\pi_\mu = -i\dot{A}_\mu + \text{space derivatives}\]  

(1.222)

we can write instead of Eq. (1.221)

\[
[A_\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = [\dot{A}_\mu(\vec{x}, t), \dot{A}_\mu(\vec{y}, t)] = 0 \\
[A_\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = ig_{\mu\nu}\delta^3(\vec{x} - \vec{y})
\]  

(1.223)

If we compare these relations with the corresponding ones for the real scalar field, where the only one non-vanishing is,

\[\dot{\varphi}(\vec{x}, t), \varphi(\vec{y}, t)] = -i\delta^3(\vec{x} - \vec{y})\]  

(1.224)

we see \((g_{\mu\nu} = \text{diag}(+, -, -, -)\) that the relations for space components are equal but they differ for the time component. This sign will be the source of the difficulties previously mentioned.

If, for the moment, we do not worry about this sign, we expand \(A_\mu(x)\) in plane waves,

\[
A_\mu(x) = \int \frac{dk}{k} \sum_{\lambda=0}^{3} \left[ a(k, \lambda)\varepsilon^\mu(k, \lambda)e^{-ik \cdot x} + a^\dagger(k, \lambda)\varepsilon^{\mu\dagger}(k, \lambda)e^{ik \cdot x} \right]
\]  

(1.225)

where \(\varepsilon^\mu(k, \lambda)\) are a set of four independent 4-vectors that we assume to real, without loss of generality. We will now make a choice for these 4-vectors. We choose \(\varepsilon^\mu(1)\) and \(\varepsilon^\mu(2)\) orthogonal to \(k^\mu\) and \(n^\mu\), such that

\[\varepsilon^\mu(k, \lambda)\varepsilon_\mu(k, \lambda') = -\delta_{\lambda\lambda'} \text{ for } \lambda, \lambda' = 1, 2\]  

(1.226)

After, we choose \(\varepsilon^\mu(k, 3)\) in the plane \((k^\mu, n^\mu)\) orthogonal to \(n^\mu\) and normalized, that is

\[\varepsilon^\mu(k, 3)n_\mu = 0 \quad ; \quad \varepsilon^\mu(k, 3)\varepsilon_\mu(k, 3) = -1\]  

(1.227)

Finally we choose \(\varepsilon^\mu(k, 0) = n^\mu\). The vectors \(\varepsilon^\mu(k, 1)\) and \(\varepsilon^\mu(k, 2)\) are called transverse polarizations, while \(\varepsilon^\mu(k, 3)\) and \(\varepsilon^\mu(k, 0)\) longitudinal and scalar polarizations, respectively. We can give an example. In the frame where \(n^\mu = (1, 0, 0, 0)\) and \(\vec{k}\) is along the z axis we have

\[\varepsilon^\mu(k, 0) \equiv (1, 0, 0, 0) \quad ; \quad \varepsilon^\mu(k, 1) \equiv (0, 1, 0, 0)\]
\[ \varepsilon^\mu(k,2) \equiv (0,0,1,0); \varepsilon^\mu(k,3) \equiv (0,0,0,1) \]  

(1.228)

In general we can show that

\[ \varepsilon(k,\lambda) \cdot \varepsilon^*(k,\lambda') = g^{\lambda\lambda'} \]

\[ \sum_\lambda g^{\lambda\lambda} \varepsilon^\mu(k,\lambda) \varepsilon^{*\nu}(k,\lambda) = g_{\mu\nu} \]  

(1.229)

Inserting the expansion (1.225) in (1.223) we get

\[ [a(k,\lambda),a^\dagger(k',\lambda')] = -g^{\lambda\lambda'} 2k^0(2\pi)^3\delta^3(\vec{k} - \vec{k}') \]  

(1.230)

showing, once more, that the quanta associated with \( \lambda = 0 \) has a commutation relation with the wrong sign. Before addressing this problem, we can verify that the generalization of Eq. (1.223) for arbitrary times is

\[ [A_\mu(x),A_\nu(y)] = -ig_{\mu\nu}\Delta(x,y) \]  

(1.231)

showing the covariance of the theory. The function \( \Delta(x - y) \) is the same that was introduced before for scalar fields.

Therefore, up to this point, everything is as if we had 4 scalar fields. There is, however, the problem of the sign difference in one of the commutators. Let us now see what are the consequences of this sign. For that we introduce the vacuum state defined by

\[ a(k,\lambda)|0\rangle = 0 \quad \lambda = 0, 1, 2, 3 \]  

(1.232)

To see the problem with the sign we construct the one-particle state with scalar polarization, that is

\[ |1\rangle = \int d\tilde{k} f(k)a^\dagger(k,0)|0\rangle \]  

(1.233)

and calculate its norm

\[ \langle 1|1 \rangle = \int d\tilde{k}_1d\tilde{k}_2f^*(k_1)f(k_2)\langle 0|a(k_1,0)a^\dagger(k_2,0)|0\rangle \]

\[ = -(0|0\rangle \int d\tilde{k}|f(k)|^2 \]  

(1.234)

where we have used Eq. (1.230) for \( \lambda = 0 \). The state \( |1\rangle \) has a negative norm. The same calculation for the other polarization would give well behaved positive norms. We therefore conclude that the Fock space of the theory has indefinite metric. What happens then to the probabilistic interpretation of quantum mechanics?

To solve this problem we note that we are not working anymore with the classical Maxwell theory because we modified the Lagrangian. What we would like to do is to impose the condition \( \partial \cdot A = 0 \), but that is impossible as an equation for operators, as that would bring us back to the initial problems with \( \pi^0 = 0 \). We can, however, require that condition on a weaker form, as a condition only to be verified by the physical states.
CHAPTER 1. FREE FIELD QUANTIZATION

More specifically, we require that the part of $\partial \cdot A$ that contains the annihilation operator (positive frequencies) annihilates the physical states,

$$\partial^\mu A^{(+)}_\mu |\psi\rangle = 0 \quad (1.235)$$

The states $|\psi\rangle$ can be written in the form

$$|\psi\rangle = |\psi_T\rangle |\phi\rangle \quad (1.236)$$

where $|\psi_T\rangle$ is obtained from the vacuum with creation operators with transverse polarization and $|\phi\rangle$ with scalar and longitudinal polarization. This decomposition depends, of course, on the choice of polarization vectors. To understand the consequences of Eq. (1.235) is enough to analyze the states $|\phi\rangle$ as $\partial^\mu A^{(+)}_\mu$ contains only scalar and longitudinal polarizations,

$$i\partial \cdot A^{(+)} = \int \tilde{d}k \ e^{-ik \cdot x} \sum_{\lambda=0,3} a(k, \lambda) \varepsilon(k, \lambda) \cdot k \quad (1.237)$$

and therefore Eq. (1.235) becomes

$$\sum_{\lambda=0,3} k \cdot \varepsilon(k, \lambda) a(k, \lambda) |\phi\rangle = 0 \quad (1.238)$$

Condition (1.238) does not determine completely $|\phi\rangle$. In fact, there is much arbitrariness in the choice of the transverse polarization vectors, to which we can always add a term proportional to $k^\mu$ because $k \cdot k = 0$. This arbitrariness must reflect itself on the choice of $|\phi\rangle$. Condition (1.238) is equivalent to,

$$[a(k, 0) - a(k, 3)] |\phi\rangle = 0 \quad (1.239)$$

We can construct $|\phi\rangle$ as a linear combination of states $|\phi_n\rangle$ with $n$ scalar or longitudinal photons:

$$|\phi\rangle = C_0 |\phi_0\rangle + C_1 |\phi_1\rangle + \cdots + C_n |\phi_n\rangle + \cdots \quad (1.240)$$

where

$$N' |\phi_n\rangle = n |\phi_n\rangle \quad (1.241)$$

and

$$N' = \int \tilde{d}k \ [a^\dagger(k, 3)a(k, 3) - a^\dagger(k, 0)a(k, 0)] \quad (1.242)$$

Then

$$n \langle \phi_n |\phi_n\rangle = \langle \phi_n |N' |\phi_n\rangle = 0 \quad (1.243)$$

where we have used Eq. (1.239). This means that

$$\langle \phi_n |\phi_n\rangle = \delta_{n0} \quad (1.244)$$
that is, for \( n \neq 0 \), the state \( |\phi_n\rangle \) has zero norm. We have then for the general state \( |\phi\rangle \),

\[
\langle \phi | \phi \rangle = |C_0|^2 \geq 0
\]

and the coefficients \( C_i, i = 1, \ldots, n \ldots \) are arbitrary. We have to show that this arbitrariness does not affect the physical observables. The Hamiltonian is

\[
H = \int d^3x : \pi^\mu \dot{A}_\mu - \mathcal{L} : \\
= \frac{1}{2} \int d^3x : \sum_{i=1}^3 \left[ \dot{A}_i^2 + (\nabla A_i)^2 \right] - \dot{A}_0^2 - (\nabla A_0)^2 :
\]

(1.246)

It is easy to check that if \( |\psi\rangle \) is a physical state we have

\[
\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \psi_T | \int \tilde{d}k k^0 \sum_{\lambda=1}^2 a^\dagger(k, \lambda)a(k, \lambda) | \psi_T \rangle}{\langle \psi_T | \psi_T \rangle}
\]

(1.247)

and the arbitrariness on the physical states completely disappears when we take average values. Besides that, only the physical transverse polarizations contribute to the result. One can show (see Problem 1.10) that the arbitrariness in \( |\phi\rangle \) is related with a gauge transformation within the class of Lorenz gauges.

It is important to note that although for the average values of the physical observables only the transverse polarizations contribute, the scalar and longitudinal polarizations are necessary for the consistency of the theory. In particular they show up when we consider complete sums over the intermediate states.

Invariance for translations is readily verified. For that we write,

\[
P^\mu = \int \tilde{d}k \ k^\mu \sum_{\lambda=0}^3 (-g^{\lambda\lambda}) a^\dagger(k, \lambda)a(k, \lambda)
\]

(1.248)

Then

\[
i[P^\mu, A^\nu] = \int \tilde{d}k \ \tilde{d}k' ik^\mu \sum_{\lambda, \lambda'} (-g^{\lambda\lambda'}) \left\{ [a^\dagger(k, \lambda)a(k, \lambda), a(k', \lambda')] \varepsilon^{\nu(k', \lambda')e^{-ik'x}} + [a^\dagger(x, \lambda)a(k, \lambda), a^\dagger(k', \lambda')] \varepsilon^{\nu(k', \lambda')e^{ik'x}} \right\}
\]

\[
= \int \tilde{d}k \ ik^\mu \sum_{\lambda} \left[ a(k, \lambda)\varepsilon^{\nu(k, \lambda)e^{-ikx}} - a^\dagger(k, \lambda)\varepsilon^{\nu(k, \lambda)e^{ikx}} \right]
\]

(1.249)

showing the invariance under translations. In a similar way, it can be shown the invariance for Lorentz transformations (see Problem 1.11). For that we have to show that

\[
M^{jk} = \int d^3x : [x^j T^{0k} - x^k T^{0j} + E^j A^k - E^k A^j] :
\]

(1.250)
\[ M^{0i} = \int d^3x : \left[ x^0 T^{0i} - x^i T^{00} - (\partial \cdot A) A^i - E^i A^0 \right] : \] (1.251)

where \((\xi = 1)\)

\[ T^{0i} = -(\partial \cdot A) \partial^i A^0 - E^k \partial^i A^k \]

\[ T^{00} = \sum_{i=1}^{3} \left[ \dot{A}^2_i + (\nabla A_j)^2 \right] - \dot{A}_0^2 - (\nabla A_0)^2 \] (1.252)

Using these expressions one can show that the photon has helicity \(\pm 1\), corresponding therefore to spin one. For that we start by choosing the direction of \(\mathbf{k}\) along the axis 3 (z axis) and take the polarization vector with the choice of Eq. (1.228). A one-photon physical state will then be (not normalized),

\[ |k,\lambda \rangle = a_\dagger(k,\lambda) |0 \rangle \quad \lambda = 1, 2 \] (1.253)

Let us now calculate the angular momentum along the axis 3. This is given by

\[ M^{12} |k,\lambda \rangle = M^{12} a_\dagger(k,\lambda) |0 \rangle \]

\[ = [M^{12}, a_\dagger(k,\lambda)] |0 \rangle \] (1.254)

where we have used the fact that the vacuum state satisfies \(M^{12} |0 \rangle = 0\). The operator \(M^{12}\) has one part corresponding the orbital angular momenta and another corresponding to the spin. The contribution of the orbital angular momenta vanishes (angular momenta in the direction of motion) as one can see calculating the commutator. In fact the commutator with the orbital angular momenta is proportional to \(k^1\) or \(k^2\), which are zero by hypothesis. Let us then calculate the spin part. Using the notation,

\[ A^\mu = A^{\mu(+)} + A^{\mu(-)} \] (1.255)

where \(A^{\mu(+)}(A^{\mu(-)})\) correspond to the positive (negative) frequencies, we get

\[ E^1 A^2 - E^2 A^1 := E^{1(+)} A^{2(+)} + E^{1(-)} A^{2(+)} + A^{2(-)} E^{1(+)} + E^{1(-)} A^{2(-)} - (1 \leftrightarrow 2) \] (1.256)

Then

\[ \left[ : E^1 A^2 - E^2 A^1 :, a_\dagger(k,\lambda) \right] = \]

\[ = E^{1(+)} \left[ A^{2 (+)}, a_\dagger(k,\lambda) \right] + \left[ E^{1 (+)}, a_\dagger(k,\lambda) \right] A^{2 (+)} \]

\[ + E^{1 (-)} \left[ A^{2 (+)}, a_\dagger(k,\lambda) \right] + A^{2 (-)} \left[ E^{1 (+)}, a_\dagger(k,\lambda) \right] - (1 \leftrightarrow 2) \]

\[ = E^1 \left[ A^{2 (+)}, a_\dagger(k,\lambda) \right] + A^2 \left[ E^{1 (+)}, a_\dagger(k,\lambda) \right] - (1 \leftrightarrow 2) \] (1.257)

Now (recall that \(\lambda = 1, 2\))

\[ [A^{2 (+)}, a_\dagger(k,\lambda)] = \int d\mathbf{k} \sum_{\lambda'} \varepsilon^{2}(k', \lambda') \left[ a(k', \lambda'), a_\dagger(k,\lambda) \right] e^{-i\mathbf{k}' \cdot \mathbf{x}} \]
\[ = \varepsilon^2(k, \lambda)e^{-ik \cdot x} \]

\[ \{E^1(+), a^\dagger(k, \lambda)\} = \int \tilde{d}k' \sum_{\lambda'} (ik'\varepsilon^0(k', \lambda') + ik'\varepsilon^0(k', \lambda')) \left[a(k', \lambda'), a^\dagger(k, \lambda)\right] e^{-ik' \cdot x} \]

\[ = ik\varepsilon^1(k, \lambda)e^{-ik \cdot x} \] (1.258)

Therefore

\[ \int d^3x \left[ :E^1 A^2 - E^2 A^1: , a^\dagger(k, \lambda)\right] \]

\[ = \int d^3x e^{-ik \cdot x} \left[ E^1\varepsilon^2(k, \lambda) + A^2ik\varepsilon^1(k, \lambda) - E^2\varepsilon^1(k, \lambda) + A^1ik\varepsilon^2(k, \lambda)\right] \]

\[ = \int d^3x e^{-ik \cdot x} \left[ \varepsilon^1(k, \lambda)\partial_0 A^2(x) - \varepsilon^2(k, \lambda)\partial_0 A^1(x)\right] \] (1.259)

where we have used the fact that \( E^i = -\dot{A}^i, \ i = 1, 2, \) for our choice of frame and polarization vectors. On the other hand

\[ a(k, \lambda) = -i \int d^3x e^{ik \cdot x} \partial_0 \varepsilon^\mu(k, \lambda)A_\mu(x) \]

\[ a^\dagger(k, \lambda) = i \int d^3x e^{-ik \cdot x} \bar{\partial}_0 \varepsilon^\mu(k, \lambda)A_\mu(x) \] (1.260)

For our choice we get

\[ a^\dagger(k, 1) = -i \int d^3x e^{-ik \cdot x} \bar{\partial}_0 A^1(x) \]

\[ a^\dagger(k, 2) = -i \int d^3x e^{-ik \cdot x} \bar{\partial}_0 A^2(x) \] (1.261)

and therefore

\[ [M^{12}, a^\dagger(k, \lambda)] = i\varepsilon^1(k, \lambda)a^\dagger(k, 2) - i\varepsilon^2(k, \lambda)a^\dagger(k, 1) \] (1.262)

We find that the state \( a^\dagger(k, \lambda)|0\rangle, \lambda = 1, 2 \) is not an eigenstate of the operator \( M^{12} \).

However the linear combinations,

\[ a^\dagger_R(k) = \frac{1}{\sqrt{2}} \left[ a^\dagger(k, 1) + ia^\dagger(k, 2)\right] \]

\[ a^\dagger_L(k) = \frac{1}{\sqrt{2}} \left[ a^\dagger(k, 1) - ia^\dagger(k, 2)\right] \] (1.263)

which correspond to right and left circular polarization, verify

\[ [M^{12}, a^\dagger_R(k)] = a^\dagger_L(k); [M^{12}, a^\dagger_L(k)] = -a^\dagger_R(k) \] (1.264)

showing that the photon has spin 1 with right or left circular polarization (negative or positive helicity).
1.4.3 Feynman propagator

The Feynman propagator is defined as the vacuum expectation value of the time ordered product of the fields, that is

\[ G_{\mu\nu}(x, y) \equiv \langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle \]

\[ = \theta(x^0 - y^0) \langle 0 | A_\mu(x) A_\nu(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | A_\nu(y) A_\mu(x) | 0 \rangle \]  

(1.265)

Inserting the expansions for \( A_\mu(x) \) and \( A_\nu(y) \) we get

\[ G_{\mu\nu}(x - y) = -g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 + i\varepsilon} e^{-ik \cdot (x - y)} \]

\[ = \int \frac{d^4k}{(2\pi)^4} G_{\mu\nu}(k) e^{-ik \cdot (x - y)} \]  

(1.266)

where \( G_{\mu\nu}(k) \) is the Feynman propagator on the momentum space

\[ G_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \]  

(1.267)

It is easy to verify that \( G_{\mu\nu}(x - y) \) is the Green’s function of the equation of motion, that for \( \xi = 1 \) is the wave equation, that is

\[ \square_x G_{\mu\nu}(x - y) = ig_{\mu\nu}\delta^4(x - y) \]  

(1.268)

These expressions for \( G_{\mu\nu}(x - y) \) and \( G_{\mu\nu}(k) \) correspond to the particular case of \( \xi = 1 \), the so-called Feynman gauge. For the general case where \( \xi \neq 0 \) the equation of motion reads

\[ \left[ \square_x g^\mu_\rho - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial_\rho \right] A^\rho(x) = 0 \]  

(1.269)

For this case the equal times commutation relations are more complicated (see Problem 1.12). Using those relations one can show that the Feynman propagator is still the Green’s function of the equation of motion, that is

\[ \left[ \square_x g^\mu_\rho - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial_\rho \right] \langle 0 | T A^\rho(x) A^\nu(y) | 0 \rangle = ig^{\mu\nu}\delta^4(x - y) \]  

(1.270)

Using this equation we can then obtain in an arbitrary \( \xi \) gauge (of the Lorenz type),

\[ G_{\mu\nu}(k) = -i \frac{g_{\mu\nu}}{k^2 + i\varepsilon} + i(1 - \xi) \frac{k_\mu k_\nu}{(k^2 + i\varepsilon)^2} \]  

(1.271)
1.5 Discrete Symmetries

We know from the study of the Dirac equation the transformations like space inversion (Parity) and charge conjugation, are symmetries of the Dirac equation. More precisely, if $\psi(x)$ is a solution of the Dirac equation, then

$$\psi'(x) = \psi(-\vec{x}, t) = \gamma_0 \psi(\vec{x}, t)$$

and

$$\psi^c(x) = C\psi^T(x)$$

are also solutions (if we take the charge $-e$ for $\psi^c$). Similar operations could also be defined for scalar and vector fields.

With second quantization the fields are no longer functions, they become operators. We have therefore to find unitary operators $P$ and $C$ that describe those operations within this formalism. There is another discrete symmetry, time reversal, that in second quantization will be described by an anti-unitary operator $T$. We will exemplify with the scalar field how to get these operators. We will leave the Dirac and Maxwell fields as exercises.

1.5.1 Parity

To define the meaning of the Parity operation we have to put the system in interaction with the measuring system, considered to be classical. This means that we will consider the system described by

$$\mathcal{L} \rightarrow \mathcal{L} - j'_\mu(x)A^\mu_{ext}(x)$$

where we have considered that the interaction is electromagnetic. $j'_\mu(x)$ is the electromagnetic current that has the form,

$$j'_\mu(x) = ie : \phi^* \partial_\mu \phi : \text{ scalar field}$$

$$j'_\mu(x) = e : \bar{\psi}\gamma_\mu \psi : \text{ Dirac field}$$

In a Parity transformation we invert the coordinates of the measuring system, therefore the classical fields are now

$$A^\mu_{ext} = (A^0_{ext}(-\vec{x}, t)), -A^\mu_{ext}(-\vec{x}, t) = A^\mu_{ext}(-\vec{x}, t)$$

For the dynamics of the new system to be identical to that of the original system, which should be the case if Parity is conserved, it is necessary that the equations of motion remain the same. This is true if

$$\mathcal{P} \mathcal{L}(\vec{x}, t) \mathcal{P}^{-1} = \mathcal{L}(-\vec{x}, t)$$

$$\mathcal{P} j'_\mu(\vec{x}, t) \mathcal{P}^{-1} = j'_\mu(-\vec{x}, t)$$

Eqs. (1.277) and (1.278) are the conditions that a theory should obey in order to be invariant under Parity. Furthermore $\mathcal{P}$ should leave the commutation relations unchanged, so that the quantum dynamics is preserved. For each theory that conserves Parity should be possible to find an unitary operator $\mathcal{P}$ that satisfies these conditions.
Now we will find such an operator $\mathcal{P}$ for the scalar field. It is easy to verify that the condition
\[ \mathcal{P} \varphi(\vec{x}, t) \mathcal{P}^{-1} = \pm \varphi(-\vec{x}, t) \] (1.279)
satisfies all the requirements. The sign $\pm$ is the intrinsic parity of the particle described by the field $\varphi$, (+ for scalar and $-$ for pseudo-scalar). In terms of the expansion of the momentum, Eq. (1.279) requires
\[ \mathcal{P} a(k) \mathcal{P}^{-1} = \pm a(-k) \; ; \; \mathcal{P} a^\dagger(k) \mathcal{P}^{-1} = \pm a^\dagger(-k) \] (1.280)
where $-k$ means that we have changed $\vec{k}$ into $-\vec{k}$ (but $k^0$ remains intact, that is, $k^0 = +\sqrt{\vec{k}^2 + m^2}$). It is easier to solve Eq. (1.280) in the momentum space. As $\mathcal{P}$ should be unitary, we write
\[ \mathcal{P} = e^{i\mathcal{P}} \] (1.281)
Then
\[ \mathcal{P} a(k) \mathcal{P}^{-1} = a(k) + i[P, a(k)] + \cdots + \frac{i^n}{n!} [P, \cdots, [P, a(k)] \cdots] + \cdots \]
\[ = -a(-k) \] (1.282)
where we have chosen the case of the pseudo-scalar field.

Eq. (1.282) suggests the form
\[ [P, a(k)] = \frac{\lambda}{2} [a(k) + \varepsilon a(-k)] \] (1.283)
where $\lambda$ and $\varepsilon = \pm 1$ are to be determined. We get
\[ [P, [P, a(k)]] = \frac{\lambda^2}{2} [a(k) + \varepsilon a(-k)] \] (1.284)
and therefore
\[ \mathcal{P} a(k) \mathcal{P}^{-1} = a(k) + \frac{1}{2} \left[ i\lambda + \frac{(i\lambda)^2}{2!} + \cdots + \frac{(i\lambda)^4}{4!} + \cdots \right] (a(k) + \varepsilon a(-k)) \]
\[ = \frac{1}{2} [a(k) - \varepsilon a(-k)] + \frac{1}{2} i\lambda [a(k) + \varepsilon a(-k)] \]
\[ = -a(-k) \] (1.285)
We solve Eq. (1.285) if we choose $\lambda = \pi$ and $\varepsilon = +1$ ($\lambda = \pi$ and $\varepsilon = -1$ for the scalar case). It is easy to check that
\[ P_{ps} = -\frac{\pi}{2} \int \tilde{dk} \left[ a^\dagger(k)a(k) + a^\dagger(k)a(-k) \right] = P_{ps}^\dagger \] (1.286)
and it is solution of Eq. (1.283) for $\lambda = \pi$ and $\varepsilon = +1$. Therefore,
\[ \mathcal{P}_{ps} = \exp \left\{ -i\frac{\pi}{2} \int \tilde{dk} \left[ a^\dagger(k)a(k) + a^\dagger(k)a(-k) \right] \right\} \] (1.287)
and for the scalar field
\[ P_s = \exp \left\{ -\frac{i\pi}{2} \int \frac{dk}{k} \left[ a^\dagger(k)a(k) - a^\dagger(k)a(-k) \right] \right\} \quad (1.288) \]

For the case of the Dirac field, the condition equivalent to Eq. (1.279) is now
\[ P\psi(\vec{x}, t)P^{-1} = \gamma^0 \psi(-\vec{x}, t) \quad (1.289) \]

Repeating the same steps we get
\[ P_{\text{Dirac}} = \exp \left\{ -\frac{i\pi}{2} \int \frac{dp}{p} \sum_s \left[ b^\dagger(p, s)b(p, s) - b^\dagger(p, s)b(-p, s) \right. \right. \]
\[ + \left. \left. d^\dagger(p, s)d(p, s) + d^\dagger(p, s)d(-p, s) \right] \right\} \quad (1.290) \]

The case of the Maxwell field is left as an exercise.

### 1.5.2 Charge conjugation

The conditions for charge conjugation invariance are now
\[ \mathcal{C}\mathcal{L}(x)\mathcal{C}^{-1} = \mathcal{L} ; \quad \mathcal{C}j_\mu\mathcal{C}^{-1} = -j_\mu \quad (1.291) \]

where \( j_\mu \) is the electromagnetic current. Conditions (1.291) are verified for the charged scalar fields if
\[ \mathcal{C}\phi(x)\mathcal{C}^{-1} = \phi^*(x) ; \quad \mathcal{C}\phi^*(x)\mathcal{C}^{-1} = \phi(x) \quad (1.292) \]

and for the Dirac field if
\[ \mathcal{C}\psi_\alpha(x)\mathcal{C}^{-1} = C_{\alpha\beta}\psi_\beta(x) \]
\[ \mathcal{C}\psi_\beta(x)\mathcal{C}^{-1} = -\psi_\beta(x)C_{\beta\alpha}^{-1} \quad (1.293) \]

where \( C \) is the charge conjugation matrix.

Finally from the invariance of \( j_\mu A^\mu \) we obtain the condition for the electromagnetic field,
\[ \mathcal{C}A_\mu\mathcal{C}^{-1} = -A_\mu \quad (1.294) \]

By using a method similar to the one used in the case of the Parity we can get the operator \( \mathcal{C} \) for the different theories. For instance, for the scalar field we get
\[ \mathcal{C}_s = \exp \left\{ i\frac{\pi}{2} \int \frac{dk}{k} (a^\dagger_+ - a^\dagger_-)(a_+ - a_-) \right\} \quad (1.295) \]

and for the Dirac field
\[ \mathcal{C} = C_1C_2 \quad (1.296) \]
with

\[ C_1 = \exp \left\{ -i \int \overrightarrow{dp} \sum_s \phi(p, s) \left[ b^{\dagger}(p, s)b(p, s) - d^{\dagger}(p, s)d(p, s) \right] \right\} \]

\[ C_2 = \exp \left\{ \frac{i\pi}{2} \int \overrightarrow{dp} \sum_s \left[ b^{\dagger}(p, s) - d^{\dagger}(p, s) \right] \left[ b(p, s) - d(p, s) \right] \right\} \] (1.297)

with

\[ v(p, s) = e^{i\phi(p, s)} u^c(p, s) \]

\[ u(p, s) = e^{i\phi(p, s)} v^c(p, s) \] (1.298)

where the phase \( \phi(p, s) \) is arbitrary (see [5]).

### 1.5.3 Time reversal

Classically the meaning of the time reversal invariance it is clear. We change the sign of the time, the velocities change direction and the system goes from what was the final state to the initial state. This exchange between the initial and final state has as consequence, in quantum mechanics, that the corresponding operator must be anti-linear or anti-unitary. In fact \( \langle f | i \rangle = \langle i | f \rangle^* \) and therefore if we want \( \langle T\varphi_f | T\varphi_i \rangle = \langle \varphi_i | \varphi_f \rangle \) then \( T \) must include the complex conjugation operation. We can write

\[ T = UK \] (1.299)

where \( U \) is unitary and \( K \) is the instruction to take the complex conjugate of all c-numbers. Then

\[ \langle T\varphi_f | T\varphi_i \rangle = \langle UK\varphi_f | UK\varphi_i \rangle \]

\[ = \langle U\varphi_f | U\varphi_i \rangle^* \]

\[ = \langle \varphi_i | \varphi_f \rangle^* = \langle \varphi_f | \varphi_i \rangle \] (1.300)

as we wanted. A theory will be invariant under time reversal if

\[ T\mathcal{L}(\vec{x}, t)T^{-1} = \mathcal{L}(\vec{x}, -t) \]

\[ Tj_{\mu}(\vec{x}, t)T^{-1} = j^\mu(\vec{x}, -t) \] (1.301)

For the scalar field this condition will be verified if

\[ T\varphi(\vec{x}, t)T^{-1} = \pm\varphi(\vec{x}, -t) \] (1.302)

and for the electromagnetic field we must have.

\[ TA^\mu(\vec{x}, t)T^{-1} = A^\mu(\vec{x}, -t) \] (1.303)
1.5. DISCRETE SYMMETRIES

making \( j^\mu A_\mu \) invariant. For the case of the Dirac field the transformation is

\[
T\psi_\alpha(\vec{x}, t)T^{-1} = T_{\alpha\beta}\psi_\beta(\vec{x}, -t)
\]  
(1.304)

In order that Eq. (1.301) is satisfied, the \( T \) matrix must satisfy

\[
T\gamma_\mu T^{-1} = \gamma^T_\mu = \gamma^{\mu*}
\]  
(1.305)

with a solution, in the Dirac representation,

\[
T = i\gamma^1\gamma^3
\]  
(1.306)

Applying the same type of reasoning already used for \( \mathcal{P} \) and \( \mathcal{C} \) we can find \( \mathcal{T} \), or equivalently, \( \mathcal{U} \). For the Dirac field, noticing that

\[
Tu(p, s) = u^*(−p, −s)e^{i\alpha_+(p, s)}
\]

\[
Tv(p, s) = v^*(−p, −s)e^{i\alpha-(p, s)}
\]  
(1.307)

we can write \( \mathcal{U} = \mathcal{U}_1\mathcal{U}_2 \) and obtain

\[
\mathcal{U}_1 = \exp\left\{-i\int \frac{dp}{p} \sum_s \left[ \alpha_+ b^\dagger(p, s)b(p, s) - \alpha_- d^\dagger(p, s)d(p, s) \right]\right\}
\]  
(1.308)

and

\[
\mathcal{U}_2 = \exp\left\{-i\frac{\pi}{2}\int \frac{dp}{p} \sum_s \left[ b^\dagger(p, s)b(p, s) + b^\dagger(p, s)b(−p − s)
\right.
\]

\[
\left. - d^\dagger(p, s)d(p, s) - d^\dagger(p, s)d(−p, −s) \right]\right\}
\]  
(1.309)

1.5.4 The \( \mathcal{TCP} \) theorem

It is a fundamental theorem in Quantum Field Theory that the product \( \mathcal{TCP} \) is an invariance of any theory that satisfies the following general conditions:

- The theory is local and covariant for Lorentz transformations.
- The theory is quantized using the usual relation between spin and statistics, that is, commutators for bosons and anti-commutators for fermions.

This theorem due to Lüdus, Zumino, Pauli e Schwinger has an important consequence that if one of the discrete symmetries is not preserved then another one must also be violated to preserve the invariance of the product. For a proof of the theorem see the books of Bjorken and Drell[1, 6] and Itzykson and Zuber[7].
1.1 Verify, for the scalar field, the covariant relations for translations and Lorentz transformations,

\[ i[P^\mu, \varphi] = \partial^\mu \varphi \]
\[ i[M^{\mu\nu}, \varphi] = (x^\mu \partial^\nu - x^\nu \partial^\mu) \varphi \] (1.310)

1.2 Show that

\[ \partial^0 \Delta(x - y)|_{x^0 = y^0} = -\delta^3(\vec{x} - \vec{y}) \] (1.311)

1.3 Show that

\[
\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)} = \int \tilde{dk} \left[ \theta(x^0 - y^0)e^{-ik(x-y)} + \theta(y^0 - x^0)e^{ik(x-y)} \right]
\] (1.312)

where \( \tilde{dk} \equiv \frac{d^3k}{(2\pi)^32k^0} \). **Hint:** Integrate in the complex plane of the variable \( dk^0 \) and use the prescription \( i\varepsilon \) to define the contours.

1.4 Show that for the Dirac theory the requirements of Lorentz invariance are satisfied,

\[ i[M^{\mu\nu}, \varphi] = (x^\mu \partial^\nu - x^\nu \partial^\mu) \varphi + \Sigma^{\mu\nu} \varphi \quad ; \quad \Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \] (1.313)

1.5 Show that

\[ \{\psi_\alpha(\vec{x}, t), \psi_\beta^\dagger(\vec{y}, t)\} = \delta_{\alpha\beta} \delta^3(\vec{x} - \vec{y}) \] (1.314)

1.6 Show that

\[
S_F(x - y)_{\alpha\beta} = \theta(x^0 - y^0) \langle 0 | \overline{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle - \theta(y^0 - x^0) \langle 0 | \overline{\psi}_\alpha(x) \psi_\beta(y) | 0 \rangle
\] (1.315)
corresponds to
Problems

\[ S_F(x - y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i(p + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon} e^{-ip \cdot (x - y)} \]  

(1.316)

**Hint:** Expand \( \psi_\alpha e^{-i \psi_\beta} \) in plane waves.

1.7 Show that

\[ (i\partial_x - m)_{\alpha\beta} S_F(x - y)_{\beta\gamma} = i\delta_{\alpha\gamma} \delta^4(x - y) \]  

(1.317)

1.8 Show that it is always possible to choose the electromagnetic potential \( A^\mu \) such that

\[ A^0 = 0, \nabla \cdot \vec{A} = 0 \quad \text{(Radiation gauge)} \]  

(1.318)

1.9 Show that we have

\[ [A^\mu(x), A^\nu(y)] = -ig_{\mu\nu} \Delta(x - y) \]  

(1.319)

1.10 Consider the indefinite metric formalism for the electromagnetic field.

a) Consider the expectation value of \( A_\mu \) in the state \( |\phi\rangle \). Show that

\[ \langle \phi | A_\mu | \phi \rangle = C_0^* C_1 \int \tilde{dk} \ e^{-ik \cdot x} \langle 0 | [\epsilon^\mu_\nu(k,3) a(k,3) + \epsilon^\mu_\nu(k,0) a(k,0)] | \phi_1 \rangle \]  

+ h.c.  

(1.320)

b) Choose the state \( |\phi_1\rangle \) in the form

\[ |\phi_1\rangle = \int \tilde{dk} \ f(k) \left[ a^\dagger(k,3) - a^\dagger(k,0) \right] |0\rangle \]  

(1.321)

Show that

\[ \langle \phi | A_\mu | \phi \rangle = \int \tilde{dk} \ [\epsilon^\mu_\nu(k,3) + \epsilon^\mu_\nu(k,0)] (C_0^* C_1 e^{-ik \cdot x} f(k) + c.c.) \]  

(1.322)

c) Choose \( \epsilon^\mu(k,\lambda) \) to be real. Show that

\[ \epsilon^\mu(k,3) + \epsilon^\mu(k,0) = \frac{k^\mu}{(n \cdot k)} \]  

(1.323)

d) Show that

\[ \langle \phi | A_\mu | \phi \rangle = \partial_\mu \Lambda(x) \]  

(1.324)

where

\[ \Box \Lambda = 0 \]  

(1.325)

Comment the result.
1.11 Show the covariance of the electromagnetism for the Lorentz transformations,

\[ i[M^{\mu\nu}, A^\lambda] = (x^\mu \partial^\nu - x^\nu \partial^\mu)A^\lambda + \Sigma^{\mu\nu,\lambda}_\sigma A^\sigma \tag{1.326} \]

where

\[ \Sigma^{\mu\nu,\lambda}_\sigma = g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda} \tag{1.327} \]

1.12 Show that for the general case of \( \xi \neq 1 \) we have

\[ [A_\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = 0 \]

\[ [\dot{A}_i(\vec{x}, t), \dot{A}_j(\vec{y}, t)] = [\dot{A}_0(\vec{x}, t), \dot{A}_0(\vec{y}, t)] = 0 \]

\[ [\dot{A}_0(\vec{x}, t), \dot{A}_i(\vec{y}, t)] = i(1 - \xi)\partial_i \delta^3(\vec{x} - \vec{y}) \tag{1.328} \]

1.13 Use the results of Problem 1.12 to show that, in the general gauge with \( \xi \neq 1 \) we have

\[ \left[ \Box g^\mu_\rho - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial_\rho \right] \langle 0 | TA^\rho(x)A^\nu(y) | 0 \rangle = ig^{\mu\nu} \delta^3(x - y) \tag{1.329} \]

where

\[ \left( \Box g^\mu_\rho - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial_\rho \right) A^\rho = 0 \tag{1.330} \]

1.14 Find the operator \( P \) for the Dirac and Maxwell fields.

1.15 Find the operator \( C \) for the Dirac and Maxwell fields.

1.16 Show that

\[ T \psi_\alpha(\vec{x}, t)T^{-1} = T_{\alpha\beta} \psi_\beta(\vec{x}, -t) \tag{1.331} \]

ensures that

\[ T \mathcal{L}(\vec{x}, t)T^{-1} = \mathcal{L}(\vec{x}, -t) \tag{1.332} \]

if there is a matrix \( T \) such that \( T \gamma_\mu T^{-1} = \gamma^{\mu*} \). Find \( T \) in the Dirac representation.

1.17 Find the operator \( T \) for the Dirac and Maxwell fields.

1.18 Consider the Lagrangian

\[ \mathcal{L} = \bar{\psi} i\gamma^\mu D_\mu P_L \psi - m \bar{\psi} \psi \tag{1.333} \]

where

\[ D_\mu = \partial_\mu + iA_\mu^a T^a \]

\[ P_L = \frac{1 - \gamma_5}{2} \tag{1.334} \]

Show that the theory is neither invariant under \( P \) nor under \( C \) but it is invariant for the product \( CP \).
Chapter 2

Physical States. S Matrix. LSZ Reduction.

2.1 Physical states

In the previous chapter we saw, for the case of free fields, how to construct the space of states, the so-called Fock space of the theory. When we consider the real physical case, with interactions, we are no longer able to solve the problem exactly. For instance, the interaction between electrons and photons is given by a set of nonlinear coupled equations,

\[
(i\partial - m)\psi = eA\psi \\
\partial_\mu F^{\mu\nu} = e\gamma^\nu \psi
\]

that do not have an exact solution. In practice we have to resort to approximation methods. In the following chapter we will learn how to develop a covariant perturbation theory. Here we are going just to study the general properties of the theory.

Let us start by the physical states. As we do not know how to solve the problem exactly, we cannot prove the assumptions we are going to make about these states. However, these are reasonable assumptions, based essentially on Lorentz covariance. We choose our states to be eigenstates of energy and momentum, and of all the other observables that commute with \( P^\mu \). Besides that, we will also assume that

i) The eigenvalues of \( p^2 \) are non-negative and \( p^0 > 0 \).

ii) There exists one non-degenerate base state, with the minimum of energy, which is Lorentz invariant. This state is called the vacuum state \( |0\rangle \) and by convention

\[
p^\mu |0\rangle = 0
\]

iii) There exist one particle states \( |p^{(i)}\rangle \), such that,

\[
p^{(i)}_\mu p^{(i)\mu} = m_i^2
\]
for each stable particle with mass $m_i$.

iv) The vacuum and the one-particle states constitute the discrete spectrum of $p'$.

## 2.2 In states

As we are mainly interested in scattering problems, we should construct states that have a simple interpretation in the limit $t \to -\infty$. At that time, the particles that are going to participate in the scattering process have not interacted yet (we assume that the interactions are adiabatically switched off when $|t| \to \infty$ which is appropriate for scattering problems).

We look for operators that create one particle states with the physical mass. To be explicit, we start by an hermitian scalar field given by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial \mu \varphi \partial_{\mu} \varphi - \frac{1}{2} m^2 \varphi^2 - V(x) \quad (2.4)$$

where $V(x)$ is an operator made of more than two interacting fields $\varphi$ at point $x$. For instance, those interactions can be self-interactions of the type

$$V(x) = \frac{\lambda}{4!} \varphi^4(x) \quad (2.5)$$

The field $\varphi$ satisfies the following equation of motion

$$(\Box + m^2) \varphi(x) = -\frac{\partial V}{\partial \varphi(x)} \equiv j(x) \quad (2.6)$$

and the equal time canonical commutation relations,

$$[\varphi(\vec{x}, t) \varphi(\vec{y}, t)] = [\pi(\vec{x}, t) \pi(\vec{y}, t)] = 0$$

$$[\pi(\vec{x}, t), \varphi(\vec{y}, t)] = -i \delta^3(\vec{x} - \vec{y}) \quad (2.7)$$

where

$$\pi(x) = \dot{\varphi}(x) \quad (2.8)$$

if we assume that $V(x)$ has no derivatives. We designate by $\varphi_{in}(x)$ the operator that creates one-particle states. It will be a functional of the fields $\varphi(x)$. Its existence will be shown by explicit construction. We require that $\varphi_{in}(x)$ must satisfy the conditions:

i) $\varphi_{in}(x)$ and $\varphi(x)$ transform in the same way for translations and Lorentz transformations. For translations we have then

$$i [P^\mu, \varphi_{in}(x)] = \partial^\mu \varphi_{in}(x) \quad (2.9)$$

ii) The spacetime evolution of $\varphi_{in}(x)$ corresponds to that of a free particle of mass $m$, that is

$$(\Box + m^2) \varphi_{in}(x) = 0 \quad (2.10)$$
2.2. IN STATES

From these definitions it follows that \( \varphi_{in}(x) \) creates one-particle states from the vacuum. In fact, let us consider a state \( |n\rangle \), such that,

\[
P^\mu |n\rangle = p_n^\mu |n\rangle.
\]

Then

\[
\partial^\mu \langle n|\varphi_{in}(x)|0\rangle = i \langle n|[P^\mu, \varphi_{in}(x)]|0\rangle
\]

and therefore

\[
\Box \langle n|\varphi_{in}(x)|0\rangle = -p_n^2 \langle n|\varphi_{in}(x)|0\rangle
\]

Then

\[
(\Box + m^2) \langle n|\varphi_{in}(x)|0\rangle = (m^2 - p_n^2) \langle n|\varphi_{in}(x)|0\rangle = 0
\]

where we have used the fact that \( \varphi_{in}(x) \) is a free field, Eq. (2.10). Therefore the states created from the vacuum by \( \varphi_{in} \) are those for which \( p_n^2 = m^2 \), that is, the one-particle states of mass \( m \).

The Fourier decomposition of \( \varphi_{in}(x) \) is then the same as for free fields, that is,

\[
\varphi_{in}(x) = \int \frac{dk}{2\pi} \left[ a_{in}(k)e^{-ik\cdot x} + a_{in}^\dagger(k)e^{ik\cdot x} \right]
\]

where \( a_{in}(k) \) and \( a_{in}^\dagger(k) \) satisfy the usual algebra for creation and annihilation operators. In particular, by repeated use of \( a_{in}^\dagger(k) \) we can create one state of \( n \) particles.

To express \( \varphi_{in}(x) \) in terms of \( \varphi(x) \) we start by introducing the retarded Green’s function of the Klein-Gordon operator,

\[
(\Box_x + m^2)\Delta_{ret}(x - y; m) = \delta^4(x - y)
\]

where

\[
\Delta_{ret}(x - y; m) = 0 \quad \text{if} \quad x^0 < y^0
\]

We can then write

\[
\sqrt{Z}\varphi_{in}(x) = \varphi(x) - \int d^4y \Delta_{ret}(x - y; m)j(y)
\]

The field \( \varphi_{in}(x) \), defined by Eq. (2.15), satisfies the two initial conditions. The constant \( \sqrt{Z} \) was introduced to normalize \( \varphi_{in} \) in such a way that it has amplitude 1 to create one-particle states from the vacuum. The fact that \( \Delta_{ret} = 0 \) for \( x_0 \to -\infty \), suggests that \( \sqrt{Z}\varphi_{in}(x) \) is, in some way, the limit of \( \varphi(x) \) when \( x_0 \to -\infty \). In fact, as \( \varphi \) and \( \varphi_{in} \) are operators, the correct asymptotic condition must be set on the matrix elements of the operators. Let \( |\alpha\rangle \) and \( |\beta\rangle \) be two normalized states. We define the operators

\[
\varphi^f(t) = i \int d^3xf^*(x)\partial_0^\dagger \varphi(x)
\]

\[
\varphi_{in}^f = i \int d^3xf^*(x)\partial_0^\dagger \varphi_{in}(x)
\]
where $f(x)$ is a normalized solution of the Klein-Gordon equation. By Green’s theorem, $\varphi_{in}^f$ does not depend on time (for plane waves $f = e^{-ik\cdot x}$ and $\varphi_{in}^f = a_{in}$). Then the asymptotic condition of Lehmann, Symanzik e Zimmermann (LSZ) \cite{8}, is

$$\lim_{t \to -\infty} \langle \alpha | \varphi^f(t) | \beta \rangle = \sqrt{Z} \langle \alpha | \varphi_{in}^f | \beta \rangle \quad (2.20)$$

### 2.3 Spectral representation for scalar fields

We saw that $Z$ had a physical meaning as the square of the amplitude for the field $\varphi(x)$ to create one-particle states from the vacuum. Let us now find a formal expression for $Z$ and show that $0 \leq Z \leq 1$.

We start by calculating the expectation value in the vacuum of the commutator of two fields,

$$i \Delta'(x, y) \equiv \langle 0 | [\varphi(x), \varphi(y)] | 0 \rangle \quad (2.21)$$

As we do not know how to solve the equations for the interacting fields $\varphi$, we can not solve exactly the problem of finding the $\Delta'$, in contrast with the free field case. We can, however, determine its form using general arguments of Lorentz invariance and the assumed spectra for the physical states. We introduce a complete set of states between the two operators in Eq. (2.21) and we use the invariance under translations in order to obtain,

$$\langle n | \varphi(y) | m \rangle = \langle n | e^{iP \cdot y} \varphi(0) e^{-iP \cdot y} | m \rangle = e^{i(p_n - p_m) \cdot y} \langle n | \varphi(0) | m \rangle \quad (2.22)$$

Therefore we get

$$\Delta'(x, y) = -i \sum_n \langle 0 | \varphi(0) | n \rangle \langle n | \varphi(0) | 0 \rangle (e^{-ip_n \cdot (x-y)} - e^{ip_n \cdot (x-y)})$$

$$\equiv \Delta'(x - y) \quad (2.23)$$

that is, like in the free field case, $\Delta'$ is only a function of the difference $x - y$. Introducing now

$$1 = \int d^4q \, \delta^4(q - p_n) \quad (2.24)$$

we get

$$\Delta'(x - y) = -i \int \frac{d^4q}{(2\pi)^3} \left[ (2\pi)^3 \sum_n \delta^4(p_n - q) \langle 0 | \varphi(0) | n \rangle \langle n | \varphi(0) | 0 \rangle \right] (e^{-iq \cdot (x-y)} - e^{iq \cdot (x-y)})$$

$$= -i \int \frac{d^4q}{(2\pi)^3} \rho(q) (e^{-iq \cdot (x-y)} - e^{iq \cdot (x-y)}) \quad (2.25)$$

where we have defined the density $\rho(q)$ (spectral amplitude),

$$\rho(q) = (2\pi)^3 \sum_n \delta^4(p_n - q) \langle 0 | \varphi(0) | n \rangle \langle n | \varphi(0) | 0 \rangle \quad (2.26)$$
This spectral amplitude measures the contribution to \( \Delta' \) of the states with 4-momentum \( q^\mu \). \( \rho(q) \) is Lorentz invariant (as can be shown using the invariance of \( \varphi(x) \) and the properties of the vacuum and of the states \( |n\rangle \)) and vanishes when \( q \) is not in future light cone, due to the assumed properties of the physical states. Then we can write

\[
\rho(q) = \overline{\rho}(q^2) \theta(q^0)
\] (2.27)

and we get

\[
\Delta'(x - y) = -i \int \frac{d^4q}{(2\pi)^3} \overline{\rho}(q^2) \theta(q^0) \left( e^{-iq(x-y)} - e^{iq(x-y)} \right)
\]

\[
= -i \int \frac{d^4q}{(2\pi)^3} \int d\sigma^2 \delta(q^2 - \sigma^2) \overline{\rho}(\sigma^2) \theta(q^0) \left[ e^{-iq(x-y)} - e^{iq(x-y)} \right]
\]

\[
= \int_0^\infty d\sigma^2 \overline{\rho}(\sigma^2) \Delta(x - y; \sigma)
\] (2.28)

where

\[
\Delta(x - y; \sigma) = -i \int \frac{d^4q}{(2\pi)^3} \delta(q^2 - \sigma^2) \theta(q^0) \left( e^{-iq(x-y)} - e^{iq(x-y)} \right)
\] (2.29)

is the invariant function defined for the commutator of free fields with mass \( \sigma \).

The Eq. (2.28) is known as the spectral decomposition of the commutator of two fields. This expression will allow us to show that \( 0 \leq Z < 1 \). To show that, we separate the states of one-particle from the sum in Eq. (2.26). Let \( |p\rangle \) be a one-particle state with momentum \( p \). Then

\[
\langle 0|\varphi(x)|p\rangle = \sqrt{Z} \langle 0|\varphi_\text{in}(x)|p\rangle + \int d^4y \Delta_{\text{ret}}(x - y; m) \langle 0|j(y)|p\rangle
\]

\[
= \sqrt{Z} \langle 0|\varphi_\text{in}(x)|p\rangle
\] (2.30)

where we have used

\[
\langle 0|j(y)|p\rangle = \langle 0|(\Box + m^2)\varphi_\text{in}(y)|p\rangle =
\]

\[
= (\Box + m^2)e^{-ip\cdot y} \langle 0|\varphi_\text{in}(0)|p\rangle
\]

\[
= (m^2 - p^2)e^{-ip\cdot y} \langle 0|\varphi_\text{in}(0)|p\rangle = 0
\] (2.31)

On the other hand

\[
\langle 0|\varphi_\text{in}(x)|p\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot x} \langle 0|a_\text{in}(k)|p\rangle
\]

\[
= e^{-ip\cdot x}
\] (2.32)

and therefore

\[
\rho(q) = (2\pi)^3 \int dp \, \delta^4(p - q)Z + \text{contributions from more than one particle}
\]
\[ Z \delta(q^2 - m^2) \theta(q^0) + \cdots \] (2.33)

Therefore
\[ \Delta'(x - y) = Z \Delta(x - y; m) + \int_{m_1^2}^{\infty} d\sigma^2 \overline{p}(\sigma^2) \Delta(x - y; \sigma) \] (2.34)

where \( m_1 \) is the mass of the lightest state of two or more particles. Finally noticing that
\[ \frac{\partial}{\partial x^0} \Delta'(x - y)|_{x^0 = y^0} = \frac{\partial}{\partial x^0} \Delta(x - y; \sigma)|_{x^0 = y^0} = -\delta^3(\vec{x} - \vec{y}) \] (2.35)

we get the relation
\[ 1 = Z + \int_{m_1^2}^{\infty} d\sigma^2 \overline{p}(\sigma^2) \] (2.36)

which means
\[ 0 \leq Z < 1 \] (2.37)

where this last step results from the assumed positivity of \( \overline{p}(\sigma^2) \).

### 2.4 Out states

In the same way as we reduced the dynamics of \( t \to -\infty \) to the free fields \( \varphi_{in} \), it is also possible to define in the limit \( t \to +\infty \) the corresponding free fields, \( \varphi_{out}(x) \). These free fields will be the final state of a scattering problem. The formalism is copied from the case of \( \varphi_{in} \), and therefore we will present the results without going into the details of the derivations. \( \varphi_{out}(x) \) obey the following relations:
\[ i \left[ P^\mu, \varphi_{out} \right] = \partial^\mu \varphi_{out} \]
\[ \Box + m^2 \varphi_{out} = 0 \] (2.38)

and has the expansion
\[ \varphi_{out}(x) = \int d^4k \left[ a_{out}(k)e^{-ik \cdot x} + a_{out}^\dagger(k)e^{ik \cdot x} \right] \] (2.39)

The asymptotic condition is now
\[ \lim_{t \to \infty} \langle \alpha | \varphi^f(t) | \beta \rangle = \sqrt{Z} \langle \alpha | \varphi_{out}^f | \beta \rangle \] (2.40)

and
\[ \sqrt{Z} \varphi_{out}(x) = \varphi(x) - \int d^4y \Delta_{adv}(x - y; m)j(y) \] (2.41)

where the Green’s functions \( \Delta_{adv} \) satisfy
\[ \Box + m^2 \Delta_{adv}(x - y; m) = \delta^4(x - y) \]
\[ \Delta_{adv}(x - y; m) = 0 ; \quad x^0 > y^0 . \] (2.42)

For one-particle states we get
\[ \langle 0 | \varphi(x) | p \rangle = \sqrt{Z} \langle 0 | \varphi_{out}(x) | p \rangle = \sqrt{Z} \langle 0 | \varphi_{in}(x) | p \rangle = \sqrt{Z} e^{-ip \cdot x} \] (2.43)
2.5 S matrix

We have now all the formalism needed to study the transition amplitudes from one initial state to a given final state, the so-called S matrix elements. Let us start by an initial state with \( n \) non-interacting particles (we suppose that initially they are well separated),

\[
|p_1 \cdots p_n; \text{in}\rangle \equiv |\alpha; \text{in}\rangle \quad (2.44)
\]

where \( p_1 \cdots p_n \) are the 4-momenta of the \( n \) particles. Other quantum numbers are assumed but not explicitly written. The final state will be, in general, a state with \( m \) particles

\[
|p'_1 \cdots p'_m; \text{out}\rangle \equiv |\beta; \text{out}\rangle \quad (2.45)
\]

The S matrix element \( S_{\beta\alpha} \) is defined by the amplitude

\[
S_{\beta\alpha} \equiv \langle \beta; \text{out}|\alpha; \text{in}\rangle \quad (2.46)
\]

The S matrix is an operator that induces an isomorphism between the in and out states, that by assumption are a complete set of states,

\[
\langle \beta; \text{out} = \langle \beta; \text{in}|S \quad (2.47)
\]

i) \( \langle 0|S|0\rangle = \langle 0|0\rangle = 1 \) (stability and unicity of the vacuum)

ii) The stability of the one-particle states gives

\[
\langle p; \text{in}|S|p; \text{in}\rangle = \langle p; \text{out}|p; \text{in}\rangle = \langle p; \text{in}|p; \text{out}\rangle = 1 \quad (2.48)
\]

because \( |p; \text{in}\rangle = |p; \text{out}\rangle \).

iii) \( \varphi_{\text{in}}(x) = S\varphi_{\text{out}}(x)S^{-1} \)

iv) The S matrix is unitary. To show this we have

\[
\delta_{\beta\alpha} = \langle \beta; \text{out}|\alpha; \text{out}\rangle = \langle \beta; \text{in}|SS^\dagger|\alpha; \text{in}\rangle \quad (2.49)
\]

and therefore

\[
SS^\dagger = 1 \quad (2.50)
\]
v) The $S$ matrix is Lorentz invariant. In fact we have

$$\varphi_{in}(ax + b) = U(a, b)\varphi_{in}(x)U^{-1}(a, b) = US\varphi_{out}(x)S^{-1}U^{-1}$$

But

$$\varphi_{in}(ax + b) = S\varphi_{out}(ax + b)S^{-1},$$

and therefore we get finally

$$S = U(a, b)SU^{-1}(a, b).$$

### 2.6 Reduction formula for scalar fields

The $S$ matrix elements are the quantities that are directly connected to the experiment. In fact, $|S_{\beta\alpha}|^2$ represents the transition probability from the initial state $|\alpha; in\rangle$ to the final $|\beta; out\rangle$. We are going in this section to use the previous formalism to express these matrix elements in terms of the so-called Green functions for the interacting fields. In this way the problem of the calculation of these probabilities is transferred to the problem of calculating these Green functions. These, of course, can not be evaluated exactly, but we will learn in the next chapter how to develop a covariant perturbation theory for that purpose.

Let us then proceed to the derivation of the relation between the $S$ matrix elements and the Green functions of the theory. This technique is known as the **LSZ reduction** from the names of Lehmann, Symanzik and Zimmermann [8] that have introduced it. By definition

$$\langle p_1 \cdots ; out|q_1 \cdots ; in \rangle = \langle p_1, \cdots ; out|a_{in}^{\dagger}(q_1)|q_2, \cdots ; in \rangle$$

Using

$$a_{in}^{\dagger}(q_1) = -i \int_t d^3xe^{-iq_1 \cdot x} \partial_0 \varphi_{in}(x)$$

where the integral is time-independent, and therefore can be calculated for an arbitrary time $t$. If we take $t \to -\infty$ and use the asymptotic condition for the $in$ fields, Eq. (2.20), we get

$$\langle p_1 \cdots ; out|q_1 \cdots ; in \rangle = - \lim_{t \to -\infty} iZ^{-1/2} \int_t d^3xe^{-iq_1 \cdot x} \partial_0 \langle p_1 \cdots ; out|\varphi(x)|q_2 \cdots ; in \rangle$$

In a similar way one can show that

$$\langle p_1 \cdots ; out|a_{out}^{\dagger}(q_1)|q_2 \cdots ; in \rangle =$$
2.6. REDUCTION FORMULA FOR SCALAR FIELDS

\[ = - \lim_{t \to \infty} iZ^{-1/2} \int d^3x e^{-iq_1 \cdot x} \hat{\partial}_0 \langle p_1 \cdots; out|\varphi(x)|q_2 \cdots; in \rangle. \] (2.57)

Then, using the result,

\[ \left( \lim_{t \to \infty} - \lim_{t \to -\infty} \right) \int d^3x f(\vec{x}, t) = \int d^4x \partial_0 f(\vec{x}, t) \] (2.58)

and subtracting Eq. (2.57) from Eq. (2.56) we get

\[ \langle p_1 \cdots; out|q_1 \cdots; in \rangle = \langle p_1 \cdots; out|a_{out}^\dagger(q_1)|q_2 \cdots; in \rangle \]

\[ + iZ^{-1/2} \int d^4x \partial_0 \left[ e^{-iq_1 \cdot x} \hat{\partial}_0 \langle p_1 \cdots; out|\varphi(x)|q_2 \cdots; in \rangle \right] \] (2.59)

The first term on the right-hand side of Eq. (2.59) corresponds to a sum of disconnected terms, in which at least one of the particles is not affected by the interaction (it will vanish if none of the initial momenta coincides with one of the final momenta). Its form is

\[ \langle p_1 \cdots; out|a_{out}^\dagger(q_1)|q_2 \cdots; in \rangle = \sum_{k=1}^n (2\pi)^3 2p_k^0 \delta^3(\vec{p}_k - \vec{q}_1) \langle p_1, \cdots, \hat{p}_k, \cdots; out|q_2, \cdots; in \rangle \] (2.60)

where \( \hat{p}_k \) means that this momentum was taken out from that state. For the second term we write,

\[ \int d^4x \partial_0 \left[ e^{-iq_1 \cdot x} \hat{\partial}_0 \langle p_1 \cdots; out|\varphi(x)|q_2 \cdots; in \rangle \right] \]

\[ = \int d^4x \left[ -\hat{\partial}_0^2 e^{-iq_1 \cdot x} \langle \cdots \rangle + e^{-iq_1 \cdot x} \partial_0^2 \langle \cdots \rangle \right] \]

\[ = \int d^4x \left[ (-\Delta^2 + m^2) e^{-iq_1 \cdot x} \langle \cdots \rangle + e^{-iq_1 \cdot x} \partial_0^2 \langle \cdots \rangle \right] \]

\[ = \int d^4x e^{-iq_1 \cdot x} (\Box + m^2) \langle p_1 \cdots; out|\varphi(x)|q_2 \cdots; in \rangle \] (2.61)

where we have used \( (\Box + m^2)e^{-iq_1 \cdot x} = 0 \), and have performed an integration by parts (whose justification would imply the substitution of plane waves by wave packets).

Therefore, after this first step in the reduction we get,

\[ \langle p_1, \cdots; p_n; out|q_1 \cdots; q_6; in \rangle = \]

\[ = \sum_{k=1}^n 2p_k^0 (2\pi)^3 \delta^3(\vec{p}_k - \vec{q}_1) \langle p_1, \cdots, \hat{p}_k, \cdots; p_n; out|q_2 \cdots; q_6; in \rangle \]
+ iZ^{-1/2} \int d^4 x e^{-i q \cdot x} (\Box + m^2) \langle p_1 \cdots p_n; \text{out} | \varphi(x) | q_2 \cdots q_\ell; \text{in} \rangle \quad (2.62)

We will proceed with the process until all the momenta in the initial and final state are exchanged by the field operators. To be specific, let us now remove one momentum in the final state. From now on we will no longer consider the disconnected terms, because in practice we are only interested in the cases where all the particles interact\(^1\). We have then

\[
\langle p_1 \cdots p_n; \text{out} | \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle = \langle p_2 \cdots p_n; \text{out} | a_{\text{out}}(p_1) \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

\[
= \lim_{y_i^0 \to \infty} iZ^{-1/2} \int d^3 y_1 e^{i p_1 \cdot y_1} \langle p_2 \cdots p_n; \text{out} | \varphi(y_1) \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

\[
= \langle p_2 \cdots p_n; \text{out} | \varphi(x_1) a_{\text{in}}(p_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

\[
+ \lim_{y_i^0 \to \infty} iZ^{-1/2} \int d^3 y_1 e^{i p_1 \cdot y_1} \frac{\partial}{\partial y_1^0} \langle p_2 \cdots p_n; \text{out} | \varphi(y_1) \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

\[
- \lim_{y_i^0 \to -\infty} iZ^{-1/2} \int d^3 y_1 e^{i p_1 \cdot y_1} \frac{\partial}{\partial y_1^0} \langle p_2 \cdots p_n; \text{out} | \varphi(y_1) \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

\[
= \langle p_2 \cdots p_n; \text{out} | \varphi(x_1) a_{\text{in}}(p_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

\[
+iZ^{-1/2} \left( \lim_{y_i^0 \to \infty} - \lim_{y_i^0 \to -\infty} \right) \int d^3 y_1 e^{i p_1 \cdot y_1} \frac{\partial}{\partial y_1^0} \langle p_2 \cdots p_n; \text{out} | T \varphi(y_1) \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

(2.63)

where we have used the properties of the time-ordered product, Eq. (1.161). Applying the same procedure that lead to Eq. (2.61) we obtain,

\[
\langle p_1 \cdots p_n; \text{out} | \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle = \text{disconnected terms}
\]

\[
+iZ^{-1/2} \int d^4 y_1 e^{i p_1 \cdot y_1} (\Box_{y_1} + m^2) \langle p_2 \cdots p_n; \text{out} | T \varphi(y_1) \varphi(x_1) | q_2 \cdots q_\ell; \text{in} \rangle
\]

(2.64)

It is not very difficult to generalize this method to obtain the final reduction formula for scalar fields,

\[
\langle p_1 \cdots p_n; \text{out} | q_1 \cdots q_\ell; \text{in} \rangle = \text{disconnected terms}
\]

\[
+ \left( \frac{i}{\sqrt{Z}} \right)^{n+\ell} \int d^4 y_1 \cdots d^4 y_n d^4 x_1 \cdots d^4 x_\ell e^{i \sum_{k=1}^n p_k \cdot y_k - i \sum_{r=1}^\ell q_r \cdot x_r}
\]

\[
(\Box_{y_1} + m^2) \cdots (\Box_{x_\ell} + m^2) \langle 0 | T \varphi(y_1) \cdots \varphi(y_n) \varphi(x_1) \cdots \varphi(x_\ell) | 0 \rangle
\]

(2.65)

This last equation is the fundamental equation in quantum field theory. It allows us to relate the transition amplitudes to the Green functions of the theory. The quantity

\[
\langle 0 | T \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle \equiv G(x_1 \cdots x_n)
\]

(2.66)

\(^1\)Once we know the cases where all the particles interact, we can always calculate situations where some of the particles do not participate in the scattering.
is known as the complete green function for \( r = m + \ell \) particles and we will introduce the following diagrammatic representation for it,

\[
G(x_1 \cdots x_n) = \quad (2.67)
\]

The factors \((\Box + m^2)\) in Eq. (2.65) force the external particles to be on-shell. In fact, in momentum space \((\Box + m^2) \to (-p^2 + m^2)\). Therefore, Eq. (2.65) will vanish unless the propagators of the external legs are on-shell, as in that case they will have a pole, \(\frac{1}{p^2 - m^2}\). Eq. (2.65) will then give the residue of that pole. We conclude that for the transition amplitudes only the truncated Green functions will contribute, that is the ones with the external legs removed. In the next chapter we will learn how to evaluate these Green functions in perturbation theory.

### 2.7 Reduction formula for fermions

#### 2.7.1 States in and out

The definition of the \textit{in} and \textit{out} follows exactly the same steps as in the case of the scalar fields. We will therefore, for simplicity, just state the results with the details.

The states \(\psi_{\text{in}}(x)\) satisfy the conditions,

\[
(i\not{\partial} - m)\psi_{\text{in}}(x) = 0
\]

\[
[P_{\mu}, \psi_{\text{in}}(x)] = -i\partial_{\mu}\psi_{\text{in}}(x) .
\]

(2.68)

The states \(\psi_{\text{in}}(x)\) will create one-particle states and they are related with the fields \(\psi(x)\) by,

\[
\sqrt{Z_2}\psi_{\text{in}}(x) = \psi(x) - \int d^4y S_{\text{ret}}(x - y, m)j(y)
\]

(2.69)

where \(\psi(x)\) satisfies the Dirac equation,

\[
(i\not{\partial} - m)\psi(x) = j(x)
\]

(2.70)

and \(S_{\text{ret}}\) is the retarded Green function for the Dirac equation,

\[
(i\not{\partial}_x - m)S_{\text{ret}}(x - y, m) = \delta^4(x - y)
\]
\[ S_{\text{ret}}(x - y) = 0 \ ; \ x^0 < y^0 \] (2.71)

The fields \( \psi_{in}(x) \), as free fields, have the Fourier expansion,

\[
\psi_{in}(x) = \int \frac{dp}{2\pi} \sum_s \left[ b_{in}(p, s) u(p, s) e^{-ip \cdot x} + d_{in}^\dagger(p, s) v(p, s) e^{ip \cdot x} \right] \] (2.72)

where the operators \( b_{in}, d_{in} \) satisfy exactly the same algebra as in the free field case. The asymptotic condition is now,

\[
\lim_{t \to -\infty} \langle \alpha | \psi^f(t) | \beta \rangle = \sqrt{Z_2} \langle \alpha | \psi_{in}^f | \beta \rangle \] (2.73)

where \( \psi^f(t) \) and \( \psi_{in}^f \) have a meaning similar to Eq. (2.19).

For the \( \psi_{out} \) fields we get essentially the same expressions with \( \psi_{in} \) substituted by \( \psi_{out} \). The main difference is in the asymptotic condition that now reads,

\[
\lim_{t \to \infty} \langle \alpha | \psi^f(t) | \beta \rangle = \sqrt{Z_2} \langle \alpha | \psi_{out}^f | \beta \rangle \] (2.74)

implying the following relation between the fields \( \psi_{out} \) and \( \psi \),

\[
\sqrt{Z_2} \psi_{out} = \psi(x) - \int d^4y S_{\text{adv}}(x - y; m) j(y) \] (2.75)

where

\[
(i\partial_x - m) S_{\text{adv}}(x - y; m) = \delta^4(x - y) \]

\[
S_{\text{adv}}(x - y; m) = 0 \quad x^0 > y^0 . \] (2.76)

### 2.7.2 Spectral representation fermions

Let us consider the vacuum expectation value of the anti-commutator of two Dirac fields,

\[
S'_{\alpha\beta}(x, y) \equiv i \langle 0 | \{ \psi_{\alpha}(x), \overline{\psi}_{\beta}(y) \} | 0 \rangle \]

\[
= i \sum_n \left[ \langle 0 | \psi_{\alpha}(0) | n \rangle \langle n | \overline{\psi}_{\beta}(0) | 0 \rangle e^{-ip_n(x-y)} + \langle 0 | \overline{\psi}_{\beta}(0) | n \rangle \langle n | \psi_{\alpha}(0) | 0 \rangle e^{ip_n(x-y)} \right] \]

\[
\equiv S'_{\alpha\beta}(x - y) \] (2.77)

where we have introduced a complete set of eigen-states of the 4-momentum. As before we introduce the spectral amplitude \( \rho_{\alpha\beta}(q) \),

\[
\rho_{\alpha\beta}(q) \equiv (2\pi)^3 \sum_n \delta^4(p_n - q) \langle 0 | \psi_{\alpha}(0) | n \rangle \langle n | \overline{\psi}_{\beta}(0) | 0 \rangle \] (2.78)
2.7. REDUCTION FORMULA FOR FERMIONS

We will now find the most general form for $\rho_{\alpha\beta}(q)$ using Lorentz invariance arguments. $\rho_{\alpha\beta}(q)$ is a $4 \times 4$ matrix in Dirac space, and can therefore be written as

$$\rho_{\alpha\beta}(q) = \overline{\tau}(q) \gamma_{\alpha\beta} + \rho_{\mu}(q) \sigma_{\alpha\beta}^{\mu} + \tilde{\rho}_{\mu}(q) \gamma_{\alpha\beta}^5 + \tilde{\rho}_{\mu}(q)(\gamma_{\mu}\gamma^5)_{\alpha\beta}$$  \hspace{1cm} (2.79)

Lorentz invariance arguments restrict the form of the coefficients $\overline{\tau}(q), \rho_{\mu}(q), \rho_{\mu\nu}(q), \tilde{\rho}(q)$ and $\tilde{\rho}_{\mu}(q)$. Under Lorentz transformations the fields transform as

$$U(a)\psi_\alpha(0)U^{-1}(a) = S^{-1}_{\alpha\lambda}(a)\overline{\psi}_\lambda(0)$$

and

$$U(a)\overline{\psi}_\alpha(0)U^{-1}(a) = \overline{\psi}_\lambda(0)S_{\lambda\alpha}(a)$$

Then we can show that the matrix (in Dirac space), $\rho_{\alpha\beta}$ must obey the relation,

$$\rho(q) = S^{-1}(a)\rho(qa^{-1})S(a)$$  \hspace{1cm} (2.81)

where we have used a matrix notation. This relation gives the properties of the different coefficients on Eq. (2.79). For instance,

$$\rho_{\mu}(q) = a_{\mu\nu}\rho_{\nu}(qa^{-1})$$  \hspace{1cm} (2.82)

which means that $\rho_{\mu}$ transform as a $4-$vector.

Using the fact that $\rho_{\alpha\beta}$ is a function of $q$ and vanishes outside the future light cone, we can finally write

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\eta_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} + \tilde{\rho}_1(q^2)(\gamma_{\alpha\beta}^5) + \tilde{\rho}_2(q^2)\sigma_{\alpha\beta}$$  \hspace{1cm} (2.83)

that is, $\rho_{\alpha\beta}(q)$ is determined up to four scalar functions of $q^2$. Requiring invariance under parity transformations we get, instead of Eq. (2.81),

$$\rho_{\alpha\beta}(q, q_0) = \rho_0(q_{\alpha\beta})(\delta_{\alpha\beta}^0 - q_{\alpha\beta}^0)$$  \hspace{1cm} (2.84)

and inserting in Eq. (2.83) we obtain,

$$\tilde{\rho}_1 = \tilde{\rho}_2 = 0$$  \hspace{1cm} (2.85)

Therefore for the Dirac theory, that preserves parity, we get,

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\eta_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta}$$  \hspace{1cm} (2.86)

Repeating the steps of the scalar case we write,

$$S'_{\alpha\beta}(x - y) = \int_0^\infty d\sigma^2 \{ \rho_1(\sigma^2)S_{\alpha\beta}(x - y; \sigma) +$$

$$+ [\sigma \rho_1(\sigma^2) - \rho_2(\sigma^2) \Delta(x - y; \sigma)] \delta_{\alpha\beta} \Delta(x - y; \sigma) \}$$  \hspace{1cm} (2.87)

where $\Delta$ and $S_{\alpha\beta}$ are the functions defined for free fields. We can then show that
CHAPTER 2. PHYSICAL STATES. S MATRIX. LSZ REDUCTION.

i) $\rho_1$ e $\rho_2$ are real

ii) $\rho_1(\sigma^2) \geq 0$

iii) $\sigma \rho_1(\sigma^2) - \rho_2(\sigma^2) \geq 0$

Using the previous relations we can extract of the one-particle states from Eq. (2.87). We get,

\[
S'_{\alpha \beta}(x - y) = Z_2 S_{\alpha \beta}(x - y; m) + \int_{m_1^2}^{\infty} d\sigma^2 \left\{ \rho_1(\sigma^2) S_{\alpha \beta}(x - y; \sigma) + [\sigma \rho_1(\sigma^2) - \rho_2(\sigma^2)] \delta_{\alpha \beta} \Delta(x - y; \sigma) \right\}
\]

(2.88)

where $m_1$ is the threshold for the production of two or more particles. Evaluating Eq. (2.88) at equal times we can obtain

\[
1 = Z_2 + \int_{m_1^2}^{\infty} d\sigma^2 \rho_1(\sigma^2)
\]

(2.89)

that is

\[
0 \leq Z_2 < 1
\]

(2.90)

2.7.3 Reduction formula fermions

To get the reduction formula for fermions we will proceed as in the scalar case. The only difficulty has to do with the spinor indices. The creation and annihilation operators can be expressed in terms of the fields $\psi_{in}$ by the relations,

\[
b_{in}(p,s) = \int d^3x \bar{\psi}(p,s) e^{ip \cdot x} \gamma_0 \psi_{in}(x)
\]

\[
d^\dagger_{in}(p,s) = \int d^3x \bar{\psi}(p,s) e^{-ip \cdot x} \gamma_0 \psi_{in}(x)
\]

\[
b_{in}^\dagger(p,s) = \int d^3x \bar{\psi}^\dagger_{in}(x) \gamma_0 e^{-ip \cdot x} u(p,s)
\]

\[
d_{in}(p,s) = \int d^3x \bar{\psi}^\dagger_{in}(x) \gamma_0 e^{ip \cdot x} v(p,s)
\]

(2.91)

with the integrals being time independent. In fact, to be more rigorous we should substitute the plane wave solutions by wave packets, but as in the scalar case, to simplify matter we will not do it. To establish the reduction formula we start by extracting one electron from the initial state,

\[
\langle \beta; out | (ps) \alpha; in \rangle = \langle \beta; out | b_{in}^\dagger(p,s) | \alpha, in \rangle
\]
2.7. REDUCTION FORMULA FOR FERMIONS

\[
\langle \beta - (p, s); out|\alpha; in \rangle + \langle \beta; out|b_{in}^{\dagger}(p, s) - b_{out}^{\dagger}(p, s)|\alpha; in \rangle \\
= \text{disconnected terms} \\
+ \int d^3x \langle \beta; out|\psi_{in}(x) - \psi_{out}(x)|\alpha; in \rangle \gamma^0 e^{-ip \cdot x} u(p, s) \\
= \text{disconnected terms} \\
- \left( \lim_{t \to -\infty} - \lim_{t \to +\infty} \right) \frac{1}{\sqrt{Z_2}} \int d^3x \langle \beta; out|\psi(x)|\alpha; in \rangle \gamma^0 e^{-ip \cdot x} u(p, s) \\
= \text{disconnected terms} \\
-Z_2^{-1/2} \int d^4x \left[ \langle \beta; out|\partial_0 \psi(x)|\alpha; in \rangle \gamma^0 e^{-ip \cdot x} u(p, s) \\
+ \langle \beta; out|\psi(x)|\alpha; in \rangle \gamma^0 \partial_0 (e^{-ip \cdot x} u(p, s)) \right] \tag{2.92}
\]

Using now

\[
(i\gamma^0 \partial_0 + i\gamma^j \partial_j - m)(e^{-ip \cdot x} u(p, s)) = 0 \tag{2.93}
\]

we get, after an integration by parts,

\[
\langle \beta; out|b_{in}^{\dagger}(p, s)|\alpha; in \rangle = \text{disconnected terms} \\
-iZ_2^{-1/2} \int d^4x \langle \beta; out|\psi(x)|\alpha; in \rangle (-i\partial_x - m) e^{-ip \cdot x} u(p, s) \tag{2.94}
\]

In a similar way the reduction of an anti-particle from the initial state gives,

\[
\langle \beta; out|d_{in}^{\dagger}(p, s)|\alpha; in \rangle = \text{disconnected terms} \\
+iZ_2^{-1/2} \int d^4x e^{-ip \cdot x} \tau(p, s) (i\partial_x - m) \langle \beta; out|\psi(x)|\alpha; in \rangle \tag{2.95}
\]

while the reduction of a particle or of an anti-particle from the final state give, respectively,

\[
\langle \beta; out|b_{out}(p, s)|\alpha; in \rangle = \text{disconnected terms} \\
-iZ_2^{-1/2} \int d^4x e^{ip \cdot x} \bar{u}(p, s) (i\partial_x - m) \langle \beta; out|\psi(x)|\alpha; in \rangle \tag{2.96}
\]

and

\[
\langle \beta; out|d_{out}(p, s)|\alpha; in \rangle = \text{disconnected terms} \\
+iZ_2^{-1/2} \int d^4x \langle \beta; out|\psi(x)|\alpha; in \rangle (-i\partial_x - m) v(p, s) e^{ip \cdot x} \tag{2.97}
\]

Notice the formal relation between one electron in the initial state and a positron in the final state. To go from one to the other one just has to do,

\[
u(p, s) e^{-ip \cdot x} \rightarrow -v(p, s) e^{ip \cdot x} \tag{2.98}
\]
The following steps in the reduction are similar, one only has to pay attention to signs because of the anti-commutation relations for fermions. To write the final expression we denote the momenta in the state \(|in⟩\) by \(p_1\) or \(\overline{p}_1\), respectively for particles or anti-particles, and those in the state \(|out⟩\) by \(p'_1\) or \(\overline{p}'_1\). We also make the following conventions (needed to define the global sign),

\[
|\langle p_1, s_1⟩; \cdots; \langle \overline{p}_1, \overline{s}_1⟩; \cdots; \langle in⟩⟩ = b^\dagger_{in} (p_1, s_1) \cdots d^\dagger_{in} (\overline{p}_1, \overline{s}_1) \cdots |0⟩
\]  

and

\[
\langle out; \langle p'_1, s'_1⟩; \cdots; \langle \overline{p}'_1, \overline{s}'_1⟩; \cdots; \langle in⟩⟩ = \langle 0⟩ \cdots d_{out} (\overline{p}'_1, \overline{s}'_1), \cdots b_{out} (p'_1, s'_1) \]  

(2.99)

(2.100)

Then, if \(n(n')\) denotes the total number of particles (anti-particles), we get

\[
\langle out; \langle p'_1, s'_1⟩; \cdots; \langle \overline{p}'_1, \overline{s}'_1⟩; \cdots |\langle p_1, s_1⟩; \cdots \langle \overline{p}_1, \overline{s}_1⟩; \cdots; \langle in⟩⟩⟩ = \text{disconnected terms}
\]

\[
+(-iZ_2^{-1/2})^n iZ_2^{-1/2}^{n'} \int d^4x_1 \cdots d^4y_1 \cdots d^4x'_1 \cdots d^4y'_1 \cdots
\]

\[
e^{-i \sum(p_1 \cdot x_1) - i \sum(\overline{p}_1 \cdot \overline{y}_1)} e^{+i \sum(p'_1 \cdot x'_1) + i \sum(\overline{p}'_1 \cdot \overline{y}'_1)}
\]

\[
\mathcal{T}(p'_1, s'_1)(i\partial \cdot \overline{p}_1 - m) \cdots \mathcal{T}(\overline{p}'_1, \overline{s}'_1)(i\partial \cdot \overline{y}_1 - m)
\]

\[
\langle 0⟩ T(\cdots \overline{\psi}(y'_1) \cdots \overline{\psi}(x'_1) \overline{\psi}(x_1) \cdots \overline{\psi}(y_1) \cdots |0⟩)
\]

\[
(-i\partial \cdot \overline{p}_1 - m) u(p_1, s_1) \cdots (-i\partial \cdot \overline{y}_1 - m) v(\overline{p}'_1, \overline{s}'_1)
\]  

(2.101)

Eq. (2.101) is the fundamental expression that allows to relate the elements of the \(S\) matrix with the Green functions of the theory. The operators within the time-ordered product can be reordered, modulo some minus sign. The sign and ordering shown correspond to the conventions in Eqs. (2.99) and (2.100). In terms of diagrams, we represent the Green function,

\[
\langle 0⟩ T [\overline{\psi}(y'_m) \cdots \overline{\psi}(y'_1) \overline{\psi}(x'_m) \cdots \overline{\psi}(x'_1) \overline{\psi}(x_1) \cdots \overline{\psi}(x_1) \overline{\psi}(y_1) \cdots \overline{\psi}(y_m)] |0⟩
\]

(2.102)

by the diagram\(^2\) of Fig. (2.1).

The operators \((i\partial - m)\) force the particles to be on-shell and remove the propagators from the external lines (truncated Green functions). In the next chapter we will learn how to determine these functions in perturbation theory.

### 2.8 Reduction formula for photons

The LSZ formalism for photons, has some difficulties connected with the problems in quantizing the electromagnetic field. When one adopts a formalism (radiation gauge)

\[\ell - m = \ell' - m'\]

\(^2\)With lepton number conservation, the number of particles minus anti-particles is conserved, that is
where the only components of the field $A^\mu$ are transverse (as in Ref.[6]), the problems arise in showing the Lorentz and gauge invariance of the $S$ matrix. In the formalism of the undefined metric, that we adopted in section 1.4.2, the difficulties are connected with the states of negative norm, besides the gauge invariance.

Here we are going to ignore these difficulties and assume that we can define the $in$ fields by the relation,

$$\sqrt{Z_3} A^\mu_{in}(x) = A^\mu(x) - \int d^4 y D^{\mu\nu}_{ret}(x - y) j^\nu(y)$$ (2.103)

and in the same way for the $out$ fields,

$$\sqrt{Z_3} A^\mu_{out}(x) = A^\mu(x) - \int d^4 y D^{\mu\nu}_{adv}(x - y) j^\nu(y)$$ (2.104)

where

$$\square A^\mu_{in} = \square A^\mu_{out} = 0$$

$$\square A^\mu = j^\mu$$

$$\square D^{\mu\nu}_{adv, ret} = \delta^{\mu\nu} \delta^\mu(x - y)$$ (2.105)

The fields $in$ and $out$ are free fields, and therefore they have a Fourier expansion in plane waves and creation and annihilation operators of the form

$$A^\mu_{in}(x) = \int \frac{d^3 k}{(2\pi)^3} \sum_{\lambda=0}^3 \left[ a_{in}(k,\lambda) \varepsilon^\mu(k,\lambda) e^{-ik \cdot x} + a_{in}^\dagger(k,\lambda) \varepsilon^{\mu\ast}(k,\lambda) e^{ik \cdot x} \right]$$ (2.106)

and therefore

$$a_{in}(k,\lambda) = -i \int d^3 x e^{ik \cdot x} \partial_\alpha \varepsilon^\mu(k,\lambda) A^{in}_\mu(x)$$

---

3We will see in chapter 6 a more satisfactory procedure to quantize all gauge theories, including Maxwell theory of the electromagnetic field. We will see that the resulting perturbation theory coincides with the one we get here. This is our justification to be less precise here.
where, as usual, \( a_{in}(k,\lambda) \) and \( a_{out}^{-1}(k,\lambda) \) are time independent. In Eq. (2.106) all the polarizations appear, but as the elements of the \( S \) matrix are between physical states, we are sure that the longitudinal and scalar polarizations do not contribute. In this formalism what is difficult to show is the spectral decomposition. We are not going to enter those details, just state that we can show that \( Z_3 \) is gauge independent and satisfies \( 0 \leq Z_3 < 1 \). The reduction formula is easily obtained. We get

\[
\langle \beta; out|(k\lambda)\alpha; in \rangle = \langle \beta - (k,\lambda); out|\alpha; in \rangle + \langle \beta; out|a_{in}^\dagger(k,\lambda) - a_{out}^{-1}(k,\lambda)|\alpha; in \rangle
\]

\[
= \text{disconnected terms}
\]

\[
+i \int d^3x \varepsilon_{\mu}^{*}(k,\lambda) \langle \beta; out|A_{in}^{\mu}(x) - A_{out}^{\mu}(x)|\alpha; in \rangle
\]

\[
= \text{disconnected terms}
\]

\[
-i \left( \lim_{t \to +\infty} - \lim_{t \to -\infty} \right) Z_3^{-1/2} \int d^3x \varepsilon_{\mu}^{*}(k,\lambda) \langle \beta; out|A^{\mu}(x)|\alpha; in \rangle
\]

\[
= \text{disconnected terms}
\]

\[
-i Z_3^{-1/2} \int d^4x \varepsilon_{\mu}^{*}(k,\lambda) \langle \beta; out|A^{\mu}(x)|\alpha; in \rangle
\]

\[
= \text{disconnected terms}
\]

\[
-i Z_3^{-1/2} \int d^4x \varepsilon_{\mu}^{*}(k,\lambda)
\]

\[
(2.108)
\]

The final formula for photons is then

\[
\langle k'_1 \cdots k'_n; out|k_1 \cdots k_\ell; in \rangle = \text{disconnected terms}
\]

\[
+ \left( \frac{-i}{\sqrt{Z_3}} \right)^{n+\ell} \int d^4y_1 \cdots d^4y_n d^4x_1 \cdots d^4x_\ell e^{i \left( \sum k'_i y_i - i \sum k_i x_i \right)}
\]

\[
\varepsilon^{\mu_1}(k_1,\lambda_1) \cdots \varepsilon^{\mu_\ell}(k_\ell,\lambda_\ell) \varepsilon^{*\nu_1}(k'_1,\lambda'_1) \cdots \varepsilon^{*\nu_n}(k'_n,\lambda'_n)
\]

\[
\Box_{y_1} \cdots \Box_{x_\ell} \langle 0| T(A_{\mu'_1}(y_1) \cdots A_{\mu'_n}(y_n) A_{\mu_1}(x_1) \cdots A_{\mu_\ell}(x_\ell) | 0 \rangle
\]

\[
(2.109)
\]

and corresponds to the diagram of Fig. (2.2).

## 2.9 Cross sections

The reduction formulas, Eqs.(2.65), (2.101) and (2.109), are the fundamental results of this chapter. They relate the transition amplitudes from the initial to the final state with the Green functions of the theory. In the next chapter we will show how to evaluate these Green functions setting up the so-called covariant perturbation theory. Before we close this chapter, let us indicate how these transition amplitudes

\[
S_{fi} \equiv \langle f; out|i; in \rangle
\]

\[
(2.110)
\]
2.9. CROSS SECTIONS

are related with the quantities that are experimentally accessible, the cross sections. Then the path between experiment (cross sections) and theory (Green functions) will be established.

As we have seen in the reduction formulas there is always a trivial contribution to the $S$ matrix, that corresponds to the so-called disconnected terms, when the system goes from the initial to the final state without interaction. The subtraction of this trivial contribution leads us to introduce the $T$ matrix with the relation,

$$S_{fi} = 1_{fi} - i(2\pi)^4 \delta^4(P_f - P_i)T_{fi}$$ (2.111)

where we have factorized explicitly the delta function expressing the 4-momentum conservation. If we neglect the trivial contribution, the transition probability from the initial to the final state will be given by

$$W_{f \leftarrow i} = \left|(2\pi)^4 \delta^4(P_f - P_i)T_{fi}\right|^2$$ (2.112)

To proceed we have to deal with the meaning of a square of a delta function. This appears because we are using plane waves. To solve this problem we can normalize in a box of volume $V$ and consider that the interaction has a duration of $T$. Then

$$(2\pi)^4 \delta^4(P_f - P_i) = \lim_{V \to \infty, T \to \infty} \int_V d^3x \int_{-T/2}^{T/2} dx e^{i(P_f - P_i) \cdot x}. (2.113)$$

However

$$F \equiv \int_V d^3x \int_{-T/2}^{T/2} dx e^{i(P_f - P_i) \cdot x} = V \delta_{\vec{P}_f, \vec{P}_i} \frac{2}{|E_f - E_i|} \sin \frac{T}{2} (E_f - E_i)$$ (2.114)

and the square of the last expression can be done, giving,

$$|F|^2 = V^2 \delta_{\vec{P}_f, \vec{P}_i} \frac{4}{|E_f - E_i|^2} \sin^2 \frac{T}{2} (E_f - E_i)$$ (2.115)

If we want the transition rate by unit of volume (and unit of time) we divide by $VT$. Then

$$\Gamma_{fi} = \lim_{V \to \infty, T \to \infty} V \delta_{\vec{P}_f, \vec{P}_i} \frac{2}{|E_f - E_i|^2} \sin^2 \frac{T}{2} (E_f - E_i)^2 |T_{fi}|^2$$ (2.116)
Using now the results

\[
\lim_{V \to \infty} V \delta \vec{p}_j = (2\pi)^3 \delta^3(\vec{p}_j - \vec{p}_i)
\]

\[
\lim_{T \to \infty} 2 \sin^2 \frac{T}{2}(E_f - E_i) = (2\pi) \delta(E_f - E_i)
\]  

we get for the transition rate by unit volume and unit time,

\[
\Gamma_{fi} \equiv (2\pi)^4 \delta^4(P_f - P_i)|T_{fi}|^2
\]  

To get the cross section we have to further divide by the incident flux, and normalize the particle densities to one particle per unit volume. Finally, we sum (integrate) over all final states in a certain energy-momentum range. We get,

\[
d\sigma = \frac{1}{\rho_1 \rho_2} \frac{1}{|\vec{v}_{12}|} \Gamma_{fi} \prod_{j=3}^n \frac{d^3p_j}{2p_j^0(2\pi)^3}
\]  

where

\[
\rho_1 = 2E_1; \quad \rho_2 = 2E_2
\]  

An equivalent way of writing this equation is

\[
d\sigma = \frac{1}{4 \left[(p_i \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2}} (2\pi)^4 \delta^4(P_f - P_i)|T_{fi}|^2 \prod_{j=3}^n dp_j
\]  

that exhibits well the Lorentz invariance of each part that enters the cross section\(^4\). The incident flux and phase space factors are purely kinematics. The physics, with its interactions, is in the matrix element \(T_{fi}\).

We note that with our conventions, fermion and boson fields have the same normalization, that is, the one-particle states obey

\[
\langle p|p' \rangle = 2p^0(2\pi)^3 \delta^3(p - p')
\]  

differing in this way from some older books like Ref.[1].

\(^4\)It is assumed that, in the case of two beams they are in the same line. Then the cross section, being a transverse area, is invariant for Lorentz transformations along that direction.
Problems for Chapter 2

2.1 Show that the spectral representation for fermions, $\rho_{\alpha\beta}(q)$, satisfies,

a) $\rho(q) = S^{-1}(a)\rho(qa^{-1})S(a)$

b) $\rho_{\alpha\beta}(\vec{q}, q^0) = \gamma^0_{\alpha\lambda} \rho_{\lambda\delta}(-\vec{q}, q^0) \gamma^0_{\delta\beta}$

2.2 Use the results of the previous problem to show that, in a theory that preserves Parity, like QED, we have

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\delta_{\alpha\beta} + \rho_2(q^2)\sigma_{\alpha\beta} \quad \text{(2.123)}$$

2.3 Show that the functions $\rho_1$ and $\rho_2$ defined in problem 2.2 satisfy the following properties:

i) $\rho_1$ and $\rho_2$ are real

ii) $\rho_1(\sigma^2) \geq 0$

iii) $\sigma \rho_1(\sigma^2) - \rho_2(\sigma^2) \geq 0$

2.4 Show that for the Dirac field we have

$$1 = Z_2 + \int_{m^2}^{\infty} d\sigma^2 \rho_1(\sigma^2) \quad \text{(2.124)}$$

2.5 Show that

$$\langle 0 | [\varphi_{in}(x), \varphi_{out}(y)] | 0 \rangle = i\Delta(x - y; m) \quad \text{(2.125)}$$
CHAPTER 2. PHYSICAL STATES. S MATRIX. LSZ REDUCTION.
Chapter 3

Covariant Perturbation Theory

3.1 $U$ matrix

In this chapter we are going to develop a method to evaluate the Green functions of a given theory. From what we have seen in the two previous chapters, we realize that we only know how to calculate for free fields, like the $in$ and $out$ fields. However, the Green functions we are interested in, are given in terms of the physical interacting fields, and we do not know how to operate with these. We are going to see how to express the physical fields as perturbative series in terms of free $in$ fields. In this way we will be able to evaluate the Green functions in perturbation theory.

We start by defining the $U$ matrix. To simplify matters, we will be considering, for the moment, only scalar fields. In the end we will return to the other cases. The physical interacting fields $\varphi(\vec{x}, t)$ and their conjugate momenta $\pi(\vec{x}, t)$, satisfy the same equal time commutation relations than the $in$ fields, $\varphi_{in}(\vec{x}, t)$ and their $\pi_{in}(\vec{x}, t)$. Also, both $\varphi$ and $\varphi_{in}$ form a complete set of operators, in the sense that any state, free or interacting, can be obtained by application of $\varphi_{in}$ or $\varphi$ in the vacuum. This implies that there should be an unitary transformation $U(t)$ that relates $\varphi$ with $\varphi_{in}$, that is,

$$\varphi(\vec{x}, t) = U^{-1}(t)\varphi_{in}(\vec{x}, t)U(t)$$
$$\pi(\vec{x}, t) = U^{-1}(t)\pi_{in}(\vec{x}, t)U(t)$$

(3.1)

The dynamics of $U$ can be obtained from the equations of motion for $\varphi(x)$ and $\varphi_{in}(x)$. These are,

$$\frac{\partial \varphi_{in}}{\partial t}(x) = i[H_{in}(\varphi_{in}, \pi_{in}), \varphi_{in}]$$
$$\frac{\partial \pi_{in}}{\partial t}(x) = i[H_{in}(\varphi_{in}, \pi_{in}), \pi_{in}]$$

(3.2)

and

$$\frac{\partial \varphi}{\partial t}(x) = i[H(\varphi, \pi), \varphi]$$
$$\frac{\partial \pi}{\partial t}(x) = i[H(\varphi, \pi), \pi]$$

(3.3)
Then from Eqs. (3.2) and (3.1) we get,

\[
\dot{\varphi}_{in}(x) = \frac{\partial}{\partial t} \left[ U(t) \varphi(x) U^{-1}(t) \right] \\
= \left[ \dot{U}(t) U^{-1}(t), \varphi_{in} \right] + i \left[ H(\varphi_{in}, \pi_{in}), \varphi_{in}(x) \right] \\
= \dot{\varphi}_{in}(x) + \left[ \dot{U} U^{-1} + i H_I(\varphi_{in}, \pi_{in}), \varphi_{in} \right] 
\]

(3.4)

where

\[
H_I(\varphi_{in}, \pi_{in}) = H(\varphi_{in}, \pi_{in}) - H_{in}(\varphi_{in}, \pi_{in}) \equiv H_I(t) 
\]

(3.5)

and in a similar way

\[
\dot{\pi}_{in}(x) = \dot{\pi}_{in} + \left[ \dot{U} U^{-1} + i H_I(\varphi_{in}, \pi_{in}), \pi_{in} \right] 
\]

(3.6)

From Eqs. (3.4) and (3.6) we obtain,

\[
i \dot{U} U^{-1} = H_I(t) + E_0(t) 
\]

(3.7)

where \(E_0(t)\) commutes with \(\varphi_{in}\) and \(\pi_{in}\) and is therefore a time dependent c-number, not an operator. Defining

\[
H_I'(t) = H_I(t) + E_0(t) 
\]

(3.8)

we get a differential equation for \(U(t)\), that reads,

\[
i \frac{\partial U(t)}{\partial t} = H_I'(t) U(t) 
\]

(3.9)

The solution of this equation in terms of the \(in\) fields, is the basis of the covariant perturbation theory.

To integrate Eq. (3.9) we need an initial condition. For that we introduce the operator

\[
U(t, t') \equiv U(t) U^{-1}(t') 
\]

(3.10)

where \(t \geq t'\), and that obviously satisfies

\[
U(t, t) = 1 
\]

(3.11)

It is easy to see that \(U(t, t')\) also satisfies Eq. (3.9), that is,

\[
i \frac{\partial U(t, t')}{\partial t} = H_I'(t) U(t, t') 
\]

(3.12)

and has the initial condition, Eq. (3.11). To proceed we start by transforming Eq. (3.12) in an equivalent integral equation, that is,

\[
U(t, t') = 1 - i \int_t^{t'} dt_1 H_I'(t_1) U(t_1, t') 
\]

(3.13)
Notice that we have not solved the problem because $U(t, t')$ appears on both sides of the equation. However, we can iterate the equation to get the expansion,

$$U(t, t') = 1 - i \int_{t'}^t dt_1 H'_I(t_1) + (-i)^2 \int_{t'}^t dt_2 H'_I(t_2)$$

$$+ \cdots + (-i)^n \int_{t'}^t dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n H'_I(t_1) \cdots H'_I(t_n)$$

$$+ \cdots$$  \hfill (3.14)

Of course this expansion can only be useful if $H_I$ contains a small parameter and, because of that, we can truncate the expansion at certain order in that parameter. Coming back to Eq. (3.14), as $t_1 \geq t_2 \geq \cdots \geq t_n$, the product is time-ordered and we can therefore write

$$U(t, t') = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t'}^t dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n T(H'_I(t_1) \cdots H'_I(t_n))$$  \hfill (3.15)

Using the symmetry $t_1, t_2$ we can write,

$$\int_{t'}^t dt_1 \int_{t'}^{t_1} dt_2 T(H'_I(t_1)H'_I(t_2)) = \int_{t'}^t dt_2 \int_{t'}^{t_2} dt_1 T(H'_I(t_1)H'_I(t_2))$$

$$= \frac{1}{2} \int_{t'}^t dt_1 \int_{t'}^{t} dt_2 T(H'_I(t_1)H'_I(t_2))$$  \hfill (3.16)

which can be seen from the illustration in Fig. 3.1.

![Figure 3.1: Integration regions in Eq.(3.16)](image)

In general, for $n$ integrations, instead of $\frac{1}{2}$ we will have $\frac{1}{n!}$, and we get,

$$U(t, t') = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^t dt_1 \cdots \int_{t'}^{t} dt_n T(H'_I(t_1) \cdots H'_I(t_n))$$

$$\equiv T \left( \exp[-i \int_{t'}^t dt H'_I(t)] \right)$$

$$= T \left( \exp[-i \int_{t'}^t d^4x H_I(\varphi_{in})] \right)$$ \hfill (3.17)
where the time-ordered product is to be interpreted expanding the exponential.

The operators $U$ satisfy the following multiplication rule

$$U(t, t') = U(t, t'')U(t'', t')$$

(3.18)

which can be seen using the definition, Eq. (3.10), or from the explicit expression, Eq. (3.17). From Eq. (3.18), we can obtain,

$$U(t, t') = U^{-1}(t', t)$$

(3.19)

### 3.2 Perturbative expansion of Green functions

As we saw in the previous chapter, the LSZ technique reduces the evaluation of the elements of the $S$ matrix to a basic ingredient, the so-called Green functions of the theory. These are expectation values of time-ordered products of the Heisenberg fields, $\varphi(x)$,

$$G(x_1, \cdots, x_n) \equiv \langle 0 \mid T\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n) \mid 0 \rangle \quad (3.20)$$

The basic idea for the evaluation of the Green functions consists in expressing the fields $\varphi(x)$ in terms of the fields $\varphi_{in}(x)$, using the operator $U$. We get

$$G(x_1, \cdots, x_n) = \langle 0 \mid T(U^{-1}(t_1)\varphi_{in}(x_1)U(t_1, t_2)\varphi_{in}(x_2)U(t_2, t_3)\cdots$$

$$
\cdots U(t_{n-1}, t_n)\varphi_{in}(x_n)U(t_n)) \mid 0 \rangle \\
= \langle 0 \mid T(U^{-1}(t)U(t, t_1)\varphi_{in}(x_1)U(t_1, t_2)\cdots$$

$$
\cdots U(t_{n-1}, t_n)\varphi_{in}(x_n)U(t_n, -tU(-t)) \mid 0 \rangle$$

(3.21)

where $t$ is a time that we will let go to $\infty$. When $t \to \infty$, $t$ is later than all the $t_i$ and $-t$ is earlier than all the times $t_i$. Therefore we can take $U^{-1}(t) e U(-t)$ out of the time-ordered product. Using the multiplicative property of the operator $U$ we can then write,

$$G(x_1, \cdots, x_n) = \langle 0 \mid U^{-1}(t)T\left(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp[-i\int_{-t}^t H'_{\text{f}}(t')dt']\right)U(-t) \mid 0 \rangle$$

(3.22)

where the time-ordered product $T$ is meant to be applied after expanding the exponential. If it were not for the presence of the operators $U^{-1}(t)$ and $U(-t)$, we would have been successful in expressing the Green function $G(x_1 \cdots x_n)$ completely in terms of the in fields. Now we are going to show that the vacuum is an eigenstate of the operator $U(t)$. For that we consider an arbitrary state $\langle \alpha p; \text{in} \rangle$ that contains one particle of momentum $p$, all the other quantum numbers being denoted collectively by $\alpha$. To simplify, we continue considering the case of the scalar field. We have then,

$$\langle \alpha p; \text{in} \mid U(-t) \mid 0 \rangle = \langle \alpha; \text{in} \mid a_{in}(p)U(-t) \mid 0 \rangle$$

$$= -i \int d^3x f_p^+(\vec{x}, t') \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t'} \right) \langle \alpha; \text{in} \mid \varphi_{in}(\vec{x}, -t')U(-t) \mid 0 \rangle$$
where \( f_p(\vec{x}, t) = e^{-ip\cdot x} \). We use now Eq. (3.1) to express \( \varphi_{in}(\vec{x}, -t) \) in terms of \( \varphi(\vec{x}, -t) \). We get,

\[
\langle \alpha p; \text{in}|U(-t)|0 \rangle = \]
\[
= -i \int d^3x f_p^*(\vec{x}, -t') \partial_0' \langle \alpha; \text{in}|U(-t')\varphi(\vec{x}, -t')U^{-1}(-t')U(-t)|0 \rangle
\]
\[
= -i \int d^3x f_p^*(\vec{x}, -t') \left[ -\partial_0' \langle \alpha; \text{in}|U(-t')\varphi(\vec{x}, -t')U^{-1}(-t')U(-t)|0 \rangle \right]
\]
\[
+ \langle \alpha; \text{in}|U(-t')\varphi(\vec{x}, -t')U^{-1}(-t')U(-t)|0 \rangle
\]
\[
+i \int d^3x f_p^*(\vec{x}, -t') \langle \alpha; \text{in}|\dot{U}(-t')\varphi(\vec{x}, -t')U^{-1}(-t')U(-t)|0 \rangle
\]
\[
+i \int d^3x f_p^*(\vec{x}, -t') \langle \alpha; \text{in}|U(-t')\varphi(\vec{x}, -t')U^{-1}(-t')U(-t)|0 \rangle \tag{3.24}
\]

We take now the \( t = t' \to \infty \) limit. Then

\[
\langle \alpha p; \text{in}|U(-t)|0 \rangle = \sqrt{Z} \langle \alpha; \text{in}|U(-t)a_{m}(p)|0 \rangle
\]
\[
+ \langle \alpha; \text{in}|\dot{U}(-t)\varphi(\vec{x}, -t) + U(-t)\varphi(\vec{x}, -t)\dot{U}^{-1}(-t)U(-t)|0 \rangle \tag{3.25}
\]

Now the first term in Eq. (3.25) vanishes because \( a_{m}(p)|0 \rangle = 0 \). The second term also vanishes because we have (we omit the arguments to simplify the notation),

\[
\dot{U}\varphi + U\varphi\dot{U}^{-1}U = \dot{U}U^{-1}\varphi_{in}U + \varphi_{in}U\dot{U}^{-1}U
\]
\[
= \dot{U}U^{-1}\varphi_{in}U - \varphi_{in}\dot{U}U^{-1}U
\]
\[
= [\dot{U}U^{-1}, \varphi_{in}]U = -i[H_1, \varphi_{in}]U = 0 \tag{3.26}
\]

where we have used Eq. (3.7) and assumed that the interactions have no derivative\(^1\). We conclude then that,

\[
\lim_{t \to \infty} \langle \alpha p; \text{in}|U(-t)|0 \rangle = 0 \tag{3.27}
\]

for all states \( \text{in} \) that contain at least one particle. This means that,

\[
\lim_{t \to \infty} U(-t)|0 \rangle = \lambda_-|0 \rangle \tag{3.28}
\]

In a similar way we could show that,

\[
\lim_{t \to \infty} U(t)|0 \rangle = \lambda_+|0 \rangle \tag{3.29}
\]

\(^1\)The study of theories with derivatives was not trivial before the quantization via path integral was introduced. As we will be viewing this method for gauge theories, we can avoid here the complications of the derivatives. The quantization via path integral is the only method that is available for non-abelian gauge theories as we will be discussing in chapter 6.
Returning now to the expression for the Green function, we can write,

$$G(x_1, \cdots, x_n) = \lambda_-\lambda_+^* \langle 0 | T \left( \varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp \left[ -i \int_{-t}^{t} dt' H'_I(t') \right] \right) | 0 \rangle$$  \hspace{1cm} (3.30)$$

The dependence in the operator $U$ disappeared from the expectation value. To proceed, let us evaluate the constants $\lambda_-\lambda_+^*$ or more to the point, the combination $\lambda_-\lambda_+^*$ that appears in Eq. (3.30). We get (in the limit $\to \infty$),

$$\lambda_-\lambda_+^* = \langle 0 | U(-t) | 0 \rangle \langle 0 | U^{-1}(t) | 0 \rangle$$

$$= \langle 0 | U(-t)U^{-1}(t) | 0 \rangle = \langle 0 | U(-t, t) | 0 \rangle$$

$$= \langle 0 | T \left( \exp \left[ +i \int_{-t}^{t} dt' H'_I(t') \right] \right) | 0 \rangle$$

$$= \langle 0 | T \left( \exp \left[ -i \int_{-t}^{t} dt' H'_I(t') \right] \right) | 0 \rangle^{-1}$$  \hspace{1cm} (3.31)$$

Using this result we can write the Green function of Eq. (3.30) in the form,

$$G(x_1, \cdots, x_n) = \frac{\langle 0 | T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp[-i \int_{-t}^{t} dt' H'_I(t')] | 0 \rangle}{\langle 0 | T(\exp[-i \int_{-t}^{t} dt' H'_I(t')] | 0 \rangle}$$  \hspace{1cm} (3.32)$$

when $t \to \infty$. Before we write the final expression, we can now introduce the number $E_0(t)$. For that we recall that,

$$H'_I = H_I + E_0$$  \hspace{1cm} (3.33)$$

and noticing that $E_0$ is not an operator, we get a factor $\exp[-i \int_{-t}^{t} dt'E_0(t')]$ both in the numerator and denominator, canceling out in the final result. The final result can then be obtained from Eq. (3.32), just substituting $H'_I$ by $H_I$. We get,

$$G(x_1 \cdots, x_n) = \frac{\langle 0 | T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp[-i \int_{-t}^{t} dt' H_I(t')] | 0 \rangle}{\langle 0 | T(\exp[-i \int_{-t}^{t} dt' H_I(t')] | 0 \rangle}$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} d^4 y_1 \cdots d^4 y_m \langle 0 | T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) H_I(y_1) \cdots H_I(y_m) | 0 \rangle}{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} d^4 y_1 \cdots d^4 y_m \langle 0 | T(H_I(y_1) \cdots H_I(y_m) | 0 \rangle}$$  \hspace{1cm} (3.34)$$

This equation is the fundamental result. The Green functions have been expressed in terms of the $in$ fields whose algebra we know. It is therefore possible to reduce Eq. (3.34) to known quantities. In this reduction plays an important role the Wick’s theorem, to which we now turn.

### 3.3 Wick’s theorem

To evaluate the amplitudes that appear in Eq. (3.34) we have to move the annihilation operators to the right until they act on the vacuum. The final result from these manipulations can be stated in the form of a theorem, known as Wick’s theorem, which reads,

$$T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n)) =$$
3.3. WICK’S THEOREM

\[
\begin{align*}
&= \varphi_{in}(x_1) \cdots \varphi_{in}(x_n) : +[0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0] : \varphi_{in}(x_3) \cdots \varphi_{in}(x_n) : +\text{perm.} \\
&+ (0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0) (0| T(\varphi_{in}(x_3)\varphi_{in}(x_4)) |0) : \varphi_{in}(x_5) \cdots \varphi_{in}(x_n) : +\text{perm.} \\
&+ \cdots \\
&\begin{cases}
(0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0) \cdots (0| T(\varphi_{in}(x_{n-1})\varphi_{in}(x_n)) |0) + \text{perm.} & \text{n even} \\
(0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0) \cdots (0| T(\varphi_{in}(x_{n-2})\varphi_{in}(x_{n-1})) |0) \varphi_{in}(x_n) + \text{perm.} & \text{n odd}
\end{cases}
\end{align*}
\]

(3.35)

**Proof:**

The proof of the theorem is done by induction. For \( n = 1 \) it is certainly true (and trivial). Also for \( n = 2 \) we can shown that

\[ T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : +\text{c-number} \quad (3.36) \]

where the \textit{c-number} comes from the commutations that are needed to move the annihilation operators to the right. To find this constant, we do not have to do any calculation, just to notice that

\[ (0| : \cdots : |0) = 0 \quad (3.37) \]

Then

\[ T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : + (0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0) \quad (3.38) \]

which is in agreement with Eq. (3.35).

Continuing with the induction, let us assume that Eq. (3.35) is valid for a given \( n \). We have to show that it remains valid for \( n + 1 \). Let us consider then \( T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_{n+1})) \) and let us assume that \( t_{n+1} \) is the earliest time. Then

\[
T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_{n+1})) = \\
= T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n))\varphi_{in}(x_{n+1}) \\
= : \varphi_{in}(x_1) \cdots \varphi_{in}(x_n) : \varphi_{in}(x_{n+1}) \\
+ \sum_{\text{perm}} (0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0) : \varphi_{in}(x_3) \cdots \varphi_{in}(x_n) : \varphi_{in}(x_{n+1}) \\
+ \cdots 
\]

(3.39)

To write Eq. (3.39) in the form of Eq. (3.35) it is necessary to find the rule showing how to introduce \( \varphi_{in}(x_{n+1}) \) inside the normal product. For that, we introduce the notation,

\[ \varphi_{in}(x) = \varphi_{in}^{(+)}(x) + \varphi_{in}^{(-)}(x) \quad (3.40) \]
where \( \varphi^{(+)}_m(x) \) contains the annihilation operator and \( \varphi^{(-)}_m(x) \) the creation operator. Then we can write,

\[
\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) := \sum_{A,B} \prod_{i \in A} \varphi^{(-)}_m(x_i) \prod_{j \in B} \varphi^{(+)}_m(x_j) \quad (3.41)
\]

where the sum runs over all the sets \( A, B \) that constitute partitions of the \( n \) indices. Then

\[
\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) : \varphi_{in}(x_{n+1}) =
\]

\[
= \sum_{A,B} \prod_{i \in A} \varphi^{(-)}_m(x_i) \prod_{j \in B} \varphi^{(+)}_m(x_j) [\varphi^{(+)}_m(x_{n+1}) + \varphi^{(-)}_m(x_{n+1})]
\]

\[
= \sum_{A,B} \prod_{i \in A} \varphi^{(-)}_m(x_i) \prod_{j \in B} \varphi^{(+)}_m(x_j) \varphi^{(+)}_m(x_{n+1})
\]

\[
+ \sum_{A,B} \prod_{i \in A} \varphi^{(-)}_m(x_i) \varphi^{(-)}_m(x_{n+1}) \prod_{j \in B} \varphi^{(+)}_m(x_j)
\]

\[
+ \sum_{A,B} \prod_{i \in A} \varphi^{(-)}_m(x_i) \prod_{j \in B, j \neq k} \varphi^{(+)}_m(x_j) \langle 0 | \varphi^{(+)}_m(x_k) \varphi^{(-)}_m(x_{n+1}) | 0 \rangle \quad (3.42)
\]

we can now write,

\[
\langle 0 | \varphi^{(+)}_m(x_k) \varphi^{(-)}_m(x_{n+1}) | 0 \rangle = \langle 0 | \varphi_{in}(x_k) \varphi_{in}(x_{n+1}) | 0 \rangle
\]

\[
= \langle 0 | T(\varphi_{in}(x_k) \varphi_{in}(x_{n+1})) | 0 \rangle \quad (3.43)
\]

where we have used the fact that \( t_{n+1} \) is the earliest time. We can then write Eq. (3.42) in the form,

\[
\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) : \varphi_{in}(x_{n+1}) =: \varphi_{in}(x_1) \cdots \varphi_{in}(x_{n+1}) :
\]

\[
+ \sum_k : \varphi_{in}(x_1) \cdots \varphi_{in}(x_{k-1}) \varphi_{in}(x_k) \cdots \varphi_{in}(x_n) : (0 | T(\varphi_{in}(x_k) \varphi_{in}(x_{n+1})) | 0 \rangle \quad (3.44)
\]

With this result, Eq. (3.39) takes the general form of Eq. (3.35) for the \( n + 1 \) case, ending the proof of the theorem. To fully understand the theorem, it is important to do in detail the case \( n = 4 \), to see how things work. The importance of the Wick’s theorem for the applications comes from the following two corollaries.

**Corollary 1**: If \( n \) is odd, then \( \langle 0 | T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n)) | 0 \rangle = 0 \), as results trivially from Eqs. (3.35) and (3.37) and from,

\[
\langle 0 | \varphi_{in}(x) | 0 \rangle = 0 \quad (3.45)
\]

**Corollary 2**: If \( n \) is even

\[
\langle 0 | T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n)) | 0 \rangle =
\]
\[
\sum_{\text{perm}} \delta_p \langle 0 | T(\varphi_{\text{in}}(x_1)\varphi_{\text{in}}(x_2)) | 0 \rangle \cdots \langle 0 | T(\varphi_{\text{in}}(x_{n-1})\varphi_{\text{in}}(x_n)) | 0 \rangle \tag{3.46}
\]

where \(\delta_p\) is the sign of the permutation that is necessary to introduce in case of fermion fields. This result, that in practice is the most important one, also results from Eqs. (3.35), (3.37) and (3.45).

Therefore the vacuum expectation value of the time-ordered product of \(n\) operators that appear in the general formula, Eq. (3.34), are obtained considering all the vacuum expectation values of the fields taken two by two \((\text{contractions})\) in all possible ways. Now these \textit{contractions} are nothing else than the Feynman propagators for free fields. For instance,

\[
\langle 0 | T(\varphi_{\text{in}}(x_1)\varphi_{\text{in}}(x_2)\varphi_{\text{in}}(x_3)\varphi_{\text{in}}(x_4)) | 0 \rangle = \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3) \tag{3.47}
\]

where

\[
\Delta_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \tag{3.48}
\]

is the Feynman propagator for the free field theory in the case of scalar fields.

It is convenient to use a graphical (diagrammatic) representation for these propagators. We have in configuration space,

\[
\Delta_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \tag{3.49}
\]

\[
S_F(x - y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{3.50}
\]

\[
D_{\mu\nu}^F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} e^{-ik(x-y)} \tag{3.51}
\]

respectively for scalar, spinor and photon (in the Feynman gauge) fields.

As the interaction Hamiltonian is normal ordered, there will be no contractions between the fields that appear in \(H_I\). The fields in \(H_I\) can only contract with fields outside. In this
way the contractions will connect the points corresponding to $\mathcal{H}_I$, the so-called vertices, to either external points or points in another $\mathcal{H}_I$, corresponding to another vertex. To illustrate this point let us consider the $\lambda\phi^4$ theory where,

$$\mathcal{H}_I(x) = \frac{1}{4!}\lambda : \phi^4_{\text{in}}(x) :$$

Then a contribution of order $\lambda^2$ to $G(x_1, x_2, x_3, x_4)$ comes from the term,

$$\frac{\lambda^2}{(4!)^2} \langle 0 | T(\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2)\phi_{\text{in}}(x_3)\phi_{\text{in}}(x_4) : \phi^4_{\text{in}}(y_1) :: \phi^4_{\text{in}}(y_2) : | 0 \rangle$$

and leads to the diagrams in Fig. (3.2). In these diagrams, the interaction is represented by four lines coming from one point, $y_1$ or $y_2$. These lines are contractions between one field from one $\mathcal{H}_I$ with other field that might belong either to another $\mathcal{H}_I$, or be one of the external fields in $G(x_1 \cdots x_4)$. To obtain the Feynman rules we are left with a combinatorial problem. We are not going to find them here, as they are much easier to express in momentum space, as we will see in the following.

In Fig. (3.2) the diagrams a), b) and d) are called \textit{connected} while the diagram c) is called \textit{disconnected}. One diagram is \textit{disconnected} when there is a part of the diagram that is not connected in any way to an external line. We will see in the following that these diagrams do not contribute to the Green functions. Diagram d) is \textit{connected} but is also called \textit{reducible} because it can be obtained by multiplication of simpler Green functions. As we will see only the \textit{irreducible} diagrams are important.
3.4 Vacuum–Vacuum amplitudes

We have seen in the previous section examples of the numerator of Eq. (3.34). Let us now look at the denominator, the so-called vacuum-vacuum amplitudes. Continuing with the example of $\lambda \varphi^4$, some of the diagrams contributing for these amplitudes are shown in Fig. (3.3). The diagrams associated with the numerator of Eq. (3.34) can be separated into connected and disconnected parts. For all diagrams that have as connected part a contribution of order $s$ in the interaction $H_I$, the numerator of $G(x_1 \cdots x_n)$ takes the form,

$$
\sum_{p=0}^{\infty} \frac{(-i)^p}{p!} \int d^4 y_1 \cdots d^4 y_p \langle 0 | T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) H_I(y_1) \cdots H_I(y_s)) | 0 \rangle_c
$$

where the subscript $c$ indicates that only the connected parts are included. The combinatorial factor

$$
\binom{p}{s} = \frac{p!}{s!(p-s)!}
$$

is the number of ways in which we can extract $s$ terms $H_I$ from a set of $p$ terms. We write then Eq. (3.54) in the form ($r = p - s$),

$$
\frac{(-i)^s}{s!} \int d^4 y_1 \cdots d^4 y_s \langle 0 | T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) H_I(y_1) \cdots H_I(y_s)) | 0 \rangle_c
$$

This equation has the form of a connected diagram of order $s$ times an infinite series of vacuum-vacuum amplitudes, that cancels exactly against the denominator. This is true for all orders, and therefore we can write,

$$
G(x_1, \cdots x_n) = \frac{\sum_i G_i(x_1 \cdots x_n)}{\sum_k D_k} = \frac{\langle \sum_i G_i^c(x_1, \cdots x_n) \rangle}{\sum_k D_k}
$$
where $G^c_i$ are the connected diagrams and $D_k$ the disconnected ones. This result means that we can simply ignore completely the disconnected diagrams and consider only the connected ones when evaluating the Green functions. These are simply the sum of all connected diagrams, simplifying enormously the structure of Eq. (3.34).

3.5 Feynman rules for $\lambda \varphi^4$

To understand how the Feynman rules appear, let us consider the case of a real scalar field with an interaction of the form,

\[ \mathcal{H}_I = \frac{\lambda}{4!} \varphi^4 \]

To be more precise we consider two particles in the initial and final state. Then the $S$ matrix element is,

\[ S_{fi} = \langle p_1'p_2': \text{out} | p_1p_2: \text{in} \rangle \]

\[ = (-i\lambda)^4 \int d^4x_1d^4x_2d^4x_3d^4x_4 e^{-ip_1\cdot x_1-ip_2\cdot x_2+ip'_1\cdot x_3+ip'_2\cdot x_4} \]

\[ (\Box x_1 + m^2) \cdots (\Box x_4 + m^2) \langle 0 | T(\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)) | 0 \rangle \]

As the case $p = 0$ is trivial (there is no interaction) we begin by the $p = 1$ case.

• $p = 1$

Then the Green function is,

\[ G(x_1, x_2, x_3, x_4) = (-i\lambda) \int d^4z \langle 0 | T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi_{in}(z_1)}{4!} : \cdots : \frac{\varphi_{in}(z_p)}{4!} :) | 0 \rangle \]

\[ = (-i\lambda)^4 \int d^4z \Delta_F(x_1-z)\Delta_F(x_2-z)\Delta_F(x_3-z)\Delta_F(x_4-z) \]

(3.61)

to which corresponds, in the configuration space, the diagram of Fig. (3.4). To proceed,
3.5. FEYNMAN RULES FOR $\lambda \phi^4$

we introduce the Fourier transform of the propagators, that is,

$$\Delta_F(x_1 - z) = \int \frac{d^4q_1}{(2\pi)^4} e^{-i q_1 \cdot (x_1 - z)} \Delta_F(q_1)$$  \hspace{1cm} (3.62)

where

$$\Delta_F(q_1) = \frac{i}{q_1^2 - m^2}$$  \hspace{1cm} (3.63)

then

$$G(x_1, \cdots, x_4) = (-i\lambda) \int d^4z \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} e^{-i q_1 \cdot x_1 - i q_2 x_2 - i q_3 x_3 - i q_4 x_4 + i(q_1 + q_2 + q_3 + q_4) \cdot z} \Delta_F(q_1) \Delta_F(q_2) \Delta_F(q_3) \Delta_F(q_4)$$

$$= (-i\lambda) \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} e^{-i q_1 \cdot x_1 - i q_2 x_2 - i q_3 x_3 - i q_4 x_4} (2\pi)^4 \delta^4(q_1 + q_2 + q_3 + q_4) \Delta_F(q_1) \Delta_F(q_2) \Delta_F(q_3) \Delta_F(q_4)$$  \hspace{1cm} (3.64)

If we now introduce the $T$ matrix transition amplitude, defined by

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta(P_f - P_i) T_{fi}$$  \hspace{1cm} (3.65)

we obtain

$$-iT_{fi} = (-i\lambda)$$  \hspace{1cm} (3.66)

for this amplitude we draw the Feynman diagram of Fig. (3.5), and we associate to the vertex the number $(-i\lambda)$.

• $p = 2$

Let us consider now a more complicated case, the evaluation of $G(x_1 \cdots x_4)$ in second order in the coupling $\lambda$. After this exercise we will be in position to be able to state the
Figure 3.5: Vertex in momentum space.

Feynman rules in momentum space with all generality. From Eq. (3.60) we get in second order in $\lambda$,

$$G(x_1, \cdots x_4) = \frac{(-i\lambda)^2}{2!} \int d^4 z_1 d^4 z_2 \langle 0 | T \left( \varphi_{in}(x_1) \varphi_{in}(x_2) \varphi_{in}(x_3) \varphi_{in}(x_4) : \frac{\varphi_{in}^4(z_1)}{4!} : \frac{\varphi_{in}^4(z_2)}{4!} : \right) | 0 \rangle_c$$

$$= \frac{(-i\lambda)^2}{2!} \int d^4 z_1 d^4 z_2 \left( \frac{4 \times 3}{4!} \right) \times \left( \frac{4 \times 3}{4!} \right) \times 2 \left\{ \Delta_F(x_1 - z_1) \Delta_F(x_2 - z_2) \Delta_F(z_1 - z_2) \Delta_F(z_1 - z_2) \Delta_F(z_2 - x_3) \Delta_F(z_2 - x_4) + \Delta_F(x_1 - z_1) \Delta_F(x_2 - z_2) \Delta_F(z_1 - z_2) \Delta_F(z_1 - z_2) \Delta_F(z_1 - x_3) \Delta_F(z_2 - x_4) + \Delta_F(x_1 - z_1) \Delta_F(x_2 - z_2) \Delta_F(z_1 - z_2) \Delta_F(z_1 - z_2) \Delta_F(z_1 - x_3) \Delta_F(z_2 - x_3) + \Delta_F(x_1 - z_2) \Delta_F(x_2 - z_2) \Delta_F(z_2 - z_1) \Delta_F(z_2 - z_1) \Delta_F(z_1 - x_3) \Delta_F(z_1 - x_4) + \Delta_F(x_1 - z_2) \Delta_F(x_2 - z_1) \Delta_F(z_1 - z_2) \Delta_F(z_2 - z_1) \Delta_F(z_1 - x_3) \Delta_F(z_2 - x_4) + \Delta_F(x_1 - z_2) \Delta_F(x_2 - z_1) \Delta_F(z_1 - z_2) \Delta_F(z_1 - z_2) \Delta_F(z_1 - x_4) \Delta_F(z_1 - x_3) \right\}$$

$$= \frac{(-i\lambda)^2}{2!} \int d^4 z_1 d^4 z_2 \left\{ \right\}$$
3.5. FEYNMAN RULES FOR $\lambda \varphi^4$

Let us now go into momentum space, by introducing the Fourier transform of the propagators. We start by diagram a),

$$G^{(a)}(x_1, x_2, x_3, x_4) =$$

$$\int \frac{(-i\lambda)^2}{2!} \frac{1}{2} \int d^4z_1 d^4z_2 \Delta_F(x_1 - z_1) \Delta_F(x_2 - z_1) \Delta_F(z_1 - z_2) \Delta_F(z_2 - x_3) \Delta_F(z_2 - x_4)$$

$$= \frac{(-i\lambda)^2}{2!} \frac{1}{2} \int d^4z_1 d^4z_2 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \frac{d^4q_5}{(2\pi)^4} \frac{d^4q_6}{(2\pi)^4}$$

$$e^{i((q_1 \cdot x_1 + q_2 \cdot x_2 + q_3 \cdot x_3 + q_4 \cdot x_4) + z_1 \cdot (q_5 - q_1 - q_2 + q_3) + z_2 \cdot (q_3 + q_4 - q_5 - q_6))}$$

$$\Delta_F(q_1) \Delta_F(q_2) \Delta_F(q_3) \Delta_F(q_4) \Delta_F(q_5) \Delta_F(q_6)$$

$$= \frac{(-i\lambda)^2}{2!} \frac{1}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \frac{d^4q_5}{(2\pi)^4} \frac{d^4q_6}{(2\pi)^4} \delta^4(q_1 + q_2 - q_3 - q_4) e^{i[q_1 \cdot x_1 + q_2 \cdot x_2 + q_3 \cdot x_3 + q_4 \cdot x_4]}$$

$$\Delta_F(q_1) \Delta_F(q_2) \Delta_F(q_3) \Delta_F(q_4) \Delta_F(q_5) \Delta_F(q_1 + q_2 - q_3 - q_4)$$

(3.68)

Now we insert the last equation into the reduction formula. We get

$$S^{(a)}_{fi} = (i)^4 \int d^4x_1 \cdots d^4x_4 e^{-i[p_1 \cdot x_1 + p_2 \cdot x_2 - p'_1 \cdot x_3 - p'_2 \cdot x_4]}$$

$$\left(\Box x_1 + m^2 \right) \cdots \left(\Box x_4 + m^2 \right) G^{(a)}(x_1, \cdots, x_4)$$

(3.69)

The only dependence of $G^{(a)}$ in the coordinates, $x_i (i = 1, \cdots, 4)$, is in the exponential, therefore,

$$\left(\Box x_i + m^2 \right) \rightarrow (-q_i^2 + m^2)$$

(3.70)

and using

$$(-q_i^2 + m^2) \Delta_F(q_i) = -i$$

(3.71)

we get

$$S^{(a)}_{fi} = \frac{(-i\lambda)^2}{2!} \frac{1}{2} \int d^4x_1 \cdots d^4x_4 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \delta^4(q_1 + q_2 - q_3 - q_4)$$

+ ($z_1 \leftrightarrow z_2$)

(3.67)
we get, exactly the same result as in Eq. (3.73). Therefore, matrix, we associate to each vertex a factor \((-i\lambda)^2\) and for each loop the integral (3.74)

\[
\int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(q_1 + q_2 - q_5)
\]

This expression can be written in the form

\[
S_{fi}^{(a)} = (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \frac{(-i\lambda)^2}{2!} \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(p_1 + p_2 - q)
\]

If we denote by \(a'\) the diagram \(a\) with the interchange \(z_1 \leftrightarrow z_2\) and redo the calculation we get exactly the same result as in Eq. (3.73). Therefore,

\[
S_{fi}^{(a+a')} = (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2)(-i\lambda)^2 \frac{1}{2} \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(p_1 + p_2 - q)
\]

or in terms of the \(T_{fi}\) matrix,

\[
-i T_{fi}^{(a+a')} = (-i\lambda)^2 \frac{1}{2} \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(p_1 + p_2 - q)
\]

To encode this result we draw the Feynman diagram of Fig. (3.6), that has the same topology as \(a\) and \(a'\) but in momentum space. We find that in order to evaluate the \(-iT\) matrix, we associate to each vertex a factor \((-i\lambda)\), to each internal line a propagator \(\Delta_F\) and for each loop the integral \(\int \frac{d^4q}{(2\pi)^4}\). Besides that we have 4-momentum conservation at each vertex. Finally there is a symmetry factor (see below) which takes the value \(\frac{1}{2}\) for this diagram.

If we repeat the calculations for diagrams \(b) + b'\) and \(c) + c'\) it is easy to see that we get,

\[
-i T_{fi}^{(b+b')} = (-i\lambda)^2 \frac{1}{2} \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(q - p_1 + p'_1)
\]

\[
-i T_{fi}^{(c+c')} = (-i\lambda)^2 \frac{1}{2} \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(q - p_1 + p'_1)
\]
3.6. **Feynman Rules for QED**

We now turn to the case of QED. Like $\lambda \phi^4$, it is a theory without derivatives and therefore,

$$\mathcal{L}_I = -\mathcal{H}_I = -e Q \bar{\psi}_m \gamma^\mu \psi_m A_{\mu}^m$$

(3.79)

where $e$ is the absolute value of the electron charge, or the proton charge. For the electron the sign enters explicitly in $Q = -1$. This way of writing in Eq. (3.79), allows for obvious

$$-iT_{fi}^{(c+c')} = (-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(q - p_1 + p_2')$$

(3.77)

to which correspond the diagrams of Fig. (3.7).

After this exercise we are in position to state the Feynman rules with all generality for the $\lambda \phi^4$ theory. These are rules for the $-iT$ matrix, that is, after we factorize $(2\pi)^4 \delta^4(\cdots)$. These are (for a process with $n$ external legs):

1. Draw all topologically distinct diagrams with $n$ external legs.
2. At each vertex multiply by the factor $(-i\lambda)$.
3. To each internal line associate a propagator $\Delta_F(q)$.
4. For each loop include an integral $\int \frac{d^4q}{(2\pi)^4}$. The direction of this momentum is irrelevant, but we have to respect 4-momentum conservation at each vertex.
5. Multiply by the symmetry factor of the diagram. This is defined by,

$$S = \frac{\# \text{ of distinct ways of connecting the vertices to the external legs}}{\text{Permutations of each vertex} \times \text{Permutations of equal vertices}}$$

(3.78)

6. Add the contributions of all the topologically distinct diagrams. The result is the $-iT$ matrix amplitude that enters the formula for the cross section.
CHAPTER 3. COVARIANT PERTURBATION THEORY

Figure 3.8: Kinematics for the Compton scattering.

generalizations for particles with other charges, like for instance the quarks. For QED we have then,

$$\mathcal{L}_{\text{QED}}^I = e \bar{\psi}_i \gamma^\mu \gamma^\nu A_{\mu i}^\nu$$

Due to the electric charge conservation, the Green functions that we have to deal with have an equal number of $\psi$ and $\bar{\psi}$ fields. In general we have,

$$G(x_1 \cdots x_n x_{n+1} \cdots x_{2n}; y_1 \cdots y_p) =$$

$$= \langle 0 | T(\psi(x_1) \cdots \psi(x_n) \bar{\psi}(x_{n+1}) \cdots \bar{\psi}(x_{2n}) A_{1 i}^1(y_1) \cdots A_{p i}^p(y_p)) | 0 \rangle$$

where, for simplicity, we omit the spinorial indices in the fermion fields. This equation is written in terms of the physical fields. Following a similar procedure to the scalar field case, we can obtain an expression for $G$ in terms of the in fields. This will be,

$$G(x_1 \cdots x_{2n}; y_1 \cdots y_p) =$$

$$= \frac{\langle 0 | T\bar{\psi}_i(x_1) \cdots \bar{\psi}_i(x_{2n}) A_{1 i}^1(y_1) \cdots A_{p i}^p(y_p) e^{i \int d^4 z \mathcal{L}_I(z)}}{\langle 0 | T \exp{i \int d^4 z \mathcal{L}_I(z)} | 0 \rangle}$$

$$= \langle 0 | T\bar{\psi}_i(x_1) \cdots \bar{\psi}_i(x_{2n}) A_{1 i}^1(y_1) \cdots A_{p i}^p(y_p) e^{i \int d^4 z \mathcal{L}_I(z)} | 0 \rangle_c$$

where the fields in $\mathcal{L}_I$ are normal ordered, and $\langle 0 | \cdots | 0 \rangle_c$ means that we only consider the connected diagrams. To get the Feynman rules we will evaluate a few simple processes.

3.6.1 Compton scattering

Compton scattering corresponds to the following process,

$$e^- + \gamma \to e^- + \gamma$$

and we choose the kinematics in Fig. (3.8). The $S$ matrix element to evaluate is therefore,

$$S_{fi} = \langle \langle p', s' \rangle, k' \rangle_{\text{out}}|\langle p, s \rangle, k \rangle_{\text{in}} \rangle$$
Using the LSZ reduction formula Eq. (2.101), and Eq. (2.109) we can write,
\[S_{fi} = \int d^4x' d^4x d^4y d^4y' e^{-i[p'x+k'x'-k'y' + p'y]} \bar{\psi}(k) \gamma^\mu (k) \gamma^\mu (k')\]
\[\propto \langle p', s')_{\alpha'} (i\partial_{x'} - m)_{\alpha'} | \bar{\psi}_{\gamma} (x') A_{\mu}(x) A_{\mu'}(y') \rangle \langle 0 | T(\psi_{\beta'}(x') | \bar{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) \rangle \langle 0 | (-i\partial_{y} - m)_{\beta} u_{\alpha}(p, s)\]
\[\equiv (3.85)\]

Our task is therefore to evaluate the Green function
\[G(x', x, y, y') \equiv \langle 0 | T(\psi_{\beta'}(x') | \bar{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) \langle 0 \rangle\]
\[\equiv (3.86)\]

If we use Eq. (3.82) and the fact that the interaction has an odd number of fields, we find that the lowest contribution is quadratic in the interaction\(^2\). We get
\[G(x, x', y, y') = \frac{(ie)^2}{2!} \int d^4z_1 d^4z_2 \langle 0 | T(\psi_{\beta'}(x') | \bar{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) \langle 0 \rangle\]
\[\equiv (3.87)\]

Now we use Wick’s theorem to write \(\langle 0 | T(\cdots) \rangle \langle 0 \rangle\) in terms of the propagators. We get,
\[\langle 0 | T(\psi_{\beta'}(x') | \bar{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) \langle 0 \rangle\]
\[\equiv (3.88)\]

\(^2\)By Wick’s theorem the expectation value of an odd number of fields vanishes.
CHAPTER 3. COVARIANT PERTURBATION THEORY

Figure 3.9: Diagrams for Compton scattering in configuration space.

\[
+ S_{F \beta \gamma} (x' - z_1) S_{F \gamma' \gamma} (z_1 - z_2) D_{F \mu \rho} (y - z_2) D_{F \mu' \sigma} (y' - z_1) \\
+ S_{F \beta' \gamma'} (x' - z_2) S_{F \delta \gamma} (z_2 - z_1) D_{F \mu \sigma} (y - z_1) D_{F \mu' \rho} (y' - z_2) \\
+ S_{F \beta' \gamma'} (x' - z_2) S_{F \delta \gamma} (z_2 - z_1) D_{F \mu \sigma} (y - z_1) D_{F \mu' \rho} (y' - z_2)
\]

(3.88)

To better understand Eq. (3.88) it is useful to draw the corresponding diagrams in configuration space. We show them in Fig. (3.9). From this figure it is clear that \( b \equiv c \) and \( a \equiv d \) because \( z_1 \) and \( z_2 \) are irrelevant labels. From this we get a factor of 2 that is going to cancel the \( \frac{1}{n!} \) in Eq. (3.87)\(^3\). We have then only two distinct diagrams that we take as c) and d). Then, including already the factor of 2, we get for diagram c)

\[
G^{(c)} (x,x',y,y') = (ie)^2 \gamma^\sigma \gamma^\rho S_{F \beta' \gamma'} (x' - z_2) S_{F \delta \gamma} (z_2 - z_1) D_{F \mu \sigma} (y - z_1) D_{F \mu' \rho} (y' - z_2)
\]

(3.89)

To proceed we could, like in the case of \( \lambda \phi^4 \), introduce the Fourier transform of the propagators. However, it is easier to get rid of the external legs using the results,

\[
(i \partial_x - m)_{\alpha \lambda} S_{F \lambda \beta} (x - y) = i \delta_{\alpha \beta} \delta^4 (x - y)
\]

\[
S_{F \alpha \lambda} (x - y) (-i \partial_y - m)_{\lambda \beta} = i \delta_{\alpha \beta} \delta^4 (x - y)
\]

(3.90)

and

\[
\Box x D_{F \mu \nu} (x - y) = ig_{\mu \nu} \delta^4 (x - y)
\]

(3.91)

\(^3\)In fact this result is general, for \( n \) vertices we have \( n! \) that cancels against the factor \( \frac{1}{n!} \) from the expansion of the exponential.
Therefore, the spinor to an electron in the initial state we associate a spinor to get the quantity $(ie\gamma)$ corresponding to the diagram on the left panel of Fig. (3.10). In Eq. (3.95) we factor out the quantity $(ie\gamma)$, because it will be clear that this quantity will be the Feynman rule for the vertex. The arrows in these diagrams correspond to the flow of electric charge. Notice that to an electron in the initial state we associate a spinor $u(p, s)$ and for an electron in

\[
S_{fi}^{(c)} = (ie)^2 \int d^4x d^4y d^4z e^{-i(p+k-q)} S_F(q) (ie\gamma) u(p, s)
\]

Finally we use

\[
S_F(z_2 - z_1) = \int \frac{d^4q}{(2\pi)^4} \frac{i(q + m)}{q^2 - m^2 + i\epsilon} e^{-iq(z_2 - z_1)}
\]

\[
\equiv \int \frac{d^4q}{(2\pi)^4} S_F(q) e^{-iq(z_2 - z_1)}
\]

(3.93)

to get

\[
S_{fi}^{(c)} = \int \frac{d^4q}{(2\pi)^4} d^4z e^{-i(z_1 + k - q) + i(z_2 + p + k' - q)}
\]

\[
\zeta(k)\zeta^{(c)}(k') \eta(p', s') (ie\gamma) S_F(p + k)(ie\gamma) u(p, s)
\]

(3.94)

Therefore, the $T$ matrix transition amplitude is,

\[
-T_{fi}^{(c)} = \zeta(k)\zeta^{(c)}(k') \eta(p', s') (ie\gamma) S_F(p + k)(ie\gamma) u(p, s)
\]

(3.95) corresponding to the diagram on the left panel of Fig. (3.10). In Eq. (3.95) we factor out the quantity $(ie\gamma)$, because it will be clear that this quantity will be the Feynman rule for the vertex. The arrows in these diagrams correspond to the flow of electric charge. Notice that to an electron in the initial state we associate a spinor $u(p, s)$ and for an electron in

![Figure 3.10: Diagrams for Compton scattering.](image-url)
the final state we associate the spinor $\bm{\pi}(p', s')$. Since the electron line as to be a c-number, we start writing the line in the reverse order of that of the arrows.

In a similar way for diagram d) we will get the diagram represented in Fig. (3.10), that corresponds to the following expression,

$$-iT^{(d)}_{fi} = \epsilon^\mu(k)\epsilon^{*\nu'}(k')\bm{\pi}(p', s')(ie\gamma_\mu)SF(p - k')(ie\gamma_{\nu'})u(p, s)$$  \hspace{1cm} (3.96)

Looking at Eqs. (3.95) and (3.96) we are almost in a position to state the Feynman rules for QED. Before that we will look at a case where we have positrons.

### 3.6.2 Electron–positron elastic scattering (Bhabha scattering)

We will consider electron-positron elastic scattering, the so-called Bhabha scattering,

$$e^-(p) + e^+(q) \rightarrow e^-(p') + e^+(q')$$  \hspace{1cm} (3.97)

This example will teach us two things. First, how positrons (that is the anti-particles) enter in the amplitudes. Secondly we will learn that, sometimes, due the anti-commutation rules of the fermions, we will get relative minus signs between different diagrams. We have,

$$S_{fi} = \langle (p', s'), (q', s'); \text{out} | (p, s), (q, s); \text{in} \rangle$$  \hspace{1cm} (3.98)

corresponding to the kinematics in Fig. (3.11). Notice that the arrows are in the direction of flow of charge of the electron, but the momenta do correspond to the real momenta of the particles or antiparticles in that frame: $p$ entering and $p'$ exiting for the electron, and $q$ entering and $q'$ exiting for the positron. In the following we will not look the spin dependence in order to simplify the notation. Then using Eq. (3.98) we write,

$$S_{fi} = \int d^4x'd^4x d^4y'd^4y e^{-i[p-x+q-y-p'-x'-q']} \pi(p')_\alpha (i\not{p'} - m)_{\alpha\beta} \pi(q)(i\not{q} - m)_{\gamma\delta}$$

$$\langle 0 | T\overline{\psi}(y')\psi_\beta(x')\overline{\psi}(x)\psi_\delta(y) | 0 \rangle$$

$$(-i\not{\partial} - m)_{\beta'\alpha'}u_{\alpha'}(p)(-i\not{\partial} - m)_{\delta'\gamma'}v_{\gamma'}(q')$$  \hspace{1cm} (3.99)

We have, therefore, to evaluate the Green function

$$G(y', x', x, y) \equiv \langle 0 | T\overline{\psi}(y')\psi_\beta(x')\overline{\psi}(x)\psi_\delta(y) | 0 \rangle$$  \hspace{1cm} (3.100)
The lowest order contribution is of second order in the coupling\(^4\). We have (to simplify we omit the label \(m\)),

\[
G(y', x', x, y) = \frac{(ie)^2}{2} (\gamma^\mu)_{\epsilon \epsilon'} (\gamma^\nu)_{\phi \phi'} \int d^4z_1 d^4z_2 \\
\langle 0 | T \bar{\psi}_\beta(y') \gamma_\epsilon(x') \bar{\psi}_\beta(x) \gamma_\epsilon(y) : \bar{\psi}_\epsilon(z_1) \gamma_\epsilon(z_1) A_\mu(z_1) : \bar{\psi}_\phi(z_2) \gamma_\phi(z_2) A_\nu(z_2) : | 0 \rangle \\
= \frac{(ie)^2}{2} (\gamma^\mu)_{\epsilon \epsilon'} (\gamma^\nu)_{\phi \phi'} \int d^4z_1 d^4z_2 \\
- S_{F\beta\epsilon}(x' - z_1) S_{F\epsilon'\beta'}(z_1 - x) S_{F\delta\epsilon'}(y - z_2) S_{F\epsilon'\delta'}(z_2 - y') D_{F\mu\nu}(z_1 - z_2) \\
+ S_{F\delta\epsilon}(y - z_1) S_{F\epsilon'\beta'}(z_1 - x) S_{F\beta\epsilon}(x' - z_2) S_{F\phi'\delta'}(z_2 - y') D_{F\mu\nu}(z_1 - z_2) \\
+ (z_1 \leftrightarrow z_2) \tag{3.101}
\]

Once more the exchange \((z_1 \leftrightarrow z_2)\) compensates for the factor \(\frac{1}{\mathcal{D}}\) and we have two diagrams with a relative minus sign, as it is shown in Fig. (3.12). Let us look at the contribution of diagram a),

\[
S^{(a)}_{f_1} = - \int d^4x d^4y d^4x' d^4y' d^4z_1 d^4z_2 (ie)^2 (\gamma^\mu)_{\epsilon \epsilon'} (\gamma^\nu)_{\phi \phi'} e^{-i[p_\epsilon + q_\epsilon - p'_\epsilon - q'_\epsilon]} \\
\bar{u}(p') \gamma_\epsilon(p) u_\alpha(m - \gamma_\delta) \gamma_\delta \gamma_\epsilon \gamma_\nu(q') \gamma_\delta \gamma_\epsilon \gamma_\nu(q) \\
S_{F\beta\epsilon}(x' - z_1) S_{F\epsilon'\beta'}(z_1 - x) S_{F\delta\epsilon'}(y - z_2) S_{F\epsilon'\delta'}(z_2 - y') \\
(\bar{\psi}_x - m)_{\beta \epsilon'} u_\alpha(p) (\bar{\psi}_y - m)_{\phi \phi'} v_\gamma(q') D_{F\mu\nu}(z_1 - z_2) \\
= - \int d^4z_1 d^4z_2 e^{-i[p_\epsilon + q_\epsilon - p'_\epsilon - q'_\epsilon]} \\
\bar{u}(p') (ie\gamma^\mu) u(p) \bar{\psi}(q)(ie\gamma^\nu) v(q') D_{F\mu\nu}(z_1 - z_2) \tag{3.102}
\]

\(^4\)There is, of course, a contribution without interaction, but that corresponds to disconnected terms in which we are not interested.
Using now the Fourier transform of the photon propagator,

\[ D_{F\mu\nu}(z_1 - z_2) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}e^{-ik(z_1 - z_2)}}{k^2 + i\varepsilon} \equiv \int \frac{d^4k}{(2\pi)^4} D_{F\mu\nu}(k)e^{-ik(z_1 - z_2)} \]  

we get

\[
S_{fi}^{(a)} = -\bar{\psi}(p')(ie\gamma^\mu)u(p)\bar{\psi}(q)(ie\gamma^\nu)v(q') \int d^4z_1 d^4z_2 \frac{d^4k}{(2\pi)^4} D_{F\mu\nu}(k)e^{-iz_1:(p' - p') + k}e^{-iz_2:(q - q' - k)}
\]

\[ = -(2\pi)^4\delta^4(p + q - p' - q')\bar{\psi}(p')(ie\gamma^\nu)u(p)\bar{\psi}(q)(ie\gamma^\mu)v(q')D_{F\mu\nu}(p' - p) \]  

(3.104)

and therefore the T matrix element is,

\[ -iT_{fi}^{(a)} = -\bar{\psi}(p')(ie\gamma^\mu)u(p)D_{F\mu\nu}(p' - p)\bar{\psi}(q)(ie\gamma^\nu)v(q') \]  

(3.105)

to which corresponds the Feynman diagram of Fig. (3.13).

In a similar way we would get

\[ -iT_{fi}^{(b)} = \bar{\psi}(q)(ie\gamma^\mu)u(p)D_{F\mu\nu}(p + q)\bar{\psi}(p')(ie\gamma^\nu)v(q') \]  

(3.106)

that corresponds to the diagram of Fig. (3.14). Which of the diagrams has the minus sign is irrelevant, because this is the lowest order diagram. It depends on the conventions determining how to build the in state that lead to Eq. (3.98). Only the relative sign is important. However, higher order terms have to respect the same conventions.

### 3.6.3 Fermion Loops

Before we summarize the Feynman rules for QED let us look at what happens with fermion loops. One such example is the second order correction to the photon propagator shown...
in Fig. (3.15). First of all, the loop orientation it is only relevant if leads to topologically different diagrams. Therefore the diagrams of Fig. (3.16) are topologically equivalent and only one should be considered. However the diagrams in Fig. (3.17) are topologically distinct and both should be considered.

The second aspect that is relevant is a possible sign coming from the anti-commutation of the fermion fields, that should affect some diagrams, and in particular the fermion loop. To understand this sign we should note that by definition of loop, the internal lines are not connected to external fermion lines, they should originate only in the interaction. Therefore they should come from terms of the form

\[
\langle 0 | T \psi(z_1) \bar{A}(z_1) \psi(z_1) : \cdots : \bar{\psi}(z_n) A(z_n) \psi(z_n) : \cdots | 0 \rangle .
\]  

Now it is clear that in order to make the appropriate contractions of the fermion fields to bring them to the form of the Feynman propagator, \( \langle 0 | T \psi(z_1) \bar{\psi}(z_2) | 0 \rangle \), it is necessary...
Figure 3.17: Topologically distinct diagrams

to make an odd number of permutations of the fermion fields, and therefore we get a \((-\) sign for the loops. This sign is physically relevant because there is a lowest order diagram where the photons do not interact, corresponding to the free propagator. So the minus sign is defined in relation to this lowest order diagram and therefore it is not arbitrary (see the difference with respect to the discussion of the Bahbha scattering).

3.6.4 Feynman rules for QED

We are now in position to state the Feynman rules for \textbf{QED}

1. For a given process, draw all topologically distinct diagrams.

2. For each electron entering a diagram a factor \(u(p, s)\). If it leaves the diagram a factor \(\bar{u}(p, s)\).

3. For each positron leaving the diagram (final state) a factor \(v(p, s)\). It it enters the diagram (initial state) then we have a factor \(\bar{v}(p, s)\).

4. For each photon in the initial state we have the vector \(\varepsilon^\mu(k)\) and in the final state \(\varepsilon^{*\mu}(k)\).

5. For each internal fermionic line the propagator

\[
S_{\alpha\beta}(p) = i\frac{(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon} \tag{3.108}
\]

6. For each virtual photon the propagator (Feynman gauge)

\[
D_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2} \tag{3.109}
\]

7. For each vertex the factor
3.7 General formalism for getting the Feynman rules

After showing how to obtain the Feynman rules for $\lambda \phi^4$ and QED, we are going to present here, without proof, a general method to obtain the Feynman rules of any theory, including the case when the interactions have derivatives, that we have excluded up to now, and that is very important for the Standard Model. This method can only be fully justified with the methods of Chapter 5. For simplicity we will consider only scalar fields.

The starting point is the action taken as a functional of the fields,

$$ \Gamma_0[\varphi] \equiv \int d^4x \mathcal{L}[\varphi]. $$

3.7. GENERAL FORMALISM FOR GETTING THE FEYNMAN RULES

8. For each internal momentum, not fixed by conservation of momenta, as in the case of loops, a factor

$$ \int \frac{d^4q}{(2\pi)^4} $$

9. For each loop of fermions a $-1$ sign.

10. A factor of $-1$ between diagrams that differ by exchange of fermionic lines. In doubt, revert to Wick’s theorem.

Comments

- In QED there are no symmetry factors, that is, they are always equal to 1.

- In our discussion we did not consider the $Z$ factors that come in the reduction formulas, like in Eq. (2.65). This is true in lowest order in perturbation theory. They can be calculated also in perturbation theory. Their definition is (for instance for the electron),

$$ \lim_{\not{p} \to m} S'_F(p) = Z_2 S_F(p) $$

where $S'_F(p)$ is the propagator of the theory with interactions. Then we can obtain, in perturbation theory,

$$ Z_2 = 1 + O(\alpha) + \cdots $$

In higher orders it is necessary to correct the external lines with these $\sqrt{Z}$ factors.
In fact, $\Gamma_0[\varphi]$ is the generating functional of the one particle irreducible Green functions in lowest order, as we will see in Chapter 5. The rules are as follows:

**Propagators**

1. Start by evaluating $\Gamma_0^{(2)}(x_i, x_j) \equiv \frac{\delta^2 \Gamma_0[\varphi]}{\delta \varphi(x_i) \delta \varphi(x_j)}$

2. Then evaluate the Fourier Transform (FT) to get $\Gamma_0^{(2)}(p_i, p_j)$ defined by the relation

   $$ (2\pi)^4 \delta^4(p_i + p_j) \Gamma_0^{(2)}(p_i, p_j) \equiv \int d^4x_i d^4x_j e^{-i(p_i \cdot x_i + p_j \cdot x_j)} \Gamma_0^{(2)}(x_i, x_j) \tag{3.115} $$

   where all the momenta are incoming.

3. The Feynman propagator is then

   $$ G_{Fij}^{(0)} = i[\Gamma_0^{(2)}(p_i, p_j)]^{-1} . \tag{3.116} $$

   Do not forget that $p_i = -p_j$.

**Vertices**

1. Evaluate $\Gamma_0^{(n)}(x_1 \cdots x_n) = \frac{\delta^n \Gamma_0[\varphi]}{\delta \varphi(x_1) \cdots \delta \varphi(x_n)}$

2. Then take the Fourier Transform to obtain

   $$ (2\pi)^4 \delta^4(p_1 + p_2 + \cdots + p_n) \Gamma_0^{(n)}(p_1 \cdots p_n) \equiv \int d^4x_1 \cdots d^4x_n e^{-i(p_1 \cdot x_1 + \cdots + p_n \cdot x_n)} \Gamma_0^{(n)}(x_1 \cdots x_n) \tag{3.117} $$

3. The vertex in momenta space is then given by the rule

   $$ i\Gamma_0^{(n)}(p_1, \cdots p_n) \tag{3.118} $$

**Comments**
• For fermionic fields it is necessary to take care with the order of the derivation. The convention that we take is

\[
\frac{\delta^2}{\delta \psi_\alpha(x) \delta \overline{\psi}_\beta(y)} (\overline{\psi}(z) \Gamma \psi(z)) \equiv \Gamma_{\alpha \beta} \delta^4(z - x) \delta^4(z - y)
\]

(3.119)

\(\psi_\alpha(x) e \psi_\beta(x)\) are here taken as classical anti-commuting fields (Grassmann variables, see Chapter 5).

• The functional derivatives are defined by

\[
\frac{\delta \varphi_i(x)}{\delta \varphi_k(y)} \equiv \delta_{ik} \delta^4(x - y)
\]

(3.120)

3.7.1 Example: scalar electrodynamics

The Lagrangian is

\[
\mathcal{L} = (\partial_\mu - i e Q A_\mu) \varphi^* (\partial^\mu + i e Q A^\mu) \varphi - m \varphi^* \varphi + \mathcal{L}_{\text{Maxwell}} - \frac{\lambda}{4} (\varphi^* \varphi)^2
\]

(3.121)

Therefore

\[
\mathcal{L}_{\text{int}} = -i e Q \varphi^* \overleftrightarrow{\partial_\mu} \varphi A^\mu + e^2 Q^2 \varphi^* \varphi A_\mu A^\mu
\]

(3.122)

The propagators are the usual ones, let us consider only the vertices. There are two vertices. The cubic vertex is

\[
\Gamma_{(3)}(x_1, x_2, x_3) = -i e Q \int d^4 z \delta^4(z - x_1) (\overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial_\mu}) \delta^4(z - x_2) \delta^4(z - x_3)
\]

(3.123)

therefore

\[
(2\pi)^4 \delta^4(p + k + q) \Gamma_{(3)}(p, q, k) \equiv -i e Q \int d^4 z d^4 x_1 d^4 x_2 d^4 x_3 e^{-i (x_1 \cdot p + x_2 \cdot q + x_3 \cdot k)}
\]

\[
\delta^4(z - x_1) (\overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial_\mu}) \delta^4(z - x_2) \delta^4(z - x_3)
\]
\[
\begin{align*}
&= -ieQ \int d^4 z d^4 x_2 e^{-i[(p+k)\cdot z + q \cdot x_2]} \partial_\mu \delta^4(z - x_2) \\
&\quad + ieQ \int d^4 z d^4 x_1 e^{-i[p \cdot x_1 + (q+k)\cdot z]} \partial_\mu \delta^4(z - x_1) \\
&= -ieQ (i p_\mu - i q_\mu) (2\pi)^4 \delta^4(p + q + k) \quad (3.124)
\end{align*}
\]

Therefore we obtain for this vertex
\[
i\Gamma_\mu(p,q,k) = ieQ (p_\mu - q_\mu) = -ieQ (q_\mu - p_\mu) \quad (3.125)
\]

The other vertex is

![Quartic vertex in scalar QED (seagull).](image)

We obtain,
\[
\Gamma^{(4)}_{\mu\nu}(x_1,x_2,x_3,x_4) = 2e^2 Q^2 \delta^4(x_1 - x_2) \delta^4(x_1 - x_3) \delta^4(x_1 - x_4) g_{\mu\nu} \quad (3.126)
\]
and
\[
\Gamma^{(4)}_{\mu\nu}(p,q,k_1,k_2) = 2(eQ)^2 g_{\mu\nu} \quad (3.127)
\]
and we finally get for the Feynman rule.
\[
i2e^2 Q^2 \ g_{\mu\nu} \quad (3.128)
\]

Comment

- From the above results we can enunciate a simple rule for interactions that have derivatives of fields.

Consider that we have one field in the Lagrangian that has a derivative, say \(\partial_\mu \phi\). Then the rule is
\[
\partial_\mu \phi \rightarrow -i \ (\text{incoming momentum})_\mu \quad (3.129)
\]
In the end do not forget to multiply the result by \(i\).
As an example consider the following term in the Lagrangian for scalar electrodynamics

$$\mathcal{L} = i e Q \partial_\mu \varphi^* \varphi A^\mu + \cdots$$  

(3.130)

If $p$ is the incoming momentum of the line associated with the field $\varphi^*$, see Fig. 3.18, we have

$$\text{Vertex} = i \times (i e Q) \times (-i p_\mu) = i e Q p_\mu$$  

(3.131)

in agreement with Eq. (3.125).
Problems for Chapter 3

3.1 Show explicitly that Wick’s theorem is valid for the case of 4 fields, that is

\[ T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) =: \phi(x_1)\cdots\phi(x_4) : + \cdots \] (3.132)

3.2 For the case of the $\lambda\phi^4$ theory verify the Feynman rules for the diagrams

3.3 Consider a theory with the following interaction Lagrangian

\[ L_I = -\frac{\lambda}{3!}\phi^3 \] (3.133)

- a) Find the Feynman rules for this theory.
- b) Find the symmetry factor for the diagram

3.4 Verify that for Compton scattering the diagram
gives the result of Eq. (3.96).

3.5 Verify Eq. (3.106).

3.6 Show that in QED the symmetry factors are always 1.

3.7 Explicitly calculate the $T$ matrix element for the process $e^+e^- \rightarrow \gamma\gamma$ and verify that is in agreement with the general rules.

3.8 Show that the amplitudes for $e^+e^- \rightarrow \gamma\gamma$ and $e^-\gamma \rightarrow e^-\gamma$ are related. How can one obtain one from the other?
Chapter 4

Radiative Corrections

4.1 QED Renormalization at one-loop

We will consider the theory described by the Lagrangian

\[
\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 + \overline{\psi}(i\gamma^\mu + eA^\mu - m)\psi .
\]  (4.1)

The free propagators are

\[
\frac{i}{p - m + i\varepsilon} \equiv S_{\beta\alpha}^0(p) \quad (4.2)
\]

\[
-\frac{1}{k^2 + i\varepsilon} \left( g_{\mu\nu} + (1 - \xi) \frac{k_\mu k_\nu}{(k^2 + i\varepsilon)^2} \right)
\]

\[
= -i \left\{ \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2 + i\varepsilon} + \xi \frac{k_\mu k_\nu}{k^4} \right\}
\]

\[
\equiv G_{\mu\nu}^0(k) \quad (4.3)
\]

and the vertex

...
We will now consider the one-loop corrections to the propagators and to the vertex. We will work in the Feynman gauge ($\xi = 1$).

### 4.1.1 Vacuum Polarization

In first order the contribution to the photon propagator is given by the diagram of Fig. 4.1 that we write in the form

\[
G^{(1)}_{\mu\nu}(k) = G_{\mu\nu'}^0(k) \Pi^{\mu'\nu'}(k) G_{\nu\nu'}^0(k) \tag{4.5}
\]

where

\[
\Pi^{\mu\nu}(k) = -\frac{i}{e^2} \int \frac{d^4p}{(2\pi)^4} \frac{\text{Tr}[\gamma_\mu(\not{p} + m)\gamma_\nu(\not{p} + \not{k} + m)]}{(p^2 - m^2 + i\epsilon)((p + k)^2 - m^2 + i\epsilon)} \tag{4.6}
\]

Simple power counting indicates that this integral is quadratically divergent for large values of the internal loop momenta. In fact the divergence is milder, only logarithmic. The integral being divergent we have first to regularize it and then to define a renormalization procedure to cancel the infinities. For this purpose we will use the method of dimensional regularization. For a value of $d$ small enough the integral converges. If we define $\epsilon = 4 - d$, 

\[
\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} \frac{\text{Tr}[\gamma_\mu(\not{p} + m)\gamma_\nu(\not{p} + \not{k} + m)]}{(p^2 - m^2 + i\epsilon)((p + k)^2 - m^2 + i\epsilon)} = \gamma_{\mu\nu}(k)
\]
in the end we will have a divergent result in the limit \( \epsilon \to 0 \). We get therefore\(^1\)

\[
i \Pi_{\mu \nu}(k, \epsilon) = -4e^2 \mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{\left[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu \nu}(p^2 + p \cdot k - m^2)\right]}{(p^2 - m^2 + i\epsilon)((p + k)^2 - m^2 + i\epsilon)}
\]

\[
= -4e^2 \mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu \nu}(p, k)}{(p^2 - m^2 + i\epsilon)((p + k)^2 - m^2 + i\epsilon)} \tag{4.7}
\]

where

\[
N_{\mu \nu}(p, k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu \nu}(p^2 + p \cdot k - m^2) \tag{4.8}
\]

To evaluate this integral we first use the Feynman parameterization to rewrite the denominator as a single term. For that we use (see Appendix)

\[
\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1 - x)]^2} \tag{4.9}
\]

to get

\[
i \Pi_{\mu \nu}(k, \epsilon) = -4e^2 \mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu \nu}(p, k)}{[x(p + k)^2 - x m^2 + (1 - x)(p^2 - m^2) + i\epsilon]^2}
\]

\[
= -4e^2 \mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu \nu}(p, k)}{[p^2 + 2k \cdot px + x k^2 - m^2 + i\epsilon]^2}
\]

\[
= -4e^2 \mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu \nu}(p, k)}{[(p + kx)^2 + k^2 x(1 - x) - m^2 + i\epsilon]^2} \tag{4.10}
\]

For dimension \( d \) sufficiently small this integral converges and we can change variables

\[
p \to p - kx \tag{4.11}
\]

We then get

\[
i \Pi_{\mu \nu}(k, \epsilon) = -4e^2 \mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu \nu}(p - kx, k)}{[p^2 - C + i\epsilon]^2} \tag{4.12}
\]

where

\[
C = m^2 - k^2 x(1 - x) \tag{4.13}
\]

\( N_{\mu \nu} \) is a polynomial of second degree in the loop momenta as can be seen from Eq. (4.8). However as the denominator in Eq. (4.12) only depends on \( p^2 \) it is easy to show that

\[
\int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{[p^2 - C + i\epsilon]^2} = 0
\]

\[
\int \frac{d^dp}{(2\pi)^d} \frac{p^\mu p^\nu}{[p^2 - C + i\epsilon]^2} = \frac{1}{d} g^{\mu \nu} \int \frac{d^dp}{(2\pi)^d} \frac{p^2}{[p^2 - C + i\epsilon]^2} \tag{4.14}
\]

\(^1\)Where \( \mu \) is a parameter with dimensions of a mass that is introduced to ensure the correct dimensions of the coupling constant in dimension \( d \), that is, \( |\epsilon| = \frac{\mu}{d} \to \frac{1}{2} \). We take then \( \epsilon \to e\mu \frac{1}{2} \). For more details see the Appendix.
CHAPTER 4. RADIATIVE CORRECTIONS

This means that we only have to calculate integrals of the form

\[ I_{r,m} = \int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^r}{|p^2 - C + i\epsilon|^m} \]

\[ = \int \frac{d^{d-1} p}{(2\pi)^d} \int dp^0 \frac{(p^2)^r}{|p^2 - C + i\epsilon|^m} \quad (4.15) \]

To make this integration we will use integration in the plane of the complex variable \( p^0 \) as described in Fig. 4.2. The deformation of the contour corresponds to the so called Wick rotation,

\[ p^0 \rightarrow ip^0_E \quad ; \quad \int_{-\infty}^{+\infty} \rightarrow i \int_{-\infty}^{+\infty} dp^0_E \quad (4.16) \]

and \( p^2 = (p^0)^2 - |\vec{p}|^2 = -(p^0_E)^2 - |\vec{p}|^2 \equiv -p^2_E \), where \( p_E = (p^0_E, \vec{p}) \) is an euclidean vector, that is

\[ p^2_E = (p^0_E)^2 + |\vec{p}|^2 \quad (4.17) \]

We can then write (see the Appendix for more details),

\[ I_{r,m} = i(-1)^{r-m} \int \frac{d^d p_E}{(2\pi)^d} \frac{p^2_E^r}{|p^2_E + C|^m} \quad (4.18) \]

where we do not need the \( i\epsilon \) anymore because the denominator is positive definite\(^2 \)\((C > 0)\).

To proceed with the evaluation of \( I_{r,m} \) we write,

\[ \int d^d p_E = \int d\mathbf{p} \quad \mathbf{p}^{d-1} \quad d\Omega_{d-1} \quad (4.19) \]

where \( \mathbf{p} = \sqrt{(p^0_E)^2 + |\vec{p}|^2} \) is the length of of vector \( p_E \) in the euclidean space with \( d \) dimensions and \( d\Omega_{d-1} \) is the solid angle that generalizes spherical coordinates. We can show (see Appendix) that

\[ \int d\Omega_{d-1} = 2 \frac{\pi^{d-2}}{\Gamma(\frac{d}{2})} \quad (4.20) \]

\(^2\)The case when \( C < 0 \) is obtained by analytical continuation of the final result.
The $\mathbf{p}$ integral is done using the result,
\begin{equation}
\int_0^\infty dx \frac{x^p}{(x^2 + C)^m} = \frac{\Gamma\left(\frac{p+1}{2}\right) C^{\frac{p}{2}(p-2m+1)} \Gamma\left(-\frac{p}{2} + m - \frac{1}{2}\right)}{2 \Gamma(m)} \tag{4.21}
\end{equation}
and we finally get
\begin{equation}
I_{r,m} = iC^{r-m+\frac{d}{2}} \frac{(-1)^{r-m}}{(4\pi)^\frac{d}{2}} \frac{\Gamma(r + \frac{d}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(m - r - \frac{d}{2})}{\Gamma(m)} \tag{4.22}
\end{equation}
Note that the integral representation of $I_{r,m}$, Eq. (4.15) is only valid for $d < 2(m - r)$ to ensure the convergence of the integral when $\mathbf{p} \to \infty$. However the final form of Eq. (4.22) can be analytically continued for all the values of $d$ except for those where the function $\Gamma(m - r - d/2)$ has poles, which are (see section C.6),
\begin{equation}
m - r - \frac{d}{2} \neq 0, -1, -2, \ldots \tag{4.23}
\end{equation}
For the application to dimensional regularization it is convenient to write Eq. (4.22) after making the substitution $d = 4 - \epsilon$. We get
\begin{equation}
I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^\frac{d}{2}} \frac{4\pi}{\epsilon} \frac{\Gamma(2 + r - \frac{\epsilon}{2})}{\Gamma(2 - \frac{\epsilon}{2})} \frac{\Gamma(m - r - 2 + \frac{\epsilon}{2})}{\Gamma(m)} \tag{4.24}
\end{equation}
that has poles for $m - r - 2 \leq 0$ (see section C.6).

We now go back to calculate $\Pi_{\mu\nu}$. First we notice that after the change of variables of Eq. (4.11) we get, neglecting terms that vanish due to Eq. (4.14),
\begin{equation}
N_{\mu\nu}(p - kx, k) = 2p_\mu p_\nu + 2x^2 k_\mu k_\nu - 2xk_\mu k_\nu - g_{\mu\nu} \left(p^2 + x^2 k^2 - xk^2 - m^2\right) \tag{4.25}
\end{equation}
and therefore
\begin{equation}
N_{\mu\nu} \equiv \mu^\epsilon \int d^dp \frac{N_{\mu\nu}(p - kx, k)}{[p^2 - C + i\epsilon]^2}
= \left(\frac{2}{d-1}\right) g_{\mu\nu} \mu^\epsilon I_{1,2} + \left[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu} m^2\right] \mu^\epsilon I_{0,2} \tag{4.26}
\end{equation}
Using now Eq. (4.24) we can write
\begin{equation}
\mu^\epsilon I_{0,2} = \frac{i}{16\pi^2} \left(\frac{4\pi \mu^2}{C}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)} \tag{4.27}
\end{equation}
where we have used the expansion of the $\Gamma$ function, Eq. (C.47),
\begin{equation}
\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon) \tag{4.28}
\end{equation}
\[ \gamma \text{ being the Euler constant and we have defined, Eq. (C.50),} \]
\[ \Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \]  

(4.29)

In a similar way
\[ \mu^2 I_{1,2} = -\frac{i}{16\pi^2} \left( \frac{4\pi \mu^2}{C} \right)^{\frac{3}{2}} \frac{C \Gamma(3 - \frac{\mu}{2}) \Gamma(-1 + \frac{\mu}{2})}{\Gamma(2 - \frac{\mu}{2}) \Gamma(2)} \]
\[ = \frac{i}{16\pi^2} C \left( 1 + 2\Delta_\epsilon - 2 \ln \frac{C}{\mu^2} \right) + O(\epsilon) \]  

(4.30)

Due to the existence of a pole in $1/\epsilon$ in the previous equations we have to expand all quantities up to $O(\epsilon)$. This means for instance, that
\[ \frac{2}{d-1} = \frac{2}{4-\epsilon} - 1 = -\frac{1}{2} + \frac{1}{8}\epsilon + O(\epsilon^2) \]  

(4.31)

Substituting back into Eq. (4.26), and using Eq. (4.13), we obtain
\[ \mathcal{N}_{\mu\nu} = g_{\mu\nu} \left[ -\frac{1}{2} + \frac{1}{8}\epsilon + O(\epsilon^2) \right] \left[ \frac{i}{16\pi^2} C \left( 1 + 2\Delta_\epsilon - 2 \ln \frac{C}{\mu^2} \right) + O(\epsilon) \right] \]
\[ + \left[ -2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu} m^2 \right] \left[ \frac{i}{16\pi^2} \left( \Delta_\epsilon - \ln \frac{C}{\mu^2} \right) + O(\epsilon) \right] \]
\[ = -\frac{i}{16\pi^2} k_\mu k_\nu \left[ \left( \Delta_\epsilon - \ln \frac{C}{\mu^2} \right) 2x(1-x) \right] \]
\[ + \frac{i}{16\pi^2} g_{\mu\nu} k^2 \left[ \Delta_\epsilon \left( x(1-x) + x(1-x) \right) + \ln \frac{C}{\mu^2} \left( -x(1-x) - x(1-x) \right) \right. \]
\[ \left. + x(1-x) \left( \frac{1}{2} - \frac{1}{2} \right) \right] \]
\[ + \frac{i}{16\pi^2} g_{\mu\nu} m^2 \left[ \Delta_\epsilon(-1 + 1) + \ln \frac{C}{\mu^2} (1 - 1) + \left( \frac{1}{2} + \frac{1}{2} \right) \right] \]  

(4.32)

and finally
\[ \mathcal{N}_{\mu\nu} = \frac{i}{16\pi^2} \left( \Delta_\epsilon - \ln \frac{C}{\mu^2} \right) \left( g_{\mu\nu} k^2 - k_\mu k_\nu \right) 2x(1-x) \]  

(4.33)

Now using Eq. (4.7) we get
\[ \Pi_{\mu\nu}(k) = -4\epsilon^2 \frac{1}{16\pi^2} \left( g_{\mu\nu} k^2 - k_\mu k_\nu \right) \int_0^1 dx \ 2x(1-x) \left( \Delta_\epsilon - \ln \frac{C}{\mu^2} \right) \]
\[ = - \left( g_{\mu\nu} k^2 - k_\mu k_\nu \right) \Pi(k^2, \epsilon) \]  

(4.34)

where
\[ \Pi(k^2, \epsilon) \equiv \frac{2\alpha}{\pi} \int_0^1 dx \ x(1-x) \left[ \Delta_\epsilon - \ln \frac{m^2 - x(1-x)k^2}{\mu^2} \right] \]  

(4.35)
This expression clearly diverges as $\epsilon \to 0$. Before we show how to renormalize it let us discuss the meaning of $\Pi_{\mu\nu}(k)$. The full photon propagator is given by the series represented in Fig. 4.3, where

$$\equiv i \Pi_{\mu\nu}(k) = \text{sum of all one-particle irreducible (proper) diagrams to all orders} \quad (4.36)$$

In lowest order we have the contribution represented in Fig. 4.4, which is what we have just calculated. To continue it is convenient to rewrite the free propagator of the photon (in an arbitrary gauge $\xi$) in the following form

$$iG^{0}_{\mu\nu} = \left( g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^{2}} \right) \frac{1}{k^{2}} + \xi \frac{k_{\mu}k_{\nu}}{k^{4}} = P^{T}_{\mu\nu} \frac{1}{k^{2}} + \xi \frac{k_{\mu}k_{\nu}}{k^{4}}$$

$$\equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu} \quad (4.37)$$

where we have introduced the transversal projector $P^{T}_{\mu\nu}$ defined by

$$P^{T}_{\mu\nu} = \left( g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^{2}} \right) \quad (4.38)$$
obviously satisfying the relations,

\[
\begin{cases}
  k^\mu P^T_{\mu\nu} = 0 \\
  P^T_{\mu\nu} P^T_{\nu\rho} = P^T_{\mu\rho}
\end{cases}
\]  

The full photon propagator can also in general be written separating its transversal and longitudinal parts

\[
G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu}
\]  

where \( G^T_{\mu\nu} \) satisfies

\[
G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu}
\]  

Eq. (4.34) means that, to first order, the vacuum polarization tensor is transversal, that is

\[
i \Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu} \Pi(k)
\]  

This result is in fact valid to all orders of perturbation theory, a result that can be shown using the Ward-Takahashi identities. This means that the longitudinal part of the photon propagator is not renormalized,

\[
G^L_{\mu\nu} = G^{0L}_{\mu\nu}
\]  

For the transversal part we obtain from Fig. 4.3,

\[
i G^T_{\mu\nu} = \frac{P^T_{\mu\nu}}{k^2} + \frac{P^T_{\mu\rho}}{k^2} (-i)k^2 P^T_{\rho\nu} \Pi(k^2) (-i) P^T_{\nu\sigma} \frac{1}{k^2} + \cdots
\]

\[
= \frac{P^T_{\mu\nu}}{k^2} \left[ 1 - \Pi(k^2) + \Pi^2(k^2) + \cdots \right]
\]  

which gives, after summing the geometric series,

\[
i G^T_{\mu\nu} = \frac{1}{P^T_{\mu\nu} k^2 \left[ 1 + \Pi(k^2) \right]}
\]  

All that we have done up to this point is formal because the function \( \Pi(k) \) diverges. The most satisfying way to solve this problem is the following. The initial lagrangian from which we started has been obtained from the classical theory and nothing tell us that it should be exactly the same in quantum theory. In fact, as we have just seen, the normalization of the wave functions is changed when we calculate one-loop corrections, and the same happens to the physical parameters of the theory, the charge and the mass. Therefore we can think that the correct lagrangian is obtained by adding corrections to the
classical lagrangian, order by order in perturbation theory, so that we keep the definitions of charge and mass as well as the normalization of the wave functions. The terms that we add to the lagrangian are called counterterms\(^3\). The total lagrangian is then,

\[
\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta \mathcal{L}
\]  

Counterterms are defined from the normalization conditions that we impose on the fields and other parameters of the theory. In QED we have at our disposal the normalization of the electron and photon fields and of the two physical parameters, the electric charge and the electron mass. The normalization conditions are, to a large extent, arbitrary. It is however convenient to keep the expressions as close as possible to the free field case, that is, without radiative corrections. We define therefore the normalization of the photon field as,

\[
\lim_{k \to 0} k^2 i G_{\mu \nu}^{RT} = 1 \cdot P^{T}_{\mu \nu}
\]  

where \(G_{\mu \nu}^{RT}\) is the renormalized propagator (the transversal part) obtained from the lagrangian \(L_{\text{total}}\). The justification for this definition comes from the following argument. Consider the Coulomb scattering to all orders of perturbation theory. We have then the situation described in Fig. 4.5. Using the Ward-Takahashi identities one can show that

\[
\text{Figure 4.5: Corrections to Coulomb scattering.}
\]

the last three diagrams cancel in the limit \(q = p' - p \to 0\). Then the normalization condition, Eq. (4.47), means that we have the situation described in Fig. 4.6, that is, the experimental value of the electric charge is determined in the limit \(q \to 0\) of the Coulomb scattering.

\(^3\)This interpretation in terms of quantum corrections makes sense. In fact we can show that an expansion in powers of the coupling constant can be interpreted as an expansion in \(\hbar^L\), where \(L\) is the number of the loops in the expansion term.
The counterterm lagrangian has to have the same form as the classical lagrangian to respect the symmetries of the theory. For the photon field it is traditional to write

$$\Delta \mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\delta Z_3 F_{\mu\nu}F_{\mu\nu}$$

(4.48)

corresponding to the following Feynman rule

$$\mu \nu kk \quad -i\delta Z_3 k^2 \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

(4.49)

We have then

$$i\Pi_{\mu\nu} = i\Pi_{\mu\nu}^{\text{loop}} - i\delta Z_3 k^2 \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

$$= -i \left( \Pi(k, \epsilon) + \delta Z_3 \right) k^2 P_{\mu\nu}$$

(4.50)

Therefore we should make the substitution

$$\Pi(k, \epsilon) \rightarrow \Pi(k, \epsilon) + \delta Z_3$$

(4.51)

in the photon propagator. We obtain,

$$iG_{\mu\nu}^{T} = P_{\mu\nu}^{T} \frac{1}{k^2} \frac{1}{1 + \Pi(k, \epsilon) + \delta Z_3}$$

(4.52)

The normalization condition, Eq. (4.47), implies

$$\Pi(0, \epsilon) + \delta Z_3 = 0$$

(4.53)

from which one determines the constant $\delta Z_3$. We get

$$\delta Z_3 = -\Pi(0, \epsilon) = -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \left[ \Delta_\epsilon - \ln \frac{m^2}{\mu^2} \right]$$

$$= -\frac{\alpha}{3\pi} \left[ \Delta_\epsilon - \ln \frac{m^2}{\mu^2} \right]$$

(4.54)
The renormalized photon propagator can then be written as

$$iG_{\mu\nu}(k) = \frac{pT_{\mu\nu}}{k^2[1 + \Pi(k, \epsilon) - \Pi(0, \epsilon)]} + iG_{\mu\nu}^L$$

(4.55)

The finite radiative corrections are given through the function

$$\Pi^R(k^2) = \Pi(k^2, \epsilon) - \Pi(0, \epsilon)$$

$$= -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \ln \left[ \frac{m^2 - x(1-x)k^2}{m^2} \right]$$

$$= -\frac{\alpha}{3\pi} \left\{ \frac{1}{3} + 2 \left( 1 + \frac{2m^2}{k^2} \right) \left[ \left( \frac{4m^2}{k^2} - 1 \right)^{1/2} \cot^{-1} \left( \frac{4m^2}{k^2} - 1 \right)^{1/2} - 1 \right] \right\}$$

(4.56)

where the last equation is valid for $k^2 < 4m^2$. For values $k^2 > 4m^2$ the result for $\Pi^R(k^2)$ can be obtained from Eq. (4.56) by analytical continuation. Using $(k^2 > 4m^2)$

$$\cot^{-1} iz = i \left( -\tanh^{-1} z + \frac{i\pi}{2} \right)$$

(4.57)

and

$$\left( \frac{4m^2}{k^2} - 1 \right)^{1/2} \rightarrow i\sqrt{1 - \frac{4m^2}{k^2}}$$

(4.58)

we get

$$\Pi^R(k^2) = -\frac{\alpha}{3\pi} \left\{ \frac{1}{3} + 2 \left( 1 + \frac{2m^2}{k^2} \right) \left[ -1 + \sqrt{1 - \frac{4m^2}{k^2}} \tanh^{-1} \left( \frac{1 - \frac{4m^2}{k^2}}{k^2} \right)^{1/2} \right] \right\}$$

(4.59)

(4.60)

The imaginary part of $\Pi^R$ is given by

$$Im \Pi^R(k^2) = \frac{\alpha}{3} \left( 1 + \frac{2m^2}{k^2} \right) \sqrt{1 - \frac{4m^2}{k^2}} \theta \left( 1 - \frac{4m^2}{k^2} \right)$$

(4.61)

and it is related to the pair production that can occur \(^5\) for $k^2 > 4m^2$.

### 4.1.2 Self-energy of the electron

The electron full propagator is given by the diagrammatic series of Fig. 4.7, which can be written as,

\(^4\)Notice that the photon mass is not renormalized, that is the pole of the photon propagator remains at $k^2 = 0$.

\(^5\)For $k^2 > 4m^2$ there is the possibility of producing one pair $e^+e^-$. Therefore on top of a virtual process (vacuum polarization) there is a real process (pair production).
\[ S(p) = S^0(p) + S^0(p) \left( -i \Sigma(p) \right) S^0(p) + \cdots \]

\[ = S^0(p) \left[ 1 - i \Sigma(p)S(p) \right] \]  

(4.62)

where we have identified

\[ \equiv -i \Sigma(p) \]  

(4.63)

Multiplying on the left with \( S_0^{-1}(p) \) and on the right with \( S^{-1}(p) \) we get

\[ S_0^{-1}(p) = S^{-1}(p) - i \Sigma(p) \]  

(4.64)

which we can rewrite as

\[ S^{-1}(p) = S_0^{-1}(p) + i \Sigma(p) \]  

(4.65)

Using the expression for the free field propagator,

\[ S_0(p) = \frac{i}{\not{p} - m} \implies S_0^{-1}(p) = -i(\not{p} - m) \]  

(4.66)

we can then write

\[ S^{-1}(p) = S_0^{-1}(p) + i\Sigma(p) \]

\[ = -i \left[ \not{p} - (m + \Sigma(p)) \right] \]  

(4.67)

We conclude that it is enough to calculate \( \Sigma(p) \) to all orders of perturbation theory to obtain the full electron propagator. The name *self-energy* given to \( \Sigma(p) \) comes from the fact that, as can be seen in Eq. (4.67), it comes as an additional (momentum dependent) contribution to the mass.

In lowest order there is only the diagram of Fig. 4.8 contributing to \( \Sigma(p) \) and therefore we get,

\[ -i\Sigma(p) = (+ie)^2 \int \frac{d^4k}{(2\pi)^4} (-i) \frac{g_{\mu\nu}}{k^2 - \lambda^2 + i\varepsilon} \gamma^\mu \not{p} + \not{k} - m + i\varepsilon \gamma^\nu \]  

(4.68)
where we have chosen the Feynman gauge ($\xi = 1$) for the photon propagator and we have introduced a small mass for the photon $\lambda$, in order to control the infrared divergences (IR) that will appear when $k^2 \to 0$ (see below). Using dimensional regularization and the results of the Dirac algebra in dimension $d$,

$$\gamma_\mu (\not p + \not k) \gamma^\mu = -(\not p + \not k) \gamma_\mu \gamma^\mu + 2(\not p + \not k) = -(d-2)(\not p + \not k)$$

$$m \gamma_\mu \gamma^\mu = m d$$

we get

$$-i \Sigma(p) = -\mu' e^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - \lambda^2 + i\varepsilon} \gamma_\mu \frac{\not p + \not k + m}{(p + k)^2 - m^2 + i\varepsilon} \gamma^\mu$$

$$= -\mu' e^2 \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)(\not p + \not k) + m d}{[k^2 - \lambda^2 + i\varepsilon][(p + k)^2 - m^2 + i\varepsilon]}$$

$$= -\mu' e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)(\not p + \not k) + m d}{[(k + px)^2 + p^2x(1-x) - \lambda^2(1-x) - xm^2 + i\varepsilon]^2}$$

$$= -\mu' e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)(\not p(1-x) + \not k) + m d}{[k^2 + p^2x(1-x) - \lambda^2(1-x) - xm^2 + i\varepsilon]^2}$$

$$= -\mu' e^2 \int_0^1 dx \left[ -(d-2)\not p(1-x) + m d \right] I_{0,2}$$

where$^6$

$$I_{0,2} = \frac{i}{16\pi^2} \left[ \Delta_\varepsilon - \ln \left[ -p^2 x(1-x) + m^2 x + \lambda^2 (1-x) \right] \right]$$

The contribution from the loop in Fig. 4.8 to the electron self-energy $\Sigma(p)$ can then be written in the form,

$$\Sigma(p)^{\text{loop}} = A(p^2) + B(p^2) \not p$$

with

$$A = e^2 \mu' (4 - \varepsilon) m \frac{1}{16\pi^2} \int_0^1 dx \left[ \Delta_\varepsilon - \ln \left[ -p^2 x(1-x) + m^2 x + \lambda^2 (1-x) \right] \right]$$

$$B = -e^2 \mu' (2 - \varepsilon) \frac{1}{16\pi^2} \int_0^1 dx (1-x) \left[ \Delta_\varepsilon \right]$$

$^6$The linear term in $k$ vanishes.
Using now the expansions
\[
\mu^\epsilon (4 - \epsilon) = 4 \left[ 1 + \epsilon \left( \ln \mu - \frac{1}{4} \right) + O(\epsilon^2) \right]
\]
\[
\mu^\epsilon (4 - \epsilon) \Delta \epsilon = 4 \left[ \Delta \epsilon + 2 \left( \ln \mu - \frac{1}{4} \right) + O(\epsilon) \right]
\]
\[
\mu^\epsilon (2 - \epsilon) = 2 \left[ 1 + \epsilon \left( \ln \mu - \frac{1}{2} \right) + O(\epsilon^2) \right]
\]
\[
\mu^\epsilon (2 - \epsilon) \Delta \epsilon = 2 \left[ \Delta \epsilon + 2 \left( \ln \mu - \frac{1}{2} \right) + O(\epsilon) \right]
\]
we can finally write,
\[
A(p^2) = \frac{4 e^2 m}{16\pi^2} \int_0^1 dx \left[ \Delta \epsilon - \frac{1}{2} - \ln \left( \frac{-p^2 x (1-x) + m^2 x + \lambda^2 (1-x)}{\mu^2} \right) \right] \] (4.75)
and
\[
B(p^2) = -\frac{2 e^2}{16\pi^2} \int_0^1 dx (1-x) \left[ \Delta \epsilon - 1 - \ln \left( \frac{-p^2 x (1-x) + m^2 x + \lambda^2 (1-x)}{\mu^2} \right) \right] \] (4.76)
To continue with the renormalization program we have to introduce the counterterm la-
grangian and define the normalization conditions. We have
\[
\Delta L = i (Z_2 - 1) \bar{\psi} \gamma^\mu \partial_\mu \psi - (Z_2 - 1) m \bar{\psi} \psi + Z_2 \delta m \bar{\psi} \psi + (Z_1 - 1) e \bar{\psi} \gamma^\mu \psi A_\mu \] (4.77)
and therefore we get for the self-energy
\[
-i \Sigma(p) = -i \Sigma^{\text{loop}}(p) + i (\not{p} - m) \delta Z_2 + i \delta m \] (4.78)
Contrary to the case of the photon we see that we have two constants to determine. In the on-shell renormalization scheme that is normally used in QED the two constants are obtained by requiring that the pole of the propagator corresponds to the physical mass (hence the name of on-shell renormalization), and that the residue of the pole of the renormalized electron propagator has the same value as the free field propagator. This implies,
\[
\Sigma(\not{p} = m) = 0 \rightarrow \delta m = \Sigma^{\text{loop}}(\not{p} = m)
\]
\[
\left. \frac{\partial \Sigma}{\partial \not{p}} \right|_{\not{p} = m} = 0 \rightarrow \delta Z_2 = \left. \frac{\partial \Sigma^{\text{loop}}}{\partial \not{p}} \right|_{\not{p} = m}
\] (4.79)
We then get for \( \delta m \),
\[
\delta m = A(m^2) + m B(m^2)
\]
4.1. QED RENORMALIZATION AT ONE-LOOP

\[
\frac{2me^2}{16\pi^2} \int_0^1 dx \left\{ \left[ 2\Delta_\epsilon - 1 - 2 \ln \left( \frac{m^2x^2 + \lambda^2(1-x)}{\mu^2} \right) \right] 
- (1-x) \left[ \Delta_\epsilon - 1 - \ln \left( \frac{m^2x^2 + \lambda^2(1-x)}{\mu^2} \right) \right] \right\} 
= \frac{2me^2}{16\pi^2} \left[ \frac{3}{2} \Delta_\epsilon - \frac{1}{2} - \int_0^1 dx (1+x) \ln \left( \frac{m^2x^2 + \lambda^2(1-x)}{\mu^2} \right) \right] 
= \frac{3\alpha m}{4\pi} \left[ \Delta_\epsilon - \frac{1}{3} - \frac{2}{3} \int_0^1 dx (1+x) \ln \left( \frac{m^2x^2}{\mu^2} \right) \right] 
\tag{4.80}
\]}

where in the last step in Eq. (4.80) we have taken the limit \( \lambda \to 0 \) because the integral does not diverge in that limit. \(^7\) In a similar way we get for \( \delta Z_2 \),

\[
\delta Z_2 = \frac{\partial \Sigma_\text{loop}}{\partial \phi} \bigg|_{\phi=m} = \frac{\partial A}{\partial \phi} \bigg|_{\phi=m} + B + m \frac{\partial B}{\partial \phi} \bigg|_{\phi=m} 
\tag{4.81}
\]

where

\[
\frac{\partial A}{\partial \phi} \bigg|_{\phi=m} = \frac{4e^2m^2}{16\pi^2} \int_0^1 dx \frac{2(1-x)x}{-m^2x(1-x) + m^2x + \lambda^2(1-x)} 
= \frac{2m^2}{\pi} \int_0^1 dx \frac{(1-x)x}{m^2x^2 + \lambda^2(1-x)} 
\]

\[
B = -\frac{\alpha}{2\pi} \int_0^1 dx (1-x) \left[ \Delta_\epsilon - 1 - \ln \left( \frac{m^2x^2 + \lambda^2(1-x)}{\mu^2} \right) \right] 
\tag{4.82}
\]

Substituting Eq. (4.82) in Eq. (4.81) we get,

\[
\delta Z_2 = -\frac{\alpha}{2\pi} \left[ \frac{1}{2} \Delta_\epsilon - \frac{1}{2} - \int_0^1 dx (1-x) \ln \left( \frac{m^2x^2}{\mu^2} \right) \right] 
- 2 \int_0^1 dx \frac{(1+x)(1-x)xm^2}{m^2x^2 + \lambda^2(1-x)} 
= \frac{\alpha}{4\pi} \left[ -\Delta_\epsilon - 4 + \ln \frac{m^2}{\mu^2} - 2 \ln \frac{\lambda^2}{m^2} \right] 
\tag{4.83}
\]

where we have taken the \( \lambda \to 0 \) limit in all cases that was possible. It is clear that the final result in Eq. (4.83) diverges in that limit, therefore implying that \( Z_2 \) is IR divergent. This is not a problem for the theory because \( \delta Z_2 \) is not a physical parameter. We will see in section 4.4.2 that the IR divergences cancel for real processes. If we had taken a general gauge (\( \xi \neq 1 \)) we would find out that \( \delta m \) would not be changed but that \( Z_2 \) would show a gauge dependence. Again, in physical processes this should cancel in the end.

4.1.3 The Vertex

The diagram contributing to the QED vertex at one-loop is the one shown in Fig. 4.9. In

\(^7\)\( \delta m \) is not IR divergent.
the Feynman gauge ($\xi = 1$) this gives a contribution,

$$
\left. i e \mu \epsilon^2 \Lambda^{\text{loop}}_{\mu}(p', p) \right|_{p = p'} = \left. (i e \mu \epsilon^2)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\rho\sigma}}{k^2 - \lambda^2 + i\epsilon} \gamma^\sigma \frac{i[(p' + \vec{k}) + m]}{(p' + k)^2 - m^2 + i\epsilon} \frac{i[(\vec{p} + k') + m]}{(p + k)^2 - m^2 + i\epsilon} \gamma^\rho \right|_{p = p'}
$$

(4.84)

where $\Lambda_\mu$ is related to the full vertex $\Gamma_\mu$ through the relation

$$
\left. i\Gamma_\mu = i e \left( \gamma_\mu + \Lambda^{\text{loop}}_\mu + \gamma_\mu \delta Z_1 \right) \right|_{p = p'} = i e \left( \gamma_\mu + \Lambda^{R}_\mu \right)
$$

(4.85)

The integral that defines $\Lambda^{\text{loop}}_{\mu}(p', p)$ is divergent. As before we expect to solve this problem by regularizing the integral, introducing counterterms and normalization conditions. The counterterm has the same form as the vertex and is already included in Eq. (4.85). The normalization constant is determined by requiring that in the limit $q = p' - p \rightarrow 0$ the vertex reproduces the tree level vertex because this is what is consistent with the definition of the electric charge in the $q \rightarrow 0$ limit of the Coulomb scattering. Also this should be defined for on-shell electrons. We have therefore that the normalization condition gives,

$$
\left. \Pi(p) \left( \Lambda^{\text{loop}}_{\mu} + \gamma_\mu \delta Z_1 \right) u(p) \right|_{\vec{p} = m} = 0
$$

(4.86)

If we are interested only in calculating $\delta Z_1$ and in showing that the divergences can be removed with the normalization condition then the problem is simpler. It can be done in two ways.

**1st method**

We use the fact that $\delta Z_1$ is to be calculated on-shell and for $p = p'$. Then

$$
i \sigma \Lambda^{\text{loop}}_{\mu}(p, p) = e^2 \mu \epsilon \int \frac{d^4k}{(2\pi)^4 k^2 - \lambda^2 + i\epsilon} \frac{1}{\vec{p} + \vec{k} - m + i\epsilon} \frac{1}{\vec{p} + \vec{k}' - m + i\epsilon} \frac{1}{\vec{p}' + \vec{k} - m + i\epsilon} \gamma^\mu
$$

(4.87)
4.1. QED Renormalization at One-Loop

However we have

\[ \frac{1}{\not{p} + \not{k} - m + i\varepsilon} \gamma_\mu \frac{1}{\not{p} + \not{k} - m + i\varepsilon} = -\frac{\partial}{\partial p^\mu} \frac{1}{\not{p} + \not{k} - m + i\varepsilon} \]  

(4.88)

and therefore

\[ i\Lambda_{\mu}^{\text{loop}}(p,p) = -\epsilon^2 \mu^\varepsilon \frac{\partial}{\partial p^\mu} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - \lambda^2 + i\varepsilon} \gamma_\rho \frac{\not{p} + \not{k} + m}{(p + k)^2 - m^2 + i\varepsilon} \gamma_\rho \]  

(4.89)

\[ = -i \frac{\partial}{\partial p^\mu} \Sigma^{\text{loop}}(p) \]  

(4.90)

We conclude then, that \( \Lambda_{\mu}^{\text{loop}}(p,p) \) is related to the self-energy of the electron\(^8\),

\[ \Lambda_{\mu}^{\text{loop}}(p,p) = -\frac{\partial}{\partial p^\mu} \Sigma^{\text{loop}} \]  

(4.91)

On-shell we have

\[ \Lambda_{\mu}^{\text{loop}}(p,p) \bigg|_{p=m} = \frac{\partial \Sigma^{\text{loop}}}{\partial p^\mu} \bigg|_{p=m} = -\delta Z_2 \gamma_\mu \]  

(4.92)

and the normalization condition, Eq. (4.86), gives

\[ \delta Z_1 = \delta Z_2 \]  

(4.93)

As we have already calculated \( \delta Z_2 \) in Eq. (4.83), then \( \delta Z_1 \) is determined.

2nd method

In this second method we do not rely in the Ward identity but just calculate the integrals for the vertex in Eq. (4.84). For the moment we do not put \( p' = p \) but we will assume that the vertex form factors are to be evaluated for on-shell spinors. Then we have

\[ i\overline{\psi}(p')\Lambda_{\mu}^{\text{loop}} u(p) = \epsilon^2 \mu^\varepsilon \int \frac{d^d k}{(2\pi)^d} \frac{\overline{\psi}(p) \gamma_\rho [p' + \not{k} + m] \gamma_\mu [p + \not{k} + m] \gamma_\rho u(p)}{D_0 D_1 D_2} \]  

(4.94)

where

\[ N_\mu = \overline{\psi}(p) \left[ (-2 + d) k^2 \gamma_\mu + 4 p \cdot p' \gamma_\mu + 4(p + p') \cdot k \gamma_\mu + 4m k_\mu \right] \]

*This result is one of the forms of the Ward-Takahashi identity.
\[ \begin{align*}
-4\# (\mathbf{p}+\mathbf{p}^\prime)_\mu + 2(2-d)\# k_\mu \right] u(p) \quad (4.95) \\
D_0 &= k^2 - \lambda^2 + i\epsilon \quad (4.96) \\
D_1 &= (k+p')^2 - m^2 + i\epsilon \quad (4.97) \\
D_2 &= (k+p)^2 - m^2 + i\epsilon \quad (4.98)
\end{align*} \]

Now using the results of section C.7.3 with

\[ r'^\mu = p'^\mu ; \quad r''_\mu = p^\mu \quad (4.99) \]
\[ P^\mu = x_1p'^\mu + x_2p^\mu \quad (4.100) \]

\[ C = (x_1 + x_2)^2 m^2 - x_1x_2 q^2 + (1 - x_1 - x_2)\lambda^2 \quad (4.101) \]

where

\[ q = p' - p \quad (4.102) \]

we get,

\[ i \overline{u}(p') \Lambda^{loop}_{\mu} u(p) = i \frac{\alpha}{4\pi} \Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{2C} \]

\[ \begin{align*}
&\left\{ \overline{u}(p')\gamma_\mu u(p) \left[ -(-2 + d)(x_1^2 m^2 + x_2^2 m^2 + 2x_1 x_2 p' \cdot p) - 4p' \cdot p \\
&+ 4(p + p') \cdot (x_1 p' + x_2 p) + \frac{(2 - d)^2}{2} C \left( \Delta_\mu - \ln \frac{C}{\mu^2} \right) \right] \\
&+ \overline{u}(p)u(p)m \left[ 4(x_1 p' + x_2 p)_\mu - 4(p' + p)_\mu(x_1 + x_2) \\
&- 2(2 - d)(x_1 + x_2)(x_1 p' + x_2 p)_\mu \right] \right\} \quad (4.103)
\end{align*} \]

\[ = i \overline{\eta}(p) \left[ G(q^2) \gamma_\mu + H(q^2) (p + p') \right] u(p) \quad (4.104) \]

where we have defined\(^9\),

\[ G(q^2) \equiv \frac{\alpha}{4\pi} \left[ \Delta_\mu - 2 - 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln \frac{(x_1 + x_2)^2 m^2 - x_1 x_2 q^2 + (1 - x_1 - x_2)\lambda^2}{\mu^2} \right. \]

\[ \left. + \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left( \frac{-2(x_1 + x_2)^2 m^2 - x_1 x_2 q^2 - 4m^2 + 2q^2}{(x_1 + x_2)^2 m^2 - x_1 x_2 q^2 + (1 - x_1 - x_2)\lambda^2} + \frac{2(x_1 + x_2)(4m^2 - q^2)}{(x_1 + x_2)^2 m^2 - x_1 x_2 q^2 + (1 - x_1 - x_2)\lambda^2} \right) \right] \quad (4.105) \]

\[ H(q^2) \equiv \frac{\alpha}{4\pi} \left[ \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{-2m(x_1 + x_2) + 2m(x_1 + x_2)^2}{(x_1 + x_2)^2 m^2 - x_1 x_2 q^2 + (1 - x_1 - x_2)\lambda^2} \right] \quad (4.106) \]

\(^9\)To obtain Eq. (4.106) one has to show that the symmetry of the integrals in \(x_1 \leftrightarrow x_2\) implies that the coefficient of \(p\) is equal to the coefficient of \(p'\).
Now, using the definition of Eq. (4.85), we get for the renormalized vertex,
\[
\pi(p')\Lambda^R_{\mu}(p',p)u(p) = \pi(p') \left[ (G(q^2) + \delta Z_1) \gamma_\mu + H(q^2) (p + p')_\mu \right] u(p)
\] (4.107)
As \(\delta Z_1\) is calculated in the limit of \(q = p' - p \to 0\) it is convenient to use the Gordon identity to get rid of the \((p' + p)_\mu\) term. We have
\[
\pi(p') (p' + p)_\mu u(p) = \pi(p') \left[ 2m\gamma_\mu - i\sigma_{\mu\nu} q^\nu \right] u(p)
\] (4.108)
and therefore,
\[
\pi(p')\Lambda^R_{\mu}(p',p)u(p) = \pi(p') \left[ (G(q^2) + 2mH(q^2) + \delta Z_1) \gamma_\mu - iH(q^2) \sigma_{\mu\nu} q^\nu \right] u(p)
\] (4.109)
where we have introduced the usual notation for the vertex form factors,
\[
F_1(q^2) \equiv G(q^2) + 2mH(q^2) + \delta Z_1
\] (4.110)
\[
F_2(q^2) \equiv -2mH(q^2)
\] (4.111)
The normalization condition of Eq. (4.86) implies \(F_1(0) = 0\), that is,
\[
\delta Z_1 = -G(0) - 2mH(0)
\] (4.112)
We have therefore to calculate \(G(0)\) and \(H(0)\). In this limit the integrals of Eqs. (4.105) and (4.106) are much simpler. We get (we change variables \(x_1 + x_2 \to y\),
\[
G(0) = \frac{\alpha}{4\pi} \left[ \Delta_4 - 2 - 2 \int_0^1 dx_1 \int_{x_1}^1 dy \ln \frac{y^2m^2 + (1 - y)\lambda^2}{\mu^2} \right.
onumber
+ \left. \int_0^1 dx_1 \int_{x_1}^1 dy \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1 - y)\lambda^2} \right]
\] (4.113)
\[
H(0) = \frac{\alpha}{4\pi} \int_0^1 dx_1 \int_{x_1}^1 dy \frac{-2my + 2my^2}{y^2m^2 + (1 - y)\lambda^2}
\] (4.114)
Now using
\[
\int_0^1 dx_1 \int_{x_1}^1 dy \ln \frac{y^2m^2 + (1 - y)\lambda^2}{\mu^2} = \frac{1}{2} \left( \ln \frac{m^2}{\mu^2} - 1 \right)
\] (4.115)
\[
\int_0^1 dx_1 \int_{x_1}^1 dy \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1 - y)\lambda^2} = 7 + 2 \ln \frac{\lambda^2}{m^2}
\] (4.116)
\[
\int_0^1 dx_1 \int_{x_1}^1 dy \frac{-2my + 2my^2}{y^2m^2 + (1 - y)\lambda^2} = -\frac{1}{m}
\] (4.117)
(where we took the limit \(\lambda \to 0\) if possible) we get,
\[
G(0) = \frac{\alpha}{4\pi} \left[ \Delta_4 + 6 - \ln \frac{m^2}{\mu^2} + 2 \ln \frac{\lambda^2}{m^2} \right]
\] (4.118)
\[ H(0) = -\frac{\alpha}{4\pi} \frac{1}{m} \]  

Substituting the previous expressions in Eq. (4.112) we get finally,

\[ \delta Z_1 = \frac{\alpha}{4\pi} \left[ -\Delta - 4 + \ln \frac{m^2}{\mu^2} - 2 \ln \frac{\lambda^2}{m^2} \right] \]

in agreement with Eq. (4.83) and Eq. (4.93). The general form of the form factors \( F_i(q^2) \), for \( q^2 \neq 0 \), is quite complicated. We give here only the result for \( q^2 < 0 \) (in section C.10.3 we will give a general formula for numerical evaluation of these functions),

\[ F_1(q^2) = \frac{\alpha}{4\pi} \left\{ 2 \ln \frac{\lambda^2}{m^2} + 4 \right\} \left( \theta \coth \theta - 1 \right) - \theta \tanh \frac{\theta}{2} - 8 \coth \theta \int_0^{\theta/2} \beta \tanh \beta d\beta \]

\[ F_2(q^2) = \frac{\alpha}{2\pi} \frac{\theta}{\sinh \theta} \]

where

\[ \sinh^2 \frac{\theta}{2} = -\frac{q^2}{4m^2} \]

In the limit of zero transferred momenta \((q = p' - p = 0)\) we get

\[ \begin{cases} 
 F_1(0) = 0 \\
 F_2(0) = \frac{\alpha}{2\pi} 
\end{cases} \]

a result that we will use in section 4.4.1 while discussing the anomalous magnetic moment of the electron.

### 4.2 Ward-Takahashi identities in QED

In the study of the QED vertex, in one of the methods, we used the Ward identity

\[ \Lambda_\mu(p, p) = -\frac{\partial}{\partial p^\mu} \Sigma(p) \]

We are going to derive here the general form for these identities. The following discussion is formal in the sense that the various Green functions are divergent. We have to prove that we can find a regularization scheme that preserves the identities. This happens when one uses a regularization that preserves the gauge invariance of the theory. Examples are dimensional regularization and the Pauli-Villars regularization.

Ward identities are a consequence of the gauge invariance of the theory, as will be fully discussed in chapters 5 and 6. Here we are only going to use the fact that there is a conserved current,

\[ j_\mu = \bar{\psi} \gamma_\mu \psi \]

\[ \partial_\mu j^\mu = 0 \]
Using now the equal time commutation relations, and zero, respectively, we get,

\[ \partial \delta^0 \left( 0 | T_{j\mu}(x) \psi(x_1) \overline{\psi}(y_1) \cdots \psi(x_n) \overline{\psi}(y_n) A_{\nu_1}(z_1) \cdots A_{\nu_p}(z_p) | 0 \right) \]  

(4.126)

This quantity does not vanish, despite the fact that \( \partial \psi j_\mu = 0 \). This happens because in the time ordered product we have \( \theta \) functions that depend on the coordinate \( x^0 \). For instance, for the field \( \psi(x_i) \) we should have a contribution of the form,

\[ \delta^0 \left( \theta(x^0 - x_i^0) j_\mu(x) \psi(x_i) + \theta(x_i^0 - x^0) \psi(x_i) j_\mu(x) \right) \]

\[ = \delta(x^0 - x_i^0) j_\mu(x) \psi(x_i) - \delta(x^0 - x_i^0) \psi(x_i) j_\mu(x) \]

\[ = [j_\mu(x), \psi(x_i)] \delta(x^0 - x_i^0) \]  

(4.127)

In this way we get \( ( \ ^\wedge \ \text{means that we omit that term from the sum}) \),

\[ \partial \delta^0 \left( 0 | T_{j\mu}(x) \psi(x_1) \cdots \overline{\psi}(y_n) A_{\nu_1}(z_1) \cdots A_{\nu_p}(z_p) | 0 \right) \]

\[ = \sum_{i=1}^{n} \langle 0 | T \left( [j_\mu(x), \psi(x_i)] \delta(x^0 - x_i^0) \overline{\psi}(y_i) \right. \]

\[ + \psi(x_i) \left( j_\mu(x), \overline{\psi}(y_i) \right) \delta(x^0 - y_i^0) \} \psi(x_1) \overline{\psi}(y_1) \cdots \overline{\psi}(y_i) \cdots A_{\nu_p}(z_p) | 0 \rangle \]

\[ + \sum_{j=1}^{p} \langle 0 | T \overline{\psi}(x_1) \cdots \overline{\psi}(y_n) A_{\nu_1}(z_1) \cdots [j_\mu(x), A_{\nu_j}(z_j)] \delta(x^0 - z_j^0) \cdots A_{\nu_p}(z_p) | 0 \rangle \]  

(4.128)

Using now the equal time commutation relations,

\[ [j_\mu(x), \psi(x')] \delta(x^0 - x^0) = -e \psi(x) \delta^4(x - x') \]

\[ [j_\mu(x), \overline{\psi}(x')] \delta(x^0 - x^0) = e \overline{\psi}(x) \delta^4(x - x') \]  

(4.129)

\[ [j_\mu(x), A_{\nu}(x')] \delta(x^0 - x^0) = 0 \]

that express that \( \overline{\psi}, \psi \) and \( A_{\mu} \) create quanta with charge \( Q = \int d^3 x j^0(x) \) equal to \(-e, +e\) and zero, respectively, we get,

\[ \partial \delta^0 \left( 0 | T_{j\mu}(x) \psi(x_1) \cdots \overline{\psi}(y_n) A_{\nu_1}(z_1) \cdots A_{\nu_p}(z_p) | 0 \right) \]

\[ = e \langle 0 | T \overline{\psi}(y_1) \cdots A_{\nu_p}(z_p) | 0 \rangle \sum_{i=1}^{n} \left[ \delta^4(x - y_i) - \delta^4(x - x_i) \right] \]  

(4.130)

Taking different values for \( n \) and \( p \) we get different relations among the Green functions of the theory. We will consider in the following, two important cases.

### 4.2.1 Transversality of the photon propagator \( n = 0, p = 1 \)

The Green function \( \langle 0 | T_{j\mu}(x) A_{\nu}(y) | 0 \rangle \) corresponds to the Feynman diagram of Fig. 4.10, and it is related with the full photon propagator shown in Fig. 4.11, by the diagrammatic relation shown in Fig. 4.12 known as the Dyson-Schwinger equation for QED. It can be
FIGURE 4.10: Green function \( \langle 0 | T j_\mu(x) A_\nu(y) | 0 \rangle \).

FIGURE 4.11: Photon propagator written as
\[
G_{\mu \nu}(x - y) = G^0_{\mu \nu}(x - y) - i \int d^4 x' G^0_{\mu \rho}(x - x') \langle 0 | T j^\rho(x') A_\nu(y) | 0 \rangle.
\]

We apply now the derivative \( \partial_\mu x \) to get,
\[
\partial_\mu G_{\mu \nu}(x - y) = \partial_\mu G^0_{\mu \nu}(x - y) - i \int d^4 x' \partial_\mu G^0_{\mu \rho}(x - x') \langle 0 | T j^\rho(x') A_\nu(y) | 0 \rangle.
\]

The free photon propagator is given by,
\[
G^0_{\mu \rho}(x' - x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i(x-x') \cdot p} G^0_{\mu \rho}(p).
\]

where
\[
G^0_{\mu \rho}(p) = -i \left[ (g_{\mu \nu} - \frac{p_\mu p_\nu}{p^2}) \frac{1}{p^2} + \frac{\xi p_\mu p_\nu}{p^2} \right].
\]
4.2. WARD-TAKAHASHI IDENTITIES IN QED

and we get

\[ \partial^\nu G_{\mu\nu}(x - y) = \partial^\nu G^0_{\mu\nu}(x - y) + i \int d^4 x' \partial^\nu \widetilde{F}(x - x') \langle 0 | T j_\mu(x') A_\nu(y) | 0 \rangle \]

\[ = \partial^\nu G^0_{\mu\nu}(x - y) - i \int d^4 x' \widetilde{F}(x - x') \partial^\nu \langle 0 | T j_\mu(x') A_\nu(y) | 0 \rangle \]

\[ = \partial^\nu G^0_{\mu\nu}(x - y) \quad (4.136) \]

where we have made an integration by parts and used the Ward-Takahashi identity for \( n = 0, p = 1 \). We have then

\[ \partial^\nu G_{\mu\nu}(x - y) = \partial^\nu G^0_{\mu\nu}(x - y) \quad (4.137) \]

which in momenta space implies

\[ p^\mu G_{\mu\nu}(p) = p^\mu G^0_{\mu\nu}(p) \quad (4.138) \]

This means that the longitudinal part of the photon propagator is not renormalized, or in other words, that the self-energy of the photon (vacuum polarization) is transverse. In fact

\[ p^\mu G^0_{\mu\nu}(p) = -i \xi \frac{p_\nu p^2}{p^2} \quad (4.139) \]

or

\[ p^\mu = -i \xi \frac{p_\nu}{p^2} G^{-1}_{\mu\nu}(p) = -\xi \frac{p_\nu}{p^2} \Gamma_{\nu\mu}(p) \quad (4.140) \]

But, in agreement with our conventions, we have

\[ \Gamma_{\nu\mu}(p) = -(g_{\nu\mu} p^2 - p_\nu p_\mu) - \frac{1}{\xi} p_\nu p_\mu + \Pi_{\nu\mu}(p^2) \quad (4.141) \]

and therefore

\[ -\xi \frac{p_\nu}{p^2} \Gamma_{\nu\mu}(p) = p_\mu - \frac{1}{\xi} \frac{1}{p^2} p_\nu \Pi_{\nu\mu}(p^2) = p_\mu \quad (4.142) \]

which gives

\[ p^\nu \Pi_{\nu\mu}(p^2) = 0 \quad (4.143) \]

that is, the self-energy is transverse.

4.2.2 Identity for the vertex \( n = 1, p = 0 \)

We are now interested in the Green function,

\[ \langle 0 | T j_\mu(x) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle \quad (4.144) \]

to which corresponds the diagram of Fig. 4.13. This Green function can be related with the vertex \( \langle 0 | T A_\mu(x) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle \) corresponding to the diagram of Fig. 4.14, through the following diagrammatic equation,
that we can write as,

$$
\langle 0 | T A_\mu(x) \psi_\beta(x_1) \bar{\psi}_\alpha(y_1) | 0 \rangle = -i \int d^4x' G_\mu^{0}(x - x') \langle 0 | T j^\nu(x') \psi_\beta(x_1) \bar{\psi}_\alpha(y_1) | 0 \rangle \quad (4.146)
$$
Taking the Fourier transform,
\[
\int d^4x d^4x_1 d^4y_1 e^{i(p' \cdot x_1 - p \cdot y_1 - q \cdot x)} \langle 0 | T A_\mu(x) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle = -iG^0_{\mu\nu}(q) \int d^4x d^4x_1 d^4y_1 e^{i(p' \cdot x_1 - p \cdot y_1 - q \cdot x)} \langle 0 | T j^\nu(x_1) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle \tag{4.148}
\]
where the direction of the momenta are shown in Fig. 4.15, and the momentum transferred

\[ q = p' - q \]

Figure 4.15: Definition of the momenta in the vertex.

is \( q = p' - p \).

On the other side, using the definition of \( \Gamma_\mu \), we have,
\[
\int d^4x d^4x_1 d^4y_1 e^{i(p' \cdot x_1 - p \cdot y_1 - q \cdot x)} \langle 0 | T A_\mu(x) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle = (2\pi)^4 \frac{\delta^4(p' - p - q)}{p' \cdot \Gamma_\mu(p', p)} S(p) \tag{4.147}
\]

Therefore we get,
\[
(2\pi)^4 \delta(p' - p - q) G_{\mu\nu}(q) S(p') i\Gamma_\nu(p', p) S(p) = -iG^0_{\mu\nu}(q) \int d^4x d^4x_1 d^4y_1 e^{i(p' \cdot x_1 - p \cdot y_1 - q \cdot x)} \langle 0 | T j^\nu(x_1) \psi(x_1) \overline{\psi}(y_1) | 0 \rangle \tag{4.149}
\]

Multiplying by \( q^\mu \) and using the result,
\[
q^\mu G_{\mu\nu}(q) = q^\mu G^0_{\mu\nu}(q) = -i\frac{q^\nu q_\nu}{q^2} \tag{4.150}
\]

we can then write (using the Ward identity for \( n = 1, p = 0 \))
\[
(2\pi)^4 \delta(p' - q - p) S(p') q^\nu \Gamma_\nu(p', p) S(p) = \int d^4x d^4x_1 d^4y_1 \partial_\nu \langle 0 | T j^\nu(x_1) \psi(x_1) \overline{\psi}(y_1) | 0 \rangle e^{i(p' \cdot x_1 - p \cdot y_1 - q \cdot x)}
\]
\[ = i e \int d^4x_1 d^4y_1 e^{i(p'\cdot x_1 - p\cdot y_1 - q\cdot x)} \langle 0 | T\psi(x_1)\bar{\psi}(y_1) | 0 \rangle \left[ \delta(x - y_1) - \delta(x - x_1) \right] \]

\[ = i e (2\pi)^4 \delta(p' - p - q) \left[ S(p') - S(p) \right] \quad (4.151) \]

or

\[ q^\nu \Gamma_\nu(p', p) = i e \left[ S^{-1}(p) - S^{-1}(p') \right] \quad (4.152) \]

As \( q^\nu = (p' - p)^\nu \) we get in the limit \( p' = p \),

\[ \Gamma_\nu(p, p) = -i e \frac{\partial S^{-1}}{\partial p^\nu} = -e \left( \gamma_\nu - \frac{\partial \Sigma}{\partial p^\nu} \right) \quad (4.153) \]

Using \( \Gamma_\nu = -e(\gamma_\nu + \Lambda_\nu) \) we finally get the Ward identity in the form used before,

\[ \Lambda_\nu(p, p) = -\frac{\partial \Sigma}{\partial p^\nu} . \quad (4.154) \]

### 4.3 Counterterms and power counting

All that we have shown in the previous sections can be interpreted as follows. The initial Lagrangian \( \mathcal{L}(e, m, \cdots) \) has been obtained from a correspondence between classical and quantum theory. It is then natural that the initial Lagrangian has to be modified by quantum corrections. The total Lagrangian is then given by,

\[ \mathcal{L}_{\text{total}} = \mathcal{L}(e, m, \cdots) + \Delta \mathcal{L} \quad (4.155) \]

and

\[ \Delta \mathcal{L} = \Delta \mathcal{L}^{(1)} + \Delta \mathcal{L}^{(2)} + \cdots \quad (4.156) \]

where \( \Delta \mathcal{L}^{[i]} \) is the \( i \)th - loops correction. This also correspond to order \( \hbar^i \) as counting in terms of loops is equivalent to counting in terms of \( \hbar^{10} \). This interpretation is quite attractive because in the limit \( \hbar \to 0 \) the total Lagrangian reduces to the classical one. With the Lagrangian \( \mathcal{L}_{\text{tot}} \) we can then obtain finite results, although \( \mathcal{L}_{\text{tot}} \) is divergent because of the counter-terms in \( \Delta \mathcal{L} \).

With this language the results up to the first order in \( \hbar \) can be written as,

\[ \mathcal{L}(e, m, \cdots) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\lambda^2}{2} A^\mu A_\mu - \frac{1}{2\xi} (\partial \cdot A)^2 + i \bar{\psi} \not{\partial} \psi - m \bar{\psi}\psi - e \bar{\psi} A \psi \]

\[ \Delta \mathcal{L}^{(1)} = -\frac{1}{4} (Z_3 - 1) F_{\mu\nu} F^{\mu\nu} + (Z_2 - 1)(i \bar{\psi} \not{\partial} \psi - m \bar{\psi}\psi) \]

\[ ^{10}\hbar^{E-1+L} = \hbar^{\frac{E}{2} + \frac{L}{2}}. \]

We have the following relations \( L = I - V + 1 ; 3V = E + 2I \) (this only for QED).
4.3. COUNTERTERMS AND POWER COUNTING

\[ +Z_2 \delta m \bar{\psi} \psi - e(Z_1 - 1)\bar{\psi} A \psi \]  

(4.158)

The Lagrangian

\[ \mathcal{L}_{\text{total}} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + \frac{\lambda^2}{2} A_{\mu} A^\mu - \frac{1}{2\xi} (\partial \cdot A)^2 \]

\[ + Z_2(i\bar{\psi} \gamma^\mu \psi - m \bar{\psi} \psi + \delta m \bar{\psi} \psi) \]

\[ - e Z_1 \bar{\psi} A \psi \]  

(4.159)

will give the renormalized Green’s functions up to the order \( \bar{\hbar} \).

In fact, we have only shown that the two-point and three-point Green’s functions (self-energies and vertex) were finite. It is important to verify that all the Green’s functions, with an arbitrary number of external legs are finite, as we have already used all our freedom in the renormalization of those Green’s functions. This leads us to the so-called *power counting*.

Let us consider a Feynman diagram \( G \), with \( L \) loops, \( I_B \) bosonic and \( I_F \) fermionic internal lines. If there are vertices with derivatives, \( \delta_v \) is the number of derivatives in that vertex. We define then the *superficial degree of divergence* of the diagram (note that \( L = I_B + I_F + 1 - V \)) by,

\[ \omega(G) = 4L + \sum_v \delta_v - I_F - 2I_B \]

\[ = 4 + 3I_F + 2I_B + \sum_v (\delta_v - 4) \]  

(4.160)

For large values of the momenta the diagram will be divergent as

\[ \Lambda^\omega(G) \text{ if } \omega(G) > 0 \]  

(4.161)

and as

\[ \ln \Lambda \text{ if } \omega(G) = 0 \]  

(4.162)

where \( \Lambda \) is a *cutoff*. The origin of the different terms can be seen in the following correspondence,

\[ \int \frac{d^4 q}{(2\pi)^4} \text{ (for each loop) } \rightarrow 4L \]

\[ \partial_\mu \leftrightarrow k_\mu \rightarrow \delta_v \]  

(4.163)

\[ \frac{i}{q - m} \rightarrow -I_F \]

\[ \frac{i}{q^2 - m^2} \rightarrow -2I_B \]

The expression for \( \omega(G) \) is more useful when expressed in terms of the number of external legs and of the dimensionality of the vertices of the theory. Let \( \omega_v \) be the dimension, in terms of mass, of the vertex \( v \), that is,

\[ \omega_v = \delta_v + \# \text{campos bosónicos} + \frac{3}{2} \# \text{campos fermiónicos} \]  

(4.164)
Then, if we denote by \( f_v(b_v) \) the number of fermionic (bosonic) internal lines that join at the vertex \( v \), we can write,

\[
\sum_v \omega_v = \sum_v \left( \delta_v + \frac{3}{2} f_v + b_v \right) + \frac{3}{2} E_F + E_B
\]

(4.165)

where \( E_F(E_B) \) are the total number of external fermionic (bosonic) lines of the diagram. As we have,

\[
I_F = \frac{1}{2} \sum_v f_v
\]

\[
I_B = \frac{1}{2} \sum_v b_v
\]

(4.166)

we get

\[
\sum_v \omega_v = \sum_v \delta_v + 3I_F + 2I_B + \frac{3}{2} E_F + E_B
\]

(4.167)

Substituting in the expression for \( \omega(G) \) we get finally,

\[
\omega(G) = 4 - \frac{3}{2} E_F - E_B + \sum_v (\omega_v - 4)
\]

\[
= 4 - \frac{3}{2} E_F - E_B - \sum_v [g_v]
\]

(4.168)

where \([g_v]\) denotes the dimension in terms of mass of the coupling constant of vertex \( v \), satisfying,

\[
\omega_v + [g_v] = 4
\]

(4.169)

From the previous expression for the superficial degree of divergence, Eq. (4.168), we can then classify theories in three classes,

i) **Non-renormalizable Theories**

They have at least one vertex with \( \omega_v > 4 \) (or \([g_v] < 0\)). The superficial degree of divergence increases with the number of vertices, that is, with the order of perturbation theory. For an order high enough all the Green functions will diverge.

ii) **Renormalizable Theories**

All the vertices have \( \omega_v \leq 4 \) and at least one has \( \omega_v = 4 \). If all vertices have \( \omega_v = 4 \) then

\[
\omega(G) = 4 - \frac{3}{2} E_F - E_B
\]

(4.170)

and all the diagrams contributing to a given Green function have the same degree of divergence. Only a finite number of Green functions are divergent.
iii) Super-Renormalizable Theories

All the vertices have $\omega_v < 4$. Only a finite number of diagrams are divergent.\(^{11}\)

Coming back to our question of knowing which are the divergent diagrams in QED, we can now summarize the situation in Table 4.1. All the other diagrams are superficially convergent. We have therefore a situation where there are only a finite number of divergent diagrams, exactly the ones that we considered before. This analysis shows that, up to order $\hbar$, the Lagrangian

$$\mathcal{L}_{\text{total}} = -\frac{1}{4} Z_3 F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \lambda^2 A_\mu A^\mu - \frac{1}{2 \xi} (\partial \cdot A)^2$$

$$+ Z_2 (\bar{\psi} \gamma^\mu \gamma^\nu \psi - m \bar{\psi} \psi + \delta m \bar{\psi} \psi)$$

$$- e Z_1 \bar{\psi} A_\psi$$

(4.171)

gives Green functions that are finite and renormalized with an arbitrary number of external legs. It remains to be shown that this Lagrangian is still valid up an arbitrary order in $\hbar$, with the only modification that the renormalization constants $Z_1, Z_2, Z_3$ and $\delta m$ are now given by power series,

$$Z_1 = Z_1^{(1)} + Z_1^{(2)} + \cdots$$

(4.172)

The effective degree of divergence it is sometimes smaller than the superficial degree because of symmetries of the theory. This is what happens for gauge theories like QED (see Table 4.1).

<table>
<thead>
<tr>
<th>$E_F$</th>
<th>$E_B$</th>
<th>$\omega(G)$</th>
<th>Effective degree of divergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0 (Current Conservation (CC))</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td></td>
<td>0 (Furry’s Theorem)</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>0</td>
<td>Convergent (CC)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0 (Current Conservation)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.1: Superficial and effective degree of divergence in QED.
Then the Lagrangian written in terms of the bare quantities is identical to the original Lagrangian\(^{12}\)

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \lambda_0^2 A_0^\mu A_0^\mu - \frac{1}{2} \xi_0 (\partial \cdot A_0)^2 \\
+ i(\bar{\psi}_0 \not{D} \psi_0 - m_0 \bar{\psi}_0 \psi_0) - e_0 \bar{\psi}_0 A_0 \psi_0
\]

(4.174)

Finally we notice that the bare Green functions are related to the renormalized ones by

\[
G_{n,\ell}^0(p_1, \cdots, p_n, k_1, \cdots, k_\ell, \mu_0, m_0, \ell_0, \xi_0, \Lambda) = Z_n^0(\Lambda) Z_{\ell/2}^0 G_{n,\ell}^R(p_1, \cdots, p_n, k_1, \cdots, k_\ell, \mu, m, e, \xi)
\]

(4.175)

where \(p_1 \cdots p_n, (k_1 \cdots k_\ell)\) are the fermion (boson) momenta. We will come back to these relations in the study of the renormalization group, in chapter 7.

### 4.4 Finite contributions from RC to physical processes

#### 4.4.1 Anomalous electron magnetic moment

We will show here, for the case of the electron anomalous moment, how the finite part of the radiative corrections can be compared with experiment, given credibility to the renormalization program. In fact we will just consider the first order, while to compare with the present experimental limit one has to go to fourth order in QED and to include also the weak and QCD corrections. The electron magnetic moment is given by

\[
\vec{\mu} = e \frac{2g}{m} \vec{\sigma}
\]

(4.176)

where \(e = -|e|\) for the electron. One of the biggest achievements of the Dirac equation was precisely to predict the value \(g = 2\). Experimentally we know that \(g\) is close to, but not exactly, 2. It is usual to define this difference as the anomalous magnetic moment. More precisely,

\[
g = 2(1 + a)
\]

(4.177)

or

\[
a = \frac{g}{2} - 1
\]

(4.178)

Our task is to calculate \(a\) from the radiative corrections that we have computed in the previous sections. To do that let us start to show how a value \(a \neq 0\) will appear in non relativistic quantum mechanics. Schrödinger’s equation for a charged particle in an exterior field is,

\[
\frac{i}{\hbar} \frac{\partial \varphi}{\partial t} = \left[ \frac{(\vec{p} - e\vec{A})^2}{2m} + e\varphi - \frac{e}{2m} (1 + a) \vec{\sigma} \cdot \vec{B} \right] \varphi
\]

(4.179)

\(^{12}\)The terms \(\frac{\lambda^2}{2} A^2 = \frac{\lambda^2}{2} A_0^2\) and \(\frac{1}{4}(\partial \cdot A)^2 = \frac{1}{4}(\partial \cdot A_0)^2\) are not renormalized. This is a consequence of the Ward-Takahashi identities for QED. The Ward identity \(Z_1 = Z_2\) is crucial for the equality \(e_0 A_0 = eA\) giving a meaning to the electric charge independently of the renormalization scheme.
Now we consider that the external field is a magnetic field \( \vec{B} = \nabla \times \vec{A} \). Then keeping only terms first order in \( e \) we get

\[
H = \frac{p^2}{2m} - e \vec{p} \cdot \vec{A} + \frac{e}{2m} (1 + a) \vec{\sigma} \cdot \nabla \times \vec{A}
\]

\[
\equiv H_0 + H_{\text{int}} \tag{4.180}
\]

With this interaction Hamiltonian we calculate the transition amplitude between two electron states of momenta \( p \) and \( p' \). We get

\[
\langle p' \mid H_{\text{int}} \mid p \rangle = -\frac{e}{2m} \int \frac{d^3x}{(2\pi)^3} \chi^\dagger e^{-i\vec{p}' \cdot \vec{x}} \left[ \frac{1}{2m} \vec{\sigma} \cdot \vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p} + (1 + a) \vec{\sigma} \times \nabla \cdot \vec{A} \right] e^{i\vec{p} \cdot \vec{x}} \chi
\]

\[
= -\frac{e}{2m} \chi^\dagger \left[ (\vec{p}' + \vec{p}) \cdot \vec{A} + i(1 + a) \vec{\sigma} \varepsilon^{ijk} q^j A^k \right] e^{-i\vec{q} \cdot \vec{x}} \chi \tag{4.181}
\]

This is the result that we want to compare with the non relativistic limit of the renormalized vertex. The amplitude is given by,

\[
A = e \pi(p') \left( \gamma_\mu + \Lambda_\mu^R \right) u(p) A^\mu(q)
\]

\[
= e \pi(p') \left[ \gamma_\mu \left( 1 + F_1(q^2) \right) + i \sigma_\mu q^\nu F_2(q^2) \right] u(p) A^\mu(q)
\]

\[
= \frac{e}{2m} \pi(p') \left\{ (p' + p)_\mu \left[ 1 + F_1(q^2) \right] + i \sigma_\mu q^\nu \left[ 1 + F_1(q^2) + F_2(q^2) \right] \right\} u(p) A^\mu(q) \tag{4.182}
\]

where we have used Gordon’s identity. For an external magnetic field \( \vec{B} = \nabla \times \vec{A} \) and in the limit \( q^2 \to 0 \) this expression reduces to

\[
A = \frac{e}{2m} \pi(p') \left\{ (p' + p)_k \left[ 1 + F_1(0) \right] + i \sigma_{kj} q^j \left[ 1 + F_1(0) + F_2(0) \right] \right\} u(p) A^k(q)
\]

\[
= \frac{e}{2m} \pi(p') \left[ -(p' + p)^k + i \Sigma^i e^{kij} q^j \left( 1 + \frac{\alpha}{2\pi} \right) \right] u(p) A^k(q) \tag{4.183}
\]

where we have used the results of Eq. (4.123),

\[
\begin{cases}
F_1(0) = 0 \\
F_2(0) = \frac{\alpha}{2\pi}
\end{cases} \tag{4.184}
\]

Using the explicit form for the spinors \( u \)

\[
u(p) = \begin{pmatrix} \chi \\ \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \\ \frac{2m}{\alpha} \chi \end{pmatrix} \tag{4.185} \]
we can write in the non relativistic limit,
\begin{equation}
A = -\frac{e}{2m} \chi^\dagger \left[ (p' + p)^k + i \left( 1 + \frac{\alpha}{2\pi} \right) \sigma^{ijk}q^j \right] \chi A^k
\end{equation}
which after comparing with Eq. (4.181) leads to
\begin{equation}
a^e_{th} = \frac{\alpha}{2\pi}
\end{equation}
This result obtained for the first time by Schwinger and experimentally confirmed, was very important in the acceptance of the renormalization program of Feynman, Dyson and Schwinger for QED.

4.4.2 Cancellation of IR divergences in Coulomb scattering

In this section we will show how the IR divergences cancel in physical processes. We will take as an example the Coulomb scattering from a fixed nucleus. This is better done if we start from first principles. Coulomb scattering corresponds to the diagram of Fig. 4.16, which gives the following matrix element for the $S$ matrix,

\[ S_{fi} = iZe^2(2\pi)\delta(E_i - E_f) \frac{1}{|q|^2} \bar{u}(p_f)\gamma^0 u(p_i) \]  

Figure 4.16: Lowest order diagram to Coulomb scattering.

\[ S_{fi} = iZe^2(2\pi)\delta(E_i - E_f) \frac{1}{|q|^2} \bar{u}(p_f)\gamma^0 u(p_i) \]  

We will now study the radiative corrections to this result in lowest order in perturbation theory. Due to the IR divergences it is convenient to introduce a mass $\lambda$ for the photon. For a classical field, as we are considering, this means a screening. If we take,
\begin{equation}
A_0^c(x) = Ze^\frac{-\lambda|x|}{4\pi|x|}
\end{equation}
the Fourier transform will be,
\begin{equation}
A_0^c(q) = Ze^\frac{1}{|q|^2 + \lambda^2}
\end{equation}
that shows that the screening is equivalent to a mass for the photon. With these modifications we have,
\begin{equation}
S_{fi} = iZe^2(2\pi)\delta(E_f - E_i) \frac{i}{|q|^2 + \lambda^2} \bar{u}(p_f)\gamma^0 u(p_i)
\end{equation}
4.4. FINITE CONTRIBUTIONS FROM RC TO PHYSICAL PROCESSES

We are interested in calculating the corrections up to order $e^3$ in the amplitude. To this contribute, the diagrams of Fig. 4.17. Diagram 1 is of order $e^2$ while diagrams 2, 3, 4 are of order $e^4$. Therefore the interference between 1 and $(2 + 3 + 4)$ is of order $\alpha^3$ and should be added to the result of the bremsstrahlung in a Coulomb field. The contribution from $1 + 2 + 3$ can be easily obtained by noticing that

$$eA^\mu_c\gamma_\mu \to eA^\mu_c(\gamma_\mu + \Lambda^R_\mu + \Pi^R_{\mu\nu}G^{\nu\rho}\gamma_\rho)$$

where $\Lambda^R_\mu$ and $\Pi^R_{\mu\nu}$ have been calculated before. We get

$$S_{f_i}^{1+2+3} = iZe^2(2\pi)\delta(E_i - E_f) \frac{1}{|q|^2 + \lambda^2} \mp(p_f)\gamma^0 \left\{ 1 + \alpha \left[ -\frac{1}{2\varphi} \tanh \varphi \left( 1 + \ln \frac{\lambda}{m} \right) (2\varphi \coth 2\varphi - 1) - 2 \coth 2\varphi \int_0^\varphi \beta \tanh \beta d\beta + \left( 1 - \frac{\coth^2 \varphi}{\beta} \right) (\varphi \coth \varphi - 1) + \frac{1}{9} \right] - \frac{q}{2m} \frac{\alpha}{\pi} \frac{\varphi}{\sinh 2\varphi} \right\} u(p_i)$$

where

$$\frac{|q|^2}{4m} = \sinh^2 \varphi.$$  

Finally the fourth diagram gives

$$S_{f_i}^{(4)} = (iZe)^2(e)\int \frac{d^4k}{(2\pi)^4} \pm(p_f) \left[ 2\pi\delta(E_f - k^0) \frac{2\pi\delta(k^0 - E_i)}{(p_f - k)^2 - \lambda^2} \gamma^0 \frac{i}{k - m + i\varepsilon} \gamma_\mu \frac{2\pi\delta(k^0 - E_i)}{(k - p_i)^2 - \lambda^2} \right]$$

$$= -2i\frac{Ze^2}{\pi} \frac{\alpha^2}{2\pi} 2\pi\delta(E_f - E_i) \pm(p_f) \left[ m(I_1 - I_2) + \gamma_\mu E_i (I_1 + I_2) \right] u(p_i)$$

with

\(^{13}\)We do not have to consider the self-energies of the external legs of the electron because they are on-shell.
\[ I_1 = \int d^3k \frac{1}{[(\vec{p}_f - \vec{k})^2 + \lambda^2][(\vec{p}_i - \vec{k})^2 + \lambda^2][(\vec{p})^2 - (\vec{k})^2 + i\varepsilon]} \]  

(4.196)

and

\[ \frac{1}{2}(\vec{p}_i + \vec{p}_f)I_2 = \int d^3k \frac{\vec{k}}{[(\vec{p}_f - \vec{k})^2 + \lambda^2][(\vec{p}_i - \vec{k})^2 + \lambda^2][(\vec{p})^2 - (\vec{k})^2 + i\varepsilon]} \]  

(4.197)

In the limit \( \lambda \to 0 \) it can be shown that

\[ I_1 = \frac{\pi^2}{2ip^3 \sin^2 \theta/2} \ln \left( \frac{2p \sin(\theta/2)}{\lambda} \right) \]  

(4.198)

\[ I_2 = \frac{\pi^2}{2p^3 \cos^2 \theta/2} \left\{ \frac{\pi}{2} \left[ 1 - \frac{1}{\sin \theta/2} \right] - i \left[ \frac{1}{\sin^2 \theta/2} \ln \left( \frac{2p \sin \theta/2}{\lambda} \right) + \ln \frac{\lambda}{2p} \right] \right\} \]  

(4.199)

With these expressions we get for the cross section

\[ \frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2}{|\vec{q}|^2} \frac{1}{2} \sum_{pol} |\bar{u}(p_f)\Gamma u(p_i)|^2 \]  

(4.200)

where

\[ \Gamma = \gamma^0 (1 + A) + \gamma^0 \frac{\vec{q}}{2m} B + C \]  

(4.201)

and

\[ A = \frac{\alpha}{\pi} \left[ \left( 1 + \ln \frac{\lambda}{m} \right) (2\varphi \coth 2\varphi - 1) - 2 \coth 2\varphi \int_0^\varphi d\beta \tanh \beta - \frac{\varphi}{2} \tanh \varphi \right. \]  

\[ + \left. \left( 1 - \frac{1}{3} \coth^2 \varphi \right) (\varphi \coth \varphi - 1) + \frac{1}{9} \right] - \frac{Z\alpha}{2\pi^2} |\vec{q}|^2 E(I_1 + I_2) \]  

(4.202)

\[ B = -\frac{\alpha}{\pi} \frac{\varphi}{\sinh 2\varphi} \]  

(4.203)

\[ C = -\frac{Z\alpha}{2\pi^2 m} |\vec{q}|^2 (I_1 - I_2) \]  

(4.204)

Therefore

\[ \frac{1}{4} \sum_{pol} |\bar{u}(p_f)pu(p_i)|^2 = \frac{1}{4} \text{Tr}[\Gamma(\not{p}_f + m)\Gamma(\not{p}_i + m)] \]
4.4. **FINITE CONTRIBUTIONS FROM RC TO PHYSICAL PROCESSES**

\[
4E^2(1 - \beta^2 \sin^2 \theta/2) + 2E^2B\beta^2 \sin^2 \theta/2 \\
+ 2E^22\text{Re}A \left(1 - \beta^2 \sin^2 \theta/2\right) + 2\text{Re}C(2mE) + O(\alpha^2) \tag{4.205}
\]

Notice that \(A, B\) e \(C\) are of order \(\alpha\). Therefore the final result is, up to order \(\alpha^3\):

\[
\left(\frac{d\sigma}{d\Omega}\right)_{\text{elast}ic} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left\{1 + \frac{2\alpha}{\pi} \left[1 + \ln \frac{\lambda}{m}\right] (2\varphi \coth \varphi - 1) - \frac{\varphi}{2} \tanh \varphi - 2 \coth \varphi \int_0^\varphi d\beta \beta \tanh \beta + \left(-\frac{\coth^2 \varphi}{3}\right)(\varphi \coth \varphi - 1) + \frac{1}{9}
\right.
\]

\[
- \frac{\varphi}{\sinh 2\varphi} \left(\frac{B^2 \sin^2 \theta/2}{2 \beta^2 \sin^2 \theta/2}\right) + Z\alpha \pi \beta \sin \frac{\theta}{2} \left[1 - \sin \theta/2\right] \\
\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left\{1 + \ln \frac{\lambda}{m}\right\} (2\varphi \coth \varphi - 1) - \frac{\varphi}{2} \tanh \varphi - 2 \coth \varphi \int_0^\varphi d\beta \beta \tanh \beta + \left(-\frac{\coth^2 \varphi}{3}\right)(\varphi \coth \varphi - 1) + \frac{1}{9}
\right.
\]

\[
\left(\frac{d\sigma}{d\Omega}\right)_{\text{elast}ic} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left\{1 + \frac{2\alpha}{\pi} \left[1 + \ln \frac{\lambda}{m}\right] (2\varphi \coth \varphi - 1) - \frac{\varphi}{2} \tanh \varphi - 2 \coth \varphi \int_0^\varphi d\beta \beta \tanh \beta + \left(-\frac{\coth^2 \varphi}{3}\right)(\varphi \coth \varphi - 1) + \frac{1}{9}
\right.
\]

\[
\left(\frac{d\sigma}{d\Omega}\right)_{\text{elast}ic} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left\{1 + \frac{2\alpha}{\pi} \left[1 + \ln \frac{\lambda}{m}\right] (2\varphi \coth \varphi - 1) - \frac{\varphi}{2} \tanh \varphi - 2 \coth \varphi \int_0^\varphi d\beta \beta \tanh \beta + \left(-\frac{\coth^2 \varphi}{3}\right)(\varphi \coth \varphi - 1) + \frac{1}{9}
\right.
\]

As we had said before the result is IR divergent in the limit \(\lambda \to 0\). This divergence is not physical and can be removed in the following way. The detectors have an energy threshold, below which they can not detect. Therefore in the limit \(\omega \to 0\) bremsstrahlung in a Coulomb field and Coulomb scattering can not be distinguished. This means that we have to add both results. If we consider an energy interval \(\Delta E\) with \(\lambda \leq \Delta E \leq E\) we get

\[
\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \int_{\omega \leq \Delta E} d^3k 2\omega(2\pi)^3 e^2 \left[\frac{2p_i \cdot p_f}{k_i \cdot p_i} - \frac{m^2}{(k \cdot p)^2} - \frac{m^2}{(k \cdot p_f)^2}\right] \tag{4.207}
\]

Giving a mass to the photon (that is \(\omega = (|k|^2 + \lambda^2)^{1/2}\)) the integral can be done with the result,

\[
\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \frac{2\alpha}{\pi} \left\{(2\varphi \coth 2\varphi - 1) \ln \frac{2\Delta E}{\lambda} + \frac{1}{2\beta} \ln \frac{1 + \beta}{1 - \beta} - \frac{1}{2} \cosh 2\varphi \frac{1 - \beta^2}{\beta \sin \theta/2} \int_{\cos \theta/2}^1 d\xi \frac{1}{(1 - \beta^2 \xi^2)(\xi - \cos^2 \theta/2)^{1/2}} \ln \frac{1 + \beta \xi}{1 - \beta \xi}\right\} \tag{4.208}
\]

The inclusive cross section can now be written as

\[
\frac{d\sigma}{d\Omega}(\Delta E) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{elast}ic} + \left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} (1 - \delta_R + \delta_B) \tag{4.209}
\]
where $\delta_R$ and $\delta_B$ are complicated expressions that depend on the resolution of the detector $\Delta E$ but do not depend on $\lambda$ that can be finally put to zero. One can show that in QED all the IR divergences can be treated in a similar way. One should note that the final effect of the bremsstrahlung is finite and can be important.
Chapter 5

Functional Methods

5.1 Introduction

In this chapter, called *Functional Methods*, we are going to present the path integral quantization. For systems that are not described by gauge theories this method may seem unnecessary, as the canonical quantization works without problems. However, for non-abelian gauge theories, as we shall see in the next chapter, this is the only known method. Besides this fundamental point, the quantization done using functional methods and the path integral formalism is very elegant and allows us to obtain the results much faster, even for the cases where the canonical quantization works. Examples of this are the Ward-Takahashi identities and the Dyson-Schwinger equations, as we will discuss at the end of the chapter.

We are going to assume that the reader is familiar with the path-integral quantization for systems of $N$ particles in non-relativistic quantum mechanics. Therefore only a brief summary of the results will be given. A more detailed account is given in Appendix A. The step from the quantization of a system with $N$ particles to the quantization of a field theory will be done heuristically. A more rigorous treatment will be given in Appendix B.

Before we start, let us clarify some questions related with the notation. Let us assume that we have real scalar field $\phi^a(x)$ where $a = 1, \ldots N$. In the following we will encounter expressions of the type,

$$ I_1 = \int d^4x \phi^a(x)\phi^b(x) $$

or

$$ I_2 = \int d^4x d^4y \phi^a(x)M^{ab}(x,y)\phi^b(y) $$

where $M^{ab}(x,y)$ is normally a differential operator. According to the rules for functional derivation, we have,

$$ \frac{\delta I_1}{\delta \phi^b(y)} = 2\phi^b(y) $$

where we used the result

$$ \frac{\delta \phi^a(x)}{\delta \phi^b(y)} = \delta^{ab}\delta^4(x - y) $$

147
If we keep all the indices in the previous expressions (and in some much more complicated that we will encounter soon), we will get a very complicated situation with respect to the notation. Therefore it will be useful to make use of a more compact notation. To this end we identify,

$$\phi_i \iff \phi^a(x)$$

that is, the index $i$ will represent both the discrete index $a$ as well as the continuous $x$,

$$i \iff \{a, x\}$$

In the case that the fields have further indices we will assume that $i$ will always represent them collectively. We also use the Einstein convention meaning a sum for discrete indices and an integration for continuous indices. With these conventions Eq. (5.1) and Eq. (5.4) can be written as

$$I_1 = \phi_i \phi_i \quad I_2 = \phi_i M_{ij} \phi_j$$

$$\frac{\delta I_1}{\delta \phi_j} = 2\phi_j \quad \frac{\delta \phi_i}{\delta \phi_j} = \delta_{ij}$$

In the following we will use these conventions, returning to the more usual notation when convenient or in case of a possible confusion.

## 5.2 Generating functional for Green’s functions

### 5.2.1 Green’s functions

The basic objects in Quantum Field Theory are the so-called Green functions. To avoid unnecessary complications we are going to use mostly the example of the scalar field. The generalizations are however quite straightforward. The Green function of order $n$ is given by

$$G^{(n)}(x_1, \ldots, x_n) \equiv \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle .$$

The Green functions defined in the previous equation are, sometimes called complete to distinguish from the Green functions connected, truncated or one particle irreducible that we now are going to define.

### 5.2.2 Connected Green’s functions

We call connected Green functions those that in which none of the external lines goes through the diagram without interacting. As an example, in Fig. 5.1 we represent a disconnected contribution to $G^4(x_1, \ldots, x_4)$, while in Fig. 5.2 we have a connected contribution to the same Green function.

Therefore the connected Green functions are obtained summing over all the connected diagrams. The disconnected Green functions, corresponding to disconnected diagrams, can be obtained from connected Green functions of lower order, therefore the relevant quantities are the connected Green functions $G^\text{c}_n(x_1, \ldots, x_n)$. It is clear that we have

$$G^\text{c}_n(x_1, \ldots, x_n) = G^n(x_1, \ldots, x_n) - \text{partes desconexas},$$
5.2. GENERATING FUNCTIONAL FOR GREEN’S FUNCTIONS

\[ G^4(x_1, \ldots, x_4) \]

Figure 5.1: Disconnected contribution to \( G^4(x_1, \ldots, x_4) \).

\[ G^2_c(x_1, x_2) = G^2(x_1, x_2) \] \quad (5.10)

Conventionally we represent \( G^m_c(x_1, \ldots, x_n) \) by the diagram of Fig. 5.3,

\[ G^2(x_1, x_2) \]

Figure 5.2: Connected contribution to \( G^4(x_1, \ldots, x_4) \).

\[ G^2_c(x_1, x_2) = G^2(x_1, x_2) \]

Conventionally we represent \( G^m_c(x_1, \ldots, x_n) \) by the diagram of Fig. 5.3,

\[ G^2(x_1, x_2) \]

Figure 5.3: Graphical representation for \( G^m_c(x_1, \ldots, x_n) \).

Sometimes it is important to consider the Green functions in momentum space. We define then \( G^m_c(p_1, \ldots, p_n) \) through the relation (Fourier Transform)

\[ (2\pi)^4 \delta^4(p_1 + p_2 + \cdots + p_n) \ G^m_c(p_1, \ldots, p_n) \]
\[ \equiv \int d^4x_1 \cdots d^4x_n e^{-i(p_1 \cdot x_1 + \cdots + p_n \cdot x_n)} \ G^m_c(x_1, \ldots, x_n) , \quad (5.11) \]

where all momenta, \( p_1, \ldots, p_n \), are incoming, as shown in Fig. 5.4. Notice that in the definition we have factored out the delta function that ensures the conservation of 4-momentum. With these conventions \( G^2(p, -p) \equiv G^2(p) \) is the full propagator represented in Fig. 5.5.
5.2.3 Truncated Green’s functions

For \( n > 2 \) one defines the truncated Green functions through the relation,

\[
G_{\text{trunc}}^n(p_1, \ldots, p_n) = \prod_{k=1}^{n} \left[ G^2(p_k) \right]^{-1} G^n_c(p_1, \ldots, p_n)
\]  

(5.12)

that is, we multiply each external line by the inverse of the full propagator corresponding to that line. These are the functions that play a fundamental role in the Theory, as these the ones that are related to the \( S \) matrix elements. In fact the LSZ reduction formula for scalars fields is

\[
\langle p_1, \ldots, p_n \text{ out} | q_1, \ldots, q_\ell \text{ in} \rangle = \langle p_1, \ldots, p_n \text{ in} | S | q_1, \ldots, q_\ell \text{ in} \rangle = \text{disconnected terms} + (iZ^{-1/2})^{n+\ell} \int d^4 y_1 \cdots d^4 x_\ell \exp \left[ i \left( \sum_{k=1}^{n} p_k \cdot y_k - \sum_{k=1}^{\ell} q_k \cdot x_k \right) \right]
\]

\times (\Box_{y_1} + m^2) \cdots (\Box_{x_\ell} + m^2) \langle 0 | T \phi(y_1) \cdots \phi(x_\ell) | 0 \rangle_c
\]

(5.13)

which gives

\[
\langle p_1, \ldots, p_n \text{ out} | q_1, \ldots, q_\ell \text{ in} \rangle = \langle p_1, \ldots, p_n \text{ in} | S | q_1, \ldots, q_\ell \text{ in} \rangle = \text{disconnected terms}
\]

\[
+ Z^{-(n+\ell)/2} (2\pi)^4 \delta \left( \sum p_i - \sum q_j \right) G_{\text{trunc}}^{n+\ell}(-p_1, \ldots, -p_n, q_1, \ldots, q_\ell)
\]

(5.14)

5.2.4 Irreducible diagrams

We saw in 5.14 that the \( S \) elements related with the cross sections are expressed in terms of truncated diagrams. Among the truncated Green functions there is an important subset.
These are the Green functions that correspond to the subset of one particle irreducible or proper that are the truncated diagrams that remain connected when one cuts any arbitrary internal line. For instance the diagram of Fig. 5.6\(^1\) is truncated but it is not proper, while the diagram of Fig. 5.7 is proper (in \(\lambda \phi^3\)).

![Figure 5.6: Example of a truncated diagram that is not proper.](image)

![Figure 5.7: Example of a proper diagram.](image)

The reason why the non-irreducible truncated diagrams are not important is because they can always be expressed in terms of irreducible diagrams of lower order (remember the self energy series that we saw in the last lecture). It is convenient to introduce a notation for the irreducible Green Functions (sum of all the irreducible diagrams for a given number of exterior legs) where the factor \(i\) was introduced by convenience. In Fig. 5.8 the external legs are drawn to make the figure more clear. They are in fact truncated. It is also convenient to define a notation for the truncated diagrams of order \(n\). This is shown in Fig. 5.9 or, in another way, in Fig. 5.10. We can define similar diagrams in momentum space.

### 5.3 Generating functionals for Green’s functions

The Generating Functional (FG) for the Green functions plays a very important role in Quantum Field Theory. In fact starting with it, by taking appropriate functional derivatives, one can obtain all the Green functions. Therefore they can treat simultaneously an infinite number of Green functions. The FG of the full Green functions is given by,

\[
\mathcal{Z}(J) \equiv \langle 0 | T e^{iJ_\phi} | 0 \rangle
\]

\( (5.15) \)

\(^1\)The bars indicate the the external lines are cut.
\[ i\Gamma^{(n)}(x_1, \ldots, x_n) = \] 

Figure 5.8: Irreducible Green functions.

\[ G^{(n)}_{\text{trunc}}(x_1, \ldots, x_n) = \]

Figure 5.9: Graphical representation of truncated Green functions.

where we are using the compact notation explained before.

\[ J_i \phi_i \equiv \int d^4 x \ J(x) \phi(x) . \] (5.16)

\[ Z(J) \] generates all the Green functions, because if we expand the exponential in 5.15 we get

\[ Z(J) = \sum_{n=0}^{\infty} \frac{i^n}{n!} J_{i_1} \cdots J_{i_n} \langle 0 | T \phi_{i_1} \cdots \phi_{i_n} | 0 \rangle \]

\[ = \sum_{n=0}^{\infty} \frac{i^n}{n!} J_{i_1} \cdots J_{i_n} G^{n}_{i_1 \cdots i_n} \] (5.17)

The Green functions are then obtained taking derivatives

\[ G^{n}_{i_1 \cdots i_n} = \left. \frac{\delta^n Z}{i \delta J_{i_1} \cdots i \delta J_{i_n}} \right|_{J_i = 0} \] (5.18)

The generating functional of the connected Green functions is defined by the relation,

\[ Z(J) = e^{iW(J)} \] (5.19)
5.3. GENERATING FUNCTIONALS FOR GREEN’S FUNCTIONS

\begin{equation*}
\begin{split}
W(J) &= -i \ln Z(J) . \\
\end{split}
\end{equation*}

The connected Green functions are then obtained by taking functional derivatives

\begin{equation}
G_{c_{i_1 \cdots i_n}} = i \left. \frac{\delta^n W}{i \delta J_{i_1} \cdots i \delta J_{i_n}} \right|_{J_i = 0} .
\end{equation}

Before we actually prove this statement, let us define the last important generating functional, the one that generates the irreducible Green functions. This is defined through the Legendre transformation of \( W(J) \), that is

\begin{equation}
\Gamma(\phi) \equiv W(J) - J_i \phi_i .
\end{equation}

where

\begin{equation}
\begin{cases}
\phi_i & \equiv \left. \frac{\delta W(J)}{\delta J_i} \right|_{J_i = 0} \\
J_i & = - \left. \frac{\delta \Gamma(\phi)}{\delta \phi_i} \right|_{\phi_i = 0} .
\end{cases}
\end{equation}

The irreducible Green functions are then obtained through

\begin{equation}
\Gamma_{i_1 \cdots i_n} = \left. \frac{\delta^n \Gamma(\phi)}{\delta \phi_{i_1} \cdots \delta \phi_{i_n}} \right|_{\phi_i = 0} .
\end{equation}

Having given the definitions we have now to show that \( W(J) \) and \( \Gamma(\phi) \) do generate the connected and irreducible Green functions, respectively. Let us start with \( W(J) \). The proof is done calculating \( G_{c_{i_1 \cdots i_n}} \). We are going only to do only the \( n = 2 \), \( n = 3 \) and \( n = 4 \) cases. The generalizations are immediate.

\( n = 2 \)

\begin{equation}
G_{c_{i_1 i_2}} = i \left. \frac{\delta^2 W}{i \delta J_{i_1} i \delta J_{i_2}} \right|_{J_i = 0} = \left. \frac{\delta^2 \ln Z}{i \delta J_{i_1} i \delta J_{i_2}} \right|_{J_i = 0} = \left. \frac{\delta}{i \delta J_{i_1}} Z \frac{1}{i \delta J_{i_2}} \delta Z \right|_{J_i = 0} .
\end{equation}
\[ Z = \frac{\delta^2 Z}{i\delta J_1_i\delta J_2} \bigg|_{J_i=0} = \frac{\delta^2 Z}{i\delta J_1_i\delta J_2} \bigg|_{J_i=0} \]

or

\[ G^2_{c\, i_1 i_2} = G^2_{i_1 i_2} \tag{5.26} \]

To obtain this result we have used,

\[ Z(0) = 1 \quad \text{The vacuum is normalized} \]

\[ \frac{\delta Z}{i\delta J_i} = \langle 0|T\phi_i|0 \rangle = 0 \quad \text{There is no symmetry breaking} \tag{5.27} \]

\( n = 3 \)

\[ G^3_{c\, i_{12} i_3} = \left[ \frac{1}{Z} \frac{\delta^3 Z}{i\delta J_1_i\delta J_2_i\delta J_3} - \frac{1}{Z} \frac{\delta^2 Z}{i\delta J_1_i\delta J_2} \frac{\delta Z}{i\delta J_3} \right. \]

\[ \left. - \frac{1}{Z} \frac{\delta Z}{i\delta J_2_i\delta J_3} \frac{\delta Z}{i\delta J_1} - \frac{1}{Z} \frac{\delta^2 Z}{i\delta J_1_i\delta J_3} \frac{\delta Z}{i\delta J_2} + 2 \right] \frac{1}{Z} \frac{\delta Z}{i\delta J_1} \frac{1}{Z} \frac{\delta Z}{i\delta J_2} \frac{1}{Z} \frac{\delta Z}{i\delta J_3} \bigg|_{J_i=0} \tag{5.28} \]

therefore

\[ G^3_{i_{12} i_3} = G^3_{c\, i_{12} i_3} \tag{5.29} \]

The case \( n = 4 \) is left as an exercise (see Problem x.x). The extension to \( n > 4 \) is straightforward. We have therefore showed that \( W(J) \) generates all the connected Green functions. Let us now show that \( \Gamma(\phi) \) is the generating functional for the irreducible Green functions. For that we need two auxiliary results. The first one is based in the relation

\[ \frac{\delta J_i}{\delta J_k} = \delta_{ik} \tag{5.30} \]

This relation it is obvious. However one can obtain starting with it another important relation. In fact

\[ \frac{\delta J_i}{\delta J_k} = \frac{\delta J_i}{\delta \phi_\ell} \frac{\delta \phi_\ell}{\delta J_k} = -\frac{\delta^2 \Gamma}{\delta \phi_\ell \delta \phi_\ell} \frac{\delta^2 W}{\delta J_\ell \delta J_\ell} = -i\Gamma_{i\ell} G_{\ell k} \tag{5.31} \]

or

\[ \Gamma_{i\ell} G_{\ell k} = i\delta_{ik} \tag{5.32} \]

This fundamental relation expresses the fact that \( \Gamma^2 \) is the inverse of the propagator (except for the \( i \) that comes from conventions). It is useful to write it in a diagrammatic form shown in 5.11. Notice that

\[ i\Gamma_{ik}^{(2)} = \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]
5.3. GENERATING FUNCTIONALS FOR GREEN’S FUNCTIONS

\[ \frac{\delta}{i \delta J_i} = -1 \]

Figure: Graphical representation of Eq.5.32.

which explains the disappearance of the \( i \). A second result is related to the following functional derivative,

\[ \frac{\delta}{i \delta J_i} \]

We want to derive in order to \( J_i \) quantities that depend on \( J_i \) indirectly through \( \phi_k \). We get

\[ \frac{\delta}{i \delta J_i} = \frac{\delta \phi_k}{i \delta J_i} \frac{\delta}{\delta \phi_k} = \frac{\delta^2 W}{i \delta J_i \delta J_i} \frac{\delta}{\delta \phi_k} = G^{(2)}_{ik} \frac{\delta}{\delta \phi_k} \]

and therefore

\[ \frac{\delta}{i \delta J_i} = G_{ik} \frac{\delta}{\delta \phi_k} \]

Equations 5.32 and 5.36 allow us to obtain all the relation between irreducible and connected Green functions. This analysis it is easier in a diagrammatic form, if we note the following identities,

\[ \frac{\delta}{i \delta J_i} \]

\[ \frac{\delta}{\delta \phi_k} \]

and

\[ \frac{\delta}{i \delta J_i} \]
where we have used 5.36 to establish 5.39. In all these relations it is understood that at
the end (after taking the functional derivatives) we do $J = 0$ e $\phi = 0$ at the right places.
We will use now these relations to relate the connected and irreducible Green functions
for $n = 3$ and $n = 4$.

$n = 3$

The starting point is the equation 5.32. We apply $\frac{\delta}{i\delta J_{\ell}}$ to 5.32 and we get

$$\frac{\delta}{i\delta J_{\ell}} i \delta J_{\ell} = 0 \quad (5.40)$$

Using equations 5.37 and 5.39 we get the diagram of Figure 5.12.

![Diagram](image)

Figure 5.12: Graphical result of Eq.5.40.

Multiplying on the left by $G_{mi}^{(2)}$ and using 5.32 we get

$$i \Gamma_{mkl}^{(3)} = m \quad (5.41)$$

This shows that $\Gamma_{mkl}^{(3)}$ is in fact the irreducible Green function with 3 external legs, as for
3 external legs the irreducible and truncated Green functions coincide. To show that we
have really irreducible functions and not only truncated ones, one has to go to $n = 4$ as
the difference starts there.
5.4. Feynman Rules

The formalism of functional generators allows us to obtain the Feynman rules of any theory with all the correct conventions. We have already shown how to get the Feynman rules in section 3.7. There we used the result that in lowest order (tree level) we have

$$\Gamma_{\text{tree}}(\phi) = \int d^4 x \mathcal{L}[\phi] \equiv \Gamma_0(\phi)$$  \hspace{1cm} (5.42)

Here we are going just to show this result. For the interaction terms \(n > 2\) this is clear. For instance for \(n = 3\) we have

$$i \Gamma^{(3)} = G^{(3)}_{\text{tree}}$$  \hspace{1cm} (5.43)
while for \( n = 4 \) we get

\[ i\Gamma^{(4)} = G^{(4)} - \text{irreducible parts} \tag{5.44} \]

and it is obvious that \( i\Gamma^{(4)}_{\text{tree}} \) generates the vertices.

The \( i \) factor is in agreement with the usual conventions for the Feynman rules as it comes from the term \( \exp \left( i \int d^4x L_{\text{int}} \right) \) in the calculation of the Green functions. For the quadratic terms we have, Eq. 5.32,

\[ \Gamma^{(2)}_{\text{tree}}(p) = p^2 - m^2 \tag{5.45} \]

therefore, doing the inverse Fourier transform,

\[ \Gamma^{(2)}_{\text{tree}}(x, y) = \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} e^{i(p_1 \cdot x + p_2 \cdot y)} (2\pi)^4 \delta^4(p_1 + p_2) (p_1^2 - m^2) \tag{5.46} \]

and

\[ \frac{1}{2} \int d^4x d^4y \phi(x) \Gamma^{(2)}_{\text{tree}}(x, y) \phi(y) = \]

\[ = \frac{1}{2} \int d^4x d^4y \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} e^{i(p_1 \cdot x + p_2 \cdot y)} (2\pi)^4 \delta(p_1 + p_2)(p_1^2 - m^2) \phi(x) \phi(y) \]

\[ = \frac{1}{2} \int d^4x \phi(x)(-\Box - m^2) \phi(x) = \frac{1}{2} \int d^4x \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \tag{5.47} \]

which shows that \( \Gamma_{\text{tree}} \) is in fact the action. In getting to Eq. (5.47) we have done an integration by parts and discarded, usual, the boundary term. We refer the reader to Section 3.7 for the actual recipes on how to determine the Feynman rules of any theory.
5.5 Path integral for generating functionals

5.5.1 Quantum mechanics of \(n\) degrees of freedom

Let us start reviewing the results for the path integral in systems with 1 degree of freedom. In appendix A a more detailed explanation is given for the results that we are going to use. The fundamental result is the following transition amplitude,

\[
\langle q'; t' | q; t \rangle = N \int D(q) e^{i \int_{t'}^{t} dt' L(q, \dot{q})} = N \int D(q) e^{iS}
\]

where \(N\) is a normalization factor and \(D(q)\) is a symbolic form for representing the measure of integration. This is in fact a complicated limit (see appendix A). Another important result concerns the time ordered matrix elements of operators. Let

\[
O(t_1, \ldots, t_n) = T[O_1^H(t_1)O_2^H(t_2) \cdots O_n^H(t_n)]
\]

such that

\[
t' \geq (t_1, t_2, \ldots, t_n) \geq t
\]

Then

\[
\langle q'; t' | O(t_1, \ldots, t_n) | q; t \rangle = N \int D(q)O_1(q(t_1)) \cdots O_n(q(t_n)) e^{iS}
\]

where we have assumed that the operators \(O_i\) are diagonal in the coordinate space. For the generalization to Quantum Field Theory the important objects are not the transition amplitudes but the generating functionals. Consider, for instance, the Green function

\[
G(t_1, t_2) \equiv \langle 0 | T(Q^H(t_1)Q^H(t_2)) | 0 \rangle
\]

where \(|0\rangle\) is the ground state and \(Q^H(t)\) is the coordinate operator in the Heisenberg representation. To write Eq. (5.52) using the path integral, we introduce a complete set of states and we write

\[
G(t_1, t_2) = \int dq \, d\phi_0 \langle 0 | q; t \rangle \langle q'; t' | T(Q^H(t_1)Q^H(t_2)) | q; t \rangle \langle q; t | 0 \rangle
\]

\[
= \int dq d\phi_0 \phi_0(q, t) \int D(q) \phi_0(q, t) e^{i \int_{t'}^{t} dt' L_{\phi_0}}
\]

where

\[
\phi_0(q, t) = \langle 0 | q; t \rangle = \phi_0(q) e^{-iE_0t}
\]

The appearance in this expression of the wave functions of the fundamental state makes the expression not very useful. We can remove them in the following way. Consider the matrix element

\[
\langle q'; t' | O(t_1, t_2) | q; t \rangle
\]

\[
= \int dQ \, dQ' \langle q'; t' | Q'; T' \rangle \langle Q'; T' | O(t_1, t_2) | Q; T \rangle \langle Q; T | q; t \rangle
\]

where

\[
O(t_1, t_2) = T(Q^H(t_1)Q^H(t_2))
\]
\[ t' \geq T' \geq (t_1, t_2) \geq T \geq t \] (5.56)

Let \(|n\rangle\) be the eigenstates of energy \(E_n\) with wave function \(\phi_n(q)\) that is
\[
H |n\rangle = E_n |n\rangle \\
\langle q|n\rangle = \phi_n^*(q) 
\] (5.57)

Then
\[
\langle q'; t'|Q'; T' \rangle = \langle q'| e^{-iH(t'-T')} |Q' \rangle \\
= \sum_n \langle q'|n\rangle \langle n| e^{-iH(t'-T')} |Q' \rangle \\
= \sum_n \phi_n^*(q') \phi_n(Q') e^{-iE_n(t'-T')} 
\] (5.58)

We consider now the limit \(t' \to -i\infty\). We obtain then
\[
\lim_{t' \to -i\infty} \langle q'; t'|Q'; T' \rangle = \phi_0^*(q') \phi_0(Q') e^{-E_0|t'|} e^{-iE_0T'} 
\] (5.59)

In a similar way
\[
\lim_{t \to -i\infty} \langle Q; T|q; t \rangle = \phi_0(q) \phi_0(Q) e^{-E_0|t|} e^{-iE_0T} 
\] (5.60)

Applying these limits to Eq. (5.55) we get
\[
\lim_{t' \to -i\infty} \lim_{t \to i\infty} \langle q'; t'| O(t_1, t_2) | q; t \rangle \\
= \int dQdQ' \phi_0^*(q') \phi_0(Q') e^{-E_0|t'|} e^{-iE_0T'} \\
\langle Q'; T'| O(t_1, t_2) | Q; T \rangle \phi_0(q) \phi_0^*(Q) e^{-E_0|t|} e^{-iE_0T} \\
= \phi_0^*(q) \phi_0(q) e^{-E_0|t'|} e^{-E_0|t|} \\
\int dQdQ' \phi_0^*(Q', T') \phi_0(Q, T) \langle Q'; T'| O(t_1, t_2) | Q; T \rangle 
\] (5.61)

Using the definition of Green function in Eq.5.53 we obtain the important result
\[
\lim_{t' \to -i\infty} \lim_{t \to i\infty} \langle q'; t'| O(t_1, t_2) | q; t \rangle = \phi_0^*(q') \phi_0(q) e^{-E_0|t'|} e^{-E_0|t|} G(t_1, t_2) 
\] (5.62)

On the other hand
\[
\lim_{t' \to -i\infty} \lim_{t \to i\infty} \langle q'; t'| q; t \rangle = \phi_0^*(q') \phi_0(q) e^{-E_0|t'|} e^{E_0|t|} 
\] (5.63)

Therefore we can finally write
\[
G(t_1, t_2) = \lim_{t' \to -i\infty} \lim_{t \to i\infty} \left[ \frac{\langle q'; t'| T(Q^H(t_1)Q^H(t_2)) | q; t \rangle}{\langle q'; t'| q; t \rangle} \right] 
\] (5.64)
Using 5.51 we can finally write \( G(t_1, t_2) \) in terms of a path integral, as
\[
G(t_1, t_2) = \lim_{t' \to -i\infty} \lim_{t \to i\infty} \frac{1}{\langle q'; t'| q; t \rangle} \int \mathcal{D}(q) q(t_1) q(t_2) e^{i \int_{t'}^{t} L dt} \tag{5.65}
\]
This result is easily generalized to Green functions with \( n \)-points
\[
G(t_1, \ldots, t_n) = \langle 0 | T(q(t_1) \cdots q(t_n)) | 0 \rangle = \lim_{t' \to -i\infty} \lim_{t \to i\infty} \frac{1}{\langle q'; t'| q; t \rangle} \int \mathcal{D}(q) q(t_1) \cdots q(t_n) e^{i \int_{t'}^{t} L dt} \tag{5.66}
\]
We can now see that all the Green functions can be obtained from the generating functional
\[
Z[J] = \lim_{t' \to -i\infty} \lim_{t \to i\infty} \frac{1}{\langle q'; t'| q; t \rangle} \int \mathcal{D}(q) e^{i \int_{t'}^{t} [L(q, \dot{q}) + Jq] dt} \tag{5.67}
\]
by functional derivation
\[
G(t_1, \ldots, t_n) = \left. \frac{\delta^n Z[J]}{i \delta J(t_1) \cdots i \delta J(t_n)} \right|_{J=0} \tag{5.68}
\]
The expression 5.67 for the generating functional shows that its the transition amplitude between the ground state at time \( t \) and the ground state at time \( t' \), in the presence of an exterior source and with normalization such that \( Z[J=0] = 1 \)
\[
Z[J] = \langle 0 | 0 \rangle_J \tag{5.69}
\]
For a system with \( n \) degrees of freedom, we have the generalization of 5.67
\[
Z[J_1, \ldots, J_n] = \lim_{t' \to -i\infty} \lim_{t \to i\infty} N \int \mathcal{D}(q_i) e^{i \int_{t'}^{t} [L(q_i, \dot{q}_i) + \sum_{i=1}^{N} J_i q_i] dt} \tag{5.70}
\]
Comments
- In the previous equation the time limits for times \( t \) and \( t' \) are imaginary. This means that these Green functions are in Euclidean space. For field theory this corresponds to the prescription \( m^2 \to m^2 - i\epsilon \).
- In equation 5.70 we do not explicitly wrote the normalization. It should be chosen in such a way that \( Z[0, \ldots, 0] = 1 \). However, as we will see, for the connected Green functions in Quantum Field Theory the normalization it is not relevant, and therefore we will not worry about it.

5.5.2 Field theory
To get the generating functional in Quantum Field Theory we proceed in the usual heuristic way, by making the following equivalences,
\[
t \to x^\mu
\]
$q(t) \rightarrow \phi(x)$
$\mathcal{D}(q) \rightarrow \mathcal{D}(\phi)$
$L(q, \dot{q}_i) \rightarrow \int d^3x \mathcal{L}(\phi, \partial_\mu \phi)$

We get therefore

$$Z[J] = N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(\phi)\phi(x)]}$$

(5.72)

This is the starting point in the path integral quantization. We will see in the next lecture how to use it. A more rigorous derivation of the above result, Eq. 5.72, be found in appendix B.

### 5.5.3 Applications

Once we know the generating functional $Z[J]$ we also know all the Green functions of the theory and therefore how to address any problem in Quantum Field Theory. We can ask in which conditions can we evaluate $Z[J]$? The answer is that as we only know how to do Gaussian path integrals we can only do either free fields or perturbation theory. However there are two main advantages in the method that we will now discuss:

- **Perturbation Theory**
  The expression for $Z[J]$ allows to establish the perturbative expansion and find the Feynman rules for any theory.

- **Formal Manipulations**
  Relations among the Green functions that are a consequence of symmetry properties of the theory (for instance Ward identities) are much simpler in terms of the generating functionals. Here the expression of $Z[J]$ in terms of a path integral, Eq. 5.72 is particularly useful as we will discuss later.

We will use the scalar field as an example to illustrate the first point, that is how to establish the perturbative expansion. We consider a real scalar field, described by the Lagrangian,

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi)$$

(5.73)

where $\mathcal{L}_0(\phi)$ is quadratic in the fields, that is

$$\mathcal{L}_0(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

(5.74)

We can then write,

$$Z[J] = N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}_0(\phi) + \mathcal{L}_I(\phi)]}$$

(5.75)

This last integral can formally be written in the form,

$$Z[J] = \exp \left[ i \int d^4x \mathcal{L}_I \left( \frac{\delta}{\delta \phi} \right) \right] Z_0[J]$$

(5.76)
where
\[ Z_0[J] = N \int \mathcal{D}(\phi)e^{i \int d^4x [\mathcal{L}_0 + J\phi]} . \] (5.77)

The usefulness of this expression results from the two following points:

- \( Z_0[J] \) can be exactly calculated because is quadratic in the fields (Gaussian integral).
- If \( \mathcal{L}_I(\phi) \) has a small parameter, the exponential can be developed in a power series in this parameter and the generating functional \( Z[J] \) can be obtained order by order in perturbation theory, as the integrals will be Gaussian integrals with polynomials.

### 5.5.4 Example: perturbation theory for \( \lambda \phi^4 \)

To see the connection with the usual results let us consider as an example the derivation of the Feynman rules for a real scalar theory with the interaction,
\[ \mathcal{L}_I = -\frac{\lambda}{4!} \phi^4 . \] (5.78)

The generating functional \( Z[J] \) is
\[ Z[J] = \mathcal{N} \exp \left\{ (-i\lambda) \frac{1}{4!} \int d^4x \left( \frac{\delta}{i\delta J} \right)^4 Z_0[J] \right\} \] (5.79)

where (see Problems)
\[ Z_0[J] = \exp \left\{ -\frac{1}{2} \int d^4xd^4y J(x)\Delta(x,y)J(y) \right\} \] (5.80)

The normalization \( \mathcal{N} \) is to be chosen such that \( Z[0] = 1 \), as we will see later. We expand in a series in the coupling constant,
\[ Z[J] = \mathcal{N} Z_0[J] \left\{ 1 + (-i\lambda)Z_1[J] + (-i\lambda)^2 Z_2[J] + \cdots \right\} \] (5.81)

where
\[ Z_1[J] \equiv Z_0^{-1}[J] \left\{ \frac{1}{4!} \int d^4x \left( \frac{\delta}{\delta J} \right)^4 Z_0[J] \right\} \] (5.82)

and
\[ Z_2[J] = \frac{1}{2} Z_0^{-1}[J] \left\{ \frac{1}{4!} \int d^4x \left( \frac{\delta}{\delta J} \right)^4 \right\}^2 Z_0[J] \]
\[ = \frac{1}{2} Z_0^{-1}[J] \left\{ \frac{1}{4!} \int d^4x \left( \frac{\delta}{\delta J} \right)^4 \right\} (Z_0Z_1') \]
\[ = \frac{1}{2} \left( Z_1[J] \right)^2 + \frac{1}{2} Z_0^{-1}[J] \frac{1}{4!} \int d^4x \left\{ 4 \frac{\delta^3 Z_0}{\delta J^3(x)} \frac{\delta Z_1}{\delta J(x)} \right\} \]
This result can be represented diagrammatically in the form:

\[
Z' = \frac{1}{8} \quad \quad \quad - \frac{1}{4} \quad \quad \quad + \frac{1}{4!}
\]

(5.85)

We get for the first order

\[
Z'[J] = \frac{1}{4!} \int d^4x \left[ 3\Delta(x,x_1)\Delta(x,x_2) - 3!\Delta(x,x_1) \int d^4y_1 d^4y_2 \Delta(x,y_1)\Delta(x,y_2)J(y_1)J(y_2) 
\right.
\]

\[
+ \int d^4y_1 \cdots d^4y_4 \Delta(x,y_1)\Delta(x,y_2)\Delta(x,y_3)\Delta(x,y_4)J(y_1)J(y_2)J(y_3)J(y_4) \right]
\]

(5.84)

This result can be represented diagrammatically in the form,

\[
\begin{align*}
Z' = & \quad \frac{1}{8} \quad \quad \quad - \frac{1}{4} \quad \quad \quad + \frac{1}{4!} \\
\end{align*}
\]

(5.85)

For \( Z'_2 \) we get

\[
Z'_2[J] = \frac{1}{2} \left( Z'[J] \right)^2 + \frac{1}{2} \left( \frac{1}{4!} \right)^2 4! \int d^4x_1 d^4x_2 \Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2)
\]

\[
+ \left( \frac{1}{4!} \right)^2 \left[ -72 \int d^4x_1 \int d^4x_2 \Delta(x_1,x_2) \int d^4y_1 \Delta(x_1,y_1)J(y_1)
\right.
\]

\[
\Delta(x_1,x_2)\Delta(x_2,x_2) \int d^4y_2 \Delta(x_2,y_2)J(y_2)
\]

\[
+24 \int d^4x_2 d^4x_1 \Delta(x_1,x_1) \int d^4y_1 \Delta(x_1,y_1)J(y_1)\Delta(x_1,y_2)
\]

\[
\int d^4y_2 \Delta(x_2,y_1)J(y_2) \int d^4y_3 \Delta(x_2,y_3)J(y_2) \int d^4y_4 \Delta(x_2,y_4)J(y_4)
\]

\[
+24 \int d^4x_2 \int d^4x_1 \int d^4y_1 d^4y_2 d^4y_3 \Delta(x_1,x_2)\Delta(x_1,y_1)
\]

\[
\Delta(x_1,y_2)\Delta(x_1,y_2)\Delta(x_2,x_2)\Delta(x_2,y_2)J(y_1) \cdots J(y_4)
\]

\[
-8 \int d^4x^2 \int d^4x_1 \int d^4y_1 \cdots d^4y_6 \Delta(x_1,x_2)\Delta(x_1,y_1)\Delta(x_1,y_2)
\]

\[
\Delta(x_1,y_3)\Delta(x_2,y_4)\Delta(x_2,y_5)\Delta(x_2,y_6)J(y_1) \cdots J(y_6)
\]

\[
+36 \int d^4x_2 d^4x_1 \Delta(x_1,x_1)\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_2,x_2)
\]

(5.83)
\[-36 \int d^4 x_2 d^4 x_1 \Delta(x_1, x_1) \Delta(x_1, x_2) \int d^4 y_1 d^4 y_2 \Delta(x_2, y_1) \Delta(x_2, y_2) J(y_1) J(y_2)\]

\[-36 \int d^4 x_2 d^4 x_1 d^4 y_1 d^4 y_2 \Delta(x_1, x_2) \Delta(x_1, x_2) \Delta(x_2, x_1)
\Delta(x_1, y_1) \Delta(x_1, y_2) J(y_1) J(y_2)\]

\[+36 \int d^4 x_2 d^4 x_1 d^4 y_1 \cdots d^4 y_4 \Delta(x_1, x_2) \Delta(x_1, x_2) \Delta(x_1, y_1)
\Delta(x_1, y_2) \Delta(x_2, y_3) \Delta(x_2, y_4) J(y_1) \cdots J(y_4)\]

\[-48 \int d^4 x_2 d^4 y_1 d^4 y_2 \Delta(x_1, x_2) \Delta(x_1, x_2) \Delta(x_1, x_2)
\Delta(x_1, y_1) \Delta(x_2, y_2) J(y_1) J(y_2)\]  

(5.86)

This can also be written as,

\[Z_2'[J]\]

\[= \frac{1}{2} (Z_1'[J])^2 + \frac{1}{2 \cdot 4!} \int d^4 x_1 d^4 x_2 \Delta^4(x_1, x_2)\]

\[+ \frac{3}{2 \cdot 4!} \int d^4 x_1 d^4 x_2 \Delta(x_1, x_1) \Delta^2(x_1, x_2) \Delta(x_2, x_2)\]

\[- \frac{1}{2 \cdot 3! \cdot 3!} \int d^4 x_1 d^4 x_2 d^4 y_1 \cdots d^4 y_6 \Delta(y_1, x_1) \Delta(y_2, x_1) \Delta(y_3, x_1) \Delta(y_4, x_1) \Delta(y_5, x_1) \Delta(y_6, x_1) J(y_1) \cdots J(y_6)\]

\[+ \frac{2}{4!} \int d^4 x_1 d^4 x_2 d^4 y_1 \cdots d^4 y_4 \Delta(y_1, x_1) \Delta(x_1, x_1) \Delta(x_1, x_2) \Delta(x_2, y_1) \Delta(x_2, y_2) J(y_1) \cdots J(y_4)\]

\[+ \frac{3}{2 \cdot 4!} \int d^4 x_1 d^4 x_2 d^4 y_1 \cdots d^4 y_4 \Delta(y_1, x_1) \Delta(y_2, x_1) \Delta^2(x_1, x_2) \Delta(x_2, y_1) \Delta(x_2, y_2) J(y_1) \cdots J(y_4)\]
Let us now evaluate the normalization up to second order in perturbation theory. The condition $Z[0] = 1$ gives,

$$1 = \mathcal{N} \left[ 1 + (-i\lambda)n_1 + (-i\lambda)^2 n_2 + \cdots \right]$$

(5.88)

where

$$n_1 = \frac{1}{8}$$

(5.89)

$$n_2 = \frac{1}{2} n_1^2 + \frac{1}{2 \cdot 4!} + \frac{3}{2 \cdot 4!}$$

(5.90)

We then get,

$$\mathcal{N} = \frac{1}{1 + (-i\lambda)n_1 + (-i\lambda)^2 n_2 + \cdots}$$

$$= 1 - (-i\lambda)n_1 - (-i\lambda)^2 (n_2 - n_1^2) + \cdots$$

(5.91)

Putting everything together we have

$$Z[J] = Z_0[J] \left\{ 1 - (-i\lambda)n_1 - (-i\lambda)^2 (n_2 - n_1^2) + \cdots \right\}$$

$$\left\{ 1 + (-i\lambda)Z'_1 + (-i\lambda)^2 Z'_2 + \cdots \right\}$$

$$= Z_0[J] \left\{ 1 + (-i\lambda)(Z'_1 - n_1) + (-i\lambda)^2 (Z'_2 - n_2 + n_1^2 - n_1 Z'_1) + \cdots \right\}$$

(5.92)

Defining now

$$Z_1 \equiv Z'_1 - n_1$$
5.5. PATH INTEGRAL FOR GENERATING FUNCTIONALS

\[
Z_2 \equiv Z_2' - n_2 + n_1^2 - n_1 Z_1' = Z_2' - n_2 - n_1 Z_1
\]

we get

\[
Z_1[J] = -\frac{1}{4} \quad + \quad \frac{1}{4!}
\]

and

\[
Z_2[J] = \frac{1}{2} (Z_1[J])^2 + \frac{1}{2!} \frac{1}{3!} \frac{1}{3!}
\]

\[
+ \frac{3}{2!} \frac{1}{4!}
\]

\[
+ \frac{2}{4!}
\]

\[
- \frac{1}{8}
\]

\[
- \frac{1}{8}
\]

\[
- \frac{1}{12}
\]

with \(Z_1[0] = Z_2[0] = 0\). Therefore the generating functional

\[
Z[J] = Z_0[J] \left\{ 1 + (-i\lambda) Z_1[J] + (-i\lambda)^2 Z_2[J] + \cdots \right\}
\]

is automatically correctly normalized if we neglect all the vacuum amplitudes, known as bubbles. To verify that this expression reproduces the perturbation theory results, let us evaluate the propagator up to second order in \(\lambda^2\). We get

\[
\Delta'(x_1, x_2) = \left. \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0}
\]

\[
= - \left. \frac{\delta^2 Z_0[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} - (-i\lambda) \left. \frac{\delta^2 Z_1[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} - (-i\lambda)^2 \left. \frac{\delta^2 Z_2[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0}
\]
\[ \Delta(x_1, x_2) + (-i\lambda) \frac{1}{2} \int d^4y \Delta(x_1, y) \Delta(x_2, y) \Delta(y, y) \]
\[ + (-i\lambda)^2 \int d^4y_1d^4y_2 \left[ \frac{1}{4} \Delta(x_1, y_1) \Delta(y_1, y_2) \Delta(y_1, y_2) \Delta(y_2, x_2) \right. \]
\[ + \frac{1}{4} \Delta(x_1, y_1) \Delta^2(y_1, y_2) \Delta(y_2, y_2) \Delta(y_1, x_2) + \frac{1}{6} \Delta(x_1, y_1) \Delta^3(y_1, y_2) \Delta(y_2, x_2) \right] \]  

(5.97)

In diagrammatic form we have the situation of Figure 5.15.

Figure 5.15:

Continuing with the scalar \( \lambda \phi^4 \) theory as an example, let analyze the generating functional for the connected Green functions, \( W[J] \). It is easy to see that the terms like \( Z_2^1[J] \) correspond to disconnected diagrams which are part of \( Z[J] \). Let us see how they disappear in \( W[J] \). We have

\[ iW[J] = \ln Z[J] = \]
\[ = \ln Z_0[J] + \ln \left\{ 1 + (-i\lambda)Z_1[J] + (-i\lambda)^2 Z_2[J] + \cdots \right\} \]
\[ = iW_0[J] + (-i\lambda)Z_1[J] - \frac{1}{2} (-i\lambda)^2 (Z_1[J])^2 + (-i\lambda)^2 Z_2[J] + \cdots \]
\[ = iW_0[J] + (-i\lambda)Z_1[J] + \left\{ (-i\lambda)^2 (Z_2[J] - \frac{1}{2} (Z_1[J])^2) \right\} + \cdots \]
\[ \equiv i \left\{ W_0[J] + (-i\lambda)W_1[J] + (-i\lambda)^2 W_2[J] + \cdots \right\} \]  

(5.98)

with

\[ iW_1[J] = Z_1[J], \quad iW_2[J] = Z_2[J] - \frac{1}{2} (Z_1[J])^2 \]  

(5.99)

Therefore the disconnected diagrams contained in \( Z_2[J] \) are subtracted and \( W_1 \) and \( W_2 \) have only the connected diagrams.

### 5.5.5 Symmetry factors

After doing the derivatives in order to \( J \) to obtain a given Green function, we have some diagrams multiplied by numbers known as symmetry factors. For instance for the one-loop
correction to the propagator we get,
\[
\Delta'(x_1, x_2) = \left. \frac{\delta^2 Z}{i \delta J(x_1) i \delta J(x_2)} \right|_{J=0} = \\
= \left. \frac{\delta^2 Z_0}{i \delta J(x_1) i \delta J(x_2)} \right|_{J=0} + (-i \lambda) \left. \frac{\delta^2 Z_1}{i \delta J(x_1) i \delta J(x_2)} \right|_{J=0} + \cdots \\
= \Delta(x_1, x_2) + \frac{1}{2} + \cdots 
\tag{5.100}
\]

The factor \( \frac{1}{2} \) is the symmetry factor that corresponds to that diagram. The method of the generating functional gives automatically the correct symmetry factors. However in practical applications it is normally easier to have a rule to obtain these symmetry factors.

**Rule for the Symmetry Factors**

The symmetry factor \( S \) of a diagram it is given by
\[
S = \frac{N}{D} 
\tag{5.101}
\]
where \( N \) is the \# of different ways of forming the diagram, and \( D \) it is the product of the symmetry factors of each vertex by the number of permutations of equal vertices.

As an example take the diagram that contributes to the propagator at one-loop represented in Fig. 5.16. Then according to the rule we have,

\[
S = \frac{4 \times 3}{4!} = \frac{1}{2} 
\tag{5.102}
\]

### 5.5.6 A comment on the normal ordering

In the previous example we have diagrams like in Fig. 5.16, that we generically denote by “tadpoles”, that connect fields that belong to the same vertex, and that we saw that were excluded by the normal ordering. This difference comes from the fact that we were
not very rigorous in the definition of the Lagrangian to include in the path formalism. If we would we would concluded that we should have to do the normal ordering in the Lagrangian to include in \( e^{i \int d^4x \mathcal{L}(\phi)} \) This would make the Lagrangian \( \mathcal{L}(\phi) \) to include in the path integral different from the classical Lagrangian. Let us look at \( \phi^4 \) as an example. We will use the relations

\[
\hat{\phi}(x)\hat{\phi}(x) =: \hat{\phi}(x)\hat{\phi}(x) : + \langle 0 | \hat{\phi}(x)\hat{\phi}(x) | 0 \rangle
\]  

(5.103)

or symbolically

\[
: \hat{\phi}^2 := \hat{\phi}^2 - \langle 0 | \hat{\phi}^2 | 0 \rangle .
\]  

(5.104)

In a similar way

\[
\hat{\phi}^4 =: \hat{\phi}^4 : + 6 : \hat{\phi}^2 : \langle 0 | \hat{\phi}^2 | 0 \rangle + 6 \langle 0 | \hat{\phi}^2 | 0 \rangle \langle 0 | \hat{\phi}^2 | 0 \rangle
\]  

(5.105)

Therefore we get

\[
: \hat{\phi}^4 := \hat{\phi}^4 - 6 : \hat{\phi}^2 : \langle 0 | \hat{\phi}^2 | 0 \rangle - 6 \langle 0 | \hat{\phi}^2 | 0 \rangle \langle 0 | \hat{\phi}^2 | 0 \rangle
\]  

(5.106)

or

\[
: \hat{\phi}^4 := \hat{\phi}^4 - 6 \hat{\phi}^2 \langle 0 | \hat{\phi}^2 | 0 \rangle
\]  

(5.107)

This means that the quantum Lagrangian should be written as

\[
\mathcal{L}^Q_{\text{Int}} = -\frac{\lambda}{4!} : \hat{\phi}^4 : = -\frac{\lambda}{4!} \hat{\phi}^4 + \frac{\lambda}{4} \hat{\phi}^2 I
\]  

(5.108)

where

\[
I = \langle 0 | \hat{\phi}^2 (x) | 0 \rangle
\]

\[
= \int d\vec{k}_1 d\vec{k}_2 \langle 0 | \left( a^\dagger(k_1) e^{ik_1 \cdot x} + a(k_1) e^{-ik_1 \cdot x} \right) \left( a^\dagger(k_2) e^{ik_2 \cdot x} + a(k_2) e^{-ik_2 \cdot x} \right) | 0 \rangle
\]

\[
= \int d\vec{k}_1 d\vec{k}_2 \langle 0 | a(k_1) a^\dagger(k_2) | 0 \rangle e^{i(k_2-k_1) \cdot x}
\]

\[
= \int d\vec{k}_1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k}
\]  

(5.109)

In the previous expression we have used the relations

\[
[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')
\]  

(5.110)

\[
\omega_k = \sqrt{k_0^2 + |\vec{k}|^2}
\]  

(5.111)

The integral \( I \) is divergent and in fact it is equal to the one-loop integral. In fact

\[
\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}
\]
\[= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} dk_0 \frac{i}{(k_0 - \omega_k)(k_0 + \omega_k)} \]
\[= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} = I \quad (5.112)\]

Therefore if we had been rigorous we should have to include the term, \(\frac{1}{4}\phi^2 I\) in the interaction. The Lagrangian to use in the path integral should then be,
\[\mathcal{L}_{Int}^{IC} = -\lambda \frac{\phi^4}{4!} + \frac{\lambda}{4} \phi^2 I \quad (5.113)\]

It is easy to verify that the additional term cancels the tadpole. In fact we have
\[= \frac{i}{p^2 - m^2} \left[ -i \frac{\lambda}{2} I + i \frac{\lambda}{2} I \right] \frac{i}{p^2 - m^2} \]
\[= 0 \quad (5.114)\]
and therefore the tadpoles would not appear. However many times we do not worry about this considerations and just use the classical Lagrangian in the path integral. This can be done because the tadpole gives an infinite contribution to the mass, that can be absorbed in the renormalization process.

In QED the same occurs, we should have as interaction Lagrangian
\[\mathcal{L}_{Int}^{IC} = -e \bar{\psi} \gamma^\mu \psi A_\mu + e A_\mu \langle 0 | \bar{\psi} \gamma^\mu \psi | 0 \rangle \quad (5.115)\]
and the second term would remove the tadpole shown in Fig. 5.17,

![Figure 5.17: Tadpole for QED.](image)

However, due the Lorentz invariance of the theory, one can show that this tadpole vanishes to all orders and therefore we do have to worry, just use the classical Lagrangian.
5.5.7 Generating functionals for fermions

For theories with fermion fields we introduce Grassmann variables. These anti-commuting variables are in some sense the classical limit of fermionic quantum fields. The details of this construction are explained in the Appendices A and B. Here we just review our conventions. Due to the anti-commuting character it is necessary to specify the order of the derivatives.

We will take the convention that derivatives are left derivatives, that is they obey,

\[ \frac{\delta}{\delta \eta(x)} \int d^4 y \eta(y) \psi(y) = \psi(x) \]
\[ \frac{\delta}{\delta \eta(x)} \int d^4 y \psi(y) \eta(y) = -\overline{\psi}(x) \] (5.116)

In the Green functions the order of the derivatives is such that

\[ G^{2n}(x_1, \ldots, y_n) = \langle 0 | T \psi(x_1) \cdots \psi(x_n) \overline{\psi}(y_1) \cdots \overline{\psi}(y_n) | 0 \rangle \equiv \int \delta^{2n} Z[\eta, \overline{\eta}] i\delta \eta(x_1) \cdots i\delta \eta(x_n) i\delta \overline{\eta}(y_1) \cdots i\delta \overline{\eta}(y_n) \] (5.117)

where we have defined the generating functional for fermion fields as

\[ Z[\eta, \overline{\eta}] = \langle 0 | T e^{i \int d^4 x [\overline{\eta}(x) \psi(x) + \overline{\psi}(x) \eta(x) + L]} | 0 \rangle \]
\[ = \int D(\psi, \overline{\psi}) e^{i \int d^4 x [\overline{\eta}(x) \psi(x) + \overline{\psi}(x) \eta(x) + L]} \] (5.118)

Examples of these results will be given in the Problems at the end of the chapter.

5.6 Change of variables in path integrals. Applications

5.6.1 Introduction

One of the great advantages of having an expression for the generating functional \( Z[J] \) in terms of a path integral is that a great number of manipulations that are familiar for the usual integrals, (change of variables, integration by parts, ...) can also be applied here. Let us see the implications of changing integration variables.

Let us consider an infinitesimal transformation of the form,

\[ \phi_i \to \phi_i + \varepsilon F_i(\phi) \] (5.119)

where

\[ F_i(\phi) = f_i + f_{ij} \phi_j + \cdots \] (5.120)
5.6. Change of variables in path integrals. Applications

Then we should have, in first order,

\[
\mathcal{D}(\phi) \rightarrow \mathcal{D}(\phi) \det \left| \delta_{ij} + \varepsilon \frac{\delta F_i}{\delta \phi_j} \right| = \mathcal{D}(\phi) \left( 1 + \varepsilon \frac{\delta F_i}{\delta \phi_i} \right)
\]  
(5.121)

On the other hand

\[
e^{i(S(\phi) + J_i \phi_i)} \rightarrow e^{i(S(\phi) + J_i \phi_i)} \left[ 1 + i\varepsilon \left( \frac{\delta S}{\delta \phi_i} + J_i \right) F_i(\phi) \right]
\]  
(5.122)

As the integral for \( Z(J) \) should be independent of the change of variables we get

\[
0 = \int \mathcal{D}(\phi) \left[ i \left( \frac{\delta S}{\delta \phi_i} + J_i \right) F_i(\phi) + \frac{\delta F_i}{\delta \phi_i} \right] e^{i(S[\phi] + J_i \phi_i)}
\]  
(5.123)

Using \( \phi_i \rightarrow \frac{\delta}{\delta \phi_i} \), we get a more compact expression,

\[
\left\{ i \left[ \frac{\delta S}{\delta \phi_i} \left( \frac{\delta}{\delta \phi_i} \right) + J_i \right] F_i \left( \frac{\delta}{\delta \phi_i} \right) + \frac{\delta F_i}{\delta \phi_i} \left( \frac{\delta}{\delta \phi_i} \right) \right\} Z(J) = 0
\]  
(5.124)

This the general expression that we are going to apply to two important particular cases, the Dyson-Schwinger equations and the Ward identities.

5.6.2 Dyson-Schwinger equations

Let \( F_i = f_i \) independent of \( \phi_i \), that is a simple translation of the fields. Then the previous master equation simplifies to

\[
\left( \frac{\delta S}{\delta \phi_i} \left[ \frac{\delta}{\delta \phi_i} \right] + J_i \right) Z(J) = 0
\]  
(5.125)

We will see below that this the expression for the Dyson-Schwinger (DS) equations for the generating functional of the full Green functions. In this way, the DS equations are a consequence of the path integral for constant field translations. This equations can further be written as

\[
J_k = -\frac{1}{Z} E \left[ \frac{\delta}{\delta \phi_k} \right] Z[J]
\]  
(5.126)

where the functional \( E[\phi] \) is the Euler-Lagrange equation of motion,

\[
E[\phi_k] \equiv \frac{\delta S}{\delta \phi_k}.
\]  
(5.127)

For many applications it is more convenient to write the Dyson-Schwinger equations for the connected and proper (one particle irreducible) Green functions. For this we have to write the corresponding equations for the functional \( W \) and \( \Gamma \). Using the identity

\[
\frac{1}{Z} \frac{\delta}{\delta \phi_k} \left( Z[J] f[J] \right) = \left( \frac{\delta W}{\delta \phi_k} + \frac{\delta}{\delta \phi_k} \right) f[J]
\]  
(5.128)
we can write

\[
\frac{1}{Z} E \left[ \frac{\delta}{i \delta J_k} \right] Z[J] = E \left[ i \frac{\delta W}{i \delta J_k} + \frac{\delta}{i \delta J_k} \right] 1 \quad (5.129)
\]

Therefore the DS equation for the generating functional of the connected Green functions can be written as,

\[
J_k = -E \left[ i \frac{\delta W}{i \delta J_k} + \frac{\delta}{i \delta J_k} \right] 1 \quad (5.130)
\]

Let us now find the corresponding equation for the proper, or one particle irreducible, Green functions. To obtain this equation we use the following relations

\[
\begin{aligned}
\phi_k &= \frac{i \delta W}{i \delta J_k} \\
\frac{\delta}{i \delta J_k} &= G_{km} \frac{\delta}{\delta \phi_m} \\
\frac{\delta}{\delta \phi_k} &= -i \Gamma_{kr} \frac{\delta}{i \delta J_r}
\end{aligned}
\]

We get

\[
\frac{\delta \Gamma}{\delta \phi_k} = E \left[ \phi_k + G_{km} \frac{\delta}{\delta \phi_m} \right] 1 \quad (5.131)
\]

It is in the form of Eq. 5.131, that the DS are more useful.

**Example : Self-energy in \( \phi^3 \)**

Let us start with the example of the self-energy in \( \phi^3 \). The action for this theory is, using our compact notation,

\[
S[\phi] = \frac{1}{2} \phi_k (\Box - m^2) \delta_{km} \phi_m - \frac{\lambda}{3!} (\phi_k)^3 \quad (5.132)
\]

Therefore the equation of motion is

\[
E[\phi_k] = (\Box - m^2) \phi_k - \frac{\lambda}{2} (\phi_k)^2 \quad (5.133)
\]

We get therefore, expanding the functional,

\[
E \left[ \phi_k + G_{km} \frac{\delta}{\delta \phi_m} \right] 1 = -(\Box + m^2) \phi_k - \frac{\lambda}{2} \left( \phi_k + G_{kr} \frac{\delta}{\delta \phi_r} \right) \phi_k \quad (5.134)
\]

We get therefore for the DS equation

\[
\frac{\delta \Gamma}{\delta \phi_k} = -(\Box + m^2) \phi_k - \frac{\lambda}{2} \left( \phi_k^2 + G_{kr} \delta_{rk} \right) \quad (5.135)
\]

By taking functional derivatives with respect to \( \phi_m \) we get the DS for the various Green functions, all derived from just one master equation. For instance, for the self-energy we get,

\[
\frac{\delta^2 \Gamma}{\delta \phi_k \delta \phi_m} = -(\Box + m^2) \delta_{km} - \frac{\lambda}{2} (2 \phi_k \delta_{km} - i \Gamma_{mn} G_{kr} \delta_{rk}) \quad (5.136)
\]
Setting \( \phi_k = 0 \) we get

\[
\Gamma_{km} - (-\Box - m^2)\delta_{km} = \frac{i\lambda}{2} \Gamma_{mn} G_{krn} \delta_{rk}
\]

\[
= \frac{\lambda}{2} \Gamma_{mn} G_{krn} \Gamma_{rs} G_{sk}
\]

\[
= \frac{i\lambda}{2} \Gamma_{mn} \Gamma_{rs} G_{kk'} G_{rr'} G_{mm'} \Gamma_{k'l'm'}
\]

\[
= -\frac{i\lambda}{2} G_{kk'} G_{ks} \Gamma_{k'sm}
\]

(5.137)

where we have repeatedly used the relation

\[
\Gamma_{ij} G_{jk} = i\delta_{ik}
\]

(5.138)

By definition of self-energy we have,

\[
\Gamma_{km} - (-\Box - m^2)\delta_{km} \equiv -\Sigma_{km}
\]

(5.139)

Therefore

\[
-i\Sigma_{km} = -\frac{i\lambda}{2} G_{kk'} G_{ks} \Gamma_{k'sm}
\]

(5.140)

as shown in the Fig. 5.18, We see that the DS equation is no more than the statement

\[ \text{Figure 5.18: Dyson-Schwinger equation for } \phi^3. \]

that the vertex of the theory is \( \frac{\lambda}{3!} \phi^3 \).

**Example: Self-energy in } \phi^4**

In the case the action is

\[
S[\phi] = \frac{1}{2} \phi_k(-\Box - m^2)\delta_{km} \phi_m - \frac{\lambda}{4!} (\phi_k)^4.
\]

(5.141)

Therefore the equation of motion is

\[
E[\phi] = (-\Box - m^2)\phi_k - \frac{\lambda}{3!} (\phi_k)^3.
\]

(5.142)

We then get

\[
E \left[ \phi_k + G_{km} \frac{\delta}{\delta \phi_m} \right] 1 = -(\Box + m^2)\phi_k - \frac{\lambda}{3!} \left( \phi_k + G_{km} \frac{\delta}{\delta \phi_m} \right) \left( \phi_k + G_{kn} \frac{\delta}{\delta \phi_n} \right) \phi_k
\]
\[ = -(\Box + m^2)\phi_k - \frac{\lambda}{3!} \left( \phi_k + G_{km} \frac{\delta}{\delta \phi_m} \right) \left( \phi_k^2 + G_{kn} \delta_{nk} \right) \]
\[ = -(\Box + m^2)\phi_k - \frac{\lambda}{3!} \left( \phi_k^3 + \phi_k G_{kn} \delta_{nk} + 2G_{km}\phi_k \delta_{km} \right. \]
\[ \left. - iG_{km} \Gamma_{mj} G_{knlj} \delta_{nk} \right) \]  
(5.143)

The master equation for the DS equations then reads,
\[ \frac{\delta \Gamma}{\delta \phi_k} = -(\Box + m^2)\phi_k - \frac{\lambda}{3!} \left( \phi_k^3 + \phi_k G_{kn} \delta_{nk} + 2G_{km}\phi_k \delta_{km} - iG_{km} \Gamma_{mj} G_{knlj} \delta_{nk} \right) \]  
(5.144)

To obtain the DS for the self-energy we take the derivative with respect to \( \phi_j \) make all \( \phi = 0 \) after derivation. We obtain,
\[ \Gamma_{kj} - (\Box - m^2)\delta_{kj} \]
\[ = -\frac{\lambda}{3!} \left( G_{km} \delta_{nk} \delta_{kj} + 2G_{km} \delta_{km} \delta_{kj} - iG_{km} \Gamma_{mj} G_{knlj} \delta_{nk} \right. \]
\[ \left. -G_{kmnp} \Gamma_{pj} G_{nmlj} G_{knlj} \delta_{nk} - G_{kmnp} \Gamma_{pj} G_{nkmp} \Gamma_{pj} \delta_{nk} \right) \]  
(5.145)

This equation can in turn be written as,
\[ -i\Sigma_{kj} = -i\frac{\lambda}{2} G_{kk}\delta_{kj} + i\frac{\lambda}{3!} G_{km} \Gamma_{mj} G_{kk'} G_{mm'} G_{l'n'} \Gamma_{k'n'l'} \delta_{nk} \]
\[ + i\frac{\lambda}{3!} G_{kk'} G_{mm'} G_{pp'} \Gamma_{k'n'l'} \Gamma_{pjm} G_{kk''} G_{mm''} G_{l'n''} \Gamma_{k'n''l'} \delta_{nk} \]
\[ + i\frac{\lambda}{3!} G_{km} \Gamma_{ml} G_{knlp} \Gamma_{pj} \delta_{nk} \]
\[ = -i\frac{\lambda}{2} G_{kk}\delta_{kj} + i\frac{\lambda}{3!} \delta_{kj} G_{knlj} \Gamma_{pj} \]  
(5.146)

For the \( \phi^4 \) theory we have \( \Gamma_{ijk} = 0 \) and therefore
\[ G_{knlp} = G_{kk'} G_{mm'} G_{l'n'} G_{pp'} \Gamma_{k'n''l''} G_{knlj} \Gamma_{pj} \]  
(5.147)

We finally get,
\[ -i\Sigma_{kj} = -i\frac{\lambda}{2} G_{kk}\delta_{kj} - i\frac{\lambda}{3!} G_{kk'} G_{knlj} \Gamma_{pj} G_{knlp} \delta_{nj} \]  
(5.148)

that we represent diagrammatically in Fig. 5.19 Again the DS equation for the self-energy

Figure 5.19: Dyson-Schwinger equation for \( \phi^4 \).

is nothing else than the identification of the vertex of the theory.
5.6.3 Ward identities

Consider a theory with some symmetry. This symmetry is expressed as an invariance of the action, that is

\[ \frac{\delta S[\phi]}{\delta \phi_i} F_i(\phi) = 0 \] (5.149)

where we considered the previously defined infinitesimal transformations. If this symmetry transformation also leaves invariant the integration measure \( D(\phi) \) then we get simply the expression,

\[ J_i F_i \left[ \frac{\delta}{i\delta J} \right] Z(J) = 0 \] (5.150)

This expression is known as Ward Identity. Derivation in order to the external sources will lead to relations among different Green functions as a consequence of the symmetry of the theory. For gauge theories the correct expression is a bit more complicated. The reason being, as we shall see, that in the quantization of gauge theories one normally has to introduce terms that break the symmetry, known as gauge fixing terms. In this case we can write,

\[ S_{eff} = S_I + S_{NI} \] (5.151)

where \( \frac{\delta S_I}{\delta \phi_i} F_i = 0 \) and \( \frac{\delta S_{NI}}{\delta \phi_i} F_i \neq 0 \). Then, if the measure is still invariant, we should have the more complicated expression for the Ward identities,

\[ \left( \frac{\delta S_{NI}}{\delta \phi_i} \left[ \frac{\delta}{i\delta J} \right] + J_i \right) F_i \left[ \frac{\delta}{i\delta J} \right] Z(J) = 0 \] (5.152)

In the next chapter we will apply this expression to obtain the Ward identities for QED and for the non-Abelian gauge theories. For these last ones the question of the invariance of the measure is more subtle and will be discussed there, after we have learned how to quantize these theories.
Problems for Chapter 5

5.1 Evaluate $G^4_\epsilon$ starting from 5.21 and show that it is indeed the connected Green function with four external legs.

5.2 Show that for a real scalar field we have

$$Z_0[J] = \exp\left\{-\frac{1}{2} \int d^4xd^4y J(x)\Delta^{(0)}(x,y) J(y)\right\}$$

(5.153)

where

$$\Delta^{(0)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}$$

(5.154)

**Hint:** Use a convenient generalization of the result

$$\int_{-\infty}^{+\infty} dx_1 \cdots dx_N e^{-\frac{1}{2} \sum_{i< j} M_{ij} x_i x_j + b_i x_i} = \pi^{N/2} (\det M)^{-1/2} e^{\frac{1}{2} b_i (M^{-1})_{ij} b_j}$$

(5.155)

5.3 Determine the symmetry factors for the following diagrams:

![Diagrams](image)

5.4 Consider the theory $\phi^3$, that is, $V(\phi) = \frac{1}{3!}\phi^3$. Using

$$Z[J] = \exp\left\{-i \int d^4x V \left[ \frac{\delta}{i\delta J} \right]\right\} Z_0(J)$$

(5.156)

where

$$Z_0(J) = \exp\left\{-\frac{1}{2} \int d^4xd^4x' J(x)G_F^{(0)}(x-x') J(x')\right\}$$

(5.157)

and

$$G_F^{(0)}(x-x') = i \int d^4k e^{ik(x-x')} \frac{1}{k^2 - m^2 + i\epsilon}$$

(5.158)
show that the symmetry factor of the diagram

\[ S = \frac{1}{2} \]

is \( S = \frac{1}{2} \).

**5.5** Given the Lagrangian for the free Dirac field

\[ \mathcal{L}_0 = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \]  

(5.159)

show that the generating functional for the Green functions is

\[ Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y \, \overline{\eta}(x) S_0^0(x,y) \eta(y)} \]  

(5.160)

where

\[ S_0^0(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \left( \frac{i}{\slashed{p} - m + i\varepsilon} \right)_{\alpha\beta} \]

\[ = \frac{\delta^2 Z_0}{i\delta \eta_\alpha(y) \delta \overline{\eta}_\beta(x)} \]

\[ = \langle 0 | T\psi_\beta(x)\overline{\psi}_\alpha(y) | 0 \rangle \].

**5.6** As we will show in Chapter 6 the generating functional for the Green functions in QED is given by,

\[ Z(J^\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}) \, e^{i \int d^4x (\mathcal{L}_{QED} + \mathcal{L}_{GF} + J^\mu A_\mu + \overline{\eta} \psi + \psi \overline{\eta})} \]  

(5.161)

where

\[ \mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\overline{D} - m)\psi \]

\[ \mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial \cdot A)^2 \]

\[ D_\mu = \partial_\mu + ie A_\mu \].

a) Determine \( Z_0[J^\mu, \eta, \overline{\eta}] \)

b) Show that

\[ Z[J^\mu, \eta, \overline{\eta}] = \exp \left\{ (-ie) \int d^4x \frac{\delta}{\delta \eta_\alpha(x)} (\gamma^\mu)_{\alpha\beta} \frac{\delta}{\delta \overline{\eta}_\beta(x)} \frac{\delta}{\delta J_\mu(x)} \right\} Z_0[J^\mu, \eta, \overline{\eta}] \]  

(5.162)

c) Expand

\[ Z = Z_0 \left[ 1 + (-ie) Z_1 + (-ie)^2 Z_2 + \cdots \right] \]  

(5.163)

where we have subtracted the vacuum-vacuum amplitudes in \( Z_i \), that is, \( Z_i[0] = 0 \rightarrow Z[0] = 1 \). Show that
d) Discuss the numerical factors and signs of the previous diagrams.

e) Evaluate in lowest order

\[ \langle 0 | T A^\mu(x) \psi_\beta(y) \bar{\psi}_\alpha(z) | 0 \rangle = \frac{\delta^3 Z}{i \delta \eta_\alpha(z) i \delta \bar{\eta}_\beta(y) i \delta J_\mu(x)} \]  

(5.166)

and verify that it coincides with the Feynman rules for the vertex

f) Determine the amplitude for the Compton scattering in lowest order, that is,

\[ \langle 0 | T A^\mu(x) A^\nu(y) \psi_\beta(z) \bar{\psi}_\alpha(w) | 0 \rangle = \frac{\delta^4 Z}{i \delta \eta_\alpha(w) i \delta \bar{\eta}_\beta(z) i \delta J_\nu(y) i \delta J_\mu} \]  

(5.167)

and verify that it reproduces the result obtained from the usual Feynman rules.

5.7 The Ward identities for QED derived in section 5.7 have not the form

\[ J_i F_i \left( \frac{\partial}{i \partial J} \right) Z(J) = 0 \]

(5.168)

where \( \delta \phi_i = F_i[\phi] \) because

\[ S_{GF} = \int d^4x \left( -\frac{1}{2\xi} (\partial \cdot A)^2 \right) \]

(5.169)

it is not gauge invariant. Introduce the functional

\[ Z'(J_\mu, \eta, \bar{\eta}) = \int D(A_\mu, \psi, \bar{\psi}, \omega, \bar{\omega}) e^{i \int d^4x (\mathcal{L}_{eff} + J^\mu A_\mu + \bar{\psi} \psi + \bar{\eta} \eta)} \]

(5.170)

where

\[ \mathcal{L}_{eff} = \mathcal{L}_{QED} + \mathcal{L}_{GF} + \mathcal{L}_G \]

(5.171)

and

\[ \mathcal{L}_G = -\bar{\omega} \mathcal{L}_G \omega \]

(5.172)
where $\omega$ and $\overline{\omega}$ are anti-commutative scalar fields.

a) Show that

$$Z'(J_\mu, \eta, \overline{\eta}) = N Z(J_\mu, \eta, \overline{\eta}) \quad (5.173)$$

where $N$ do not depend neither on the fields nor on the sources. Explain why this normalization does not affect the Green functions. Therefore either $Z$ or $Z'$ are good for its calculation. 

b) Show that the integration measure $D(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega})$ and $\int d^4x L_{\text{eff}}$ are invariants under the transformation

$$\delta \psi = -ie\omega \theta \psi \quad \delta \overline{\psi} = ie\overline{\psi} \omega \theta$$

$$\delta A_\mu = \partial_\mu \omega \theta$$

$$\delta \omega = \frac{1}{\xi} (\partial \cdot A) \theta$$

where $\theta$ is an anti-commutative parameter (Grassmann variable). 

c) Introduce the anti-commutative sources for the fields, $\omega$ and $\overline{\omega}$, that is

$$\overline{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = \int D(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega}) e^{i \int d^4x (L_{\text{eff}} + J_\mu A_\mu + \overline{\eta} \psi + \overline{\psi} \eta + \overline{\omega} + \zeta + \overline{\zeta})} \quad (5.175)$$

Show that

$$\overline{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = Z_G(\zeta, \overline{\zeta}) Z(J_\mu, \eta, \overline{\eta}) \quad (5.176)$$

where

$$Z(J_\mu, \eta, \overline{\eta}) = \int D(A_\mu, \psi, \overline{\psi}) e^{i \int d^4x (L_{\text{QED}} + L_{\text{GF}} + J_\mu A_\mu + \overline{\eta} \psi + \overline{\psi} \eta)} \quad (5.177)$$

Consider the functionals $\overline{W}$, $W_G$ and $W$ as well $\overline{\Gamma}, \Gamma_G$ and $\Gamma$ defined in a similar way. What is the relation between $\overline{W}$, $W_G$ and $W$ and the set $\overline{\Gamma}$, $\Gamma_G$ and $\Gamma$. 

d) Show that the Dyson-Schwinger equation for the fields $\omega$ and $\overline{\omega}$ is

$$\frac{\delta \overline{\Gamma}}{\delta \omega} = -\Box \omega \quad (5.178)$$

e) Show that the Ward identities can now be written as

$$J_i F_i \left[ \frac{\delta}{\delta \overline{J}} \right] \overline{Z} = 0 \quad (5.179)$$

Write the Ward identities for $\overline{\Gamma}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega})$. Show that you recover the known results. 

f) Show that a mass term for the photon, although it breaks the gauge symmetry does not spoil the Ward identities, if the ghosts also have mass. If the photon mass term were $\frac{1}{2} \mu^2 A_\mu A^\mu$ what would be the mass of the ghosts?
Chapter 6

Non-Abelian Gauge Theories

6.1 Classical theory

6.1.1 Introduction

We will start by reviewing briefly how to construct the classical action for a non-abelian (Yang-Mills) theory. Let us consider a compact group $G$ corresponding to some internal symmetry. Let $\phi_i, (i = 1, \cdots, N)$ be a set of fields that transform under $G$ in a representation of dimension $N$.

\[
\phi(x) \rightarrow \phi'(x) = U(g)\phi(x) \tag{6.1}
\]

where $U(g)$ is a $N \times N$ matrix. In an infinitesimal transformation

\[
g = 1 - i\alpha^a t^a \quad a = 1, \cdots, r \tag{6.2}
\]

where $\alpha^a$ are infinitesimal parameters and $t^a$ are the generators of the group. For the fundamental representation they satisfy

\[
[f^a, f^b] = i f^{abc} f^c \quad \text{Tr} \left( f^a f^b \right) = \frac{1}{2} \delta^{ab} \tag{6.3}
\]

Examples of these generators are

\[
\begin{align*}
SU(2) & \quad t^a = \frac{\sigma^a}{2} ; \quad a = 1, 2, 3 \\
SU(3) & \quad t^a = \frac{\lambda^a}{2} ; \quad a = 1, \cdots, 8
\end{align*} \tag{6.4}
\]

where $\sigma^a$ and $\lambda^a$ are the Pauli and Gell-Mann matrices, respectively. In the representation associated with the fields $\phi_i$, the matrices $T^a$ are of dimension $(N \times N)$ and they form a representation of the Lie algebra, that is

\[
[T^a, T^b] = i f^{abc} T^c \tag{6.5}
\]
Its normalization is given by
\[ \text{Tr}(T^a T^b) = \delta^{ab} T(R) \] (6.6)
where \( T(R) \) is a number that characterizes the representation \( R \). For a given representation one can show the identity (see Problem 6.1),
\[ T(R) \, r = d(R) C_2(R) \] (6.7)
where \( r \) is the dimension of the group \( G \) and \( d(R) \) is the dimension of the representation \( R \). In an infinitesimal transformation
\[ \delta \phi = -i \alpha^a T^a \phi \equiv -i \zeta \phi \] (6.8)
where we have introduced the useful notation \( \alpha \equiv \alpha^a T^a \).

### 6.1.2 Covariant derivative

In a local gauge theory we have the usual problem that the derivative does not transform as the fields, that is,
\[ \partial \mu \phi' \neq U \partial \mu \phi \] (6.9)
because the parameters depend on the coordinates, \( x^a \). To solve this we introduce the covariant derivative,
\[ \hat{D} \mu \phi = (\partial \mu - igA_\mu) \phi \quad ; \quad A_\mu = A^a_\mu T^a \] (6.10)
where \( A^a_\mu \) are the gauge fields, in equal number to the generators of the group. The transformation properties of \( A^a_\mu \) are obtained requiring that \( \hat{D} \mu \phi \) transforms in the same way as \( \phi \), that is,
\[ (\hat{D} \mu \phi)' = (\partial \mu - igA_\mu)' \phi = (\partial \mu - igA_\mu') U \phi \]
\[ = \partial \mu U \phi + U \partial \mu \phi - igA_\mu' U \phi \]
\[ = UD \mu \phi + (igU A_\mu - igA_\mu' U + \partial \mu U) \phi \] (6.11)
Therefore \( (\hat{D} \mu \phi)' = U(\hat{D} \mu \phi) \) requires
\[ A_\mu' = UA_\mu U^{-1} - \frac{i}{g} \partial \mu U U^{-1} \] (6.12)
For infinitesimal transformations \( U \approx 1 - i \zeta \) and we get
\[ \delta A_\mu \equiv A_\mu' - A_\mu = -i [\zeta, A_\mu] - \frac{1}{g} \partial \mu \zeta \] (6.13)
This can be written in components
\[ \delta A^a_\mu = -\frac{1}{g} \partial \mu \alpha^a + f^{bca} \alpha^b A^c_\mu = -\frac{1}{g} (\partial \mu \alpha^a - g f^{bca} \alpha^b A^c_\mu) \] (6.14)
As in the adjoint representation, \( (T^c)_{ab} \equiv -if^{bca} \), we get
\[ \delta A^a_\mu = -\frac{1}{g} (\partial \mu \delta_{ab} - ig(T^c)_{ab} A^c_\mu) \alpha^b = -\frac{1}{g} (D_\mu \alpha)^a \] (6.15)
that is, the gauge fields transform proportionally to the covariant derivative of the parameters of the gauge transformation.
6.1. CLASSICAL THEORY

6.1.3 Tensor $F_{\mu\nu}$

Let us calculate the commutator of two covariant derivatives,

$$[D_\mu, D_\nu] \phi = \left[ \partial_\mu - igA_\mu, \partial_\nu - igA_\nu \right] \phi$$

$$= -ig \left( \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \right) \phi$$

$$\equiv -ig F_{\mu\nu} \phi$$ \hspace{1cm} (6.16)

We have defined the tensor $F_{\mu\nu} \equiv F^a_{\mu\nu} T^a$, known as curvature,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$$ \hspace{1cm} (6.17)

In components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$ \hspace{1cm} (6.18)

which shows that it is a generalization of the Maxwell tensor. Let us see how $F_{\mu\nu}$ transforms under gauge transformations,

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu - ig [A'_\mu, A'_\nu]$$

Using $\partial_\mu U^{-1} = -U^{-1} \partial_\mu U U^{-1}$ \hspace{1cm} (6.20)

we get

$$F'_{\mu\nu} = UF_{\mu\nu} U^{-1}$$ \hspace{1cm} (6.21)

For infinitesimal transformations this gives,

$$\delta F_{\mu\nu} = -i [\alpha, F_{\mu\nu}]$$ \hspace{1cm} (6.22)

It easy to see that one can construct an invariant with the tensor $F_{\mu\nu}$. In fact the quantity,

$$\text{Tr}(F'_{\mu\nu} F'^{\mu\nu}) = \text{Tr}(F_{\mu\nu} F^{\mu\nu}) = \frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu}$$ \hspace{1cm} (6.23)

is an invariant and can be used to construct the action. Generalizing the Maxwell action for the Yang-Mills theories we get,

$$L_{YM} = -\frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$ \hspace{1cm} (6.24)
6.1.4 Choice of gauge

As we will see later in this chapter, gauge invariance can be used to choose some particular configuration, or gauge for the gauge fields. We call **pure gauge** to the field $A_\mu^{\sim}$ such that $F_{\mu\nu} = 0$. One can easily show that

$$F_{\mu\nu} = 0 \iff \exists U : A_\mu^{\sim} = \partial_\mu U U^{-1} \quad (6.25)$$

Two important examples of gauge choices are, the **Axial gauge** defined by,

$$n^\mu A_\mu^{a}(x) = 0 \quad (6.26)$$

where $n^\mu$ is a constant four vector, and the **Lorenz gauge** defined by,

$$\partial^\mu A_\mu^{a}(x) = 0 \quad (6.27)$$

6.1.5 The action and the equations of motion

The action for the pure gauge theory (without matter fields), is

$$S = -\frac{1}{2} \int d^4x \text{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{\mu\nu a} \quad (6.28)$$

and it is invariant under gauge transformations, because $\text{Tr}(F_{\mu\nu} F^{\mu\nu})$ is. The Euler-Lagrange equations

$$\partial_\mu \frac{\delta L}{\delta (\partial_\mu A_\nu^{a})} - \frac{\delta L}{\delta A_\nu^{a}} = 0 \quad (6.29)$$

can be easily obtained noticing that we have

$$\frac{\delta L}{\delta (\partial_\mu A_\nu^{a})} = \frac{\delta L}{\delta F_{\rho\sigma}^{b}} \frac{\delta F_{\rho\sigma}^{b}}{\delta (\partial_\mu A_\nu^{a})} = -F_{\mu\nu}^{a} \quad (6.30)$$

and

$$\frac{\delta L}{\delta A_\nu^{a}} = \frac{\delta L}{\delta F_{\rho\sigma}^{b}} \frac{\delta F_{\rho\sigma}^{b}}{\delta A_\nu^{a}} = gf_{bca} A_{\mu}^{b} F_{\nu}^{c} \quad (6.31)$$

We get therefore

$$\partial_\mu F^{\mu a} + gf_{bca} A_{\mu}^{b} F^{\nu c} = 0 \quad (6.32)$$

As we have in the adjoint representation, $(T^{c})_{ab} = -if^{bca}$, we get

$$(\partial_\mu \delta_{ab} - ig(T^{c})_{ab} A_{\mu}^{c}) F^{\mu b} = 0 \quad (6.33)$$

that is

$$P_{\mu}^{ab} F^{\mu b} = 0 \quad (6.34)$$

which is the equivalent of the Maxwell equations in the absence of external sources. As in the Maxwell theory, from the antisymmetry of $F^{\mu a}$ we can derive the Bianchi identities,

$$D_{\mu}^{ab} F_{\nu b} + D_{b}^{ab} F_{\mu b} + D_{\rho}^{ab} F_{\mu \rho b} = 0 \quad (6.35)$$

which are equivalent to the homogeneous Maxwell equations.
6.1.6 Energy–momentum tensor

As in the case of Electromagnetism, the canonical energy momentum tensor is not gauge invariant. In fact

\[
\tilde{\theta}^{\mu\nu} = -\frac{\delta L}{\delta (\partial_{\mu} A_{\rho})} \partial_{\nu} A_{\rho} + g^{\mu\nu} L
\]

\[
= F^{\mu\rho a} \partial_{\nu} A_{\rho}^{a} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma a} F_{\rho\sigma}^{a}
\] (6.36)

To make it gauge invariant we proceed as in the Electromagnetism. We subtract from \(\tilde{\theta}^{\mu\nu}\) a quantity that is a four divergence, in such a way that the conservation laws are not changed. The relevant quantity is,

\[
\Delta \theta^{\mu\nu} = \partial_{\rho} (F^{\mu\rho a} A_{\nu}^{a})
\]

\[
= \partial_{\rho} F^{\mu\rho a} A_{\nu}^{a} + F^{\mu\rho a} \partial_{\rho} A_{\nu}^{a}
\]

\[
= g f^{bc a} A_{\rho}^{b} F_{\rho\mu c} A_{\nu}^{a} + F^{\mu\rho a} \partial_{\rho} A_{\nu}^{a}
\]

\[
= F^{\mu\rho a} (-F_{\rho}^{\nu} + \partial_{\nu} A_{\rho}^{a})
\] (6.37)

We get therefore

\[
\theta^{\mu\nu} = \tilde{\theta}^{\mu\nu} - \Delta \theta^{\mu\nu}
\]

\[
= F^{\mu\rho a} F_{\rho}^{\nu a} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma a} F_{\rho\sigma}^{a}
\] (6.38)

which is analogous to the electromagnetism. Introducing the analog of the electric and magnetic fields,

\[
E_{a}^{i} = F_{a}^{i0} ; \quad B_{a}^{i} = -\frac{1}{2} \varepsilon_{ijk} F_{a}^{ij} \quad i, j, k = 1, 2, 3
\] (6.39)

we get

\[
\left\{
\begin{array}{l}
\theta^{00} = -\frac{1}{2} (\vec{E}^{a} \cdot \vec{E}^{a} + \vec{B}^{a} \cdot \vec{B}^{a}) \\
\theta^{0i} = (\vec{E}^{a} \times \vec{B}^{a})^{i}
\end{array}
\right.
\] (6.40)

with an interpretation similar to the case of the electromagnetism.

6.1.7 Hamiltonian formalism

From the component \(\theta^{00}\) we get for the Hamiltonian

\[
H = \int d^{3}x \frac{1}{2} (\vec{E}^{a} \cdot \vec{E}^{a} + \vec{B}^{a} \cdot \vec{B}^{a}) = \int d^{3}x \mathcal{H}
\] (6.41)

---

\(^{1}\)One should note an overall sign difference with respect to the general definition of Eq. (1.75). This is to maintain the component \(\theta^{00}\) with the meaning of a positive energy density. Obviously, the overall sign in Eq. (1.75), has no meaning prior to make contact with the model.
where $H$ is the Hamiltonian density. We are now going to show that the relation between the Hamiltonian and the Lagrangian is not the usual one. For this it is convenient to write the action using the so-called first order formalism,

$$ S = \int d^4x \left\{ -\frac{1}{2} \left( \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + g f^{abc} A^b_{\mu} A^c_{\nu} \right) F^{\mu\nu a} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu a} \right\} \quad (6.42) $$

where $A^a_{\mu}$ and $F^{a\mu}$ are now taken as independent variables. It is easy to show that the variation of $S$ in order to $F^{a\mu}$ gives back its definition,

$$ F^{a\mu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + g f^{abc} A^b_{\mu} A^c_{\nu} \quad (6.43) $$

and therefore if we substitute in $S$, Eq. 6.42, we get back the usual action. Using the definitions of $\vec{E}^a$ and $\vec{B}^a$ we get

$$ S = \int d^4x \left\{ -\partial^0 \vec{A}^a + \nabla A^0a - g f^{abc} A^0b A^c + \frac{1}{2} (\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \right\} $$

The Lagrangian density is then

$$ L = -E^ka \partial^0 A^k a - H(E^k a, A^k a) + A^0a C^a \quad (6.45) $$

where

$$ \begin{cases} 
H \equiv \frac{1}{2} (\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\
B^{ka} \equiv \frac{1}{2} \epsilon^{kmn} F^{mna} \\
C^a = \nabla \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c 
\end{cases} \quad (6.46) $$

The variables $A^k a$ and $-E^k a$ are canonical conjugate variables, $H(E^k a, A^k a)$ is the Hamiltonian density. The variables $A^0a$ play the role of Lagrange multipliers for the conditions,

$$ \nabla \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c = 0 \quad (6.47) $$

which are just the equations of motion for $\nu = 0$ (Gauss’s Law). If we introduce an equal time Poisson bracket

$$ \{ A^a(x), E^b(y) \}_{x_0 = y_0} = \delta^{ij} \delta^{ab} \delta^3(\vec{x} - \vec{y}) \quad (6.48) $$

one can show that

$$ \{ C^a(x), C^b(y) \}_{x_0 = y_0} = -g f^{abc} C^c(x) \delta^3(\vec{x} - \vec{y}) \quad (6.49) $$

$$ \{ H, C^a(x) \} = 0 \quad (6.50) $$

This shows that gauge theories, both abelian and non-abelian, are what it is known as Hamiltonian Generalized Systems, first introduced by Dirac.
To define these systems let us consider a system with canonical variables \((p_i, q_i)\) that generate the phase space \(\Gamma^{2n}\) \((i = 1, \ldots, n)\). Then the action of the Hamiltonian Generalized Systems is given by,

\[
S = \int L(t) dt \quad \text{where} \quad L(t) = \sum_{i=1}^{n} p_i \dot{q}_i - h(p, q) - \sum_{\alpha=1}^{m} \lambda^\alpha \varphi_\alpha(p, q) \quad (6.51)
\]

The variables \(\lambda^\alpha (\alpha = 1, \ldots, m)\) are Lagrange multipliers and \(\varphi_\alpha\) are the constraints. To be an Hamiltonian generalized system the following conditions should be verified,

\[
\{ \varphi^\alpha, \varphi^\beta \} = \sum_{\gamma} f^{\alpha\beta\gamma}(p, q) \varphi^\gamma \quad (6.52)
\]

\[
\{ h, \varphi^\alpha \} = f^{\alpha\beta}(p, q) \varphi^\beta \quad (6.53)
\]

The case of gauge theories is a particular case with \(f^{\alpha\beta} = 0\). Therefore to be able to quantize gauge theories we have to learn first how to quantize Hamiltonian generalized systems.

### 6.2 Quantization

#### 6.2.1 Systems with \(n\) degrees of freedom

Let us consider an Hamiltonian Generalized Systems, described before. The Lagrangian is

\[
L(t) = p_i \dot{q}_i - h(p, q) - \lambda^\alpha \varphi_\alpha(p, q) \quad (6.54)
\]

which leads to the following equations of motion,

\[
\begin{cases}
\dot{q}_i = \frac{\partial h}{\partial p_i} + \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p_i} \\
\dot{p}_i = -\frac{\partial h}{\partial q_i} - \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial q_i} \\
\varphi_\alpha(p, q) = 0 \quad \alpha = 1, \ldots, m
\end{cases} \quad (6.55)
\]

One can show that an Hamiltonian Generalized System, \((\text{HGS})\) is equivalent to an usual Hamiltonian system \((\text{HS})\) defined in a phase space \(\Gamma^{*2(n-m)}\). That is, a HGS it is equivalent to an HS with \(n - m\) degrees of freedom. To prove this we construct explicitly the HS \(\Gamma^*\). For this consider \(m\) conditions,

\[
\chi^\alpha(p, q) = 0 \quad ; \quad \alpha = 1, \ldots, m \quad (6.56)
\]

such that they satisfy,

\[
\{ \chi^\alpha, \chi^\beta \} = 0 \quad (6.57)
\]

and

\[
\det \left| \{ \varphi^\alpha, \chi^\beta \} \right| \neq 0 \quad (6.58)
\]
Then, the subspace of $\Gamma^{2n}$ defined by the conditions

$$
\begin{align*}
\chi^\alpha(p, q) &= 0 \\
\varphi^\alpha(p, q) &= 0
\end{align*}
$$

is the space $\Gamma^{*2(n-m)}$. The canonical variables $p^*$ and $q^*$ in $\Gamma^{*2(n-m)}$ can be found in the following way. Due to the requirement $\left\{ \chi^\alpha, \chi^\beta \right\} = 0$ we can choose the variables $q_i$ in $\Gamma^{2n}$ in such a way that the $\chi^\alpha$ coincide with the first $m$ variables of the coordinate type, that is

$$
q_i \equiv \left( \frac{\chi^\alpha}{m} q^* \right)
$$

Let now $p = (p^\alpha, p^*)$ be the corresponding conjugate momenta. In these variables, the condition in the determinant takes the form,

$$
\det \frac{\partial \varphi^\alpha}{\partial p^\beta} \neq 0
$$

This means that, in principle the conditions $\varphi^\alpha(p, q) = 0$ can be solved for $p^\alpha$, that is,

$$
p^\alpha = p^\alpha(p^*, q^*)
$$

The subspace $\Gamma^*$ it is therefore defined by the conditions,

$$
\begin{align*}
\chi^\alpha &\equiv q^\alpha = 0 \\
p^\alpha &= p^\alpha(p^*, q^*)
\end{align*}
$$

The variables $p^*$ and $q^*$ are canonical and the Hamiltonian is given by

$$
h^*(p^*, q^*) = h(p, q) \big|_{(\chi=0 \ ; \ \varphi=0)}
$$

The equations of motion are now

$$
\dot{q}^* = \frac{\partial h^*}{\partial p^*} \quad \dot{p}^* = -\frac{\partial h^*}{\partial q^*}
$$

in a total of $2(n-m)$ equations. The fundamental result can be formulated in a form of a theorem.

**Theorem 6.1**

The two representations lead to the same equations of motion and are therefore equivalent.

**Proof:**

The relations $q^\alpha = 0 \implies \dot{q}^\alpha = 0$, which means, in the $(p, q)$ description,

$$
\frac{\partial h}{\partial p_\alpha} + \lambda_\beta \frac{\partial \varphi_\beta}{\partial p_\alpha} = 0 \ ; \ \alpha = 1, \ldots, m
$$
Let us now consider the equations of motion in for the coordinates \( q^* \) in the two representations:

\[
\dot{q}^* = \frac{\partial h}{\partial p} + \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p^*} \\
\dot{q}^* = \frac{\partial h^*}{\partial p^*} = \frac{\partial h}{\partial p^*} + \frac{\partial h}{\partial p^\alpha} \frac{\partial p^\alpha}{\partial p^*}
\]

(6.67)

The two equations will be equivalent if

\[
\lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p} \frac{\partial \varphi_\alpha}{\partial p^*} = \frac{\partial h}{\partial p^\alpha} \frac{\partial p^\alpha}{\partial p^*}
\]

(6.68)

which means, using the previous relations,

\[
\lambda^\alpha \left( \frac{\partial \varphi_\alpha}{\partial p} + \frac{\partial \varphi_\alpha}{\partial p^\beta} \frac{\partial p^\beta}{\partial p^*} \right) = 0
\]

(6.69)

But this relation is true due to the constraints \( \varphi_\alpha = 0 \). Therefore the two representations are equivalent which proves the theorem (the equations for \( p^* \) could be treated in a similar way).

If we want to quantize these systems we can use the expressions for the evolution operator in terms of a path integral in the variables \((p^*, q^*)\) as these correspond to an usual Hamiltonian system. We have then,

\[
U(q_f^*, q_i^*) = \int \prod_t dp^* dq^* \frac{1}{(2\pi)^{2\nu}} e^{i \int dt [p^* \dot{q}^* - h(p^*, q^*)]} dt
\]

(6.70)

Although this is a possible way of quantizing the theory, it is normally not very convenient in most situations. This is because in real situations it is difficult to invert the relations \( \varphi^\alpha = 0 \) to get \( p^\alpha = p^\alpha(p^*, q^*) \). It is normally more convenient to use the variables \((p, q)\) with appropriate restrictions. This can be easily done substituting,

\[
\prod_t dp^* dq^* \rightarrow \prod_t \frac{dp dq}{2\pi} \prod_t \delta(q^*) \delta(p^\alpha - p^\alpha(p^*, q^*))
\]

(6.71)

Then

\[
U(q_f, q_i) = \int \prod_t \frac{dp dq}{2\pi} \prod_t \delta(q^*) \delta(p^\alpha - p^\alpha(p^*, q^*)) e^{i \int dt [p \dot{q} - h(p, q)]}
\]

(6.72)

This expression can be written in terms of the constraints if we recall that

\[
\delta(q^\alpha) = \delta(\chi^\alpha)
\]

\[
\delta(p^\alpha - p^\alpha(p^*, q^*)) = \delta(\varphi^\alpha) \det \left| \frac{\partial \varphi_\alpha}{\partial p^\beta} \right|
\]

(6.73)

Then we get

\[
\prod_t \delta(q^\alpha) \delta(p^\alpha - p^\alpha(p^*, q^*)) = \prod_t \delta(\varphi^\alpha) \delta(\chi^\alpha) \det |{\varphi_\alpha, \chi_\beta}| \cdot
\]

(6.74)
Finally we use the identity
\[
\delta(\varphi^\alpha) = \int \frac{d\lambda}{2\pi} e^{-i\int d\lambda^\alpha \varphi^\alpha},
\]
(6.75)
to get
\[
U(q_f, q_i) = \prod_t \int \frac{dp dq}{2\pi} \int d\lambda \prod_{t,x} \delta(\chi^\alpha) \det \{\varphi^\alpha, \chi^\beta\} e^{iS(p, q, \lambda)},
\]
(6.76)
where
\[
S(p, q, \lambda) = \int [p\dot{\varphi} - h(p, q) - \lambda \varphi] dt.
\]
(6.77)
It will be this expression in Eq. (6.76) that we will apply to the gauge theories. Note that the expression in the parenthesis is precisely the Lagrangian for generalized Hamiltonian systems, Eq. (6.54). It can be shown that the physical results do not depend on the auxiliary conditions \(\chi^\alpha = 0\). In gauge theories these are known as the gauge choice.

### 6.2.2 QED as a simple example

Let us consider the electromagnetic field coupled to an external conserved current, \(J^\mu = (\rho, \vec{J})\), with \(\partial_\mu J^\mu = 0\). The Lagrangian is,
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu.
\]
(6.78)
The action can be written using the first order formalism,
\[
S = \int d^4 x \left\{ -\vec{E} \cdot (\vec{\nabla} A^0 + \dot{\vec{A}}) - \vec{B} \cdot \vec{\nabla} \times \vec{A} + \frac{\vec{B}^2 - \vec{E}^2}{2} - \rho A^0 + \vec{J} \cdot \vec{A} \right\}.
\]
(6.79)
The equations of motion are obtained by varying with respect to \(\vec{E}\) and \(\vec{B}\)
\[
\begin{align*}
\vec{E} &= - (\vec{\nabla} A^0 + \dot{\vec{A}}), \\
\vec{B} &= \vec{\nabla} \times \vec{A}
\end{align*}
\]
\[
\begin{align*}
\vec{\nabla} \cdot \vec{B} &= 0, \\
\vec{\nabla} \times \vec{E} &= \frac{\partial \vec{B}}{\partial t}.
\end{align*}
\]
(6.80)
Varying with respect to \(A^0\) and \(\vec{A}\),
\[
\begin{align*}
\vec{\nabla} \cdot \vec{E} &= \rho, \\
\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{J}.
\end{align*}
\]
(6.81)
If we substitute \(\vec{B} = \vec{\nabla} \times \vec{A}\) we get after an integration by parts,
\[
S = \int d^4 x \left\{ -\vec{E} \cdot \dot{\vec{A}} - \left( \frac{\vec{E}^2 + (\vec{\nabla} \times \vec{A})^2}{2} - \vec{J} \cdot \vec{A} \right) + A^0 (\vec{\nabla} \cdot \vec{E} - \rho) \right\}.
\]
(6.82)
It is clear that \(A^0\) plays the role of a Lagrange multiplier. The canonical variables are \(\vec{A}\) and \(\vec{E}\) but they are not free, as there exists one constraint to be obeyed, \(\vec{\nabla} \cdot \vec{E} = \rho\). This
constraint is linear in the fields. Here resides the big simplification of the electromagnetism. If we choose a linear gauge condition, then \( \det \{ \phi^\alpha, \chi_\beta \} \) will not depend either in \( \vec{E} \) or \( \vec{A} \) and will be a constant that only will modify the normalization. Such a gauge condition is obtained, for instance, with the Lorenz gauge

\[
\chi = \partial_\mu A^\mu - c(\vec{x}, t) ,
\]

where \( c(\vec{x}, t) \) is an arbitrary function. Then the generating functional for Green functions is (the term that comes from \( \det \{ \phi^\alpha, \chi_\beta \} \) is absorbed in the normalization)

\[
Z[J^\mu] = N \int \mathcal{D}(\vec{E}, \vec{A}, A^0) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS} ,
\]

where

\[
S = \int d^4x \left\{ \frac{1}{2} (\vec{E} \cdot \vec{A} - \vec{E} \cdot (\vec{\nabla} \times \vec{A})^0 - \vec{J} \cdot \vec{A}) + A^0(\vec{\nabla} \cdot \vec{E} - \rho) \right\} + \frac{1}{2} A^\mu (\vec{\nabla} \cdot \vec{A})^\mu - J_\mu A^\mu \right\} .
\] (6.85)

The integration in \( \vec{E} \) is Gaussian and can be done immediately (we keep the notation \( N \) although this normalization will be different after the integration)

\[
Z[J^\mu] = N \int \mathcal{D}(A_\mu) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS} ,
\]

where now

\[
S = \int d^4x \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right] .
\] (6.87)

As the functions \( c(x) \) are arbitrary we can integrate over them with a weight

\[
\exp \left( -\frac{1}{2\xi} \int d^4x c^2(x) \right) .
\]

We get then the familiar result,

\[
Z[J^\mu] = N \int \mathcal{D}(A_\mu) e^{i\int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu - \frac{1}{2} \rho (\vec{\nabla} \cdot \vec{A})^2 - \frac{1}{2} J_\mu A^\mu \right]} .
\] (6.89)

As we will see later, if we had chosen a non-linear gauge condition the \( \det |\{ q, \chi \}| \) would depend on \( \vec{E} \) or \( \vec{A} \) and it would not be possible to absorb it in the normalization (which is irrelevant as we can always choose it such that \( Z[0] = 1 \)). In that case it would be necessary to use the methods of non-abelian gauge theories that we will discuss in the next section.
6.2.3 Non abelian gauge theories. Non covariant gauges

We have seen before that the action for the non-abelian gauge theories, could be written in the form,

$$ S = -2 \int d^4x \ Tr \left[ \vec{E} \cdot \partial^0 \vec{A} + \frac{1}{2} (\vec{E}^2 + \vec{B}^2) - \vec{A}^0 (\vec{\nabla} \cdot \vec{E} + g[\vec{A}, \vec{E}]) \right] $$

(6.90)

$$ = \int d^4x \left[ -E_a^k \partial^0 A^a_k - \mathcal{H}(E_k, A_k) + A^0 a C^a \right], $$

(6.91)

where

$$ C^a = \vec{\nabla} \cdot \vec{E}^a - gf^{abc} \vec{A}_b \cdot \vec{E}_c. $$

(6.92)

If we introduce the equal time Poisson brackets

$$ \{ -E_a^k(x), A^l_b(y) \}_{x^0 = y^0} = \delta^{ij} \delta_{ab} \delta^3(\vec{x} - \vec{y}), $$

(6.93)

one can then show that

$$ \{ C^a(x), C^b(y) \}_{x_0 = y_0} = -gf^{abc} C^c(x) \delta^3(\vec{x} - \vec{y}) $$

$$ \{ H, C^a(x) \} = 0, $$

(6.94)

where

$$ H = \int d^3x \mathcal{H}(E_k, A_k) = \frac{1}{2} \int d^3x \left[ (E^{ka})^2 + (B^{ka})^2 \right]. $$

(6.95)

We see then that the non-abelian gauge theories are an example of generalized Hamiltonian systems, like we saw with the electromagnetism. The variables of the type coordinate are $A^a_k$, and the conjugate momenta are $-E^a_k$. The variables $A^0 a$ are Lagrange multipliers for the constraints,

$$ \vec{\nabla} \cdot \vec{E}^a - gf^{abc} \vec{A}_b \cdot \vec{E}_c = 0, $$

(6.96)

which are part of the equations of motion.

To proceed with the quantization we have to use the formalism of the HGS. For that we have to impose $r$ auxiliary conditions (where $r$ is the dimension of the Lie group and therefore of its adjoint representation where the gauge fields are), that is, as many as the constraints $C^a(x) = 0$, $a = 1, \ldots, r$. Choose these conditions is what is known as choosing or fixing the gauge. This choice is arbitrary, and the physical results should not depend on it. However intermediate expressions as, for instance, the Feynman rules can depend on the choice.

As we saw in the case of the electromagnetism if it is possible to make a gauge fixing that is linear in the dynamical variables, $A^a_k$ and $\vec{E}^a$, then the path integral will simplify because the determinant will not depend on these variables and can be absorbed into the normalization. For the non-abelian case, a gauge where this is possible is the axial gauge that we now study.
Axial Gauge

It is always possible to make a gauge transformation such that the component of \( \vec{A}^a \) along some direction vanishes in all points, that is,

\[ A^{3a} = 0 \quad a = 1, \ldots, r, \quad (6.97) \]

where we have chosen the direction along the \( z \) axis. These \( r \) conditions are our auxiliary conditions necessary to proceed with the quantization of the theory. The advantage of this gauge choice is the following. If we calculate \( \{ C^a, A^{3b} \} \) we get

\[ \{ C^a(x), A^{3b}(y) \} = \{ \partial_k E^k_a(x), A^{3b}_b(y) \} - g f_{abc} A^k_d \{ E^k_c(x), A^{3b}_b(y) \} \]

\[ = -g \frac{\partial}{\partial x^3} \delta^3(\vec{x} - \vec{y}) = \frac{\delta}{\delta A^{3a}(x)}(\delta A^3_b(y)), \quad (6.98) \]

where we have used the fact that \( A^{3b} = 0 \). We see then that \( \{ C^a, A^{3b} \} \) does not depend on \( \vec{A}^a \) and \( \vec{E}^a \) and the determinant that appears in the expression for the path integral can be absorbed in the normalization. We can then write the generating functional for the Green functions in this gauge as

\[ Z[J^{\mu a}] = \int D(\vec{E}, \vec{A}, A^0) \prod_x \delta(A^3) e^{iS(\vec{E}, \vec{A}, A^0, J^{\mu})}, \quad (6.99) \]

where

\[ S(\vec{E}, \vec{A}, A^0, J^{\mu}) = \int d^4 x \left[ -\vec{E}^a \cdot \partial^0 \vec{A}^a - \frac{1}{2} \left[ (\vec{E}^a)^2 + (\vec{B}^a)^2 \right] + A^{0a} C^a + A^a \cdot J^a \right], \quad (6.100) \]

and

\[ C^a = \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c. \quad (6.101) \]

As the integration in \( \vec{E} \) is Gaussian we can easily get

\[ Z_A[J^{\mu a}] = \int D(A^\mu) \prod_x \delta(A^3) e^{i \int d^4 x \left[ L(x) + A^a \cdot J^a \right]}, \quad (6.102) \]

After the integration the Lagrangian is then

\[ L = -\frac{1}{4} (F^{a}_{\mu \nu} F^{a \mu \nu}) \quad (6.103) \]

The index \( A \) in \( Z_A[J^{\mu a}] \) reminds us that this generating functional corresponds to the axial gauge. Although the expression for the generating functional can be easily written in this gauge it has the disadvantage that the Feynman rules are not covariant. Before we introduce the covariant gauges, let us look at another non-covariant gauge, the so-called Coulomb gauge.
Coulomb Gauge

This gauge is defined by the auxiliary conditions,
\[ \vec{\nabla} \cdot \vec{A}_a = 0 \quad a = 1, \ldots, r . \]  
(6.104)

These auxiliary conditions have a non-trivial Poisson bracket with the constraints \( C^a(x) \). In fact one can show (see problems) that
\[ \delta \vec{A}_a = -\frac{1}{g} \int d^3 y \left\{ \vec{A}_a(x), \alpha^b(y)C_b(y) \right\}_{x_0=y_0} . \]  
(6.105)

Therefore
\[ \left\{ \vec{\nabla} \cdot \vec{A}_a(x), C_b(y) \right\}_{x_0=y_0} = -g \frac{\delta}{\delta \alpha^b(y)} (\delta \vec{A}_a(x)) , \]  
(6.106)

and
\[ \left\{ \vec{\nabla} \cdot \vec{A}_a(x), C_b(y) \right\}_{x_0=y_0} = -g \frac{\delta}{\delta \alpha^b(y)} \vec{\nabla} \cdot (\delta \vec{A}_a(x)) . \]  
(6.107)

As we have
\[ \delta \vec{A}_a(x) = \frac{1}{g} \vec{\nabla} \alpha^a(x) + f_{abc} \alpha^b(x) \vec{A}_c(x) , \]  
(6.108)

we get (with the condition \( \vec{\nabla} \cdot \vec{A}_a = 0 \))
\[ -g \vec{\nabla} \cdot (\delta \vec{A}_a(x)) = -\nabla^2 \alpha^a(x) - gf_{abc} \vec{A}_c(x) \cdot \vec{\nabla} \alpha^b(x) . \]  
(6.109)

This gives finally
\[ \left\{ \vec{\nabla} \cdot \vec{A}_a(x), C_b(y) \right\} = \left[ -\nabla^2 \delta_{ab} - gf_{abc} \vec{A}_c(x) \cdot \vec{\nabla} \right] \delta^3(\vec{x} - \vec{y}) \equiv \mathcal{M}_{ab}(x,y) . \]  
(6.110)

As \( \det \mathcal{M}_c \), although depending on \( \vec{A} \) it does not depend on \( \vec{E} \), the Gaussian integration in \( \vec{E} \) can still be done and we get
\[ Z_C[J^\mu] = \int \mathcal{D}(A_\mu) \prod_x [\det \mathcal{M}_C] \prod_x \delta (\vec{\nabla} \cdot \vec{A}) e^i \int d^4 x [\mathcal{L} + A^\mu \cdot J_\mu] . \]  
(6.111)

Now it is not possible to absorb \( \det \mathcal{M}_C \) in the normalization. The Feynman rules that can be obtained from \( Z_C[J^\mu] \) are again non-covariant.

6.2.4 Non abelian gauge theories in covariant gauges

The gauge conditions chosen up to now (axial and Coulomb gauges) lead to Feynman rules where the Lorentz covariance is lost. Of course the final physical results should not depend on this, but the non-covariance in the intermediate stages of the calculations is a complication. We are now going to generalize the previous results to covariant gauges. The method to follow will be a sub-product of the answer to the following question: How can we show the equivalence between the axial and Coulomb gauges? For the argument that follows it is convenient to work with gauge invariant quantities. Then, instead of the
functional $Z_A[J^\mu]$ we are going to consider the integral $Z_A[J = 0]$ that, as we have seen, has the meaning of a vacuum → vacuum transition in the absence of external sources,

$$Z_A[0] = \int \mathcal{D}(A_\mu) \prod_{x,a} \delta(A^{3a}(x)) \exp\{iS[A_\mu]\} , \quad (6.112)$$

where $S[A_\mu]$ is the action. In a gauge transformation,

$$A_\mu \rightarrow A'_\mu = gA_\mu U^{-1}(g) - \frac{i}{g} \partial_\mu UU^{-1} , \quad (6.113)$$

the action $S[A_\mu]$ and the measure $\mathcal{D}(A_\mu)$ are invariant, therefore we get

$$Z_A[J = 0] = \int \mathcal{D}(A_\mu) \prod_{x,a} \delta(gA^{3a}(x)) \exp\{iS[A_\mu]\} . \quad (6.114)$$

We define now the functional $\Delta_C[A_\mu]$ through the relation

$$\Delta_C^{-1}[A_\mu] = \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot g\vec{A}^a) , \quad (6.115)$$

where $\mathcal{D}(g)$ represents the infinite product of the invariant measures for the group $G$ at each space-time point, that is

$$\mathcal{D}(g) = \prod_x dg(x) . \quad (6.116)$$

The invariance of the integration measure of the group $G$, $\mathcal{D}g' = \mathcal{D}(gg')$ has the consequence that $\Delta_C$ is gauge invariant. In fact

$$\Delta_C^{-1}[gA_\mu] = \int \mathcal{D}(g') \prod_{x,a} \delta(\vec{\nabla} \cdot g'g\vec{A}^a)$$

$$= \int \mathcal{D}(g') \prod_{x,a} \delta(\vec{\nabla} \cdot g'\vec{A}^a)$$

$$= \Delta_C^{-1}[A_\mu] . \quad (6.117)$$

We introduce now in the expression for $Z_A[J = 0]$ the identity

$$1 = \Delta_C[A_\mu] \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot g\vec{A}^a) . \quad (6.118)$$

We therefore get

$$Z_A(J = 0) = \int \mathcal{D}A_\mu e^{iS[A_\mu]} \prod_{x,a} \delta(A^{3a}(x)) \Delta_C[A_\mu] \int \mathcal{D}(g) \prod_{y,b} \delta(\vec{\nabla} \cdot g\vec{B}^b)$$

$$= \int \mathcal{D}A_\mu e^{iS[A_\mu]} \Delta_C[A_\mu] \prod_{y,b} \delta(\vec{\nabla} \cdot \vec{A}^b) \int \mathcal{D}(g) \prod_{x,a} \delta(g^{-1}A^{3a}) , \quad (6.119)$$
where we have used the gauge invariance of $D$, $S[A_\mu]$ and $\Delta_C[A_\mu]$. As the measure is invariant, we can write in the last integral $g^{-1} \to gg_0$. Then

$$\int D(g) \prod_{x,a} \delta \left( g^{-1} A^3a(x) \right) = \int D(g) \prod_{x,a} \delta \left( g g_0 A^3a(x) \right) .$$ (6.120)

where $g_0$ is the gauge transformation that takes from the gauge $\vec{\nabla} \cdot \vec{A} = 0$ to the gauge $A'^3 = 0$, that is

$$A'^3 = g_0 A^3 = 0 ,$$ (6.121)

with $\vec{\nabla} \cdot \vec{A} = 0$. We still have to calculate the integral over the group, that now takes the form,

$$\int D(g) \prod_{x,a} \delta \left( g A'^3a(x) \right) ,$$ (6.122)

with $A'^3a = 0$. As $A'^3a = 0$ it is enough to consider infinitesimal transformations in the vicinity of the unit,

$$g(x) = e - i \alpha^a(x) = e - i \alpha^a(x) t^a ,$$ (6.123)

where $\alpha^a(x)$ are infinitesimal. In these conditions, the integration measure $dg(x)$ is given by,

$$dg(x) = \prod_a d\alpha^a(x) .$$ (6.124)

On the other hand to first order in $\alpha^a$ we have

$$g A'^3a(x) = \frac{1}{g} \frac{\partial \alpha^a}{\partial x^3} ,$$ (6.125)

and therefore the integral is now

$$\int D(g) \prod_{x,a} \delta \left( \frac{\partial A'^3a}{\partial x^3} \right) = \int D(\alpha) \prod_{x,a} \delta \left( \frac{1}{g} \frac{\partial \alpha^a}{\partial x^3} \right) = \mathcal{N} .$$ (6.126)

The integral is independent on $A_\mu$ and therefore it can be absorbed in the normalization. We get then

$$Z_A[J = 0] = \mathcal{N} \int D(A_\mu) \Delta_C[A_\mu] \prod_x \delta(\vec{\nabla} \cdot \vec{A}) e^{iS[A_\mu]} .$$ (6.127)

We have obtained before an expression for $Z_C[J = 0]$, which was,

$$Z_C[J = 0] = \int D(A_\mu) \prod_x \det M_C \prod_x \delta(\vec{\nabla} \cdot \vec{A}) e^{iS[A_\mu]} .$$ (6.128)

Therefore to show that the two path integrals, that represent the the $\text{vacuum} \to \text{vacuum}$ amplitudes in the absence of external sources, we still have to show that $\Delta_C[A_\mu] = \det M_C$. This can be easily shown true. In fact,

$$\Delta_C^{-1}[A_\mu] = \int D(g) \prod_{x,a} \delta \left( \vec{\nabla} \cdot g A^3a \right)$$
6.2. QUANTIZATION

\[ \int D(\alpha) \prod_{x,a} \delta \left[ \nabla \cdot \left( \frac{1}{g} \nabla \alpha^a(x) + f^{abc} \alpha^b \tilde{A}^c \right) \right] \]

\[ \int D(\alpha) \prod_{x,a} \delta \left( \frac{1}{g} \nabla \alpha^a(x) + f^{abc} \nabla \alpha^b \cdot \tilde{A}^c \right) \]

\[ \propto \det^{-1} M_C , \quad (6.129) \]

where

\[ M_C^{ab}(x,y) = -g \frac{\delta}{\delta \alpha^b(y)} \left( \nabla \cdot \eta \tilde{A}^a \right)_{\alpha=0} \]

\[ = \left( -\nabla^2 \delta_{ab} - g f^{abc} \tilde{A}^c \cdot \nabla x \right) \delta^3(\vec{x} - \vec{y}) . \quad (6.130) \]

Therefore \( \Delta_C[A_\mu] \propto \det M_C \), and except for an irrelevant normalization we have, \( Z_A[0] = Z_C[0] \).

The way we have shown this equivalence between the axial and Coulomb gauges suggests a way to define the vacuum → vacuum amplitude for an arbitrary gauge defined by the gauge conditions,

\[ F^a[A_\mu] = 0 \quad a = 1, \ldots, r \quad (6.131) \]

For that we define \( \Delta_F[A_\mu] \) by the expression

\[ \Delta_F^{-1}[A_\mu] = \int D(g) \prod_{x,a} \delta \left( F^a[A_\mu] \right) \]

and, like before, we introduce

\[ 1 = \Delta_F[A_\mu] \int D(g) \prod_{x,a} \delta \left( F^a[A_\mu] \right) , \quad (6.133) \]

in the expression for \( Z_A[J = 0] \). We therefore get

\[ Z_A[J = 0] = \int D(A_\mu) \prod_{x,a} \delta \left( A^{3a}(x) \right) e^{iS[A_\mu]} \Delta_F[A_\mu] \int D(g) \prod_{y,b} \delta \left( F^{b[y} A^a_\mu] \right) \]

\[ = \int D(A_\mu) \prod_{y,b} \delta \left( F^{b[y} A^a_\mu] \right) \Delta_F[A_\mu] e^{iS[A_\mu]} \int D(g) \prod_{x,a} \delta \left( g^{-1} A^{3a}(x) \right) \]

\[ = \mathcal{N} \int D(A_\mu) \Delta_F[A_\mu] \prod_{x,a} \delta \left( F^{b[y} A^a_\mu] \right) e^{iS[A_\mu]} \]

\[ = \mathcal{N} Z_F[J = 0] , \quad (6.134) \]

showing that the axial gauge and general gauges of the type \( F \) are equivalent. The vacuum → vacuum amplitude in the gauge \( F^a = 0 \), is therefore given by

\[ Z_F[J = 0] = \int D(A_\mu) \Delta_F[A_\mu] \prod_{x,a} \delta \left( F^a[A_\mu] \right) e^{iS[A_\mu]} . \quad (6.135) \]
To finish we still have to be able to evaluate $\Delta F[A_\mu]$. As in the definition $\Delta F[A_\mu]$ appears multiplied by $\prod \delta (F^a[A_\mu])$, we only need to know $\Delta F[A_\mu]$ for $A_\mu$ that satisfy $F^a[A_\mu] = 0$. Then for $g$ in the vicinity of the identity we have

$$F^a[A_\mu] = F^a[A_\mu] + \frac{\delta F^a}{\delta A^b_\mu} \delta A^b_\mu = -\frac{1}{g} \frac{\delta F^a}{\delta A^b_\mu} (D_\mu \alpha)^b,$$

(6.136)

where we have used $F^a[A_\mu] = 0$ and $\delta A^b_\mu = -\frac{1}{g} (D_\mu \alpha)^b$. Let us calculate $\Delta F$. We get

$$\Delta^1 F[A_\mu] = \int \mathcal{D}(g) \prod_{x,a} \delta \left( F^a[A_\mu]\right)$$

$$= \int \mathcal{D}(\alpha) \prod_{x,a} \delta \left( -\frac{1}{g} \frac{\delta F^a}{\delta A^b_\mu} (D_\mu \alpha)^b\right)$$

$$\propto \det^{-1} \mathcal{M}_F,$$

(6.137)

where

$$\mathcal{M}^{ab}_F(x, y) = \frac{\delta F^a}{\delta A^c_\mu(x)} D^{cb}_\mu \delta^4(x - y) = -g \frac{\delta F^a}{\delta \alpha^b(y)},$$

(6.138)

and therefore

$$\Delta F[A_\mu] = \det \mathcal{M}_F = \det \left(-g \frac{\delta F^a(x)}{\delta \alpha^b(y)}\right).$$

(6.139)

We have discovered how to write the vacuum $\rightarrow$ vacuum amplitude in the absence of external sources. However this is not the more interesting quantity, but rather the vacuum $\rightarrow$ vacuum amplitude in the presence of sources, $Z_F[J]$ because this is the one that generates the Green functions of the theory. In all this discussion the source terms, $\int d^4x J^a_\mu A^\mu a$, were put to zero because they are not gauge invariant, and our derivation relied upon gauge invariance. If we define $Z_F[J^\mu_\mu]$ by the relation

$$Z_F[J^\mu] \equiv \int \mathcal{D}(A_\mu) \Delta F[A_\mu] \prod_{x,a} \delta(F^a[A_\mu(x)]) e^{i(S[A_\mu] + \int d^4x J^a_\mu A^\mu a)},$$

(6.140)

then it is clear that the functional $Z_F$ will not be equivalent for different choices of $F^a = 0$. This means that the Green functions obtained from $Z_F[J^\mu]$ will depend on the gauge $F^a = 0$. In the section 6.2.5 we will show that although the Green functions depend on the gauge, this is not really a problem, because the physical results for the elements of the renormalized $S$ matrix are gauge independent and these are the ones that we compare with the experiments.

Before we finish let us make a transformation in the functional $Z_F[J^\mu_\mu]$ to get rid of the $\delta$ function. For the calculations it is important to exponentiate $\prod \delta(F^a[A_\mu])$. This can be done in the following way. We start by defining a more general gauge condition,

$$F^a[A_\mu] - c^a(x) = 0,$$

(6.141)
6.2. QUANTIZATION

where \( c^a(x) \) are arbitrary functions of space-time but that do not depend on the fields. Then \( \Delta_F[A] \) will not be changed and we write,

\[
Z_F[J^a_\mu] = N \int D(A_\mu) \Delta_F[A_\mu] \prod_x \delta(F^a[A_\mu] - c^a) e^{i(S[A_\mu] + f d^4x J^a_\mu A^\mu)} .
\] (6.142)

The left side of this equation does not depend on \( c^a(x) \) and therefore we can integrate over \( c^a(x) \) with a convenient weight, more specifically,

\[
\exp \left\{ -\frac{i}{2} \int d^4x \ c^2_a(x) \right\} ,
\] (6.143)

where \( x \) is a real parameter. We finally get

\[
Z_F[J^a_\mu] = N \int D(A_\mu) \Delta_F[A_\mu] e^{i(S[A_\mu] + f d^4x (-\frac{1}{2} F^2 + J^a_\mu A^\mu))} \] (6.144)

This expression is the starting point for the calculation of the Green functions in an arbitrary gauge defined by the gauge fixing \( F^a \). To be able to establish the Feynman rules for this theory we still have to exponentiate \( \Delta_F[A_\mu] \). This will be done in the section 6.2.6 with the introduction of the Faddeev-Popov ghosts.

### 6.2.5 Gauge invariance of the \( S \) matrix

In the previous section we have defined the generating functional for the Green functions, \( Z_F[J^a_\mu] \), for a gauge condition given by the function, \( F^a[A_\mu] \), through the relation,

\[
Z_F[J^a_\mu] \equiv N \int D(A_\mu) \Delta_F[A_\mu] \prod_x \delta(F^a[A_\mu(x)]) e^{i(S[A_\mu] + f d^4x J^a_\mu A^\mu)} .
\] (6.145)

We have shown the equivalence between different gauges in the case of vanishing sources. We are now going to show what happens when \( J^a_\mu \neq 0 \). For this we will go back and redo the proof of the equivalence in the presence of the source terms. We choose for this, the case of the Coulomb and Lorenz gauges defined by

\[
\begin{align*}
F^a &= \vec{\nabla} \cdot \vec{A}^a \quad \text{Coulomb gauge} \\
F^a &= \partial_\mu A^{\mu a} \quad \text{Lorenz gauge}.
\end{align*}
\] (6.146)

We define the generating functionals \( Z_C[J^a_\mu] \) and \( Z_L[J^a_\mu] \) by the expressions

\[
Z_C[J^a_\mu] \equiv N \int D(A_\mu) \Delta_C[A] \prod_x \delta(\vec{\nabla} \cdot \vec{A}^a) e^{i(S[A] + f d^4x J^a_\mu A^\mu)} ,
\] (6.147)

and

\[
Z_L[J^a_\mu] = N \int D(A_\mu) \Delta_L[A] \prod_x \delta(\partial_\mu A^{\mu a}) e^{i(S[A] + f d^4x J^a_\mu A^\mu)} .
\] (6.148)
Let us derive the relation between them. Following the methods of last section we introduce in \( Z_C[j_\mu^a] \) the identity given by,

\[
1 = \Delta_L[A] \int \mathcal{D}(g) \prod_{x,a} (\partial_\mu g A^{\mu a}) .
\]  

(6.149)

We get then,

\[
Z_C[j_\mu^a] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_C[A] \prod_{x,a} \delta (\vec{\nabla} \cdot \vec{A}_0^a) e^{i(S[A] + \int d^4x j_x^a A^{\mu a})} \Delta_L[A] \int \mathcal{D}(g) \prod_{y,b} \delta (\partial_\mu g A^{\mu a})
\]

\[
= \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \prod_{y,b} \delta (\partial_\mu A^{\mu b}) e^{iS[A] \Delta_C[A]} \int \mathcal{D}(g) \prod_{x,a} \delta (\vec{\nabla} \cdot g^{-1} \vec{A}_0^a) e^{i \int d^4x j_x^a g^{-1} A^{\mu a}}
\]

\[
= \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \prod_{y,b} \delta (\partial_\mu A^{\mu b}) e^{iS[A] \Delta_C[A]} \int \mathcal{D}(g) \prod_{x,a} \delta (\vec{\nabla} \cdot g g^0 \vec{A}_0^a) e^{i \int d^4x j_x^a g g^0 A^{\mu a}}
\]

(6.150)

where \( g^0 \) is the gauge transformation that goes from the gauge \( \partial_\mu A^{\mu a} = 0 \) to the gauge \( \vec{\nabla} \cdot \vec{A}_0^a = 0, \vec{A}_0^a = \delta^a_0 \vec{A} \). It is obtained solving the equation

\[
\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \left[ U(g^0) \vec{A} U^{-1}(g^0) - \frac{i}{g} \vec{\nabla} U(g^0) U^{-1}(g^0) \right] = 0
\]

(6.151)

where \( \partial_\mu A^{\mu a} = 0 \). Due to the factor \( \prod_x \delta (\vec{\nabla} \cdot \vec{A}) \) we are only interested in infinitesimal transformations, and therefore

\[
Z_C[j_\mu^a] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \prod_{y,b} \delta (\partial_\mu A^{\mu b}) e^{iS[A] \Delta_C[A]} e^{i \int d^4x j_x^a g^0 A^{\mu a}}
\]

(6.152)

where we have used, like before, the result

\[
\int \mathcal{D}(g) \prod_{x,a} \delta (\vec{\nabla} \cdot g^0 \vec{A}) = \Delta_C^{-1}[A]
\]

(6.153)

To compare with \( Z_L[j_\mu^a] \) it is necessary to write \( g^0 \vec{A} \) as a function of \( A^\mu \), solving the equation for \( g^0 \). This can be done formally in a series in the potentials \( A^\mu \). We should then have

\[
A_i' = \left( \delta_{ij} - \nabla_i \frac{1}{\sqrt{2}} \nabla_j \right) A_j + O(A_\lambda^2)
\]

(6.154)

If we restrict the Coulomb source to be transverse, \( j^0 = 0 \) and \( \vec{\nabla} \cdot \vec{j} = 0 \), we can then write,

\[
Z_C[j_\mu^a] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \prod_{y,b} \delta (\partial_\mu A^{\mu b}) e^{iS[A] + \int d^4xF_{\mu}^a j_\mu^a}
\]

(6.155)

where \( F^C_\mu[A] = A_\mu^a + O(A_\lambda^2) \). Comparing with the expression for the functional \( Z_L[j_\mu^a] \) we finally get,

\[
Z_C[j_\mu^a] = \exp \left\{ i \int d^4x j_x^a F^{\mu a} \left[ \frac{\delta}{i\delta j_x^a} \right] \right\} Z_L[j_\mu^a]
\]

(6.156)
This is the expression that relates \( Z_C \) with \( Z_L \). As \( F_\mu[A] \) is a complicated functional, we see that the expressions will be different in the two gauges. But what has a physical meaning and can be compared with the experiment are the matrix elements of the renormalized \( S \) matrix. The equivalence theorem that we will prove next shows that these matrix elements are gauge invariant. For simplicity we will make the proof for the \( \lambda \phi^4 \) case, but the reasoning also applies to the gauge theories.

**Theorem 6.2**

If two generating functionals, \( Z \) and \( \tilde{Z} \), differ only by the terms of the external sources, then they will lead to the same renormalized \( S \) matrix.

**Proof:**

Let us consider the generating functional of the Green functions,

\[
Z[J] = \mathcal{N} \int D(\phi) e^{i(S[\phi] + \int d^4 x J \phi)} ,
\]

(6.157)

where

\[
S[\phi] + \int d^4 x J \phi = \int d^4 x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + J \phi \right] .
\]

(6.158)

What happens if we couple the external source to \( \phi + \phi^3 \) instead of coupling to just \( \phi \)?

The generating functional \( \tilde{Z}[j] \) will then be,

\[
\tilde{Z}[j] = \mathcal{N} \int D(\phi) e^{i[S[\phi] + \int d^4 x j(\phi + \phi^3)]} .
\]

(6.159)

We can write \( \tilde{Z}[j] \) in terms of \( Z[J] \) using the usual trick,

\[
\tilde{Z}[j] = \exp \left\{ i \int d^4 x j(x) F \left[ \frac{\delta}{\delta J} \right] \right\} Z[J] ,
\]

(6.160)

where \( F[\phi] = \phi + \phi^3 \). Let us now consider the four-point Green function, \( \tilde{G}(1, 2, 3, 4) \) generated by \( \tilde{Z}[j] \)

\[
\tilde{G}(1, 2, 3, 4) = (-i)^4 \frac{\delta^4 \tilde{Z}[j]}{\delta j(1) \delta j(2) \delta j(3) \delta j(4)} .
\]

(6.161)

A typical diagram that contributes to \( \tilde{G}(1, 2, 3, 4) \) is shown in Fig.6.1, where the part inside the square corresponds to a Green function generated by \( Z[J] \).

Let us consider now the propagators \( \tilde{G}(1, 2) \) and \( G(1, 2) \) generated by \( \tilde{Z}[j] \) and \( Z[J] \) respectively. We get the following expansion of \( \tilde{G}(1, 2) \) in terms of \( G(1, 2) \)

\[
\tilde{G} = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array}
\]

(6.162)
CHAPTER 6. NON-ABELIAN GAUGE THEORIES

If we examine the propagators near the physical mass pole, we get ($Z_2$ and $\tilde{Z}_2$ are the renormalization constants in the schemes)

$$\lim_{p^2 \to m^2_R} \tilde{G} = \frac{i\tilde{Z}_2}{p^2 - m^2_R}; \quad \lim_{p^2 \to m^2_R} G = \frac{iZ_2}{p^2 - m^2_R}. \quad (6.163)$$

Now, if we multiply the previous expansion near the mass pole by $p^2 - m^2_R$ and take the limit $p^2 \to m^2_R$ we get,

$$\tilde{Z}_2 = Z_2 \left[ 1 + 2 + \cdots \right]. \quad (6.164)$$

From here we get

$$\sigma \equiv \left( \frac{\tilde{Z}_2}{Z_2} \right)^{1/2} = 1 + \cdots, \quad (6.165)$$

The unrenormalized $S$ matrix is given by,

$$S_{NR}(k_1, \ldots, k_n) = \prod_{i=1}^{n} \lim_{k^2_i \to m^2_R} (k^2_i - m^2_R)G(k_1, \ldots, k_n), \quad (6.166)$$

for the Green functions obtained through $Z[J]$. We define in the same way,

$$\tilde{S}_{NR}(k_1, \ldots, k_n) = \prod_{i=1}^{n} \lim_{k^2_i \to m^2_R} (k^2_i - m^2_R)\tilde{G}(k_1, \ldots, k_n), \quad (6.167)$$

for the Green functions calculated from $\tilde{Z}[j]$. In these expressions $n$ is the number of external particles. From the argument used to relate $\lim(k^2 - m^2_R)\tilde{G}(k_1, \ldots, k_n)$ with $\lim(k^2 - m^2_R)G(k_1, \ldots, k_n)$ it is easy to see that in relating $\prod \lim(k^2_i - m^2_R)\tilde{G}$ with $\prod \lim(k^2_i - m^2_R)G$ only contribute the diagrams with poles in the variables $k^2_i$.

Therefore we obtain

$$\prod_{i=1}^{n} \lim_{k^2_i \to m^2_R} (k^2_i - m^2_R)G = \left( \frac{Z}{\bar{Z}} \right)^{-\tilde{\sigma}} \prod_{i=1}^{n} \lim_{k^2_i \to m^2_R} (k^2_i - m^2_R)G \quad = \quad \sigma^n \prod_{i=1}^{n} \lim_{k^2_i \to m^2_R} (k^2_i - m^2_R)G. \quad (6.168)$$

From this we get a relation between the unrenormalized $S$ matrices in the two schemes,

$$\tilde{S}_{NR} = \sigma^n S_{NR}, \quad (6.169)$$
which can be written as,
\[
\frac{1}{Z^2} \tilde{S}_{NR}^{(k_1,\ldots,k_n)} = \frac{1}{Z^2} S_{NR}^{(k_1,\ldots,k_n)} .
\] (6.170)

But \( \frac{1}{Z^2} S_{NR}^{(k_1,\ldots,k_n)} \) it is precisely the definition of the renormalized S matrix, so we get
\[
\tilde{S}^R = S^R .
\] (6.171)

We conclude that two generating functionals that only differ by the coupling to the external source lead to the same renormalized S matrix and then to the same physical quantities. This completes the proof of the equivalence theorem.

The application of our result is now clear, because
\[
Z_C[j^a] = \exp \left\{ i \int d^4x j^a F^{\mu a} \left[ \frac{\delta}{i \delta J^a} \right] \right\} Z_L[J^a] ,
\] (6.172)

where \( F^{\mu a}[A] = A^{\mu a} + O(A^{2}) \). The difference between \( Z_C[j^a] \) and \( Z_L[J^a] \) lies in the coupling to the external source, and although the Green functions are in general gauge dependent, the renormalized S matrix is gauge independent and hence physical.

### 6.2.6 Faddeev-Popov ghosts

Having shown the gauge invariance of the renormalized S matrix, let us go back to the generating functional in an arbitrary gauge defined by the gauge condition \( F^{a}[A^b] \). We have seen in section 6.2.4 that this functional in given in the form,
\[
Z_F[J^a] = \mathcal{N} \int \mathcal{D}(A^a) \Delta_F[A] e^{i \int d^4x [\mathcal{L}(x) - \frac{1}{4}(F^a)^2 + J^a A^a]} ,
\] (6.173)

where
\[
\Delta_F[A] = \det \mathcal{M}_F = \det \left( -g \frac{\delta F^a}{\delta \alpha^b} \right) .
\] (6.174)

In this form the Feynman rules are complicated because \( \det \mathcal{M}_F \) will lead to a non-local interaction among the gauge fields. If, in some way, we could exponentiate the determinant and add it to the action, that would solve our problem.

The idea to exponentiate a determinant comes from using Gaussian integrals with Grassmann variables we have (see Appendix),
\[
\int \mathcal{D}(\varphi, \omega) e^{-\int d^4x \varphi M F \varphi} = \det \mathcal{M} .
\] (6.175)

Using this result, and making the change \( \mathcal{M}_F \rightarrow i \mathcal{M}_F \) (an irrelevant change in the normalization), we get
\[
Z_F[J^a] = \mathcal{N} \int \mathcal{D}(A^a, \varphi, \omega) e^{i \int d^4x [\mathcal{L}_{eff} + J^a A^a]} ,
\] (6.176)
where \( \varpi \) and \( \omega \) are anti-commutative scalar fields and \( \mathcal{L}_{\text{eff}} \) is given by,

\[
\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_G ,
\]
with

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} ,
\]

\[
\mathcal{L}_{GF} = -\frac{1}{2 \xi} (F^a)^2 ,
\]

\[
\mathcal{L}_G = -\varpi^a M^{ab}_{\mu} \omega^b .
\]

The fields \( \omega \) and \( \varpi \) are auxiliary fields and are called Faddeev-Popov ghosts. The name comes from the wrong spin-statistics connection, but there is no problem with this, as they are not physical fields.

Let us now evaluate more explicitly the ghost part of the Lagrangian,

\[
\mathcal{M}^{ab}_F(x, y) = -\frac{g}{\delta} \frac{\delta F^a(x)}{\delta \alpha^g(y)} = \frac{\delta F^a[A(x)]}{\delta A^a_\mu(y)} D^{\mu}_\rho \omega^b(y) ,
\]
where we have used,

\[
\delta A^a_\mu(y) = -\frac{1}{g} D^{\mu}_\rho \alpha^b .
\]

We get

\[
\int d^4x d^4y \varpi^a(x) \mathcal{M}^{ab}_F(x, y) \omega^b(y) = \int d^4x \int d^4y \varpi^a(x) \frac{\delta F^a(x)}{\delta A^a_\mu(y)} D^{\mu}_\rho \omega^b(y) ,
\]
or

\[
\mathcal{L}_G(x) = -\int d^4y \varpi^a(x) \frac{\delta F^a(x)}{\delta A^a_\mu(y)} D^{\mu}_\rho \omega^b(y) .
\]

To have a more explicit form we have to specify the gauge. In the Lorenz gauge \( F^a = \partial_\mu A^{\mu a} \) and therefore

\[
\mathcal{L}_G(x) = -\int d^4y \varpi^a(x) \partial_\mu [\delta^4(x - y)] D^{\mu}_{\rho} \omega^b(y) = \partial^\mu \varpi^a(x) D^{\mu}_{\rho} \omega^b(x) .
\]

In the last step we have used integration by parts and that the covariant derivative in the adjoint representation where the ghosts, like the gauge fields, are is given by,

\[
D^{ab}_\mu = \partial_\mu \delta^{ab} - g f^{abc} A^c_\mu .
\]

### 6.2.7 Feynman rules in the Lorenz gauge

We are now ready to write the Feynman rules that will enable us to evaluate, in perturbation theory any process in a theory that can be described by an non-abelian gauge theory. All the work done so far just lead us to an effective Lagrangian with which we can obtain
6.2. QUANTIZATION

the Feynman rules as if it was a normal theory without the problem of the mismatch of the degrees of freedom. Our effective Lagrangian is then

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{G},$$

where

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a}; \quad F_{\mu\nu}^{a} = \partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a} + g f^{bca} A_{\mu}^{b} A_{\nu}^{c},$$

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} (F_{a})^{2},$$

$$\mathcal{L}_{G} = -\varpi^{a} \int d^{4} y \frac{\delta F^{a}}{\delta A_{\mu}^{b}} D_{\mu}^{bc} \omega_{c}. \quad (6.185)$$

The structure constants $f^{abc}$ are defined by the commutation of the generators of the group $G$. Our conventions are,

$$[t^{a}, t^{b}] = if^{abc} t^{c}; \quad Tr(t^{a} t^{b}) = \frac{1}{2} \delta^{ab}. \quad (6.186)$$

To fix things, let us consider the Lorenz gauge, defined by

$$F_{a}^{\mu} A_{\mu}^{a} = 0. \quad (6.187)$$

We get then

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a} - \frac{1}{2\xi} (\partial_{\mu} A_{\mu}^{a})^{2} + \partial_{\mu} \varpi^{a} D_{\mu}^{bc} \omega^{b}. \quad (6.188)$$

Using the fact that the ghosts are in the adjoint representation of the group, we get

$$(D_{\mu} \omega)^{a} = \left(\partial_{\mu} \delta^{ab} - ig A_{\mu}^{c} (T^{c})^{ab}\right) \omega^{b}, \quad (6.189)$$

with

$$(T^{c})^{ab} \equiv -if^{bca} = -if^{abc}. \quad (6.190)$$

Therefore we have

$$D_{\mu}^{ab} \omega^{b} = (\partial_{\mu} \delta^{ab} - gf^{abc} A_{\mu}^{c}) \omega^{b}. \quad (6.191)$$

We can therefore separate the Lagrangian in kinetic and interaction parts

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}, \quad (6.192)$$

where

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} (\partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a})^{2} - \frac{1}{2\xi} (\partial_{\mu} A_{\mu}^{a})^{2} + \partial_{\mu} \varpi^{a} \partial^{\mu} \omega^{a}$$

$$= \frac{1}{2} A^{\mu a} \left[ \Box g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_{\mu} \partial_{\nu}\right] \delta^{ab} A_{\nu}^{b} - \varpi^{a} \partial^{a} \omega^{b}, \quad (6.193)$$

where we have done integrations by parts and neglected total divergences. For the interaction Lagrangian we have

$$\mathcal{L}_{\text{int}} = -gf^{abc} \partial_{\mu} A_{\nu}^{a} A_{\nu}^{b} A_{\nu}^{c} - \frac{1}{4} g^{2} f^{ade} f^{bce} A_{\mu}^{a} A_{\nu}^{d} A_{\nu}^{c} + gf^{abc} \partial_{\mu} \varpi^{a} A_{\mu}^{b} \omega^{c}. \quad (6.194)$$

We are now in position to get the Feynman rules for a non-abelian gauge theory. We get with the usual conventions,
Propagators

i) Gauge fields

\[ \frac{g^{\mu\nu}}{k^2 + i\epsilon} - (1 - \xi) \frac{k^{\mu}k^{\nu}}{(k^2 + i\epsilon)^2} \]  

(6.195)

ii) Ghosts

\[ \frac{i}{k^2 + i\epsilon} \delta_{ab} \]  

(6.196)

Vertices

i) Gauge bosons triple vertex

\[ -gf^{abc} \left( g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(p_2 - p_3)^\mu ight) 
+ g^{\rho\mu}(p_3 - p_1)^\nu \]  

(6.197)

ii) Gauge bosons quartic vertex

\[ -ig^2 \left( f^{\sigma\mu\rho}g_{\nu\sigma} - g_{\mu\rho}g_{\nu\sigma} \right) 
+ f^{\sigma\rho\mu}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} \]  

(6.198)

iii) Gauge boson-Ghost interaction

\[ gf^{abc}p_1^\mu \]  

(6.199)

Notes:

1. The dot in the vertex of the ghosts with the gauge fields corresponds to the leg where the derivative applies. This corresponds to the line exiting the diagram (the ghost lines are oriented as they have ghost number, as we will see)
2. The other rules are as usual not forgetting the minus sign for the ghost loop because of their anti-commuting character.

### 6.2.8 Feynman rules for the interaction with matter

We have just seen the Feynman rules for the pure gauge theory, without interaction with matter. This already non trivial due to the non-abelian character. Interaction with matter is done in the usual way, changing normal derivatives into covariant derivatives. In general matter is described by scalar fields, as

\[ \phi_i \quad ; \quad i = 1, \ldots, M \]  

(6.200)

We consider also fermion fields

\[ \psi_j \quad ; \quad j = 1, \ldots, N \]  

(6.201)

They belong to the representations of dimension \( M \) and \( N \), respectively. The Lagrangian will then be

\[
\mathcal{L}_{\text{matter}} = (D_\mu \phi)^\dagger D^\mu \phi - m_\phi^2 \phi \phi^\dagger - V(\phi) + i\bar{\psi}D^\mu \gamma_\mu \psi - m_\psi \bar{\psi} \psi \equiv \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}.
\]  

(6.202)

The interaction Lagrangian between matter and the gauge fields is easily obtained from the covariant derivative

\[
D^\mu_{ij} = \partial^\mu \delta_{ij} - igA^a_\mu T^a_{ij}
\]

(6.203)

where \( T^a_{ij} \) are the generators in the representations appropriate for the matter fields \( \phi \) and \( \psi \).

We get then,

\[
\mathcal{L}_{\text{int}} = ig\phi_1^\dagger (\not{\partial} - \not{A}) \phi_j T^a_{ij} A^a_\mu + g^2 \phi_1^\dagger T^a_{ij} T^b_{jk} \phi_k A^a_\mu A^{\mu b} + g\bar{\psi}^i \gamma^\mu \psi_j T^a_{ij} A^a_\mu
\]  

(6.204)

This leads to the following vertices:

**Triple Vertices**

\[ ig(\gamma^\mu)_{\beta a} T^a_{ij} \]  

(6.205)
$i g (p_1 - p_2)^\mu T^a_{ij}$  \hfill (6.206)

Quartic Vertex

$i g^2 g_{\mu\nu} \{T^a, T^b\}_{ij}$  \hfill (6.207)

**Group Factors**

The factors $f^{abc}$ and $T^a_{ij}$ that appear in the Feynman rules do not need in fact to be known. In the calculations in the end will appear combinations of those that can expressed in terms of invariant quantities that characterize the group and the representation. Our generators are hermitian ($T^a \dagger = T^a$) and satisfy the normalization conditions,

\[
[T^a, T^b] = i f^{abc} T^c \\
Tr(T^a T^b) = \delta^{ab} T(R)
\]  \hfill (6.208)

In these definitions $T(R)$ is a number that characterizes the representation $R$. Other frequently used quantity is the Casimir of the representation, defined by

\[
\sum_{a,k} T^a_{ik} T^a_{kj} = Tr[T^a T^a] = \delta_{ij} C_2(R)
\]  \hfill (6.209)

For the adjoint representation we get

\[
f^{acd} f^{bed} = \delta^{ab} C_2(G).
\]  \hfill (6.210)

$T(R)$ and $C_2(R)$ are not independent, they obey the relation

\[
T(R) r = d(R) C_2(R)
\]  \hfill (6.211)

where $r$ is the dimension of the group $G$ (number of generators) and $d(R)$ is the dimension of the representation $R$. For the adjoint representation we have

\[
d(\text{Adjoint}) = r
\]
6.3. WARD IDENTITIES

In many applications we are interested in the \( SU(N) \) family of Lie groups. For these we have the results

\[
\begin{align*}
 r &= N^2 - 1 ; \quad d(N) = N ; \quad d(adj) \equiv d(G) = r \\
 T(N) &= \frac{1}{2} ; \quad C_2(N) = \frac{N^2 - 1}{2N} \\
 T(G) &= C_2(G) = N
\end{align*}
\]

### Symmetry Factors

For the calculation of some diagrams there appear symmetry factors different from one. They were discussed before, in section 5.5.5, but for completeness we recall their definition here. The symmetry factor is given by the \# of different ways in which the lines can be connected with the same final diagram, divided by the permutations factors for the vertices involved and by the permutation factors for the number of equal vertices.

6.3 Ward Identities

### 6.3.1 BRS transformation

We are now going to study the Ward identities\(^2\) for the non-abelian gauge theories. The more convenient method is that of the Becchi, Rouet e Stora (BRS) transformations. The BRS transformations are a generalization of the gauge transformations that make invariant the effective action.

As we saw for non-abelian gauge theories the effective action is given by \((A = \text{gauge field}, \phi = \text{matter field})\)

\[
S_{\text{eff}}[A, \phi] = S[A, \phi] - \frac{1}{2\xi} \int d^4x F_a^2[A, \phi] - \int d^4x \bar{\psi} \cdot \mathcal{M}_{ab} \cdot \psi^b
\]

where \(S[A, \phi]\) is the classical action, invariant under the (infinitesimal) gauge transformations,

\[
\delta A^a_\mu = -\frac{1}{g} D^a_\mu \alpha^b, \\
\delta \phi_i = -i(T^a)_{ij} \phi_j \alpha^a,
\]

and where \(F_a[A, \phi]\) are the gauge conditions and the operator \(\mathcal{M}_{ab}\) is such that

\[
\mathcal{M}_{ab} \omega^b = \frac{\delta F_a}{\delta A^a_\mu} D^b_\mu \omega^b + \frac{\delta F_a}{\delta \phi_i} i g (T^b)_{ij} \phi_j \omega^b.
\]

The effective action, \(S_{\text{eff}}\), is not invariant under gauge transformations due to the non-invariance of the gauge fixing term and of the ghost Lagrangian. This non-invariance can

\(^2\)We use here the generic name of Ward identities for the more general identities for non-abelian gauge theories, discovered by Ward, Takahashi, Slavnov and Taylor.
disappear if we choose appropriate transformations for the ghosts in order to compensate the non-invariance of $\int d^4x F^2_a$. These transformations, known as BRS transformations, are defined by,

$$\begin{align}
\delta_{\text{BRS}} A^a_{\mu} &= D^b_\mu \omega^b \theta \\
\delta_{\text{BRS}} \phi_i &= ig (T^b)_{ij} \phi_j \omega^b \theta \\
\delta_{\text{BRS}} \omega^a &= \frac{1}{2} F^a_{\mu} [A, \phi] \theta \\
\delta_{\text{BRS}} \omega^a &= \frac{1}{2} g f^{abc} \omega^b \omega^c \theta
\end{align}$$

(6.218)

where $\theta$ is an anti-commuting parameter independent of the space-time point (Grassmann variable).

We see that for the fields $A^a_{\mu}$ and $\phi_i$ the BRS transformations are gauge transformations with parameter $\alpha^a(x) = -g \omega^a(x) \theta$. Notice the anti-commuting character of $\theta$ is necessary for the product $\omega^a \theta$ to have a bosonic (commutative) character. To show the invariance of $S_{\text{eff}}[A, \phi]$ we are going to prove a series of theorems needed for the general proof. Before we do that it is convenient to introduce the Slavnov operator, $s$, defined by the relations,

$$\begin{align}
\delta_{\text{BRS}} A^a_{\mu} &= s A^a_{\mu} \theta \\
\delta_{\text{BRS}} \phi_i &= s \phi_i \theta \\
\delta_{\text{BRS}} \omega^a &= s \omega^a \theta \\
\delta_{\text{BRS}} \omega^a &= s \omega^a \theta
\end{align}$$

(6.219)

This operator is distributive with respect to multiplication (like a derivative) and obeys the following relations,

$$\begin{align}
s(B_1 B_2) &= s B_1 B_2 + B_1 s B_2 \\
s(F_1 B_2) &= s F_1 B_2 + F_1 s B_2 \\
s(B_1 F_2) &= -s B_1 F_2 + B_1 s F_2 \\
s(F_1 F_2) &= -s F_1 F_2 + F_1 s F_2
\end{align}$$

(6.220)

**Theorem 6.3**

The Slavnov operator $s$ is nilpotent in the fields $A^a_{\mu}$, $\phi_i$, $\omega^a$, that is $s^2 A^a_{\mu} = s^2 \phi_i = s^2 \omega^a = 0$.

**Proof:** We show for each case. We have

1. $s^2 A^a_{\mu} = 0$

$$s^2 A^a_{\mu} = s (D^b_\mu \omega^b) = -\frac{\delta D^b_\mu}{\delta A^c_{\mu}} s A^c_{\mu} \omega^b + D^b_\mu s \omega^b$$

$$= -\delta^b_\mu (-g f^{abc}) D^d_\mu \omega^d \omega^b + \frac{1}{2} g f^{bcd} D^b_\mu (\omega^c \omega^d)$$

$$= \left[ g f^{abc} \partial_\mu \omega^c \omega^b + \frac{1}{2} g f^{acd} \partial_\mu \omega^c \omega^d + \frac{1}{2} g f^{abc} \omega^c \partial_\mu \omega^d \right]$$

$$+ \left[ g f^{abc} (-g) f^{cde} A^e_{\mu} \omega^d \omega^b + \frac{1}{2} g (-g) f^{bcd} f^{abe} A^e_{\mu} \omega^c \omega^d \right]$$

2. $s^2 \phi_i = 0$

$$s^2 \phi_i = s (i g (T^b)_{ij} \phi_j \omega^b \theta)$$

$$= i g (T^b)_{ij} (D^b_\mu \phi_j) \omega^b \theta + i g (T^b)_{ij} \phi_j (D^b_\mu \omega^b) \theta$$

$$= i g (T^b)_{ij} \partial_\mu \phi_j \omega^b \theta + i g (T^b)_{ij} \phi_j \partial_\mu \omega^b \theta$$

$$= i g (T^b)_{ij} \partial_\mu \phi_j \omega^b \theta + i g (T^b)_{ij} \phi_j \partial_\mu \omega^b \theta$$

$$= 0$$

3. $s^2 \omega^a = 0$

$$s^2 \omega^a = s (1/2 g f^{abc} \omega^b \omega^c \theta)$$

$$= (1/2) g f^{abc} \theta (D^b_\mu \omega^c) \omega^b + (1/2) g f^{abc} \theta \omega^b \partial_\mu \omega^c$$

$$= (1/2) g f^{abc} \theta (D^b_\mu \omega^c) \omega^b + (1/2) g f^{abc} \theta \omega^b \partial_\mu \omega^c$$

$$= 0$$

**Proof:** We show for each case. We have

1. $s^2 A^a_{\mu} = 0$

$$s^2 A^a_{\mu} = s (D^b_\mu \omega^b) = -\frac{\delta D^b_\mu}{\delta A^c_{\mu}} s A^c_{\mu} \omega^b + D^b_\mu s \omega^b$$

$$= -\delta^b_\mu (-g f^{abc}) D^d_\mu \omega^d \omega^b + \frac{1}{2} g f^{bcd} D^b_\mu (\omega^c \omega^d)$$

$$= \left[ g f^{abc} \partial_\mu \omega^c \omega^b + \frac{1}{2} g f^{acd} \partial_\mu \omega^c \omega^d + \frac{1}{2} g f^{abc} \omega^c \partial_\mu \omega^d \right]$$

$$+ \left[ g f^{abc} (-g) f^{cde} A^e_{\mu} \omega^d \omega^b + \frac{1}{2} g (-g) f^{bcd} f^{abe} A^e_{\mu} \omega^c \omega^d \right]$$

2. $s^2 \phi_i = 0$

$$s^2 \phi_i = s (i g (T^b)_{ij} \phi_j \omega^b \theta)$$

$$= i g (T^b)_{ij} (D^b_\mu \phi_j) \omega^b \theta + i g (T^b)_{ij} \phi_j (D^b_\mu \omega^b) \theta$$

$$= i g (T^b)_{ij} \partial_\mu \phi_j \omega^b \theta + i g (T^b)_{ij} \phi_j \partial_\mu \omega^b \theta$$

$$= (1/2) g f^{abc} \theta (D^b_\mu \omega^c) \omega^b + (1/2) g f^{abc} \theta \omega^b \partial_\mu \omega^c$$

$$= 0$$

3. $s^2 \omega^a = 0$

$$s^2 \omega^a = s (1/2 g f^{abc} \omega^b \omega^c \theta)$$

$$= (1/2) g f^{abc} \theta (D^b_\mu \omega^c) \omega^b + (1/2) g f^{abc} \theta \omega^b \partial_\mu \omega^c$$

$$= 0$$
\[(gf^{abc}\partial_{\mu}\omega^{c}\omega^{b} - gf^{abc}\partial_{\mu}\omega^{c}\omega^{b}) - \frac{1}{2}g^{2}(f^{abc}f^{cde} - f^{ade}f^{cbe} + f^{edb}f^{ace})A_{\mu}^{c}\omega^{d}\omega^{b} = 0 \] (6.221)

b) \(s^{2}\phi_{i} = 0\)

\[s^{2}\phi_{i} = s[ig(T^{a})_{ij}\phi_{j}\omega^{a}] = -ig(T^{a})_{ij}s\phi_{j}\omega^{a} + ig(T^{a})_{ij}\phi_{j}s\omega^{a}\]

\[= g^{2}(T^{a})_{ij}(T^{b})_{jk}\phi_{k}\omega^{b}\omega^{a} + ig(T^{a})_{ij}\phi_{j}\frac{1}{2}gf^{abc}\omega^{b}\omega^{c}\]

\[= \frac{1}{2}g^{2}(T^{a}, T^{b})_{ik}\omega^{b}\omega^{c}\phi_{k} + \frac{i}{2}g^{2}(T^{a})_{ij}\phi_{j}f^{abc}\omega^{b}\omega^{c}\]

\[= \frac{i}{2}g^{2}(T^{a})_{ij}\phi_{j}(f^{acb} + f^{abc})\omega^{b}\omega^{c}\]

\[= 0\] (6.222)

c) \(s^{2}\omega^{a} = 0\)

\[s^{2}\omega^{a} = s\left(\frac{1}{2}gf^{abc}\omega^{b}\omega^{c}\right)\]

\[= -\frac{1}{2}gf^{abc}s\omega^{b}\omega^{c} + \frac{1}{2}gf^{abc}s\omega^{b}\omega^{c}\]

\[= -gf^{abc}s\omega^{b}\omega^{c}\]

\[= -\frac{1}{2}g^{2}f^{abc}f^{bef}\omega^{e}\omega^{c}\]

\[= -\frac{1}{6}g^{2}(f^{abc}f^{bef} + f^{abe}f^{bfc} + f^{abf}f^{bce})\omega^{e}\omega^{a}\omega^{c}\]

\[= 0\] (6.223)

where we have used the anti-commutation of the ghost fields and the Jacobi identity. The theorem is then proved.

For linear gauge fixing, we can show an important result that we also give in the form of a theorem.

**Theorem 6.4**

For linear gauge fixings, the Slavnov operator verifies the relation

\[s(M_{ab}\omega^{b}) = 0\] (6.224)
Proof:
We saw before that
\[ M_{ab} \omega^b(x) = \int d^4y \left[ \frac{\delta F_a(x)}{\delta A^c_{\mu}(y)} s A^c_{\mu}(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s \phi_i(y) \right] \]  
(6.225)

If we use the definitions of \( \delta_{\text{BRS}} \) and of the Slavnov operator, we can write
\[ M_{ab} \omega^b(x) = \int d^4y \left[ \frac{\delta F_a(x)}{\delta A^c_{\mu}(y)} s A^c_{\mu}(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s \phi_i(y) \right] . \]  
(6.226)

If the gauge fixing is linear \( \frac{4F_a}{\delta A^c_{\mu}} \) and \( \frac{4F_a}{\delta \phi_i} \) do not depend on the on the fields and then
\[ s \left[ M_{ab} \omega^b(x) \right] = \int d^4y \left[ \frac{\delta F_a(x)}{\delta A^c_{\mu}(y)} s^2 A^c_{\mu}(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s^2 \phi_i(y) \right] = 0 . \]  
(6.227)

where we have used the previous results. This proves the theorem.

Using these results we can then show that the effective action is invariant under BRS transformations. We are going to show this result also in the form of a theorem.

**Theorem 6.5**
The action \( S_{\text{eff}} \) is invariant under BRS transformations.

**Proof:**
The effective action is
\[ S_{\text{eff}}[A, \phi] = S[A, \phi] + \int d^4x \left[ -\frac{1}{2\xi} F^2_a[A, \phi] - \overline{\omega} \mathcal{M}_{ab} \omega^b \right] . \]  
(6.228)

As the classical action is invariant under gauge transformations we should have
\[ s \left( S[A, \phi] \right) = 0 . \]  
(6.229)

For the other terms we have
\[ s \left( \frac{1}{2\xi} F^2_a - \overline{\omega} \mathcal{M}_{ab} \omega^b \right) = -\frac{1}{\xi} F_a s F_a + s \overline{\omega} \mathcal{M}_{ab} \omega^b - \overline{\omega} s (\mathcal{M}_{ab} \omega^b) \]  
(6.230)

But
\[ s F_a(x) = \int d^4y \left[ \frac{\delta F_a}{\delta A^c_{\mu}(y)} s A^c_{\mu}(y) + \frac{\delta F_a}{\delta \phi_i(y)} s \phi_i(y) \right] = M_{ab} \omega^b(x) , \]  
(6.231)

and using one of the previous theorems we get
\[ s (M_{ab} \omega^b) = 0 . \]  
(6.232)
Therefore
\[
\begin{align*}
    s \left( - \frac{1}{2 \xi} F_a^2 - \bar{\omega}^a \mathcal{M}_{ab} \omega^b \right) &= \left( - \frac{1}{\xi} F_a + s \bar{\omega}^a \right) \mathcal{M}_{ab} \omega^b = 0 ,
\end{align*}
\] (6.233)

where we have used the fact that \( s \bar{\omega}^a = \frac{1}{\xi} F_a \). Putting everything together we get,
\[
    s S_{\text{eff}} [A, \phi] = 0 .
\] (6.234)

For the applications we still need another result on the invariance of the integration measure that we also present as a theorem.

**Theorem 6.6**

The measure \( \mathcal{D}(A_\mu, \phi_i, \bar{\omega}, \omega^b) \) is invariant under BRS transformations.

**Proof:**

Simple calculations lead to the following relations
\[
\begin{align*}
    \frac{\delta (sA^a_\mu)}{\delta A^a_\mu} &= -gf^{aba} \delta \mu \omega^b = 0 \\
    \frac{\delta (s\phi_i)}{\delta \phi_i} &= ig(T^a)_{ii} \omega^a = 0 ; \quad (\text{Tr}(T^a) = 0) \\
    \frac{\delta (s\omega^a)}{\delta \omega^a} &= gf^{abc} \omega^c = 0 \\
    \frac{\delta (s \bar{\omega}^a)}{\delta \bar{\omega}^a} &= 0
\end{align*}
\] (6.235)

As we saw in the last chapter these relations imply that the integration measure is invariant proving the theorem.

### 6.3.2 Ward-Takahashi-Slavnov-Taylor identities

We are now in position to derive the generalization of the Ward-Takahashi identities to non-abelian gauge theories. This extension was done, among others, by Slavnov and Taylor, but frequently we use the short designation of Ward identities even for the non-abelian case. In a generic form Ward identities are relations among the Green functions that result from the gauge symmetry of the theory. As we discuss the more convenient way to describe them is through the generating functionals of the Green functions. Let us then consider a non-abelian gauge theory. For simplicity we just consider that the matter fields are scalars \( \phi_i \). Fermions can be introduced later easily. The generating functional is then
\[
    Z[J^a_\mu, J_i, \eta^a, \overline{\eta}^i] = \int \mathcal{D}(A_\mu, \phi_i, \bar{\omega}, \omega) e^{\int d^4 x \left[ L_{\text{eff}} + J^a_\mu A^{\mu a} + J_i \phi_i + \bar{\eta}^a \omega^a + \overline{\eta}^i \right]} \] (6.236)

where we have also introduced sources for the ghosts.

A BRS transformation is a change of variables in the integral. The value of the integral should not be changed by this. As \( S_{\text{eff}} \) and the measure are invariant we should have the following theorem:
Theorem 6.7
Given any Green function
\[ G(x_1, \ldots, y_1, \ldots, z_1, \ldots, w_1, \ldots) = \langle 0 | T A^a_{\mu_1}(x_1) \cdots \phi_{i_1}(y_1) \cdots \bar{\omega}^a(z_1) \cdots \omega^b(w_1) \cdots | 0 \rangle \] (6.237)
we have the following relations:

i) \( s \langle 0 | T A^a_{\mu}(x_1) \cdots \phi_{i_1}(y_1) \cdots \bar{\omega}^a(z_1) \cdots \omega^b(w_1) | 0 \rangle = 0 \)

ii) \( 0 = \langle 0 | T s A^a_{\mu}(x_1) \cdots | 0 \rangle + \cdots + \langle 0 | T s \phi_{i_1} \cdots | 0 \rangle + \cdots + \langle 0 | T \cdots s \bar{\omega}^a \cdots | 0 \rangle + \langle 0 | T \cdots s \omega^a \cdots | 0 \rangle \) (6.238)

Proof: The proof is clear if we write
\[ \langle 0 | T A^a_{\mu}(x_1) \cdots \phi_{i_1}(y_1) \cdots \bar{\omega}^a(z_1) \cdots \omega^b(w_1) | 0 \rangle = \int D(A_{\mu}, \phi, \bar{\omega}, \omega) A^a_{\mu}(x_1) \cdots \phi_{i_1}(y_1) \cdots \bar{\omega}^a(z_1) \cdots \omega^b(w_1) e^{iS_{\text{eff}}} . \] (6.239)
Then the BRS transformation should leave the integral invariant proving the first relation. Then the second relation results from the first and from the invariance of the measure and of the effective action.

This theorem constitutes a quick way to establish relations among Green functions for particular cases, as we shall see below. However to establish general results for the renormalization and gauge invariance of the S matrix, we are interested in the Ward identities expressed in terms of the generating functionals. Using the invariance of the integral for a change of variables, the invariance of the measure \( D \) and of \( S_{\text{eff}} \), we get for the Ward identity for the generating functional \( Z \),
\[ 0 = \int D(A_{\mu}, \phi, \bar{\omega}, \omega) \int d^4 x (J^\mu a s A^a_{\mu} + J_i s \phi_i + \eta^a s \omega^a - \bar{\omega}^a \eta^a) e^{i(S_{\text{eff}} + \text{sources})} . \] (6.240)
The more useful Ward identities are for the functional \( \Gamma \). The previous expression can not directly lead to \( \Gamma \) functional, because \( s A^a_{\mu}, s \phi_i \) and \( s \omega^a \) are not linear in the fields. To solve this problem we introduce sources for these non-linear operators. We generalize the effective action defining a new quantity \( \Sigma \) such that
\[ \Sigma[A_{\mu}^a, \phi_i, \bar{\omega}^a, \omega^a, K_{\mu}, K_i, L^a] \equiv S_{\text{eff}}[A_{\mu}^a, \phi_i, \bar{\omega}^a, \omega^a] + \int d^4 x (K^{a\mu} s A^a_{\mu} + K^i s \phi_i + L^a s \omega^a) , \] (6.241)
where \( K^{a\mu}, K_i \) and \( L^a \) are sources for the non-linear operators \( s A^a_{\mu}, s \phi_i \) and \( s \omega^a \) respectively. Using the previous theorems it is easy to show that \( \Sigma \) is invariant under BRS transformations, that is
\[ s \Sigma = 0 . \] (6.242)
Let us consider now the generating functional for the Green functions in the presence of the sources $J_{\mu}^a, J_i, \eta^a, \overline{\eta}, K_{\mu}^a, K_i$ and $L^a$, that is

$$Z[J_{\mu}^a, J_i, \eta, \overline{\eta}, K_{\mu}, K_i, L] = \int \mathcal{D}(A_{\mu}, \phi_i, \omega) e^{i \left[ \Sigma + \int d^4x (J_{\mu}^a A^{a\mu} + J_i \phi_i + \eta \omega^a - s\Sigma - \eta \overline{\omega}^a) \right]} .$$

We can now repeat the previous reasoning for invariance under BRS transformations. Like before we get (recall that $s\Sigma = 0$)

$$0 = \int \mathcal{D}(\cdots) \int d^4x [J_{\mu}^a s A^{a\mu} + J_i s \phi_i + \eta^a s \omega^a - s\Sigma + s\overline{\omega}^a] e^{i \Sigma + \text{sources}} ,$$

only now we have composite operators $s A, s \phi$ e $s \omega$, that is

$$s A_{\mu}^a = \frac{\delta \Sigma}{\delta K^{\mu a}} , \quad s \phi_i = \frac{\delta \Sigma}{\delta K_i} , \quad s \omega^a = \frac{\delta \Sigma}{\delta L^a} , \quad s \overline{\omega}^a = \frac{1}{\xi} F^a$$

We get then

$$\int \mathcal{D}(\cdots) \int d^4x \left[ J_{\mu}^a \frac{\delta \Sigma}{\delta K^{\mu a}} + J_i \frac{\delta \Sigma}{\delta K_i} + \overline{\eta} \frac{\delta \Sigma}{\delta L} - \frac{1}{\xi} F^a \eta^a \right] e^{i \Sigma + \text{sources}} = 0$$

or in another form

$$\int d^4x \left[ J_{\mu}^a \frac{\delta}{\delta K^{\mu a}} + J_i \frac{\delta}{\delta K_i} + \overline{\eta} \frac{\delta}{\delta L} - \frac{1}{\xi} F^a \eta^a \right] e^{i W[J_{\mu}^a, J_i, \eta, \overline{\eta}, K_{\mu}, K_i, L]} = 0 .$$

For a linear gauge condition all the differential operators in the square bracket are of first order and therefore we can write

$$\int d^4x \left[ J_{\mu}^a \frac{\delta}{\delta K^{\mu a}} + J_i \frac{\delta}{\delta K_i} + \overline{\eta} \frac{\delta}{\delta L} - \frac{1}{\xi} F^a \eta^a \right] W = 0 .$$

This is the expression of the Ward identities for the generating functional of the connected Green functions, $W$. Normally the Ward identities are more useful for generating functional of the irreducible Green functions. This defined by,

$$\Gamma[A_{\mu}, \phi_i, \overline{\eta}, \omega, K_{\mu}, K_i, L] \equiv W[J_{\mu}, J_i, \eta, \overline{\eta}, K_{\mu}, K_i, L] - \int d^4x [J_{\mu}^a A^{a\mu} + J_i \phi_i + \eta \omega + \overline{\eta} \overline{\omega}]$$

with the usual relations

$$\phi_i = \frac{\delta W}{\delta J_i} , \quad \omega^a = - \frac{\delta W}{\delta \eta^a} , \quad A_{\mu}^a = \frac{\delta W}{\delta J_{\mu}^a} , \quad \overline{\omega}^a = \frac{\delta W}{\delta \overline{\eta}^a} ,$$

as usual as the inverse relations,

$$J_i = - \frac{\delta \Gamma}{\delta \phi_i} , \quad \eta^a = \frac{\delta \Gamma}{\delta \omega^a} , \quad J_{\mu}^a = - \frac{\delta \Gamma}{\delta A_{\mu}^a} , \quad \overline{\eta} = - \frac{\delta \Gamma}{\delta \overline{\omega}}.$$
As the Legendre transform leaves the sources \( K_\mu^a, K_i \) and \( L^a \) unchanged, we should have:

\[
\frac{\delta W}{\delta K_\mu^a} = \frac{\delta \Gamma}{\delta K_\mu^a} ; \quad \frac{\delta W}{\delta K_i} = \frac{\delta \Gamma}{\delta K_i} ; \quad \frac{\delta W}{\delta L^a} = \frac{\delta \Gamma}{\delta L^a} \tag{6.252}
\]

We then get easily

\[
\int d^4x \left[ \frac{\delta \Gamma}{\delta K_\mu^a(x)} \frac{\delta \Gamma}{\delta A_\mu^a(y)} + \frac{\delta \Gamma}{\delta K_i(x)} \frac{\delta \Gamma}{\delta \phi_i(y)} - \frac{\delta \Gamma}{\delta L^a(x)} \frac{\delta \omega^a(y)}{\delta \omega^a(x)} - \frac{1}{\xi} F^a_{\mu \nu} \frac{\delta \Gamma}{\delta \omega^a(x)} \right] = 0 \tag{6.253}
\]

This is the generating functional for the Ward identities for the irreducible Green functions. The Ward identities for specific Green functions are obtained by taking appropriate functional derivatives of the fields.

In the applications the previous equation is used in connection with another functional identity, the equation of motion (or Dyson-Schwinger) for the ghosts. This can be easily obtained doing the following change of variables in the functional integral,

\[
\delta A_\mu^a = \delta \phi_i = \delta \omega^a = 0 \quad \delta \omega^a = f^a = \text{infinitesimal constant} \tag{6.254}
\]

Then

\[
\delta Z = 0 = \int \mathcal{D}(\cdots) \left( i \frac{\delta \Sigma}{\delta \omega^a} + i \eta^a \right) f^a e^{i(\Sigma + \text{sources})} \tag{6.255}
\]

but

\[
\frac{\delta \Sigma}{\delta \omega^a(x)} = -\mathcal{M}_{ab} \omega^b(x) = -sF^a(x) \quad \frac{\delta \Sigma}{\delta \omega^a(y)} = -\mathcal{M}_{ab} \omega^b(y) = -sF^a(y) \tag{6.256}
\]

We therefore obtain

\[
0 = \int \mathcal{D}(\cdots) \left\{ -i \int d^4y \left[ \frac{\delta F^a(x)}{\delta A_\mu^b(y)} \frac{\delta \Sigma}{\delta A_\mu^b(y)} + \frac{\delta F^a(x)}{\delta \phi_i(y)} \frac{\delta \Sigma}{\delta \phi_i(y)} \right] + i \eta^a(x) \right\} e^{i(\Sigma + \text{sources})} \tag{6.257}
\]

Using now

\[
\eta^a = -\frac{\delta \Gamma}{\delta \omega^a} \tag{6.258}
\]

we finally get (for linear gauges)

\[
\int d^4y \left[ \frac{\delta F^a(x)}{\delta A_\mu^b(y)} \frac{\delta \Gamma}{\delta K^{ab}(y)} + \frac{\delta F^a(x)}{\delta \phi_i(y)} \frac{\delta \Gamma}{\delta K_i(y)} \right] = -\frac{\delta \Gamma}{\delta \omega^a(x)} \tag{6.259}
\]

which the generating functional for the Dyson-Schwinger equations for the ghosts.
6.3. WARD IDENTITIES

6.3.3 Example: Transversality of vacuum polarization

We are going to give an example of the application of the Ward identities. For this we will show that the vacuum polarization is transversal. As the pure gauge theory is already non-trivial, we only consider this case, the generalizations being straightforward. To show the details of the calculations that will shed some light on the more formal expression we just proved, we are going to do this example using two methods. The first one, that we will call formal method, will use the general expression for the Ward identities satisfied by the generating functional of the irreducible Green functions, \( \Gamma \). The second method, which we call practical method, will use the results of one of the theorems on the BRS transformations that we proved before. The comparison between the two methods will be important to clarify the meaning of the expressions.

i) Formal Method

For the pure gauge theory case, the expression for the Ward identities for the generating functional \( \Gamma \) is,

\[
\int d^4x \left[ \frac{\delta \Gamma}{\delta K^a_\mu(x)} \frac{\delta \Gamma}{\delta A_{\mu a}(x)} - \frac{\delta \Gamma}{\delta L^a}(x) \frac{\delta \omega^a(x)}{\delta \omega^a(x)} + \frac{1}{\xi} F^a_\mu(x) \frac{\delta \Gamma}{\delta \omega^a(x)} \right] = 0
\]  

(6.260)

where we will choose a covariant linear gauge,

\[
F^a_\mu(x) = \partial_\mu A^a_\mu(x)
\]

(6.261)

To proceed it is necessary to know what is the meaning of the functional derivatives, \( \frac{\delta \Gamma}{\delta K^a_\mu} \) and \( \frac{\delta \Gamma}{\delta L^a} \). From their definition we have

\[
\frac{\delta \Gamma}{\delta K^a_\mu(x)} = \frac{\delta W}{\delta K^a_\mu(x)} = \frac{\delta}{\delta \eta^a_\mu(x)} \ln Z = \frac{1}{Z} \frac{\delta Z}{\delta \eta^a_\mu(x)}
\]

\[
= \frac{1}{Z} \int D(\cdots) sA^a_\mu(x) e^{i(\Sigma + \text{sourcex})}
\]

(6.262)

As \( sA^a_\mu(x) = D^a_\mu \omega^b = \partial_\mu \omega^a(x) - gf^{abc} \omega^b(x) A^c_\mu(x) \), we then get

\[
\frac{\delta \Gamma}{\delta K^a_\mu(x)} = \partial_\mu \left( \frac{1}{Z} \frac{\delta Z}{\delta \eta^a_\mu(x)} \right) - gf^{abc} \frac{1}{Z} \frac{\delta^2 Z}{\partial J^c_\mu(x) \partial \eta^b(x)}
\]

(6.263)

Introducing now \( Z \equiv \exp(iW) \), the previous expression becomes,

\[
\frac{\delta \Gamma}{\delta K^a_\mu(x)} = \partial_\mu \frac{iW}{\delta \eta^a_\mu(x)} - gf^{abc} \left[ \frac{\delta^2 iW}{\partial J^c_\mu(x) \partial \eta^b(x)} + \frac{\delta iW}{\partial J^c_\mu(x)} \frac{\delta iW}{\partial \eta^b(x)} \right],
\]

(6.264)

which has the following diagrammatic representation

\[
\frac{\delta \Gamma}{\delta K^a_\mu(x)} = \partial_\mu \left[ \cdots \right] - gf^{abc} \left[ \cdots \right]
\]

(6.265)
where $W$ is the generating functional for the connected Green functions.

In a similar way we can show that \((s\omega^a = \frac{1}{2} g f^{abc} \omega^b \omega^c)\)

\[
\frac{\delta \Gamma}{\delta L^a(x)} = \frac{1}{2} g f^{abc} \frac{1}{Z} \frac{\delta^2 Z}{i \delta \eta^i(x) i \delta \eta^j(x)} = \frac{1}{2} g f^{abc} \left[ \frac{\delta^2(iW)}{i \delta \eta^i(x) i \delta \eta^j(x)} + \frac{\delta(iW) \delta(iW)}{i \delta \omega^a(x) i \delta \omega^a(x)} \right]
\]

(6.266)

In diagrammatic form this gives

\[
\frac{\delta \Gamma}{\delta L^a(x)} = \frac{1}{2} g f^{abc} \mu \xrightarrow{b} \cdots \xrightarrow{c} i W + \frac{1}{2} g f^{abc} \mu \xrightarrow{c} \cdots \xrightarrow{b} i W
\]

(6.267)

We want to apply \(\frac{\delta^2}{\delta \omega^b(y) \delta A^c_\nu(z)}\) to the original equation. We get

\[
\frac{\delta^2}{\delta \omega^b(y) \delta A^c_\nu(z)} \left( \frac{\delta \Gamma}{\delta K^a_\mu(x)} \right) = 0 = \frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta K^a_\mu(x)} = 0 = \frac{\delta^2 \Gamma}{\delta A^c_\nu(z) \delta A^a_\mu(x)} = 0
\]

(6.268)

But we have

\[
\frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta K^a_\mu(x)} \bigg|_{x=0} = \frac{1}{d^4 w} \left( -i \frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta \omega^c(w)} \right) \left( \frac{\delta^2 \Gamma}{i \delta \eta^f(x) i \delta \eta^g(x)} \right) \bigg|_{x=0} = \partial_y^b \int d^4 w \left( -i \frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta \omega^c(w)} \right) \left( \frac{\delta^2 \Gamma}{i \delta \eta^f(x) i \delta \eta^g(x)} \right) \bigg|_{x=0}
\]

\[
= g f^{abc} \int d^4 w \left( -i \frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta \omega^c(w)} \right) \left( \frac{\delta^3 i W}{i \delta \eta^f(x) i \delta \eta^g(x) i \delta J^c_i(x)} \right) \bigg|_{x=0}
\]

\[
= \partial_x^b \delta^4(x-y) \delta^{ab} - g f^{abc} \int d^4 w \left( -i \frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta \omega^c(w)} \right) \left( \frac{\delta^3 i W}{i \delta \eta^f(x) i \delta \eta^g(x) i \delta J^c_i(x)} \right) \bigg|_{x=0}
\]

(6.269)

In a similar way we have for the second term,

\[
\frac{\delta^2}{\delta \omega^b(y) \delta A^c_\nu(z)} \left( \frac{\delta \Gamma}{\delta \omega^a(x)} \right) = 0
\]

(6.270)

\[
\frac{\delta^2}{\delta \omega^b(y) \delta A^c_\nu(z)} \left( \frac{1}{\xi} \delta_x A^{\rho a}(x) \right) = \frac{1}{\xi} \delta_x^b \delta^4(x-z) - g f^{abc} \int d^4 w \left( -i \frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta \omega^c(w)} \right) \left( \frac{\delta^3 i W}{i \delta \eta^f(x) i \delta \eta^g(x) i \delta J^c_i(x)} \right) \bigg|_{x=0}
\]

(6.271)
Using these results we get

\[
-\partial_{\mu}^{y} \frac{\delta^{2} \Gamma}{\delta A_{\mu}(y) \delta A_{\mu}(z)} - g f^{abc} \int d^{4}x d^{4}w \left( -i \frac{\delta^{2} \Gamma}{\delta \omega^{a}(y) \delta \omega^{a}(w)} \right) \\
\left( \frac{\delta^{3} i W}{i \delta \eta^{f}(w) \delta \eta^{f}(x) i \delta J_{\mu}(x)} \right) \left( \frac{\delta^{2} \Gamma}{\delta A_{\mu}(x) \delta A_{\mu}(z)} \right) + \frac{1}{\xi} \partial_{\mu}^{y} \frac{\delta^{2} \Gamma}{\delta \omega^{a}(y) \delta \omega^{a}(z)} = 0
\]

(6.272)

We now apply the Fourier transform, with the conventions shown in the Fig. 6.2, we get

\[
-\text{i} p^{\mu} G^{-1}_{\nu}(p) - g f^{abc} G^{-1}_{\nu}(p) \Delta^{-1} f X^{\mu def} + \left( -\text{i} p^{\nu} \right) \frac{\text{i}}{\xi} \Delta^{-1} c(p) = 0
\]

(6.273)

This can be written as

\[
p^{\mu} G^{-1}_{\nu}(p) = -\frac{1}{\xi} \Delta^{-1} c p_{\nu} + ig f^{abc} G^{-1}_{\nu}(p) \Delta^{-1} f X^{\mu def}
\]

(6.274)

where

\[
X^{\mu def} = \text{FT} \left[ <0| T\omega^{d}(x) \omega^{f}(w) A^{\mu}(x)|0>_c \right]
\]

(6.275)

To prove the Transversality we also need the equation of motion for the ghosts. For our case this is

\[
\frac{\delta \Gamma}{\delta \omega^{a}(y)} = -\partial_{\mu}^{y} \frac{\delta \Gamma}{\delta K^{\mu a}(z)}
\]

(6.276)

Applying the operator \( \frac{\delta}{\delta \omega^{a}(y)} \), we get

\[
\frac{\delta^{2} \Gamma}{\delta \omega^{a}(y) \delta \omega^{a}(z)} = -\Box \delta^{ab} \delta^{4}(y - z)
\]

\[
+ g f^{abc} \int d^{4}w \left( -i \frac{\delta^{2} \Gamma}{\delta \omega^{a}(y) \delta \omega^{a}(w)} \right) \partial_{\mu}^{y} \left( \frac{\delta^{3} i W}{i \delta J_{\mu}^{c}(z) i \delta \eta^{f}(w) i \delta \eta^{f}(z)} \right)
\]

(6.277)
Applying now the Fourier transform, we get

\[ i\Delta^{-1ab} = p^2\delta^{ab} + gf^{adc}(-ip^\mu)X^{def}_\mu \Delta^{-1fb} \]  

(6.278)

The previous equations allow now to complete the proof of the transversality of the vacuum polarization. For this we write,

\[ G^{-1ab}_{\mu\nu} = G^{-1ab}_{T\mu\nu} + ia_\xi \delta^{ab} p_\mu p_\nu \]  

(6.279)

where \( p_\mu G^{-1}_{T\mu\nu} = 0 \). For the free propagator we have \( a = 1 \). to show the transversality we just have to show that the longitudinal part is not renormalized and that therefore the value of \( a \) remains always \( a = 1 \). Using

\[ p_\mu G^{-1}_{T\mu\nu} = i\frac{a}{\xi} \delta^{ab} p^2 p_\nu \]  

(6.280)

and multiplying equation by \( p_\nu \) we obtain

\[ i\frac{a}{\xi} p^4 \delta^{cb} = -\frac{1}{\xi} p^2 \Delta^{-1cb} - \frac{a}{\xi} p^2 g f^{cde} p_\mu X^{\mudef} \Delta^{-1fb} \]  

(6.281)

Using now equation Eq. (6.278) we get after some trivial algebra

\[ 0 = -\frac{1}{\xi} p^2 \Delta^{-1cb} + \frac{a}{\xi} p^2 \Delta^{-1cb} \]  

(6.282)

This implies

\[ a = 1 \]  

(6.283)

as we wanted to shown.

2) Practical Method

Now we are going to use the so-called practical method based in Theorem 6.7 Using

\[ s\omega^b(x) = \frac{1}{\xi} \partial_\mu A^{\mu b}(x) \]  

(6.284)

and

\[ sA^a_\mu = \partial_\mu \omega^a - gf^{adc} \omega^d A^c_\mu \]  

(6.285)

it is easy to see that the starting Green function should be \( \langle 0|T A^a_\mu(x)\omega^b(y)|0 \rangle \). Then Theorem 6.7 tells us that

\[ s \langle 0|T A^a_\mu(x)\omega^b(y)|0 \rangle = 0 \]  

(6.286)

that is

\[ \frac{1}{\xi} \langle 0|T A^a_\mu(x)\partial_\mu A^{\mu b}(y)|0 \rangle = \langle 0|T \partial_\nu \omega^a(x)\omega^b(y)|0 \rangle - g f^{adc} \langle 0|T \omega^d(x)A^c_\mu(x)\omega^b(y)|0 \rangle \]  

(6.287)
6.3. WARD IDENTITIES

We now take the Fourier transform obtaining

\[ \frac{i}{\xi} p^a G_{\nu \rho}^{ab}(p) = -ip_\rho \Delta^{ab}(p) - gf^{abc} X_{\nu}^{dcb} \]  \hspace{1cm} (6.288)

where \( X_{\nu}^{dcb} \) has been defined before. Multiplying by \( G^{-1 \nu \mu} \Delta^{-1} \) we then get

\[ p^\mu G_{\nu \mu}^{-1 ac} = -\frac{1}{\xi} p_{\nu} \Delta^{-1 ac} + igf^{def} \chi_{\nu}^{deb} \Delta^{-1 bc} G^{-i \nu \mu f} \]  \hspace{1cm} (6.289)

which is precisely Eq. (6.274) The result in Eq. (6.278) can be easily obtained knowing that the only vertex of the ghosts is

\[ \begin{array}{c}
\mu, c \\
a \rightarrow \\
\mu, c \\
\mu, c \\
\end{array} \begin{array}{c}
\mu, c \\
b \rightarrow \\
\mu, c \\
\mu, c \\
\end{array} g f^{abc} p^\mu \]  \hspace{1cm} (6.290)

Then

\[ a \rightarrow \begin{array}{c}
\mu, c \\
\mu, c \\
\mu, c \\
\mu, c \\
\end{array} (iW) \rightarrow b = a \rightarrow b + a \rightarrow d \begin{array}{c}
\mu, c \\
\mu, c \\
\mu, c \\
\mu, c \\
\end{array} (iW) \rightarrow b \]  \hspace{1cm} (6.291)

which means

\[ \Delta^{ab}(p) = \frac{i}{p^2} \delta^{ab} + \frac{i}{p^2} g f^{abc} p^\mu X_{\mu}^{dcb} \]  \hspace{1cm} (6.292)

or in another form

\[ i\Delta^{-1 ab} = p^2 \delta^{ab} - igf^{abc} p^\mu X_{\mu}^{dcb} \Delta^{-1 b'} \]  \hspace{1cm} (6.293)

which is precisely Eq. (6.278). The proof of transversality follows now the same steps as in the formal case.

6.3.4 Gauge invariance of the \( S \) matrix

We have shown before the gauge invariance of the \( S \) matrix using the equivalence theorem and the fact that the generating functionals corresponding to different gauge conditions only differ in the source terms. The proof used some properties of the Coulomb gauge and this can raise some doubts about the generality of the argument.

We are going to show here, using the Ward identities, that the functionals \( Z_F \) and \( Z_F + \Delta F \) corresponding to the gauge conditions \( F \) and \( F + \Delta F \), respectively, only differ in the source terms. As \( F \) and \( \Delta F \) are arbitrary the proof is general. We have

\[ Z_F[J_{\mu}^a, J_i] = \int D(\cdots) \epsilon^{[S_{eff} + f d^4 x (J_{\mu}^a A_{\mu a} + J_i \phi_i)]} \]  \hspace{1cm} (6.294)
Then
\[
Z_{F+\Delta F} - Z_F = \int D(\cdots) \int d^4x \ i \left[ \frac{1}{\xi} F^a \Delta F^a - \bar{\omega} \int d^4y \ \frac{\delta \Delta F^a(x)}{\delta A^a_\mu(y)} s A^b_\mu(y) \right. \\
\left. - \bar{\omega} \int d^4y \ \frac{\delta \Delta F^a(x)}{\delta \phi_i(y)} s \phi_i(y) \right] e^{i(S_{eff}+\text{sources})} \tag{6.295}
\]

We use now the Ward identities in the form that corresponds to the generating functional \(Z\), that is
\[
0 = \int D(\cdots) \int d^4x [J^a_\mu s A^a_\mu + J^i s \phi_i + \bar{\eta}s \omega - s \bar{\omega} \eta] \ e^{i(S_{eff}+J^a_\mu A^a_\mu + J^i \phi_i + \bar{\omega} \eta + \bar{\eta} \omega)} \tag{6.296}
\]

Taking the derivative in order to \(\eta^a(x)\) and setting after the ghost sources to zero, we get
\[
0 = \int D(\cdots) \left[ \frac{1}{\xi} F^a(x) + \bar{\omega} \bar{\omega}^a(x) \int d^4y [J^a_\mu s A^a_\mu + J^i s \phi_i] \right] e^{i(S_{eff}+\int d^4x (J^a_\mu A^a_\mu + J^i \phi_i))} \tag{6.297}
\]
or
\[
-\frac{1}{\xi} F^a \left[ \frac{\delta}{\delta \omega} \right] \int D(\cdots) e^{i(S_{eff}+\text{sources})} = \\
\int D(\cdots) \bar{\omega} \bar{\omega}^a(x) \int d^4y [J^a_\mu s A^a_\mu + J^i s \phi_i] e^{i(S_{eff}+\text{sources})} \tag{6.298}
\]

Then
\[
\int D(\cdots) \left( -\frac{1}{\xi} F^a \Delta F^a \right) e^{i(S_{eff}+\text{sources})} = \\
\Delta F^a \left[ \frac{\delta}{\delta \omega} \right] \left( -\frac{1}{\xi} F^a \left[ \frac{\delta}{\delta \omega} \right] \right) \int D(\cdots) e^{i(S_{eff}+\text{sources})} \\
\Delta F^a \left[ \frac{\delta}{\delta \omega} \right] \int D(\cdots) \bar{\omega} \bar{\omega}^a(x) \int d^4y [J^a_\mu s A^a_\mu + J^i s \phi_i] e^{i(S_{eff}+\text{sources})} \tag{6.299}
\]

or
\[
\int D(\cdots) \left\{ \bar{\omega} \bar{\omega}^a(x) \int d^4y \left[ \frac{\delta \Delta F^a(x)}{\delta A^a_\mu(y)} s A^b_\mu(y) + \frac{\delta \Delta F^a(x)}{\delta \phi_i(y)} s \phi_i(y) \right] \right. \\
\left. \frac{\delta}{\delta \phi_i(y)} s \phi_i(y) \right\} e^{i(S_{eff}+\text{sources})} \\
\int D(\cdots) \bar{\omega} \bar{\omega}^a(x) \Delta F^a(x) \int d^4y [J^a_\mu s A^a_\mu + J^i s \phi_i] e^{i(S_{eff}+\text{sources})} \tag{6.300}
\]
We can then write
\[ Z_{F+\Delta F} - Z_F = \int D(\cdots) \hat{\mathcal{J}} \left[ i\omega^a(x)\Delta F^a(x) \int d^4y (J^a_{\mu b} A^b_\mu + J_i A^i_\mu) \right] e^{i \{ S_{\text{eff}} + \text{sources} \}} \]
\[ = \int D(\cdots) \hat{\mathcal{J}} \left\{ i \{ S_{\text{eff}} + \int d^4y [ J^a_\mu(y) A^a_\mu(y) + J_i \Phi_i(y)] \} \right\} \]
\[ = \int D(\cdots) \hat{\mathcal{J}} \left\{ i \{ S_{\text{eff}} + \int d^4y [ J^a_\mu(y) A^a_\mu(y) + J_i \Phi_i(y)] \} \right\} \]
where
\[ \Phi_i(y) \equiv \phi_i(y) + i \int d^4x [\omega^a(x)\Delta F^a(x) s\phi_i(y)] \] (6.302)
and
\[ A^a_\mu(y) \equiv A^a_\mu(y) + i \int d^4x [\omega^b(x)\Delta F^b(x) sA^a_\mu(y)] \] (6.303)
The difference between the generating functionals \( Z_{F+\Delta F} \) and \( Z_F \) is only in the functional form of the source terms. We can then use the equivalence theorem to show that the renormalized \( S \) matrix are equal in both cases.
\[ S_{F+\Delta F}^R = S_F^R . \] (6.304)

6.4 Ward Takahashi Identities in QED

6.4.1 Ward-Takahashi identities for the functional \( Z[J] \)

We will now derive again the Ward identities for QED, that we found in our study of renormalization, using now the functional methods. The generating functional for the Green functions for QED is given by, in a linear gauge,
\[ Z(J_\mu, \bar{\eta}, \eta) = \int D(A_\mu, \psi, \bar{\psi}) e^{i \{ S_{\text{eff}} + \int d^4x (J_\mu A_\mu + \bar{\eta} \psi + \psi \eta) \}} \] (6.305)
where \( J_\mu, \bar{\eta} \) and \( \eta \) are the sources for \( A_\mu, \psi \) and \( \bar{\psi} \) respectively. The effective action is given by,
\[ S_{\text{eff}} = \int d^4x \left[ L_{\text{QED}} - \frac{1}{2g} (\partial \cdot A)^2 \right] = S_{\text{QED}} + S_{\text{GF}} \] (6.306)
where
\[ L_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i\gamma^\mu D_\mu - m \psi . \] (6.307)
\( S_{\text{QED}} \) is invariant under local gauge transformation of the group \( U(1) \) that we write as,
\[ \left\{ \begin{array}{l}
\delta A_\mu = \partial_\mu \Lambda \\
\delta \psi = -ie\Lambda \psi \\
\delta \bar{\psi} = ie\Lambda \bar{\psi}
\end{array} \right. \]
The $S_{eff}$ contains the part of the gauge fixing that it is not invariant under these transformations. Therefore the Ward identities take the form,

$$
\left( \frac{\delta S_{GF}}{\delta \phi_i} \right) F_i \left[ \frac{\delta}{i \delta J} J_i \right] Z(J) = 0 \tag{6.308}
$$

This can be written in our case as, putting back the explicit integrations,

$$
0 = \int d^4 x \left[ \frac{1}{\xi} \partial_\mu \left( \frac{\delta}{i \delta J_\mu} \right) - \partial_\mu J^\mu - ie\bar{\eta} \frac{\delta}{i \delta \eta} + ie\eta \frac{\delta}{i \delta \eta} \right] Z(J^\mu, \eta, \eta) \tag{6.309}
$$

After an integration by parts we get,

$$
\int d^4 x \left[ -\frac{1}{\xi} \Box \partial_\mu \left( \frac{\delta}{i \delta J_\mu} \right) - \frac{\delta}{i \delta J_\mu} + ie\bar{\eta} \frac{\delta}{i \delta \eta} + ie\eta \frac{\delta}{i \delta \eta} \right] Z(J^\mu, \eta, \eta) = 0 \tag{6.310}
$$

This can be written as

$$
\left[ \frac{1}{\xi} \Box \partial_\mu \left( \frac{\delta}{i \delta J_\mu} \right) + \partial_\mu J^\mu + ie\bar{\eta} \frac{\delta}{i \delta \eta} - ie\eta \frac{\delta}{i \delta \eta} \right] Z(J, \eta, \eta) = 0 \tag{6.311}
$$

### 6.4.2 Ward-Takahashi identities for the functionals $W$ and $\Gamma$

From the point of view of the applications it is more useful the Ward identity for the generating functional of the irreducible Green functions. This problem is simpler than in the case of non-abelian gauge theories, that we just discuss, as the the previous equation is linear in the functional derivatives with respect to the different sources (we notice that if we had chosen a non-linear gauge fixing this would not be true, even in QED). The linearity allow us to write immediately

$$
\partial_\mu J^\mu + \left[ \frac{1}{\xi} \Box \partial_\mu \left( \frac{\delta}{i \delta J_\mu} \right) + ie\bar{\eta} \frac{\delta}{i \delta \eta} - ie\eta \frac{\delta}{i \delta \eta} \right] W(J, \eta, \eta) = 0 \tag{6.312}
$$

where $W$ is the generating functional for the connected Green functions,

$$
Z(J^\mu, \eta, \eta) \equiv e^{iW(J^\mu, \eta, \eta)} \tag{6.313}
$$

As we saw the generating functional for the irreducible Green functions is given by,

$$
\Gamma(A_\mu, \psi, \overline{\psi}) = W(J_\mu, \eta, \eta) - \int d^4 x [J^\mu A_\mu + \bar{\psi} \psi + \bar{\psi} \eta] \tag{6.314}
$$

We also have the relations

$$
A_\mu = \frac{\delta W}{i \delta J^\mu} ; \quad \psi = \frac{\delta W}{i \delta \eta} ; \quad \bar{\psi} = -\frac{\delta W}{i \delta \eta} \tag{6.315}
$$

and

$$
J_\mu = -\frac{\delta \Gamma}{\delta A_\mu} ; \quad \eta = -\frac{\delta \Gamma}{\delta \psi} ; \quad \bar{\eta} = \frac{\delta \Gamma}{\delta \psi} \tag{6.316}
$$

where, as usual, the fermionic derivatives are left derivatives. We can then write

$$
\frac{1}{\xi} \Box \partial_\mu A^\mu - \partial_\mu \frac{\delta \Gamma}{\delta A_\mu} + ie \frac{\delta \Gamma}{\delta \psi} \psi + ie \bar{\psi} \frac{\delta \Gamma}{\delta \psi} = 0 \tag{6.317}
$$

This equation is the starting point to generate all the Ward identities in QED. Its application it is much easier than the equivalent expression that was proved using the canonical formalism. The functional methods make this expressions particularly simple.
6.4. WARD TAKAHASHI IDENTITIES IN QED

6.4.3 Example: Ward identity for the QED vertex

To convince ourselves that this equation reproduces the Ward identities that we already know, let us derive the Ward identity for the vertex in QED. We apply \( \frac{\delta^2}{\delta \psi_\alpha(y) \delta \psi_\beta(z)} \) to the master equation. We get then

\[
\partial_\mu x \frac{\delta^3 \Gamma}{\delta \psi_\alpha(y) \delta \psi_\beta(z) \delta A_\mu(x)} = ie \left[ \frac{\delta^2 \Gamma}{\delta \psi_\alpha(y) \delta \psi_\beta(x)} \delta^4(z - x) - \frac{\delta^2 \Gamma}{\delta \psi_\alpha(x) \delta \psi_\beta(z)} \delta^4(y - x) \right]
\]

This equation means

\[
\partial_\mu \Gamma_{\mu,\beta\alpha}(x, z, y) = ie \left[ \Gamma_{\beta\alpha}(x, y) \delta^4(z - x) - \Gamma_{\beta\alpha}(z, x) \delta^4(y - x) \right]
\]

Taking now the Fourier transform to both sides of the equation, with the momenta defined as the Fig. 6.3, we get,

\[
q^\mu \Gamma_{\mu}(p', p) = ie[S^{-1}(p) - S^{-1}(p')]
\]

This is precisely the well known Ward identity.

6.4.4 Ghosts in QED

We said before that the generating functional for QED was given by,

\[
Z(J, \bar{\psi}, \eta) = \int D(A, \bar{\psi}, \psi) e^{i \int d^4x [\mathcal{L}_{QED} + \mathcal{L}_{GF} + J_\mu A_\mu + \bar{\psi} \eta + \bar{\psi} \eta]}
\]

where \( \mathcal{L}_{QED} \) is the usual Lagrangian for QED and the gauge fixing term was,

\[
\mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial \cdot A)^2.
\]
In fact this is not strictly true. If we use the prescription for the gauge theories, we would get instead,

\[ \tilde{Z}(J, \eta, \eta', \eta'', \eta'''') = \int D(A, \psi, \bar{\psi}, \psi, \bar{\psi}) e^{i \int d^4x [\mathcal{L}_{eff} + J^\mu A_\mu + \bar{\psi} i \gamma^\mu D_\mu + \bar{\psi} \gamma^\mu \gamma^5 A_\mu \psi]} \]  

(6.323)

In this expression \( \omega, \bar{\omega} \) are anti-commuting scalar fields known as the Faddeev-Popov ghosts as we saw before. Although in physical process they never appear as external states, it is useful to introduce also sources for them to discuss the Ward identities.

In the previous action, the Lagrangian \( \mathcal{L}_{eff} \) is

\[ \mathcal{L}_{eff} = \mathcal{L}_{QED} + \mathcal{L}_{GF} + \mathcal{L}_{G} \]  

(6.324)

where

\[ \mathcal{L}_{G} = -\bar{\omega} \Box \omega \]  

(6.325)

The reason why in QED we can work with the functional \( Z \) instead of \( \tilde{Z} \) is because the ghosts do not have interactions with the gauge fields and can be integrated out (Gaussian integration) and absorbed in the normalization. Nevertheless, for the Ward identities it is useful to keep them. The effective Lagrangian, \( \mathcal{L}_{eff} \), is invariant under the BRS transformations given by,

\[
\begin{align*}
\delta \psi &= -ie\omega \theta \psi \\
\delta \bar{\psi} &= ie\bar{\psi} \omega \theta \\
\delta A_\mu &= \partial_\mu \omega \theta \\
\delta \bar{\omega} &= 1/\xi (\partial \cdot A) \theta \\
\delta \omega &= 0
\end{align*}
\]  

(6.326)

The parameter \( \theta \) is an anti-commuting (Grassmann variable). The BRS transformations on the physical fields are gauge transformations with parameter \( \Lambda = \omega \theta \) and therefore \( \mathcal{L}_{QED} \) is left invariant. The transformations in the ghosts \( \omega, \bar{\omega} \) are such that the variation of \( \mathcal{L}_{GF} \) cancels that of \( \mathcal{L}_{G} \), just like in the non-abelian case. The invariance of the integration measure and of \( S_{eff} \) allows us to write immediately the Ward identities for the generating functionals. The BRS transformations allow us to obtain the Ward identities in a quick way without having to resort to the functional \( \Gamma \). This method is based on the fact, as we saw in Theorem refth:Slavnov, that the application of the operator \( \delta_{BRS} \) to any Green function gives zero, that is

\[ \delta_{BRS} \langle 0 | T A_{\mu_1} \cdots \bar{\omega} \cdots \omega \cdots \psi \cdots \bar{\psi} \cdots | 0 \rangle = 0 \]  

(6.327)

Let us show two simple applications of the method in QED.

1) The non-renormalization of the longitudinal photon propagator

This result is equivalent, as we have seen, to the statement that the vacuum polarization is transversal. It is proved easily starting with the Green function, \( \langle 0 | T A_\mu \bar{\omega} | 0 \rangle \), and using

\[ \delta_{BRS} \langle 0 | T A_\mu \bar{\omega} | 0 \rangle = 0 \]  

(6.328)

This gives

\[ \frac{1}{\xi} \langle 0 | T A_\mu \partial^\nu A_\nu | 0 \rangle \theta - \langle 0 | T \partial_\mu \omega \bar{\omega} | 0 \rangle \theta = 0 \]  

(6.329)
After taking the Fourier transform we get
\[ \frac{1}{\xi} k^\mu G_{\mu\nu}(k) = -k_\nu \Delta(k) \] (6.330)
where the ghost propagator is the free propagator
\[ \Delta(k) = \frac{i}{k^2} \] (6.331)
because the ghosts have no interactions. Multiplying by the inverse propagator of the photon we get
\[ \frac{1}{\xi} k^\mu = -\frac{k^\nu}{k^2} G^{-1\nu\mu}(k) \] (6.332)
Therefore
\[ k_\nu G^{-1\nu\mu}(k) = \frac{i}{\xi} k^\mu k^2 = k_\nu G^{-1\nu\mu}(0) \] (6.333)
This shows that the longitudinal part of the photon propagator is equal to the free longitudinal part and therefore does not get any renormalization.

2) Ward Identity for the Vertex

For the vertex we start from
\[ \delta_{BRS} \langle 0 | T \bar{\omega} \psi \psi | 0 \rangle = 0 \] (6.334)
This means
\[ \frac{1}{\xi} \langle 0 | T \partial^\mu A_\mu \bar{\psi} \psi | 0 \rangle = i e \langle 0 | T \bar{\omega} \psi \psi | 0 \rangle - i e \langle 0 | T \bar{\omega} \psi \psi | 0 \rangle \] (6.335)
After taking the Fourier transform we get
\[ \frac{i}{\xi} q^\nu T_\mu = T \] (6.336)
where we have defined
\[ \mu \quad q \quad p' \quad p \]

\[ = G_{\mu\nu}(q) S(p') i \Gamma^\nu S(p) \] (6.337)
The last equality results from the fact that ghosts have no interactions in QED in a linear gauge. Putting everything together we get

\[ i \xi q^\mu G_{\mu \nu}(q) S(p') i \Gamma^\nu S(p) = -ie \Delta(q) S(p) + ie \Delta(q) S(p') \]  \hspace{1cm} (6.339)

Using

\[ \frac{1}{\xi} k^\mu G_{\mu \nu}(k) = -k_\nu \Delta(k) \]  \hspace{1cm} (6.340)

and multiplying by the inverse of the fermion propagators we get again the well known the Ward identity for the vertex,

\[ q_\mu \Gamma^\mu(p',p) = ie \left[ S^{-1}(p) - S^{-1}(p') \right] \]  \hspace{1cm} (6.341)

### 6.5 Unitarity and Ward Identities

#### 6.5.1 Optical Theorem

The $S$ matrix, (Heisenberg 1942), can be written in the form

\[ S = 1 + iT \]  \hspace{1cm} (6.342)

Then its unitarity $SS^\dagger = 1$ implies,

\[ 2 \text{Im} T = TT^\dagger \]  \hspace{1cm} (6.343)

If we insert this relation between the same initial and final state (elastic scattering) we get

\[ 2 \text{Im} \left< \langle i | T | i \rangle \right> = \left< i | TT^\dagger | i \right> \]

\[ = \sum_f | \left< f | T | i \right> |^2 \]  \hspace{1cm} (6.344)

where we have introduced a complete set of states. This relation can still be written in the form,

\[ \sigma_{\text{total}} = 2 \text{Im} \ T_{\text{elastic}}^{\text{forward}} \]  \hspace{1cm} (6.345)

known as the optical theorem. What we call here $\sigma_{\text{total}}$ it is not exactly the cross section, because the flux factors are mixing. It is for our purpose the quantity defined by

\[ \sigma_{\text{total}} \equiv \sum_f | \left< f | T | i \right> |^2 \]  \hspace{1cm} (6.346)

Unitarity establishes therefore a relation between the total cross section and the imaginary part of the elastic amplitude in the forward direction (the initial and final state have to be the same).
6.5. UNITARITY AND WARD IDENTITIES

6.5.2 Cutkosky rules

To show that unitarity is obeyed in a given process we have to know how to calculate the imaginary part of Feynman diagrams. Of course there is always the possibility of doing explicitly the calculations and retrieve the imaginary part, but this only possible for simple diagrams (see below). Therefore it is useful to have rules, known as Cutkosky rules, that give us the imaginary part of any diagram. We will state them now.

**Rule 1**

The imaginary part of an amplitude is obtained using the expression

\[
2 \text{Im } T = - \sum_{\text{cuts}} T
\]  

(6.347)

**Rule 2**

The cut is obtained by writing the amplitude \(iT = \cdots\) and substituting in this expression the propagators of the lines we cut by the following expression,

- **Scalar fields**
  \[\Delta(p) \Rightarrow 2\pi \theta(p^0) \delta(p^2 - m^2)\]  
  (6.348)

- **Fermion fields**
  \[S(p) \Rightarrow (p + m)2\pi \theta(p^0) \delta(p^2 - m^2)\]  
  (6.349)

- **Vector gauge fields (in the Feynman gauge)**
  \[G_{\mu\nu}(p) \Rightarrow -g_{\mu\nu}2\pi \theta(p^0) \delta(p^2 - m^2)\]  
  (6.350)

In these expression the \(\theta\) functions ensure the energy flux. The Cutkosky rules are complicated to prove in general (see G. ’t Hooft, ”Diagrammar”, CERN Report 1972) but we are going to show in the two explicit examples how they work.

**Example 6.1 Free propagator**

For the free propagator of a scalar field the amplitude is

\[iT = \frac{i}{p^2 - m^2 + i\epsilon}\]  

(6.351)

The imaginary part is obtained using

\[
\frac{1}{x + i\epsilon} = P\left(\frac{1}{x}\right) - i\pi \delta(x)
\]  

(6.352)

Therefore

\[T = P\left(\frac{1}{p^2 - m^2}\right) - i\pi \delta(p^2 - m^2)\]  

(6.353)
The imaginary part is then
\[ 2 \text{Im} T = -2\pi \delta(p^2 - m^2) \] (6.354)

Using the Cutkosky rule we get
\[ 2 \text{Im} T = -2\pi \delta(p^2 - m^2) \theta(p^0) \] (6.355)

which is precisely the same result. The function \( \theta(p^0) \) tells us that the flux of energy is from left to right.

**Example 6.2 Self-energy in \( \phi^3 \)**

Let us consider the self-energy in the theory given by the Lagrangian,
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{3!} \phi^3
\] (6.356)

The self-energy is given by the diagram in the Fig. 6.4. The corresponding amplitude

![Figure 6.4: Self-energy](image)

is
\[
i T = (-i\lambda)^2 \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \frac{i}{(p - k)^2 - m^2 + i\varepsilon}
\] (6.357)

Let us calculate the imaginary part of \( T \) by two methods, first doing the explicit calculation and second using the Cutkosky rule.

i) Explicit Calculation

\[
i T = \lambda^2 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2 + i\varepsilon)((p - k)^2 - m^2 + i\varepsilon)}
\]

\[
= \lambda^2 \sqrt{\frac{d^4 p}{(2\pi)^4}} \int_0^1 dx \frac{1}{(p^2 + 2p \cdot P - M^2 + i\varepsilon)^2}
\]

\[
= \lambda^2 \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dx \frac{1}{[(p + P)^2 - \Delta]^2}
\] (6.358)

where
\[
\begin{align*}
P &= -xk \\
\Delta &= P^2 + M^2 = m^2 - k^2x(1 - x) - i\varepsilon
\end{align*}
\] (6.359)
The amplitude is then
\[ iT = \lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{1}{(p^2 - \Delta)^2} \] (6.360)

The integral is divergent. Using dimensional regularization we get
\[ T = \frac{\lambda}{16\pi^2} \mu^4 \Gamma \left( 2 - \frac{d}{2} \right) \int_0^1 dx \Delta^{-\left(2 - \frac{d}{2}\right)} \] (6.361)

Choosing on-shell renormalization, \( T_R(k^2 = m^2) = 0 \), we get
\[ T_R = T - T(k^2 = m^2) = \lambda^2 \frac{\mu^4}{16\pi^2} \Gamma \left( 2 - \frac{d}{2} \right) \int_0^1 dx \left[ \frac{\Delta(k^2)}{\mu^2} \right]^{-\frac{d}{2}} - \frac{\Delta(k^2 = m^2)}{\mu^2} \left[ \frac{d}{2} \right] \] (6.362)

In this expression \( \beta = \frac{k^2}{m^2} \) and the function \( L(\beta) \) is given by
\[ L(\beta) \equiv \int_0^1 dx \ln \left[ \frac{1 - \beta x(1-x)}{1-x(1-x)} - i\varepsilon \right] \] (6.363)

It satisfies
\[ \text{Im}L(\beta) = -\pi \sqrt{1 - \frac{4}{\beta}} \theta(\beta - 4) \] (6.364)

Therefore
\[ \text{Im}T = -\frac{\lambda^2}{16\pi^2} \left[ \text{Im}L(\beta) - \text{Im}L(1) \right] \] (6.365)

and we get finally,
\[ \text{Im}T = \frac{\lambda^2}{16\pi} \sqrt{1 - \frac{4m^2}{k^2}} \theta \left( 1 - \frac{4m^2}{k^2} \right) \] (6.366)

The \( \theta \) functions ensures that there is only imaginary part when the intermediate state could also be a final state (production of two particles of mass \( m \)).

ii) Using the Cutkosky rules

Using the rules we get
\[ 2\text{Im}T = -(i\lambda)^2 \int \frac{d^4p}{(2\pi)^4} (2\pi)^2 \theta(p^0) \theta(k^0 - p^0) \delta(p^2 - m^2) \delta((p^2 - k^2) - m^2) \]
\[ = \lambda^2 \int \frac{d^4p}{(2\pi)^4} d^4p' (2\pi)^2 \theta(p^0) \theta(k^0 - p^0) \delta(p'^2 - m^2) \delta(p'^2 - m^2) \delta^4(p' - k + p) \]
Using now the result
\[ \int d^4 p \theta(p^0)\delta(p^2 - m^2) = \int d^3 p \frac{1}{2p^0} \]  
\hfill (6.367)
We get
\[ 2\text{Im}T = \lambda^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p^0} \frac{1}{2p^0} 2\pi\delta^4(p' - k + p) \]  
\hfill (6.369)
or
\[ 2\text{Im}T = \lambda^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p^0} \frac{1}{2p^0} 2\pi\delta(k^0 - p^0 - p^0) \]  
\hfill (6.370)
\[ k = (\sqrt{s}, \vec{0}) ; \ p = (\sqrt{|\vec{p}|^2 + m^2}, \vec{p}) ; \ p' = (\sqrt{|\vec{p}'|^2 + m^2}, -\vec{p}) \]  
\hfill (6.371)
Therefore we get
\[ 2\text{Im}T = \lambda^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{4(|\vec{p}|^2 + m^2)} 2\pi\delta(\sqrt{s} - 2\sqrt{|\vec{p}|^2 + m^2}) \]  
\hfill (6.372)
Using \( s = k^2 \) we get
\[ \text{Im}T = \frac{\lambda^2}{16\pi} \sqrt{1 - \frac{4m^2}{s}} \theta \left(1 - \frac{4m^2}{s}\right) \]  
\hfill (6.373)
which is the same result as we got in the explicit calculation.

### 6.5.3 Example of Unitarity: scalars and fermions

As an example of checking the unitarity let us consider a theory described by the Lagrangian,
\[ \mathcal{L} = \bar{\psi}i\gamma^\mu \psi - m \bar{\psi} \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + g \bar{\psi} \psi \phi \]  
\hfill (6.374)
We will show unitarity in two cases (cutting fermions lines):

\( i) \) Scalar Self-energy

The self-energy of the scalars is given by the diagram in Fig. 6.5, to which corresponds the amplitude,
\[ iT = g^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \frac{i}{\vec{p} - m + i\varepsilon} \frac{i}{\vec{p}' - k - m + i\varepsilon} \right] \]  
\hfill (6.375)
Applying Cutkosky rules we get,

\[ 2 \text{Im} T = - \sum_{\text{cuts}} T \]

\[ = - g^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[(\not p + m)(\not p - \not k + m)](2\pi)\theta(p^0)\delta(p^2 - m^2) \]

\[ (2\pi)\theta(k^0 - p^0)\delta((p - k)^2 - m^2) \]  \hspace{1cm} (6.376)

To show the unitarity we calculate the cross section,

\[ \sigma = \sum_f \left| \frac{k \not p p' \not p'}{p' - p} \right|^2 \]  \hspace{1cm} (6.377)

We get

\[ \sigma = \sum_f |ig\bar{u}(p)v(p')|^2 = -g^2 \sum_f \text{Tr}[(\not p + m)(-\not p' + m)] \]  \hspace{1cm} (6.378)

where we have used \( \sum_{\text{spins}} v(p') \bar{u}(p) = -(-\not p' + m) \) and \( \sum_{\text{spins}} u(p) \bar{u}(p') = \not p + m \). Therefore

\[ \sigma = -g^2 \int dp_2 \text{Tr}[(\not p + m)(-\not p' + m)] \]  \hspace{1cm} (6.379)

where \( dp_2 \) is the phase space of two particles, that is,

\[ \int dp_2 \equiv \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{2 p^0} \frac{1}{2 p'^0} (2\pi)^4 \delta^4(k - p - p') \]

\[ = \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} (2\pi)\theta(p^0)\delta(p^2 - m^2)(2\pi)\theta(p'^0)\delta(p'^2 - m^2)(2\pi)^4 \delta^4(k - p - p') \]  \hspace{1cm} (6.380)

We conclude then that

\[ \sigma = -g^2 \int \frac{d^4 p}{(2\pi)^4} (2\pi)\theta(p^0)\delta(p^2 - m^2)(2\pi)\theta(k^0 - p^0)\delta((p - k)^2 - m^2) \]

\[ \text{Tr}[(\not p + m)(\not p - \not k + m)] \]  \hspace{1cm} (6.381)

Comparing we obtain

\[ 2 \text{Im} T = \sigma \]  \hspace{1cm} (6.382)
ii) General case

Let us consider the general case of two internal fermion lines. The amplitude $iT$ is represented by the diagram

\[
\begin{align*}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, thick, every node/.style={scale=0.7}, scale=1.0]
    \node[fill=black!20] (v1) at (0,0) {box{\rule{0.5cm}{0.5cm}}};
    \node[fill=black!20] (v2) at (1,0) {box{\rule{0.5cm}{0.5cm}}};
    \node[fill=black!20] (v3) at (0,-1) {box{\rule{0.5cm}{0.5cm}}};
    \node[fill=black!20] (v4) at (1,-1) {box{\rule{0.5cm}{0.5cm}}};
    \node at (0.5,0) {$p$};
    \node at (-0.5,0) {$-p'$};
    \draw[thick] (v1) -- (v2);
    \draw[thick] (v3) -- (v4);
    \draw[thick, ->] (v1) -- (v3);
    \draw[thick, ->] (v2) -- (v4);
    \draw[thick, dashed] (v1) -- (v4);
    \draw[thick, dashed] (v2) -- (v3);
    \node at (0,-1.5) {$k_1$};
    \node at (1,-1.5) {$k_2$};
    \node at (0,1.5) {$k_1$};
    \node at (1,1.5) {$k_2$};
    \node at (0,-2) {$k_n$};
    \node at (1,-2) {$k_n$};
\end{tikzpicture}
\end{align*}
\]

\[
p' = \sum_{i=1}^{n} k_i - p
\]

(6.383)

The amplitude $iT$ is given by

\[
iT = -\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \left[ T' S(p) T' S(-p') \right]
\]

(6.384)

Where we have defined the amplitude $iT'$ by

\[
\begin{align*}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, thick, every node/.style={scale=0.7}, scale=1.0]
    \node[fill=black!20] (v1) at (0,0) {box{\rule{0.5cm}{0.5cm}}};
    \node[fill=black!20] (v2) at (0,-1) {box{\rule{0.5cm}{0.5cm}}};
    \node at (0,0) {$p$};
    \node at (0,-1) {$-p'$};
    \draw[thick, ->] (v1) -- (v2);
    \draw[thick, dashed] (v1) -- (v2);
    \node at (0,-1.5) {$k_1$};
    \node at (0,-2) {$k_n$};
\end{tikzpicture}
\end{align*}
\]

\[
\equiv \bar{u}(p) iT' v(p')
\]

(6.385)

Therefore

\[
2 \text{Im}T = -\int \frac{d^4p}{(2\pi)^4} (2\pi)^2 \delta(p^2 - m^2) \theta(p^0) \delta(p'^2 - m^2) \theta(p'^0) \theta(p^0)
\]

\[
\mathrm{Tr} \left[ T' \left( \hat{\rho} + m \right) T' \left( -\hat{p}' + m \right) \right]
\]

\[
= -\int d\rho_2 \mathrm{Tr} \left[ T' \left( \hat{\rho} + m \right) T' \left( -\hat{p}' + m \right) \right]
\]

(6.386)

On the other hand
\[ \sigma = \sum_f \left| \prod_{f} \left| \begin{array}{c} k_1 \\ k_2 \\ \vdots \\ k_n \end{array} \right| \right|^2 \]

\[ = \sum_f \prod_{f} \left| \begin{array}{c} p \\ -p' \end{array} \right| \]

\[ = - \int d\rho \prod_{f} \text{Tr} \left[ \left( p + m \right) T' \left( -p' + m \right) T' \right] \]  

(6.387)

Therefore

\[ \sigma = 2 \text{Im} T \]  

(6.388)

If the lines to be cut were scalars the result would be the same. In this case there will be no minus sign from the loop but there will be no minus sign from the spins sum. The proof is left as an exercise.

\[ 2 \text{Im} \]

\[ = \sum_f \left| \begin{array}{c} k_1 \\ k_2 \\ \vdots \\ k_n \end{array} \right| \left| \begin{array}{c} p \\ -p' \end{array} \right| \]  

(6.389)

6.5.4 Unitarity and gauge fields

In the previous slides we have shown that unitarity holds for theories with scalar and fermion fields. We are now going to show that the proof of unitarity for gauge theories is more complicated and requires the use of the ward identities. The problem resides in the fact that the gauge fields in internal lines have unphysical polarizations while the final states should have only physical degrees of freedom. This difference would lead to a violation of unitarity in gauge theories. However we will show that the ghosts in the internal lines will compensate for this and will make the theory unitary as it should. Let us define the following amplitudes
\[ iT = \int \frac{d^4k_1}{(2\pi)^4} \left\{ \frac{1}{2} T_{\mu\nu}C_{\mu\nu}(k_1)G^{bb'}(k_2)T^{a'd'b'\mu'\nu'} - T^{ab}\Delta^{aa'}(k_1)\Delta^{bb'}(k_2)T^{a'a'b'b'} \right\} \]  

(6.392)

where

\[ k_2 = p_1 + p_2 - k_1 \]  

(6.391)

Using these definitions we can write the amplitude in the form (the factor 1/2 is a symmetry factor for the gauge fields and the minus sign is for the loop of ghosts)

Applying the Cutkosky rules we find for the imaginary part

\[ 2 \text{Im} T = \int \frac{d^4k_1}{(2\pi)^4} (2\pi)^2 \theta(k_1^0) \theta(k_2^0) \delta(k_1^2) \delta(k_2^2) \left\{ \frac{1}{2} T_{\mu\nu}T^{*a'b'\mu'\nu'} - T^{ab}T^{a'b'} \right\} \]

(6.393)

Now we have to evaluate \( \sigma_{\text{total}} \). As the ghosts are not physical we have

\[ \sigma = \sum \left| \int dp_2 \left( \frac{1}{2} T_{\mu\nu}T^{*a'b'\mu'\nu'} - T^{ab}T^{a'b'} \right) \right|^2 \]

(6.394)
where the factor $1/2$ comes now from identical particles in the final state. Writing
\[
\sum_{\text{Pol}} \varepsilon^\mu(k_1)\varepsilon^{\mu,*}(k_1) = P^{\mu\nu}(k_1)
\] (6.395)
we get
\[
\sigma = \int d\rho \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} P^{\mu\nu}(k_1) P^{\mu'\nu'}(k_2)
\] (6.396)
We now use the result (see problems)
\[
P^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu \eta^\nu + k^\nu \eta^\mu}{k \cdot \eta}
\] (6.397)
where $\eta^\mu$ is a four-vector that satisfies $\eta \cdot \varepsilon$ and $\eta^2 = 0$. We get
\[
\frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} P^{\mu\nu}(k_1) P^{\mu'\nu'}(k_2) =
\]
\[
= \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} - \frac{1}{2} T^{ab}_{\mu\nu} (k_2 \cdot \eta)\frac{1}{k_2 \cdot \eta} - \frac{1}{2} (k_1 \cdot T^{ab}_{\mu\nu}) T^{*ab}_{\mu'\nu'} - \frac{1}{2} (k_1 \cdot \eta) T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'}
\]
\[
= \frac{1}{2} (k_1 \cdot T^{ab}_{\mu\nu}) T^{*ab}_{\mu'\nu'} (k_2 \cdot \eta) + \frac{1}{2} (\eta \cdot T^{ab}_{\mu\nu})(k_1 T^{*ab}_{\mu'\nu'} T^{*ab}_{\mu'\nu'}) + \frac{1}{2} (\eta \cdot T^{ab}_{\mu\nu})(k_1 T^{*ab}_{\mu'\nu'} k_2 T^{*ab}_{\mu'\nu'})
\]
\[
\] (6.398)
Using the following Ward identities (see problems),
\[
k_1^{\mu} T^{ab}_{\mu\nu} = k_2^{\nu} T^{ab}_{\mu\nu} \quad \Rightarrow \quad k_1 \cdot T^{ab} \cdot k_2 = 0
\] (6.399)
we get
\[
\frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} P^{\mu\nu}(k_1) P^{\mu'\nu'}(k_2) =
\]
\[
= \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} - \frac{1}{2} T^{ab}_{\mu\nu} (k_1 \cdot T^{*ab}_{\mu'\nu'})\frac{1}{k_2 \cdot \eta} - \frac{1}{2} T^{*ab}_{\mu'\nu'} (k_1 \cdot \eta)\frac{1}{k_1 \cdot \eta} - \frac{1}{2} T^{ab}_{\mu'\nu'} (\eta \cdot T^{*ab}_{\mu'\nu'})\frac{1}{k_2 \cdot \eta} \quad \rightarrow \quad T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} = 0
\] (6.400)
Therefore after the sum over polarizations is correctly taken in account we obtain,

\[ \sigma = \int d\rho_2 \left[ \frac{1}{2} T_{\mu\nu}^{ab} T^{*ab\mu\nu} - T^{ab} T^{*ab} \right] \quad (6.401) \]

Comparing with the expression for \(2 \text{Im} T\) we get

\[ \sigma = 2 \text{Im} T \quad (6.402) \]

as we wanted to show.

It should be clear that the ghosts with their minus sign of the loop played a crucial role in subtracting the extra degrees of freedom. Also the Ward identities were necessary to relate the gauge field amplitudes with the ghost amplitude.
2.1 Mostre que $T(R)$ está relacionado com o operador de Casimir da representação $R$ $C_2(R)$ através de

$$T(R)r = d(R)C_2(R) \quad (6.403)$$

onde $r$ é a dimensão do Grupo $G$ e $d(R)$ é a dimensão da representação $R$. O Casimir $C_2(R)$ é definido por

$$\sum_{a,k} T^a_{ik} T^a_{kj} = \delta_{ij} C_2(R) \quad (6.404)$$

2.2 Mostrar que numa escolha diferente de condições auxiliares $\chi^\alpha = 0$ conduz ao mesmo resultado.

**Sugestão:** considere uma variação infinitesimal

$$\chi^\alpha + \delta \chi^\alpha = 0 \quad \alpha = 1, \ldots, m \quad (6.405)$$

Mostre então que

$$\pi_\alpha \delta (\varphi_\alpha) \delta (\chi_\alpha) \det(\{\varphi, \chi\}) \rightarrow \pi_\alpha \delta (\varphi_\alpha + \delta \chi_\alpha) \delta (\chi_\alpha + \delta \chi_\alpha) \det(\{\varphi, \chi + \delta \chi\}) \quad (6.406)$$

2.3 Mostre que para transformações infinitesimais

$$\delta E_\alpha(x) = -\frac{1}{y} \int_{x_n = y_n} d^3y \{E_\alpha(x), \alpha^b(y)C_b(y)\}$$

$$\delta A_\alpha(x) = -\frac{1}{y} \int_{x_n = y_n} d^3y \{A_\alpha(x), \alpha^b(y)C_b(y)\} \quad (6.407)$$

isto é, as ligações $C_\alpha$ são os geradores infinitesimais das transformações de gauge independentes do tempo.

2.4 Mostre que é sempre possível encontrar uma gauge onde $A_\alpha^3 = 0 \quad a = 1, \ldots, r$.

2.5 Mostre os resultados expressos na equação 6.94.
2.6 Mostre que a parte imaginária não depende do esquema de renormalização, calculando-a em $\overline{MS}$ e $MS$ para o exemplo 2.2, isto é para a teoria descrita pelo Lagrangeano da equação 6.356.

2.7 Considere a teoria $\frac{\lambda}{3!} \phi^3$ do problema 2.6. Faça a demonstração da unitariedade para a self-energy dessa teoria, isto é

\[ 2 \text{Im} = \sum f \]  \hspace{1cm} (6.408)

2.8 Considere a teoria descrita pelo Lagrangeano da Eq. (6.374). Refaça a demonstração geral da unitariedade no caso dos estados intermédios serem escalares, isto é

\[ 2 \text{Im} = \sum f \]  \hspace{1cm} (6.409)

2.9 Mostre que o integral que resulta de cortar $n$ linhas internas é igual ao integral do espaço de fase de $n$ partículas. Use este resultado para fazer uma demonstração geral da unitariedade.

2.10 Mostre que

\[ P^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu \eta^\nu + k^\nu \eta^\mu}{k \cdot \eta} \]  \hspace{1cm} (6.410)

onde $k^\mu, \varepsilon^\nu(k, 1), \varepsilon^\rho(k, 2)$ e $\eta^\sigma$ são quatro 4-vectores independentes e satisfazendo

\[ \eta \cdot \varepsilon(k, \sigma) = 0 \quad \sigma = 1, 2 \]
\[ \varepsilon(k, 1) \cdot \varepsilon(k, 2) = 0 \]
\[ k \cdot \varepsilon(k, \sigma) = 0 \quad \sigma = 1, 2 \]
\[ k^2 = 0 \]
\[ \eta^2 = 0 \quad \text{(escolha conveniente)} \]
\[ \varepsilon^2(k, \sigma) = -1 \quad \sigma = 1, 2 \]  \hspace{1cm} (6.411)
Sugestão: A expressão mais geral para $P_{\mu\nu}$ é

$$P_{\mu\nu} = ag_{\mu\nu} + bk^\mu k^\nu + c\eta^\mu \eta^\nu + d(k^\mu \eta^\nu + k^\nu \eta^\mu) .$$  

(6.412)

Use as relações anteriores para determinar $a, b, c, d$.

2.11 Demonstrar as identidades de Ward,

$$k_1^{\mu} T^{ab}_{\mu\nu} = k_2^{\nu} T^{ab} \quad \implies \quad k_1 \cdot T^{ab} \cdot k_2 = 0$$

(6.413)

onde $T^{ab}_{\mu\nu}$ e $T^{ab}$ são definidas em 6.390.

2.12 Mostre que o tensor $F^{a}_{\mu\nu}$ dos campos de Yang-Mills satisfaz as identidades de Bianchi:

$$D^{ab}_{\mu \rho\sigma} F^{a}_{\rho\sigma} + D^{ab}_{\rho \mu} F^{a}_{\rho\sigma} + D^{ab}_{\sigma \mu} F^{a}_{\rho\sigma} = 0$$  

(6.414)

ou

$$D^{ab}_{\mu \rho} F^{a}_{\rho\nu} = 0$$

(6.415)

onde

$$* F^{a}_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\rho\sigma} F^{a}_{\rho\sigma}$$

(6.416)

2.13 Explique o significado geométrico da Identidades de Bianchi.


2.14 Considere a teoria de Yang-Mills (YM) sem matéria.

a) Mostre que as eqs. de YM sem matéria se podem escrever na forma

$$\left\{ \begin{array}{l}
\nabla \cdot \vec{E}^a = \rho^a \\
\nabla \cdot \vec{B}^a = * \rho^a \\
\nabla \times \vec{E}^a = - \frac{\partial \vec{B}^a}{\partial t} + \vec{J}^a \\
\nabla \times \vec{B}^a = - \frac{\partial \vec{E}^a}{\partial t} + * \vec{J}^a
\end{array} \right.$$  

(6.417)

calcule $\rho^a$, $* \rho^a$, $\vec{J}^a$ e $* \vec{J}^a$.

b) Mostre que as 4-correntes $j^a_\mu \equiv (\rho^a, \vec{J}^a)$ e $* j^a_\mu \equiv (* \rho^a, * \vec{J}^a)$ são conservadas.

2.15 Mostre que $\text{Tr} (F_{\mu\nu} F^{\mu\nu})$ é uma 4-divergência. Comente sobre a sua inclusão na acção.

2.16 Mostre que o seguinte Ansatz (S. Coleman, Phys. Lett70B (77), 59)

$$A^{1a} = A^{2a} = 0$$

$$A^{0a} = - A^{3a} = x^1 f^a(x^0 + x^3) + x^2 g^a(x^0 + x^3)$$

(6.418)

onde $f^a$ e $g^a$ são funções arbitrárias, são soluções das equações de YM sem matéria. Discuta esta solução.
2.17 Considere o Ansatze de Wu-Yang para soluções estáticas em SU(2) YM.

\[ A^{0a} = x^a \frac{G(r)}{r^2} \quad A^{ia} = \varepsilon^{aij} x^j \frac{F(r)}{r^2} \]  

(a) Deduza as equações a que \( F \) e \( G \) devem obedecer.

(b) Mostre que elas são satisfeitas para \( F = -1/g \) e \( G = \) constante. Mostre que estas soluções correspondem a \( \rho^a = *\rho^a = 0 \) e \( J^a = *J^a = 0 \). \((\rho^a, \ldots \) são definidos no problema 2.14).

(c) Para as soluções da alínea b) descreva o potencial e os campos e calcule a energia.

2.18 Considere QED com a condição de gauge não linear

\[ F = \partial_\mu A^\mu + \frac{\lambda}{2} A_\mu A^\mu . \]  

(a) Escreva \( \mathcal{L}_{\text{eff}} \) e mostre que \( s\mathcal{L}_{\text{eff}} = 0 \), onde \( s \) é o operador de Slavnov.

(b) Calcule a polarização do vácuo a 1-loop. Discuta o programa de renormalização dando especial atenção aos vértices proporcionais a \( \lambda \). Pode aqui considerar a teoria sem fermiões.

(c) Demonstre a invariância da matriz \( S \) renormalizada em relação ao parâmetro \( \lambda \).

(d) Verifique o resultado anterior mostrando que o diagrama da figura junta, potencialmente perigoso para o momento magnético anômalo do elétron, não dá contribuição (seria proporcional a \( \lambda \)).

\[ \text{ localspecial attention to the vertices proportional to } \lambda. \]

(e) Deduza as identidades de Ward desta teoria para os funcionais \( Z \) e \( \Gamma \). Escreva o funcional gerador das equações de Dyson-Schwinger para os fantasmas, isto é,

\[ \frac{\delta \Gamma}{\delta \omega} = \cdots \]  

(f) Calcule ao nível árvore \( \gamma + \gamma \rightarrow \gamma + \gamma \). Compare com o resultado na gauge linear.

\[ k_1^{\mu} T^{\mu\nu} \neq 0 \text{ and } k_2^{\mu} T^{\mu\nu} \neq 0 \text{ onde } k_1 \text{ e } k_2 \text{ são os } 4- \text{ momentos dos fotões. Utilize as identidades de Ward para verificar os resultados. Há algum problema com este resultado?} \]
2.19 Considere a teoria que descreve as interações dos quarks com os gluões (Cromodinâmica Quântica) descrita pelo Lagrangeano

\[
L_{QCD} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \sum_{\alpha=1}^{n} \bar{\psi}_i^\alpha (i D - m_\alpha)_{ij} \psi_j^\alpha
\]  

onde

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c
\]

\[
(D_\mu)_{ij} = \delta_{ij} \partial_\mu - ig \left( \frac{\lambda^a}{2} \right)_{ij} A_\mu^a.
\]  

O índice \( \alpha = 1, 2, \ldots, n \) indica os diferentes sabores de quarks (up, down, ⋯, top). Para quantificar a teoria considere a condição de gauge

\[
L_{GF} = -\frac{1}{2\xi} (\partial_\mu A^{\mu a})^2,
\]

para a qual resulta o Lagrangeano dos fantasmas

\[
L_G = \partial_\mu \bar{\omega}^a \partial^\mu \omega^a + g f^{abc} \partial_\mu \bar{\omega}^a A_\mu^b \omega^c.
\]  

Para renormalizar a teoria necessitamos do seguinte Lagrangeano de contratermos:

\[
\Delta L = -\frac{1}{4} (Z_3 - 1) (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - (Z_4 - 1) g f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} - \frac{1}{4} g^2 (Z_5 - 1) f^{abc} f^{dce} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} + \sum_\alpha (Z_2 - 1) \bar{\psi}_i^\alpha \gamma^\mu \partial_\mu \psi_i^\alpha
\]

\[
- \sum_\alpha m_\alpha (Z_{m_\alpha} - 1) \bar{\psi}_i^\alpha \psi_i^\alpha + (Z_1 - 1) g \sum_\alpha \bar{\psi}_i^\alpha \gamma^\mu \left( \frac{\lambda^a}{2} \right)_{ij} \psi_i^\alpha A_\mu^c
\]

\[
+(Z_6 - 1) \partial_\mu \bar{\omega}^a \partial^\mu \omega^a + (Z_7 - 1) g f^{abc} \partial_\mu \bar{\omega}^a A_\mu^b \omega^c
\]  

a) Verifique a expressão para \( L_G \).

b) Considere a amplitude

\[
i T_{\mu\nu}^{ab} \equiv \begin{array}{c}
p_1 & k_1 & \mu, a \\
p_2 & k_2 & \nu, b \\
\end{array}
\]

Calcule ao nível árvore \( T_{\mu\nu}^{ab} \). Verifique que \( k_1^\mu T_{\mu\nu}^{ab} \neq 0 \).

c) Verifique as contas da alínea anterior calculando \( k_1^\mu T_{\mu\nu}^{ab} \) através das identidades de Ward.
d) Supondo que os gluões possam ser estados finais, a amplitude para o processo físico $q + q \to g + g$ onde $g$ é o gluão, é dada pela expressão

$$\mathcal{M} = \varepsilon^\mu(k_1) s^a T^{ab}_{\mu\nu} \varepsilon_{\nu}(k_2) s^b,$$

(6.428)

onde $\varepsilon^\mu(k_1)$ e $s^a$ são os vetores de polarização de spin e de cor, respectivamente (o mesmo para $\varepsilon_{\nu}(k_2)$ e $s^b$). Sabe-se que num processo físico, $\mathcal{M}$ se deve anular quando se faz a substituição $\varepsilon^\mu(k) \to k^\mu$. Como é que este resultado é compatível com as alíneas anteriores?

e) Mostre que se devem verificar as relações

$$\frac{Z_1}{Z_2} = \frac{Z_4}{Z_3} = \frac{Z_7}{Z_6} = \frac{\sqrt{Z_5}}{\sqrt{Z_3}},$$

(6.429)

f) Calcule $Z_1$, $Z_2$, $Z_3$, $Z_6$ e $Z_7$, usando subtração mínima (MS), (isto é, calcule só a parte divergente dos diagramas) e verifique explicitamente que $Z_1 Z_6 = Z_2 Z_7$.

g) Calcule a contribuição dos fermiões para $Z_4$ e $Z_5$ e verifique que também obedecem às relações da alínea a).

h) Calcule as funções do Grupo de Renormalização $\beta$, $\gamma_A$ e $\gamma_F$. 
Chapter 7

Renormalization Group

7.1 Callan -Symanzik equation

7.1.1 Renormalization scheme with momentum subtraction

Em teoria quântica dos campos um esquema de renormalização tem duas componentes. Primeiro há um processo de regularização que isola os infinitos que aparecem nos diagramas de Feynman. A regularização é arbitrária desde que mantenha as simetrias da teoria. Para teorias sem campos de gauge há muitos processos alternativos. Para teorias de gauge o melhor processo parece ser a regularização dimensional.

Depois de regularizada a teoria teremos que especificar um método sistemático para remover as divergências e definir os parâmetros renormalizados de teoria. A este processo chamamos esquema de renormalização. Há uma grande arbitrariedade na escolha do processo de subtração. A física contudo não pode depender desta escolha. Este é o conteúdo do grupo de renormalização: O conteúdo físico de teoria deve ser invariante para transformações que apenas mudem as condições de normalização. Esta afirmação trivial põe no entanto, como veremos, constrangimentos altamente não triviais no comportamento assintótico da teoria.

Vamos começar por estudar os chamados esquemas com subtração de momento. Conforme o ponto no espaço dos momentos externos que serve de definição às funções de Green irreductíveis, podemos ter várias formas deste esquema. Vamos exemplificar com a teoria \( \lambda \phi^4 \).

On–shell renormalization

Isto corresponde a uma série de Taylor para os momentos exteriores on - shell. Para a self-energy isto dá

\[
\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \tilde{\Sigma}(p^2)
\]

(7.1)

com as condições
\[
\left\{ \begin{array}{l}
\tilde{\Sigma}(m^2) = 0 \\
\frac{\partial \tilde{\Sigma}(p^2)}{\partial p^2} \bigg|_{p^2=m^2} = 0 
\end{array} \right. \quad (7.2)
\]

Em termos de \(\Gamma_R^{(2)}(p^2)\) dado por
\[
\Gamma_R^{(2)}(p) = p^2 - m^2 - \tilde{\Sigma}(p^2) \quad (7.3)
\]
temos
\[
\left\{ \begin{array}{l}
\Gamma_R^{(2)}(m^2) = 0 \\
\frac{\partial \Gamma_R^{(2)}}{\partial p^2} \bigg|_{p^2=m^2} = 1 
\end{array} \right. \quad (7.4)
\]

Para \(\Gamma_R^{(4)}\) uma escolha conveniente é
\[
\Gamma_R^{(4)}(p_1, p_2, p_3) = -\lambda \quad \text{para} \quad \left\{ \begin{array}{l}
p_i^2 = m^2 \\
s = t = u = \frac{4m^2}{3}
\end{array} \right. \quad (7.5)
\]

Neste caso os parâmetros \(m^2\) e \(\lambda\) são a massa física e, a menos de factores cinemáticos, a secção eficaz para \(s = t = u = \frac{4}{3} m^2\) respectivamente.

**Intermediate renormalization**

Este esquema corresponde a uma expansão de Taylor em torno de momentos nulos.

\[
\Sigma(p^2) = \Sigma(0) + \Sigma'(0)p^2 + \tilde{\Sigma}(p^2) \quad (7.6)
\]

A parte finita \(\tilde{\Sigma}(p^2)\) obedece às condições
\[
\left\{ \begin{array}{l}
\tilde{\Sigma}(0) = 0 \\
\frac{\partial \tilde{\Sigma}}{\partial p^2} \bigg|_{p^2=0} = 0 
\end{array} \right. \quad (7.7)
\]

que traduzidas em termos de \(\Gamma_R^{(2)}\) se escrevem
\[
\left\{ \begin{array}{l}
\Gamma_R^{(2)}(0) = m^2 \\
\frac{\partial \Gamma_R^{(2)}}{\partial p^2} = 1 
\end{array} \right. \quad (7.8)
\]

Para \(\Gamma_R^{(4)}\) a condição é
7.1. CALLAN-SYMANZIK EQUATION

\[ \Gamma_R^{(4)}(p_1, p_2, p_3) = -\lambda \quad \text{para} \quad p_1 = p_2 = p_3 = 0 \quad (7.9) \]

Neste esquema \( m^2 \) não é a massa física e \( \lambda \) não é nenhuma quantidade mensurável pois os pontos \( p_i = 0 \) não pertencem à região física. Podemos no entanto exprimir todas as quantidades mensuráveis em termos destes dois parâmetros, como veremos na secção 3.3.

**General case**

Os dois exemplos anteriores são casos particulares do esquema geral onde as condições de normalização podem ser funções de vários momentos de referência \( \xi_1, \xi_2 \ldots \) tais que

\[
\begin{align*}
\Gamma_R^{(2)}(\xi_1^2) &= m^2 \\
\frac{\partial \Gamma_R^{(2)}}{\partial p^2} \bigg|_{p^2=\xi_2^2} &= 1 \\
\Gamma_R^{(4)}(\xi_3, \xi_4, \xi_5) &= -\lambda
\end{align*}
\]

\[ (7.10) \]

**7.1.2 Renormalization group**

Consideremos dois esquemas de renormalização \( R \) e \( R' \). Como ambos partem do mesmo Lagrangeano não renormalizado

\[ \mathcal{L} = \mathcal{L}_R + \Delta \mathcal{L}_R = \mathcal{L}_{R'} + \Delta \mathcal{L}_{R'} \quad (7.11) \]

devemos ter

\[ \phi_R = Z_{\phi}^{-1/2}(R)\phi_0 ; \quad \phi'_{R} = Z_{\phi}^{-1/2}(R')\phi_0. \quad (7.12) \]

Logo

\[ \phi'_R = Z_{\phi}^{-1/2}(R', R)\phi_R \quad (7.13) \]

onde

\[ Z_{\phi}(R', R) = \frac{Z_{\phi}(R')}{Z_{\phi}(R)} \quad (7.14) \]

Estas relações indicam que os campos renormalizados em diferentes esquemas estão relacionados por uma constante multiplicativa. Esta constante é finita pois tanto \( \phi_{R'} \) como \( \phi_R \) são finitos. De modo semelhante

\[ \lambda_{R'} = Z_{\lambda}^{-1}(R', R)Z_{\phi}^2(R', R)\lambda_R \]

\[ m_{R'}^2 = m_R^2 + \delta m^2(R', R) \quad (7.15) \]
onde

\[ Z_\lambda(R', R) = \frac{Z_\lambda(R')}{Z_\lambda(R)} \]

\[ \delta m^2(R', R) = \delta m^2(R') - \delta m^2(R) \quad (7.16) \]

são quantidades finitas. A operação que leva as quantidades num esquema de renormalização \( R \) para outro esquema \( R' \) pode ser vista como uma transformação de \( R \) em \( R' \). O conjunto de todas estas transformações forma o Grupo de Renormalização.

### 7.1.3 Callan - Symanzik equation

Vamos agora ver como dar uma expressão analítica à invariancia para transformações do grupo de renormalização. A forma da equação do grupo de renormalização depende do esquema de renormalização utilizado. Vamos aqui obter as equações do GR para o esquema com subtração de momento, a chamada equação de Callan - Symanzik.

Notemos primeiro que

\[ \frac{\partial}{\partial m_0^2} \left( \frac{i}{p^2 - m_0^2 + i\varepsilon} \right) = \frac{i}{p^2 - m_0^2 + i\varepsilon} \frac{i}{p^2 - m_0^2 + i\varepsilon} \quad (7.17) \]

isto é, a derivação dum função de Green não renormalizada em relação à massa despida é equivalente à inserção dum operador composto \( \frac{1}{2} \phi^2 \) levando momento zero, isto é

\[ \frac{\partial \Gamma^{(n)}(p_i)}{\partial m_0^2} = -i\Gamma^{(n)}_\phi(0, p_i) \quad (7.18) \]

As funções Green irredutíveis renormalizadas são dadas por

\[
\begin{cases}
  \Gamma^{(n)}_R (p_i; \lambda; m) = Z_\phi^{-n/2} \Gamma^{(n)}_R (p_i; \lambda_0; m_0) \\
  \Gamma^{(n)}_{\phi^2 R} (p; p_i; \lambda; m) = Z_\phi^{-n/2} \Gamma^{(n)}_{\phi^2} (p; p_i; \lambda_0; m_0)
\end{cases} \quad (7.19)
\]

Então a equação anterior escreve-se

\[ \frac{\partial}{\partial m_0^2} \left[ Z_\phi^{-n/2} \Gamma^{(n)}_R (p_i, \lambda, m) \right] = -iZ_\phi^{-1} Z_\phi^{-n/2} \Gamma^{(n)}_{\phi^2 R} (0, p_i, \lambda, m) \quad (7.20) \]

e portanto

\[ -\frac{n}{2} Z_\phi^{-1} \frac{\partial Z_\phi}{\partial m_0^2} Z_\phi^{-n/2} \Gamma^{(n)}_R (p_i, \lambda, m) + Z_\phi^{-n/2} \frac{\partial}{\partial m_0^2} \Gamma^{(n)}_R (p) = -iZ_\phi^{-1} Z_\phi^{-n/2} \Gamma^{(n)}_{\phi^2 R} (0, p_i, \lambda, m) \quad (7.21) \]

ou seja
7.1. CALLAN-SYMANZIK EQUATION

\[
\left[ \frac{\partial}{\partial m} - \frac{n}{2} \frac{\partial \ln Z}{\partial m^2} \right] \Gamma_R^{(n)} = -i Z_{\phi^2} \Gamma_{\phi^2 R}^{(n)}
\]

\[
\left[ \frac{\partial^2}{\partial m^2} \frac{\partial}{\partial m} + \frac{\partial}{\partial m_0^2} \frac{\partial^2}{\partial m} + \frac{\partial}{\partial \lambda} \right] \Gamma_R^{(n)} = -i Z_{\phi^2} \Gamma_{\phi^2 R}^{(n)}
\]

ou ainda

\[
\left[ m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial \lambda} - n \gamma \right] \Gamma_R^{(n)} = -i m^2 \alpha \Gamma_{\phi^2 R}^{(n)}
\]

que é a equação de Callan-Symanzik para a teoria $\phi^4$, onde $\alpha, \beta$ e $\gamma$ são funções sem dimensões

\[
\begin{align*}
\beta &= 2 m^2 \frac{\partial \lambda}{\partial m_0^2} \\
\gamma &= m^2 \frac{\partial \ln Z_{\phi}}{\partial m_0^2} \\
\alpha &= 2 Z_{\phi^2} \frac{\partial}{\partial m^2} \frac{\partial \phi}{\partial m_0^2}
\end{align*}
\]

A função $\alpha$ está relacionada com $\gamma$. De facto se escolhermos as condições de normalização a $p_i = 0$

\[
\left\{ \begin{array}{l}
\Gamma_R^{(2)}(0, \lambda, m) = -m^2 \\
\Gamma_{\phi^2 R}^{(2)}(0, 0, \lambda, m) = i
\end{array} \right.
\]

obtemos

\[
\alpha = 2(\gamma - 1)
\]

Como as quantidades $\Gamma_R^{(n)}$ e $\Gamma_{\phi^2 R}^{(n)}$ não dependem do cut-off, esperamos também que $\alpha, \beta$ e $\gamma$ sejam independentes do cut-off. Para vermos isso pomos $n = 2$ e diferenciamos em ordem a $p^2$

\[
\left[ m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial \lambda} - 2 \gamma \right] \frac{\partial}{\partial p^2} \Gamma_R^{(2)}(p, \lambda, m) = -i m^2 \alpha \frac{\partial}{\partial p^2} \Gamma_{\phi^2 R}^{(2)}(0, p, \lambda, m)
\]

Pondo $p^2 = 0$ e usando
\[ \frac{\partial \Gamma^{(2)}_{R}}{\partial p^2} \bigg|_{p^2=0} = 1 \quad (7.30) \]

Obtemos

\[ \gamma = i m^2 (\gamma - 1) \left[ \frac{\partial}{\partial p^2} \Gamma^{(2)}_{\Phi^2 R}(0, p, \lambda, m) \right]_{p^2=0} = 1 \quad (7.31) \]

o que demonstra que \( \gamma \) é independente do cut-off. Então \( \alpha = 2(\gamma - 1) \) também é e todas as funções exceto \( \beta \) são agora independentes do cut-off. Portanto \( \beta \) também é.

Como \( \alpha, \beta \) e \( \gamma \) são sem dimensões e não dependem do cut-off, então são somente funções da constante de acoplamento que também não tem dimensões, isto é

\[ \alpha = \alpha(\lambda) \]
\[ \beta = \beta(\lambda) \]
\[ \gamma = \gamma(\lambda) \quad (7.32) \]

Nós vamos sobretudo estar interessados no esquema de subtração míni-ma, por isso não vamos agora calcular as funções \( \alpha, \beta \) e \( \gamma \) para todas as teorias, faremos isso na secção 3.3. Indicaremos no entanto um método expedito para o seu cálculo. Seja por exemplo a função \( \beta(\lambda) \). Notando que

\[ \frac{\partial \lambda}{\partial m^2}(\lambda_0, \Lambda/m) = \frac{\partial m^2}{\partial \lambda_0^2} \frac{\partial \lambda_0}{\partial m^2} \lambda(\lambda_0, \Lambda/m) \]
\[ = \frac{\partial m^2}{\partial \lambda_0^2} \frac{1}{2m} \frac{\partial \lambda}{\partial m} \lambda(\lambda_0, \Lambda/m) \quad (7.33) \]

obtemos da definição 7.24

\[ \beta = m \frac{\partial}{\partial m} \lambda(\lambda_0, \Lambda/m) = m \frac{\partial}{\partial \lambda_0} [Z(\lambda_0, \Lambda/m)] = -\lambda_0 \Lambda \frac{\partial}{\partial \Lambda} [Z(\lambda_0, \Lambda/m)] \quad (7.34) \]

ou

\[ \beta = -\lambda \frac{\partial}{\partial \ln \Lambda} [\ln Z(\lambda_0, \Lambda/m)] \quad (7.35) \]

onde\(^1\) \( Z = Z_{\lambda}^{-1} Z_{\phi}^2 \). O resultado de 1 - loop dá

\[ Z_{\lambda} = 1 + \frac{3\lambda_0}{32\pi^2} \ln \frac{\Lambda^2}{m^2} + O(\lambda_0^2) \]

\(^1\)Por definição \( \lambda = Z_{\lambda_0} \).
\[ Z_\phi = 1 + O(\lambda_0^2) \]  
(7.36)

then

\[ Z = 1 - \frac{3\lambda_0}{32\pi^2} \ln \frac{\Lambda^2}{m^2} + ... \]  
(7.37)

and

\[ \ln Z = \frac{3\lambda_0}{16\pi^2} \ln \frac{\Lambda}{m} + \cdots \]  
(7.38)

Therefore for \( \phi^4 \)

\[ \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) \]  
(7.39)

7.1.4 Weinberg’s theorem and the solution of the RG equations

The theorem of Weinberg refers to the asymptotic behavior of the Green functions in the Euclidean region \( (p_i^2 < 0) \) and for nonexceptional momenta (no partial sums are zero).

Teorema 3.1

Se os momentos não forem excepcionais e se os parametrizemos com \( p_i = \sigma k_i \) as funções de Green irreduíveis e partícula \( \Gamma^{(n)}_R \) comportam-se na região euclediana profunda \( (\sigma \to \infty e k_i \text{ fixos}, p_i^2 < 0) \) do modo seguinte

\[ \lim_{\sigma \to \infty} \Gamma^{(n)}(\sigma k_i, \lambda, m) = \sigma^{4-n}[a_0(\ln \sigma)^{b_0} + a_1(\ln \sigma)^{b_1} + \cdots] \]  
(7.40)

and

\[ \lim_{\sigma \to \infty} \Gamma^{(n)}_{\phi^2}(\sigma k_i, \lambda, m) = \sigma^{2-n}[a'_0(\ln \sigma)^{b'_0} + a'_1(\ln \sigma)^{b'_1} + \cdots] . \]  
(7.41)

We do not present the proof (see Bjorken and Drell) but note that the powers of \( \sigma \) are the canonical dimensions of the Green functions (in terms of the mass). If this behavior is verified asymptotically it depends on the sum of the series of logarithms. If the sum of a power of \( \sigma \), for example \( \sigma^{-\gamma} \), then asymptotically the behavior canonical \( \sigma^{4-n} \) is modified for \( \sigma^{4-n-\gamma} \). \( \gamma \) is called the anomalous dimension. As we will see the GR will perform this sum of logarithms and give us what the anomalous dimension.

7.1.5 Asymptotic solution of the RG equations

From the theorem of Weinberg we have that \( \Gamma^{(n)}_R \gg \Gamma^{(n)}_{\phi^2 R} \) for any order (finite) in \( \lambda \) in the Euclidean region \( (\sigma \to \infty) \). We admit that this is a true continuity of the same
depois de somar todas as ordens de teoria de perturbações, então podemos desprezar o segundo membro da equação de Callan-Symanzik e obtemos uma equação diferencial homogénea

\[
\left[ m \frac{\partial}{\partial m} + \beta(\lambda) \frac{\partial}{\partial \lambda} - n \gamma(\lambda) \right] \Gamma_{as}^{(n)}(p_i, \lambda, m) = 0
\]  

(7.42)

onde \( \Gamma_{as}^{(n)} \) é a forma assintótica de \( \Gamma_R^{(n)} \). O significado desta equação é que nesta região assintótica, uma mudança no parâmetro de massa pode ser sempre compensada por mudanças apropriadas do acoplamento e da escala dos campos.

Para resolver esta equação começamos por definir uma quantidade \( \Gamma_R^{(n)} \) sem dimensões, usando análise dimensional

\[
\Gamma_{as}^{(n)}(p_i, \lambda, m) = m^{4-n} \Gamma_R^{(n)}(p_i/m, \lambda).
\]  

(7.43)

\( \Gamma_R^{(n)} \) satisfaz

\[
\left( m \frac{\partial}{\partial m} + \sigma \frac{\partial}{\partial \sigma} \right) \Gamma_R^{(n)}(\sigma p_i/m, \lambda) = 0.
\]  

(7.44)

Então

\[
\left( m \frac{\partial}{\partial m} + \sigma \frac{\partial}{\partial \sigma} \right) m^{n-4} \Gamma_{as}^{(n)}(\sigma p_i, \lambda, m) = 0
\]  

(7.45)

ou seja

\[
\left[ m \frac{\partial}{\partial m} + \sigma \frac{\partial}{\partial \sigma} + (n - 4) \right] \Gamma_{as}^{(n)}(\sigma p_i, \lambda, m) = 0
\]  

(7.46)

Usando esta equação podemos trocar a derivação em ordem à massa pela derivação em ordem à escala na equação de Callan-Symanzik para obter

\[
\left[ \sigma \frac{\partial}{\partial \sigma} - \beta(\lambda) \frac{\partial}{\partial \lambda} + n \gamma(\lambda) + (n - 4) \right] \Gamma_{as}^{(n)}(\sigma p_i, \lambda, m) = 0
\]  

(7.47)

Para resolver esta equação removemos os termos sem derivadas com a transformação

\[
\Gamma_{as}^{(n)}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{\int_0^\lambda \frac{n(2)}{m(\sigma)} dx} F^{(n)}(\sigma p_i, \lambda, m).
\]  

(7.48)

Substituindo na equação diferencial vemos que os termos sem derivadas desaparecem e obtemos uma equação diferencial para \( F^{(n)} \)

\[
\left[ \sigma \frac{\partial}{\partial \sigma} - \beta(\lambda) \frac{\partial}{\partial \lambda} \right] F^{(n)}(\sigma p, \lambda, m) = 0
\]  

(7.49)

Introduzindo \( t = \ln \sigma \) podemos escrever
7.1. CALLAN-SYMANZIK EQUATION

\[
\left[ \frac{\partial}{\partial t} - \beta(\lambda) \frac{\partial}{\partial \lambda} \right] F^{(n)}(e^t p, \lambda, m) = 0 \tag{7.50}
\]

Para resolver esta equação introduzimos a constante de acoplamento efectiva \( \lambda(t, \lambda) \) como solução da equação

\[
\frac{\partial \lambda(t, \lambda)}{\partial t} = \beta(\lambda) \tag{7.51}
\]

com a condição fronteira \( \lambda(0, \lambda) = \lambda \). Para vermos que esta definição nos vai dar a solução, escrevemos

\[
t = \int_{\lambda}^\lambda \frac{dx}{\beta(x)} \tag{7.52}
\]

e diferenciamos em ordem a \( \lambda \)

\[
0 = \frac{1}{\beta(\lambda)} \frac{\partial \lambda}{\partial \lambda} - \frac{1}{\beta(\lambda)} \tag{7.53}
\]

ou ainda

\[
\beta(\lambda) - \beta(\lambda) \frac{\partial \lambda}{\partial \lambda} = 0 \tag{7.54}
\]

Usando agora a definição de \( \lambda \) obtemos

\[
\left[ \frac{\partial}{\partial t} - \beta(\lambda) \frac{\partial}{\partial \lambda} \right] \lambda(t, \lambda) = 0 \tag{7.55}
\]

O operador diferencial é exatamente o mesmo da equação para \( F^{(n)}(e^t p, \lambda, m) \). Portanto \( F^{(n)} \) satisfaz aquela equação se depender da \( t \) e \( \lambda \) através de \( \lambda(t, \lambda) \). Portanto a solução geral de \( \Gamma_{gs}^{(n)} \) é

\[
\Gamma_{gs}^{(n)}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{n \int_0^\lambda \frac{\gamma(x)}{\sigma(x)} dx} F^{(n)}(p_i, \lambda(t, \lambda), m) \tag{7.56}
\]

Para se obter uma interpretação física desta solução notemos que

\[
e^{n \int_0^\lambda \frac{\gamma(x)}{\sigma(x)} dx} = e^{n \int_0^\lambda \frac{\gamma(x)}{\sigma(x)} dx} e^n \int_0^\lambda \frac{\gamma(x)}{\sigma(x)} dx = e^n \int_0^\lambda \frac{\gamma(x)}{\sigma(x)} dx e^{-n \int_0^\lambda \frac{\gamma(x)}{\sigma(x)} dx} = e^n \int_0^\lambda \frac{\gamma(x)}{\sigma(x)} dx e^{-n \int_0^\lambda \gamma(t', \lambda) dt'} \tag{7.57}
\]

Portanto
\[ \Gamma^{(n)}(\sigma p_i, \lambda, m) = \sigma^{4-n}e^{-n} \int_0^t \gamma(\lambda(t'), \lambda) dt' e^{-n} \int_{\lambda(t')}^{\lambda(t)} \frac{\gamma(x')}{\mu(x')} \, dx' \Gamma^{(n)}(p_i, \lambda(t'), \lambda, m) \]  

(7.58)

Se pusermos \( \sigma = 1(t = 0) \), vemos que \( e^{n} \int_0^\gamma \beta(x) dx \) \( \Gamma^{(n)}_{as} \). Então obtemos finalmente a solução da equação do GR.

\[ \Gamma^{(n)}_{as}(\sigma p_i, \lambda, m) = \sigma^{4-n}e^{-n} \int_0^t \gamma(\lambda(t'), \lambda) dt' \Gamma^{(n)}_{as}(p_i, \lambda(t'), \lambda, m) \]  

(7.59)

Nesta forma a solução tem uma interpretação simples. O efeito de efectuar uma mudança de escala nos momentos \( p_i \) nas funções de Green \( \Gamma^{(n)}_{R} \) é equivalente a substituir a constante de acoplamento \( \lambda \), pela constante de acoplamento efectiva \( \lambda' \) a parte factores multiplicativos. O primeiro é simplesmente resultante do facto de \( \Gamma^{(n)}_{R} \) ter dimensão canónica \( 4 - n \) em unidades de massa. O factor exponencial é o termo da dimensão anómala que resultou de somar todos os logaritmos em teoria de perturbações. Este factor é controlado por \( \gamma \), a dimensão anómala. Veremos à frente como calcular a dimensão anómala, numa teoria qualquer.

### 7.2 Minimal subtraction (MS) scheme

#### 7.2.1 Renormalization group equations for MS

Vamos agora ver outras formas que pode tomar a equação do grupo de renormalização. A afirmação que a renormalização é multiplicativa pode ser escrita na forma

\[ \Gamma^{(n)}(p_i, \lambda_0, m_0) = Z_{\phi}^{-n/2} \Gamma^{(n)}_{R}(p_i, \lambda, m, \mu) \]  

(7.60)

onde \( \mu \) é a escala usada para definir a normalização das funções de Green. O lado esquerdo da equação não depende de \( \mu \), mas o lado direito depende explicitamente e implicitamente através de \( \lambda \) e \( m \). Então temos

\[ \mu \frac{\partial}{\partial \mu} \left[ Z_{\phi}^{-n/2} \Gamma^{(n)}_{R}(p_i, \lambda, m, \mu) \right] = 0 \]  

(7.61)

ou seja

\[ \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \gamma_m \frac{\partial}{\partial m} - n \gamma \right) \Gamma^{(n)}_{R} = 0 \]  

(7.62)

com

\[ \beta \left( \lambda, \frac{m}{\mu} \right) = \mu \frac{\partial \lambda}{\partial \mu} \]

\[ \gamma_m \left( \lambda, \frac{m}{\mu} \right) = \mu \frac{\partial \ln m}{\partial \mu} \]
7.2. MINIMAL SUBTRACTION (MS) SCHEME

\[ \gamma \left( \lambda, \frac{m}{\mu} \right) = \frac{1}{2 \mu} \frac{\partial \ln Z_\phi}{\partial \mu} \quad (7.63) \]

Esta equação tem a vantagem sobre a equação de Callan - Symanzik de ser homogénea. A dificuldade reside nas funções \( \beta \) e \( \gamma \) dependerem de duas variáveis \( \lambda \) e \( \frac{m}{\mu} \) e portanto a equação ser de difícil resolução. Existe contudo um esquema de renormalização em que a dependência em \( \frac{m}{\mu} \) desaparece e portanto a equação é simples de resolver. É o chamado esquema de subtracção mínima que passamos a expôr.

7.2.2 Minimal subtraction scheme

O esquema de subtração mínima (MS) está relacionado com o método de regularização dimensional. As divergências dos integrais aparecem neste método como pólos em \( \frac{1}{\varepsilon} \) onde \( \varepsilon = 4 - d \). O esquema de subtração mínima consiste em escolher os contratermos para cancelar somente os pólos.

Vamos exemplificar com a self-energy em \( \lambda \phi^4 \), a que corresponde o diagrama da Fig. 7.1

\[ \text{Figure 7.1: Diagram for self-energy in } \phi^4. \]

Temos

\[ -i \Sigma(p) = (-i \lambda) \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{i}{p^2 - m^2 + i\varepsilon} \]

\[ = -i\lambda \frac{1}{32\pi^2} \mu^\varepsilon \frac{\Gamma(1 - d/2)}{m^{2-\varepsilon}} \cdot 2^{\varepsilon/2} \pi^{\varepsilon/2} \quad (7.64) \]

onde \( \varepsilon = 4 - d \). Então

\[ \Sigma(p^2) = \lambda \frac{1}{32\pi^2} \mu^\varepsilon \frac{\Gamma(-1 + \varepsilon/2)}{m^{-2+\varepsilon}} \cdot (2\sqrt{\pi})^\varepsilon \]

\[ = \lambda \frac{m^2}{32\pi^2} \left( \frac{\mu}{m} \right)^\varepsilon \Gamma(-1 + \varepsilon/2) \cdot (2\sqrt{\pi})^\varepsilon \quad (7.65) \]

Usando\(^2\)

\(^2\gamma\) é a constante de Euler e \( \psi(x) \) a derivada logaritmica da função \( r \). Ver o Apêndice da Mecânica Quântica Relativista.
Γ\left(-1 + \frac{\varepsilon}{2}\right) = -\left[\frac{2}{\varepsilon} + \frac{\psi(2)}{\varepsilon} + 1 - \gamma + O(\varepsilon)\right] \quad (7.66)

e

\left(\frac{\mu}{m}\right)^{\varepsilon} = 1 + \varepsilon \ln\left(\frac{\mu}{m}\right) \quad (7.67)

obtemos

\Sigma(p^2) = -\frac{\lambda m^2}{32\pi^2} \left[\frac{2}{\varepsilon} + \psi(2) + 2 \ln(\mu/m) + 2 \ln 2\sqrt{\pi} + O(\varepsilon)\right] \quad (7.68)

Portanto no esquema de subtração mínima devemos adicionar um contratermo

\Delta L_{\phi^2}^{MS} = -\frac{\lambda m^2}{32\pi^2} \frac{1}{\varepsilon} \phi^2 \quad (7.69)

Se tivéssemos feito subtração de momento à escala \(\mu\), isto é \(\Sigma_R(p^2 = \mu^2) = 0\) teríamos o contratermo

\Delta L_{\phi^2}^{MOM} = -\frac{\lambda m^2}{32\pi^2} \left[\frac{1}{\varepsilon} + \frac{1}{2} \psi(2) + \ln(\mu/m) + \ln 2\sqrt{\pi}\right] \phi^2 \quad (7.70)

Vemos assim que o Lagrangeano de contratermos no esquema de subtração mínima quando expandido em série de Laurent em \(\varepsilon\) contém só termos divergentes. Portanto

\phi_0 = \sqrt{\frac{Z_\phi}{m_0}}\phi

m_0 = Z_m m

\lambda_0 = \mu^\varepsilon Z_\lambda \lambda

(tendo as constantes de renormalização \(Z_\phi\), \(Z_m\) e \(Z_\lambda\) a forma

\begin{align*}
Z_\lambda &= 1 + \sum_{r=1}^{\infty} a_r(\lambda)/\varepsilon^r \\
Z_m &= 1 + \sum_{r=1}^{\infty} b_r(\lambda)/\varepsilon^r \\
Z_\phi &= 1 + \sum_{r=1}^{\infty} c_r(\lambda)/\varepsilon^r
\end{align*} \quad (7.72)
7.2. MINIMAL SUBTRACTION (MS) SCHEME

Assim os coeficientes da equação do grupo de renormalização são independentes de \( \mu \), e como são adimensionais, também são independentes de \( m \) dependendo somente da constante de acoplamento. Isto simplifica a equação da renormalização

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \gamma_m m \frac{\partial}{\partial m} - n \gamma \right) \Gamma^{(n)}_R = 0 \tag{7.73}
\]

Usando análise dimensional

\[
\left[ m \frac{\partial}{\partial m} + (n-4) + \mu \frac{\partial}{\partial \mu} + \sigma \frac{\partial}{\partial \sigma} \right] \Gamma_R(\sigma, m, \lambda, \mu) = 0 \tag{7.74}
\]

e podemos escrever

\[
\left[ \sigma \frac{\partial}{\partial \sigma} - \beta \frac{\partial}{\partial \lambda} - (\gamma_m - 1) m \frac{\partial}{\partial m} + n \gamma + (n-4) \right] \Gamma_R(\sigma, m, \lambda, \mu) = 0 \tag{7.75}
\]

que tem a solução

\[
\Gamma_R(\sigma p_i, m, \lambda, \mu) = \sigma^{4-n} e^{-n \int_0^t \gamma(\lambda(t')) dt'} \Gamma^{(n)}_R(p_i, \overline{m}(t), \overline{\lambda}(t), \mu) \tag{7.76}
\]

onde se introduziram a massa efectiva \( \overline{m}(t) \) e a constante de acoplamento efectiva \( \overline{\lambda}(t) \) definidas por

\[
\begin{align*}
\frac{d\overline{\lambda}}{dt} &= \beta(\overline{\lambda}) ; \quad \overline{\lambda}(t = 0) = \lambda \\
\frac{d\overline{m}(t)}{dt} &= \left[ \gamma_m(\lambda) - 1 \right] \overline{m}(t) ; \quad \overline{m}(t = 0) = m
\end{align*} \tag{7.77}
\]

A solução desta equação é

\[
\overline{m}(t) = m e^{\int_0^t \left[ \gamma_m(\overline{\lambda}(t')) - 1 \right] dt'}
\]

\[
= m e^{-t e^{\int_0^t \gamma_m(\overline{\lambda}(t')) dt'}}
\]

\[
= m e^{-t e^{\int_\lambda^{(t)} dx \gamma_m(x)}} \tag{7.78}
\]

7.2.3 Physical parameters

Os parâmetros definidos pelo esquema de subtração mínima não são parâmetros físicos. Os parâmetros físicos podem no entanto ser calculados em função deles. Por parâmetro físico entendemos um elemento de matriz \( S \) ou a posição do pólo no propagador. Para eles é válido o teorema seguinte,

\textbf{Teorema 3.2:}

Qualquer parâmetro físico \( P(\lambda, m, \mu) \) satisfaz a seguinte equação do grupo de renormalização
\[ DP(\lambda, m, \mu) \equiv \left[ \mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma m \frac{\partial}{\partial m} \right] P(\lambda, m, \mu) = 0 \quad (7.79) \]

**Dem:** Consideremos primeiro o propagador escalar \( \Delta(p^2) \) que satisfaz a equação do grupo de renormalização

\[ [D + 2\gamma] \Delta(p^2, \lambda, m, \mu) = 0 \quad (7.80) \]

Podemos escrever uma série de Laurent em volta do pôlo em \( p^2 = m_p^2 \)

\[ \Delta(p^2, \lambda, m, \mu) = \frac{R^2}{p^2 - m_p^2} + \tilde{\Delta} \quad (7.81) \]

A posição do pôlo \( m_p(\lambda, m, \mu) \) e o resíduo \( R^2(\lambda, m, \mu) \) satisfazem as equações do grupo de renormalização que podem ser obtidas por aplicação do operador \( (D + 2\gamma) \) à equação anterior. Igualando os resíduos dos pôlos obtemos

\[ Dm_p(\lambda, m, \mu) = 0 \quad (7.82) \]

\[ [D + \gamma(\lambda)] R(\lambda, m, \mu) = 0 \quad (7.83) \]

Demonstrámos portanto o teorema para a massa física. Para um elemento da matriz \( S \) temos (\( S_R = R^n \Gamma^{(n)} \))

\[ D \lim_{p_i^2 \to m_p^2} R^n \Gamma^{(n)} = \lim_{p_i^2 \to m_p^2} D(R^n \Gamma^{(n)}) \]

\[ = \lim_{p_i^2 \to m_p^2} \left[ nD RR^{n-1} \Gamma^{(n)} + R^n DD^{(n)} \right] \]

\[ = \lim_{p_i^2 \to m_p^2} \left[ -n\gamma + n\gamma \right] R^n \Gamma^{(n)} = 0 \quad (7.84) \]

o que completa a demonstração.

Veremos à frente como estes resultados podem ser usados para relacionar os parâmetros físicos com os parâmetros da teoria.

### 7.2.4 Renomalization group functions in minimal subtraction

Vimos anteriormente que

\[
\begin{align*}
\phi_0 &= \sqrt{Z_\phi} \\
m_0 &= Z_m m \\
\lambda_0 &= \mu^\varepsilon Z_\lambda \lambda
\end{align*}
\]

(7.85)

e que as constantes de renormalização têm a forma.
7.2. MINIMAL SUBTRACTION (MS) SCHEME

\[
\begin{align*}
Z_\lambda &= 1 + \sum_{r=1}^{\infty} a_r(\lambda)/\varepsilon^r \\
Z_m &= 1 + \sum_{r=1}^{\infty} b_r(\lambda)/\varepsilon^r \\
Z_\phi &= 1 + \sum_{r=1}^{\infty} c_r(\lambda)/\varepsilon^r.
\end{align*}
\]

(7.86)

Vejamos como se calculam \(\beta, \gamma_m\) e \(\gamma\).

i) Cálculo de \(\beta(\lambda)\)

Por definição

\[
\beta(\lambda) = \frac{\mu}{\partial \lambda} (7.87)
\]

Esta quantidade é finita no limite \(\varepsilon \to 0\). Isto quer dizer que antes de fazermos \(\varepsilon \to 0\) deve ser uma função analítica em \(\varepsilon\). É então conveniente definir

\[
\beta(\lambda) = \hat{\beta}(\lambda, \varepsilon = 0) = d_0
\]

(7.88)

onde

\[
\hat{\beta}(\lambda, \varepsilon) = d_0 + d_1 \varepsilon + d_2 \varepsilon^2 + \cdots
\]

(7.89)

com coeficientes \(d_r\) a determinar. Posto isto, usamos o facto de \(\lambda_0\) não depender da escala \(\mu\). Então

\[
0 = \mu \frac{\partial}{\partial \mu}(\mu^\varepsilon Z_\lambda)
\]

\[
= \varepsilon \mu^\varepsilon Z_\lambda + \mu^\varepsilon \hat{\beta}(\lambda, \varepsilon) \frac{\partial Z_\lambda}{\partial \lambda} + \mu^\varepsilon Z_\lambda \hat{\beta}(\lambda, \varepsilon)
\]

(7.90)

Então

\[
\varepsilon \lambda Z_\lambda + \hat{\beta}(\lambda, \varepsilon) \left( Z_\lambda + \lambda \frac{\partial Z_\lambda}{\partial \lambda} \right) = 0
\]

(7.91)

Usando as expressões de \(Z_\lambda\) e \(\hat{\beta}\) obtemos

\[
\varepsilon \lambda + a_1 \lambda + \lambda \sum_{r=1}^{\infty} \frac{a_{r+1}}{\varepsilon^r} + (d_0 + d_1 \varepsilon + d_2 \varepsilon^2 + \cdots) \left[ 1 + \sum_{r=1}^{\infty} \frac{1}{\varepsilon^r} \left( a_r + \lambda \frac{d a_r}{d \lambda} \right) \right] = 0
\]

(7.92)

Então \(d_r = 0\) para \(r > 1\) e

\[
\varepsilon(\lambda + d_1) + \left[ a_1 \lambda + d_0 + d_1 \left( a_1 + \lambda \frac{d a_1}{d \lambda} \right) \right] + \sum_{r} \frac{1}{\varepsilon^r} \left( a_{r+1} \lambda + d_0 \left( a_r + \lambda \frac{d a_r}{d \lambda} \right) \right)
\]
\[
+ d_1 \left( a_{r+1} + \lambda \frac{da_{r+1}}{d\lambda} \right) = 0 \tag{7.93}
\]

logo

\[
\lambda + d_1 = 0
\]
\[
a_1\lambda + d_0 + d_1 \left( a_1 + \lambda \frac{da_1}{d\lambda} \right) = 0
\]
\[
a_{r+1}\lambda + d_0 \left( a_r + \lambda \frac{da_r}{d\lambda} \right) + d_1 \left( a_{r+1} + \lambda \frac{da_{r+1}}{d\lambda} \right) = 0 \tag{7.94}
\]

Estes cálculos dão

\[
d_1 = -\lambda \tag{7.95}
\]

\[
\beta(\lambda) = d_0 = \lambda^2 \frac{da_1}{d\lambda}
\]

\[
\lambda^2 \frac{d}{d\lambda} (a_{r+1}) = \beta(\lambda) \frac{d}{d\lambda} (\lambda a_r) \tag{7.96}
\]

Portanto a função \( \beta(\lambda) \) depende somente do coeficiente em \( \frac{1}{\varepsilon} \) de \( Z_\lambda \) que se calcula facilmente em teoria de perturações. Além disso vemos que os resíduos dos pólos de ordem superior se podem calcular em termos do resíduo do pólo simples. Para \( \lambda \phi^4 \) é fácil de ver que

\[
Z_\lambda = 1 + \frac{3\lambda}{16\pi^2} \frac{1}{\varepsilon} + \cdots \tag{7.97}
\]

e portanto

\[
\beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda} = \lambda^2 \frac{d}{d\lambda} \left( \frac{3\lambda}{16\pi^2} \right) = \frac{3\lambda^2}{16\pi^2} \tag{7.98}
\]

como tínhamos obtido anteriormente. Para teorias de gauge há uma pequena modificação pois \( g_0 = \mu^{\varepsilon/2} Z_g g \). Um cálculo trivial dá neste caso

\[
d_1 = -g/2 \tag{7.99}
\]

e

\[
\beta(g) = \frac{1}{2} g^2 \frac{da_1}{dg}
\]
onde, como anteriormente

\[
Z_g = 1 + \sum_{r=1}^{\infty} a_r(g)/\varepsilon^r.
\] (7.101)

**ii) Cálculo de \( \gamma_m(\lambda) \)**

Partimos de \( m_0 = Z_m m \). Aplicando \( \mu \frac{\partial}{\partial \mu} \) obtemos

\[
0 = \mu \frac{\partial Z_m}{\partial \mu} m + Z_m \mu \frac{\partial m}{\partial \mu}
\]

\[
= \beta(\lambda, \varepsilon) \frac{\partial Z_m}{\partial \lambda} m + mZ_m \mu \frac{\partial \ln m}{\partial \mu}
\] (7.102)

Como \( \mu \frac{\partial \ln m}{\partial \mu} = \gamma_m \), obtemos a equação

\[
\left[ \beta(\lambda, \varepsilon) \frac{\partial}{\partial \lambda} + \gamma_m \right] Z_m = 0
\] (7.103)

ou seja

\[
\left( \gamma_m + d_1 \frac{db_1}{d\lambda} \right) + \sum_{r=1}^{\infty} \frac{1}{\varepsilon_r} \left[ d_0 \frac{db_r}{d\lambda} + \gamma_m b_r + d_1 \frac{db_{r+1}}{d\lambda} \right] = 0
\] (7.104)

Portanto

\[
\gamma_m = -d_1 \frac{db_1}{d\lambda}
\] (7.105)

\[
-d_1 \frac{db_{r+1}}{d\lambda} = \beta(\lambda) \frac{db_r}{d\lambda} + \gamma_m b_r
\] (7.106)

onde

\[
d_1 = \begin{cases} 
-\lambda & \text{teoria } \lambda \phi^4 \\
-g/2 & \text{teorias de gauge}
\end{cases}
\] (7.107)

Mais uma vez \( \gamma_m \) depende somente do resíduo do pôlo simples.

**iii) Cálculo de \( \gamma(\lambda) \)**

Aqui é mais fácil partir da definição de \( \gamma(\lambda) \)

\[
\gamma(\lambda) = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z_\phi = \frac{1}{2} \mu \frac{\partial}{\partial \mu} Z_\phi \frac{1}{Z_\phi}
\] (7.108)
CHAPTER 7. RENORMALIZATION GROUP

\[
\left[ \hat{\beta}(\lambda, \varepsilon) \frac{\partial}{\partial \lambda} - 2\gamma(\lambda) \right] Z_\phi = 0
\]

(7.109)

logo

\[
-2\gamma(\lambda) + d_1 \frac{d c_1}{d \lambda} + \sum_{r=1}^{\infty} \frac{1}{\varepsilon^r} \left[ d_0 \frac{d c_r}{d \lambda} - 2\gamma c_r + d_1 \frac{d c_{r+1}}{d \lambda} \right] = 0
\]

(7.110)

Então

\[
\gamma(\lambda) = \frac{1}{2} d_1 \frac{d c_1}{d \lambda}
\]

(7.111)

\[
-d_1 \frac{d c_{r+1}}{d \lambda} = \beta(\lambda) \frac{d c_r}{d \lambda} - 2\gamma c_r
\]

(7.112)

sendo o coeficiente \( d_1 \) dado anteriormente, Eq. (7.107).

Podemos concluir dizendo que o coeficiente do pôlo simples nas constantes de renormalização, determina univocamente as funções \( \beta, \gamma_m \) e \( \gamma \) e também os valores dos pôlos de ordem superior.

7.2.5  \( \beta \) and \( \gamma \) properties

Nós adoptamos um esquema particular de renormalização. Se tivéssemos adoptado outro esquema teríamos outra definição dos parâmetros da teoria e funções \( \beta, \gamma_m \) e \( \gamma \) diferentes. Vamos aqui discutir os aspectos do grupo de renormalização que são independentes do esquema usado.

Consideremos então dois esquemas (ambos independentes da massa). Então

\[
g'(g) = g F_g(g) \quad F_g(g) = 1 + O(g^2)
\]

\[
Z'_m(g') = Z_m(g) F_m(g) \quad F_m(g) = 1 + O(g^2)
\]

(7.113)

\[
Z'_\phi(g') = Z_\phi(g) F_\phi(g) \quad F_\phi(g) = 1 + O(g')
\]

O 1 nas funções \( F \) expressa o facto que ao nível árvore não há ambiguidades. Usando as relações acima podemos ver como estão relacionadas as funções \( \beta, \gamma_m \) e \( \gamma \) em dois esquemas. Obtemos (estamos a considerar o caso duma teoria de gauge)

\[
\beta'(g') = \mu \frac{\partial}{\partial \mu} g' = \mu \frac{\partial}{\partial \mu} (g F_g(g)) = \beta(g) \left( F_g + g \frac{\partial F_g}{\partial g} \right)
\]

\[
\gamma'_m(g') = \mu \frac{\partial}{\partial \mu} \ln m' = \mu \frac{\partial}{\partial \mu} (F_m^{-1}(g)m) = \gamma_m(g) - \beta(g) \frac{\partial}{\partial g} \ln F_m
\]
7.2. MINIMAL SUBTRACTION (MS) SCHEME

\[
\gamma'(g') = \frac{1}{2\mu} \frac{\partial}{\partial \mu} \ln Z'(g') = \gamma(g) + \frac{1}{2} \beta(g) \frac{\partial}{\partial g} \ln F'_{\phi} \tag{7.114}
\]

As functions \(\beta, \gamma, m\) and \(\gamma\) so coincide if the schemes are the same, i.e. \(F_g = F_m = F_\phi = 1\). However, the properties following are still independent of the scheme.

i) The existence of a zero of \(\beta(g)\).

Se \(\beta(g_0) = 0\) então \(\beta'(g'_0) = 0\) para \(g'_0 = g_0 F_g(g_0)\). Note that in general \(g_0\) depends on the scheme, i.e. \(g_0 \neq g'_0\).

ii) The first derivative of \(\beta(g)\) at zero.

Seja \(\beta(g_0) = 0\). Then

\[
\frac{\partial \beta'(g'_0)}{\partial g'} = \left\{ \frac{\partial g}{\partial g'} \frac{\partial}{\partial g} \left[ \beta(g) \left( F_g + g \frac{\partial F_g}{\partial g} \right) \right] \right\}_{g_0}
\]

\[
= \left[ F_g + g \frac{\partial F_g}{\partial g} + g \frac{\partial \beta}{\partial g} + \beta(g) \frac{1}{F_g + g \frac{\partial F_g}{\partial g}} \frac{\partial}{\partial g} \left( F_g + g \frac{\partial F_g}{\partial g} \right) \right]_{g_0}
\]

\[
= \frac{\partial \beta}{\partial g}(g_0) \cdot \tag{7.115}
\]

iii) The first two terms in \(\beta(g)\).

Seja \(\beta(g) = b_0 g^3 + b_1 g^5 + O(g^7)\), e

\[
F_g(g) = 1 + ag^2 + O(g^4) \cdot \tag{7.116}
\]

Então

\[
g' = g + ag^3 + O(g^5) \tag{7.117}
\]

e

\[
g = g' - ag^3 + O(g^5) \tag{7.118}
\]

Portanto

\[
\beta'(g) = \beta(g) \frac{\partial}{\partial g} (gF_g) = (b_0 g^3 + b_1 g^5 + O(g^7))(1 + 3ag^2 + O(g^4))
\]

\[
= b_0 g^3 + (3ab_0 + b_1)g^5 + O(g^7)
\]

\[
= b_0 (g^3 - 3ag^2 + O(g^7) + (3ab_0 + b_1)(g^5 + O(g^7))
\]

\[
= b_0 g^3 + b_1 g^5 + O(g^7) \tag{7.119}
\]
iv) O primeiro termo em $\gamma(g)$ e $\gamma_m(g)$.

Seja

$$
\begin{align*}
\gamma(g) &= cg^2 + O(g^4) \\
\gamma_m(g) &= dg^2 + O(g^4)
\end{align*}
$$

(7.120)

Então como $\beta(g) = O(g^3)$ é evidente que

$$
\begin{align*}
\gamma'(g') &= cg'^2 + O(g'^4) \\
\gamma'_m(g') &= dg'^2 + O(g'^4)
\end{align*}
$$

(7.121)

v) O valor de $\gamma(g_0)$ e $\gamma_m(g_0)$ se $\beta(g_0) = 0$.

Este resultado é imediato. Como veremos na secção seguinte todos estes resultados são necessários pois eles controlam resultados físicos e estes não podem depender do esquema de renormalização.

### 7.2.6 Gauge independence of $\beta$ and $\gamma_m$ in MS

A equação do grupo de renormalização em MS foi escrita para a teoria $\lambda \phi^4$. Vamos agora considerar teorias da gauge (abelianas ou não abelianas). Para a quantificação destas teorias é necessário introduzir um termo que fixe a gauge

$$
\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2
$$

(7.122)

onde escolhemos as gauges do tipo de Lorentz. Como não há correções radiativas para a parte longitudinal do propagador, não é necessário nenhum contratermo para o termo que fixa a gauge. Portanto se pusermos, como habitualmente,

$$
A^\mu = Z_A^{-1/2}A_0^\mu
$$

(7.123)

obtemos

$$
\frac{1}{2\xi}(\partial \cdot A)^2 = \frac{1}{2\xi Z_A}(\partial \cdot A_0)^2 = \frac{1}{2\xi_0}(\partial \cdot A_0)^2
$$

(7.124)

o que quer dizer que o parâmetro de gauge é renormalizado de acordo com

$$
\xi_0 = Z_A\xi
$$

(7.125)

As funções de Green irredutíveis renormalizadas, dependem em geral de $\xi$, isto é

$$
\Gamma^{(n)}_{R}(g, m, \xi, \mu) = Z_A^{n/2}\Gamma^{(n)}_0(g_0, m_0, \xi_0, \varepsilon)
$$

(7.126)
A equação do grupo de renormalização é então

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta(g, \xi) \frac{\partial}{\partial g} + \gamma_m(g, \xi) m \frac{\partial}{\partial m} + \delta(g, \xi) \frac{\partial}{\partial \xi} - \gamma_A(g, \xi) \right] I^{(n)}_R(g, m, \xi, \mu) = 0 \quad (7.127)
\]

onde

\[
\delta(g, \xi) = \mu \frac{\partial}{\partial \mu} \xi = \mu \frac{\partial}{\partial \mu} (Z_A^{-1} \xi_0) =
\]

\[
= -\xi_0 \frac{1}{Z_A} \frac{\partial}{\partial \mu} Z_A
\]

\[
= -2 \xi \gamma_A(g, \xi)
\]

(7.128)
e se admitiu a possibilidade de \( \beta, \gamma_m, \) e \( \gamma_A \) dependerem do parâmetro \( \xi \). Contudo a dependência em \( \xi \) não é arbitrária, obedece a certos constrangimentos. Para vermos isso consideremos uma função de Green sem dimensões e correspondendo a operadores invariantes de gauge. Então

\[
\frac{\partial}{\partial \xi_0} G_0(g_0, m_0, \xi_0, \varepsilon) = 0 \quad \text{(independente de gauge)} \quad (7.129)
\]
e

\[
G_0(g_0, m_0, \xi_0, \varepsilon) = G(g, m, \xi, \mu) \quad \text{(sem dimensões) (7.130)}
\]
e portanto

\[
\frac{\partial}{\partial \xi} G = 0 \quad (7.131)
\]

ou seja

\[
\mathcal{D}_G G \equiv \left[ \frac{\partial}{\partial \xi} + \rho(g, \xi) \frac{\partial}{\partial g} + \sigma(g, \xi) m \frac{\partial}{\partial m} \right] G(g, m, \xi, \mu) = 0 \quad (7.132)
\]

onde

\[
\rho(g, \xi) = \frac{\partial g}{\partial \xi} \quad ; \quad \sigma(g, \xi) = \frac{\partial}{\partial \xi} \ln m
\]

(7.133)

Mas \( G \) obedece à equação do grupo de renormalização

\[
\mathcal{D} G \equiv \left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \gamma_m m \frac{\partial}{\partial m} + \delta \frac{\partial}{\partial \xi} \right] G = 0 \quad (7.134)
\]
Usando a equação para $D_G G = 0$ podemos substituir a derivada em ordem a $\xi$ por derivadas em ordem aos outros parâmetros, obtendo uma equação do grupo de renormalização semelhante às das teorias que não têm campos de gauge, isto é

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \gamma_m \frac{\partial}{\partial m} \right] G = 0 \tag{7.135}$$

onde

$$\bar{\beta} \equiv \beta - \rho \delta \quad \bar{\gamma}_m = \gamma_m - \sigma \delta \tag{7.136}$$

Calculemos agora o comutador $[D_G, D]G = 0$. Obtemos

$$\left\{ \frac{\partial \bar{\beta}}{\partial \xi} + \beta \frac{\partial}{\partial g} \left[ \frac{\partial \bar{\beta}}{\partial \xi} - \beta \frac{\partial}{\partial g} - \delta \frac{\partial}{\partial \xi} \right] \frac{\partial}{\partial g} + \frac{\partial \bar{\delta}}{\partial \xi} + \rho \frac{\partial}{\partial g} \right\} G = 0 \tag{7.137}$$

Introduzindo as funções $\bar{\beta}$ e $\bar{\gamma}_m$ e o operador

$$\bar{D} \equiv \frac{\partial}{\partial \xi} + \rho \frac{\partial}{\partial g} \tag{7.138}$$

a equação anterior escreve-se

$$\left[ (\bar{D}\delta) \frac{\partial}{\partial \xi} + (\bar{D} \bar{\beta} + \bar{D}(\rho \delta) - \beta \frac{\partial}{\partial g} - \delta \frac{\partial}{\partial \xi}) \frac{\partial}{\partial g} \right] G = 0 \tag{7.139}$$

Multiplicando a equação $D_G G = 0$ por $(\bar{D}\delta)$ obtemos

$$\left[ (\bar{D}\delta) \frac{\partial}{\partial \xi} + \rho(\bar{D}\delta) \frac{\partial}{\partial g} + \sigma(\bar{D}\delta) \frac{\partial}{\partial m} \right] G = 0 \tag{7.140}$$

Comparando as duas equações vemos que

$$\bar{D} \bar{\beta} = \beta \frac{\partial}{\partial g} \quad \text{e} \quad \bar{D} \bar{\gamma}_m = \beta \frac{\partial}{\partial g} \tag{7.141}$$

Estas equações asseguram que resultados físicos sejam independentes de gauge. Assim $\bar{\beta} = 0$ tem consequências físicas. Então $\bar{D} \bar{\beta} = 0$ e $\bar{D} \bar{\gamma}_m = 0$ dizendo que a existência dos zeros de $\bar{\beta}$ e a dimensão anômala da massa $\bar{\gamma}_m$ são independentes de gauge. Também se $\bar{\beta} = 0$ obtemos
\[ \mathcal{D} \left( \frac{\partial \beta}{\partial g} \right) = \frac{\partial}{\partial y} \mathcal{D} \beta + \left[ \mathcal{D}, \frac{\partial}{\partial g} \right] \beta \]

\[ = \frac{\partial}{\partial y} \mathcal{D} \beta - \frac{\partial \rho}{\partial g} \frac{\partial \beta}{\partial g} = 0 \quad (7.142) \]

e portanto a primeira derivada de \( \beta \) no zero é independente da gauge. Finalmente como \( \rho = O(g^3) \) e \( \delta = O(g^2) \) temos então

\[ \beta = \beta + O(g^5) \quad (7.143) \]

Estes resultados não dependem de se ter adoptado subtração mínima ou não. Se adoptarmos subtração mínima temos então

**Teorema 3.3**

No esquema de subtração mínima temos \( \rho = \sigma = 0 \) e portanto

\[ \mathcal{D} = \frac{\partial}{\partial \xi} ; \quad \beta = \beta \quad e \quad \gamma_m = \gamma_m \quad (7.144) \]

e \( \beta \) e \( \gamma_m \) são independentes da gauge em todas as ordens.

**Dem:** Demonstramos só para \( \rho \), para \( \sigma \) é igual.

\[ \rho = g \frac{\partial}{\partial \xi} \ln g = -g \frac{\partial Z_g}{Z_g} \frac{\partial \xi}{\partial \xi} \quad (7.145) \]

Então

\[ 0 = Z_g \rho + g \frac{\partial}{\partial \xi} \left( 1 + \frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2} + \cdots \right) \]

\[ = \rho + \frac{1}{\varepsilon} \left( \rho a_1 + g \frac{\partial a_1}{\partial \xi} \right) + O(1/\varepsilon^2) \quad (7.146) \]

obtemos portanto

\[ \rho = 0. \quad (7.147) \]
7.3 Effective gauge couplings

7.3.1 Fixed points

Como vimos na secção anterior, o comportamento assimptótico das funções de Green irredutíveis depende do comportamento assimptótico das soluções das equações para a constante de acoplamento efectivo \(\lambda(t)\) e para a massa efectiva, que como vimos são

\[
\begin{align*}
\frac{d\lambda}{dt} &= \beta(\lambda) ; \quad \lambda(0) = \lambda \\
\frac{dm}{dt} &= [\gamma_m(\lambda) - 1] m(t) ; \quad m(0) = m
\end{align*}
\]

(7.148)

Destas equações resulta que as variações da constante de acoplamento efectiva e da massa efectiva com uma variação de escala de energia são controladas pelas funções \(\beta\) e \(\gamma_m\), respectivamente. Para estudar o comportamento assimptótico de \(\lambda\) vamos admitir que \(\beta(\lambda)\) tem a forma da figura 7.2.

Os pontos, \(0, \lambda_1\) e \(\lambda_2\) onde \(\beta(\lambda)\) se anula são chamadas pontos fixos, pois se \(\lambda\) se encontra num desses pontos em \(t = 0\) então ficará aí para todos os valores do momento \(\frac{d\lambda}{dt} = 0\).

Os pontos fixos podem ser de dois tipos:

i) Ponto fixo estável ultravioleta(UV)

São aqueles em que \(\beta'(\lambda) < 0\). É o caso do ponto \(\lambda_1\) na figura 7.2. Neste caso \(\beta(\lambda) > 0\) para \(\lambda < \lambda_1\) e \(\beta(\lambda) < 0\) para \(\lambda > \lambda_1\). Então se para \(t = 0\) \(0 < \lambda < \lambda_1\) então quando \(t \to \infty\) \(\lambda \to \lambda_1\). Por outro lado se \(\lambda_1 < \lambda < \lambda_2\) quando \(t \to \infty\) também \(\lambda \to \lambda_1\). Portanto no intervalo \(0 < \lambda < \lambda_2\) a constante de acoplamento é sempre conduzida para \(\lambda_1\) quando \(t \to \infty\), isto é, para momentos grandes.

ii) Ponto fixo estável infravermelho(IR)

São aqueles em que \(\beta'(\lambda) > 0\). É o caso dos pontos \(0\) e \(\lambda_2\) da figura. É fácil de ver que quando \(t \to \infty\) a constante de acoplamento se afasta de \(0\) e \(\lambda_2\), mas que no limite \(t \to 0\) se aproxima deles.

Podemos agora estudar o comportamento assimptótico das soluções do grupo de renormalização. Supomos, por exemplo \(0 < \lambda < \lambda_2\). Então (ver figura 7.2)

\[
\lim_{t \to 0} \lambda(t, \lambda) = \lambda_1
\]

(7.149)
7.3. 

**EFFECTIVE GAUGE COUPLINGS**

A maneira como tende para \( \lambda_1 \) depende da primeira derivada de \( \beta(\lambda) \). Suponhamos que na vizinhança de \( \lambda_1 \) temos

\[
\beta(\lambda) = a(\lambda_1 - \lambda) \quad ; \quad a > 0
\]

\[
\beta'(\lambda_1) = -a < 0
\]

(7.150)

Então

\[
\overline{\lambda}(t, \lambda) = \lambda_1 + (\lambda - \lambda_1)e^{-at}
\]

(7.151)

isto é, a aproximação do ponto fixo é exponencial na variável \( t \). Será tanto maior quanto maior for \( |\beta'(\lambda_1)| = a \). Vimos anteriormente que a solução da equação da massa efectiva era

\[
\overline{m}(t) = me^{-t}e^{\int_0^t \gamma_m(\overline{\lambda})dt'}
\]

(7.152)

Se \( \lim_{t \to \infty} \overline{\lambda} = \lambda_1 \) então temos para \( t \to \infty \)

\[
\overline{m} = me^{-t(1-\gamma_m(\lambda_1))}
\]

(7.153)

o que mostra que se \( \gamma_m(\lambda_1) < 1 \) então \( m(t) \to 0 \) quando \( t \to \infty \). Na mesma aproximação

\[
\int_0^t \gamma(\overline{\lambda}(t'))dt' \simeq \gamma(\lambda_1)t
\]

(7.154)

e portanto a solução assimptótica é

\[
\lim_{\sigma \to \infty} \Gamma^m(\sigma p, m, \lambda, \mu) = a^{4-n[1+\gamma(\lambda_1)]}\Gamma^{(n)}(p, \overline{m}, \lambda_1, \mu)
\]

(7.155)

o que mostra que a dimensão dos campos não é 1 mas \( 1 + \gamma(\lambda_1) \). Daí o nome de dimensão anómala para \( \gamma(\lambda) \).

Em geral é difícil calcular os zeros da função \( \beta \), pois requer normalmente resultados para além da teoria de perturbações. Contudo \( \beta(\lambda) \), \( \gamma_m(\lambda) \) e \( \gamma(\lambda) \) têm um zero trivial na origem. Se acontecer que a origem seja um ponto fixo estável UV então quer dizer que quando a escala da energia aumenta a constante de acoplamento diminui. No limite \( t \to \infty, \overline{\lambda} \to 0 \) e por isso se diz destas teorias que são **assimptoticamente livres**. É fácil de ver que isso acontece se \( \beta'(0) < 0 \). Na secção seguinte vamos ver quais as teorias em que isso pode acontecer.

### 7.3.2 \( \beta \) function for theories with scalars, fermions and gauge fields

Vamos nesta secção mostrar que só as teorias de gauge não abelianas podem ser assimptoticamente livres, isto é, só estas verificam a propriedade \( \beta'(0) < 0 \).

i) Teorias com escalares
Já vimos anteriormente que para a teoria escalar mais simples, $\lambda \phi^4$, temos

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^4) \quad (7.156)$$

e portanto não é assimptoticamente livre. Consideramos agora a teoria escalar mais geral com campos $\phi_i$ e acoplamento

$$\mathcal{L}_I = -\lambda_{ijkl} \phi_i \phi_j \phi_k \phi_\ell \quad (7.157)$$

onde se somam os índices repetidos. Então

$$\beta_{ijkl} = \frac{d\lambda_{ijkl}(t)}{dt} = A(\lambda_{ikmn} \lambda_{jkmn} + \lambda_{ijkn} \lambda_{klnm} + \lambda_{ikmn} \lambda_{jkmn}) \quad (7.158)$$

com $A > 0$. A teoria não é assimptoticamente livre pois há sempre funções $\beta$ com derivadas positivas. Por exemplo

$$\frac{d\lambda_{1111}}{dt} = \beta_{1111} = 3A|\lambda_{11mn}|^2 > 0 \quad ; \quad \forall t \quad (7.159)$$

ii) Teorias com escalares + fermiões + acoplamentos de Yukawa

O termo da interacção mais geral para uma teoria com escalares e fermiões é

$$\mathcal{L}_I = -\sum_{i,j,k,\ell} \lambda_{ijkl} \phi_i \phi_j \phi_k \phi_\ell + \sum_{a,b,k} \overline{\psi} (A_a^k + iB_a^k \gamma_5) \psi \phi_k \quad (7.160)$$

onde $A$ e $B$ são matrizes reais. Agora já não é possível mostrar que $\lambda_{ijkl} > 0$ por causa do loop de fermiões de ordem $A^2$ ou $B^2$ com um sinal negativo. Se definirmos $(g^i)_{ab} \equiv A^i_{ab} + iB^i_{ab}$, obtemos

$$16\pi^2 \frac{dg^i}{dt} = \left(\text{Tr} g^i g^i \right) g^i + \text{Tr} (g^i g^i) g^i + M^{ij} g^j$$

$$+ \frac{1}{2} g^i g^j g^i + \frac{1}{2} g^j g^i g^i + 2 g^j g^i g^i \quad (7.161)$$

onde $M^{ij} \equiv \frac{1}{4} \lambda_{iklm} \lambda_{jklm}$. Usando este resultado é possível demonstrar o teorema seguinte:

**Teorema 3.4**

A teoria mais geral com escalares e fermiões não é assimptoticamente livre pois $\frac{d}{dt} \text{Tr} (g^i g^i) > 0$ e portanto não é possível $g_i \to 0$ quando $t \to \infty$.

**Dem:**

$$8\pi^2 \frac{d}{dt} \text{Tr} (g^i g^i) = 8\pi^2 \frac{d}{dt} \sum_{a,b,i} |g_{ab}^i|^2$$
7.3. EFFECTIVE GAUGE COUPLINGS

\[\begin{align*}
&= \text{Tr}(g^i g^{i\dagger}) \text{Tr}(g^{i\dagger} g^j) + \text{Tr}(g^i g^{j\dagger}) \langle \text{Tr}(g^j g^{i\dagger}) \rangle \\
&\quad + \frac{1}{2} \text{Tr}(g^i g^{i\dagger} g^{j\dagger} g^j) + \frac{1}{2} \text{Tr}(g^j g^{i\dagger} g^i g^{j\dagger}) \\
&\quad + 2 \text{Tr}(g^i g^{i\dagger} g^j g^{j\dagger}) + M^{ij} \text{Tr}(g^i g^{j\dagger})
\end{align*}\]

(7.162)

Agora o último termo é positivo, assim como o terceiro e o quarto. O primeiro é maior que o segundo e portanto

\[8\pi^2 \frac{d}{dt} \text{Tr}(g^i g^{i\dagger}) \geq 2 \left[ \text{Tr}(g^i g^{i\dagger}) \langle \text{Tr}(g^j g^{i\dagger}) \rangle + \text{Tr}(g^i g^{i\dagger} g^j g^{j\dagger}) \right]\]

(7.163)

e o segundo membro é positivo pois pode ser escrito

\[8\pi^2 \frac{d}{dt} \text{Tr}(g^i g^{i\dagger}) \geq (g^{i\dagger}_a g^j + g^i_a g^{j\dagger}) (g^{i\dagger}_b g^j + g^i_b g^{j\dagger}) \geq 0\]

(7.164)

como queríamos demonstrar.

iii) Teorias gauge abelianas

Consideremos o caso de QED. Temos

\[Z_3 = Z_1 Z^{-1}_2 Z^{-1/2}_3 = Z^{-1/2}_3\]

(7.165)

Z3 pode ser calculado do diagrama de polarização do vácuo representado na Fig. 7.3, e o resultado é

\[Z^{-1/2}_3 = 1 + \frac{e^2}{12\pi^2} \frac{1}{\varepsilon} + \cdots\]

(7.166)

logo

\[\beta(\varepsilon) = \frac{1}{2} e^2 \frac{da}{de} = \frac{e^3}{12\pi^2} > 0\]

(7.167)

Se tivéssemos electrodinâmica escalar Z3 seria obtido a partir dos diagramas da figura 7.4, e o resultado seria

\[Z^{-1/2}_3 = 1 + \frac{e^2}{48\pi^2} \frac{1}{\varepsilon}\]

(7.168)
o que dá neste caso $\beta(e) = \frac{e^3}{48\pi^2} > 0$. Portanto as teorias de gauge abelianas não são livres assimptoticamente.

iv) Teorias de gauge não abelianas

Comecemos pela teoria de gauge pura definida no capítulo 2. A renormalização da função de onda para os campos de gauge é obtida a partir dos diagramas da figura 7.5. Em subtração mínima obtemos,

\[
Z_A = 1 + \frac{g^2}{16\pi^2} \left( \frac{13}{3} - \xi \right) C_2(V) \frac{1}{\varepsilon} \tag{7.169}
\]

onde $C_2(V)$ é o operador de Casimir definido no capítulo 2 para a representação adjunta, a que pertencem os campos da gauge (vectores). A constante de renormalização do vértice triplo, $Z_1$, é obtida a partir dos diagramas da figura 7.6.
7.3. EFFECTIVE GAUGE COUPLINGS

\[ Z_1 = 1 + \frac{g^2}{16\pi^2} \left( \frac{17}{6} - \frac{3\xi}{2} \right) C_2(V) \frac{1}{\varepsilon} + \cdots \]  

(7.170)

Então

\[ Z_g \equiv Z_1 Z_A^{-3/2} = 1 - \frac{g^2}{16\pi^2} \left( \frac{11}{3} C_2(V) \right) \frac{1}{\varepsilon} + \cdots \]  

(7.171)

Usando \( Z_A \) e \( Z_g \) e as definições de \( \beta \) e \( \gamma \) obtemos

\[ \beta = -\frac{g^3}{16\pi^2} \frac{11}{3} C_2(V) < 0 \]  

(7.172)

e

\[ \gamma_A = -\frac{g^2}{16\pi^2} \frac{1}{2} \left( \frac{13}{3} - \xi \right) C_2(V) \]  

(7.173)

Portanto as teorias de gauge não abelianas sem campos de matéria são assimptoticamente livres. Notar que a dependência da gauge (em \( \xi \)) desapareceu de \( \beta \) de acordo com o resultado demonstrado anteriormente.

A inclusão de fermiões e escalares acoplados mínimamente é agora trivial. O lagrangeano de interacção é

\[ \mathcal{L}_{\text{int}} = g \bar{\psi}_i \gamma^\mu \psi_j T_F^{a\mu} A^a_{\mu} \]  

\[ + ig \phi_i \phi_j T_S^{a\mu} A^a_{\mu} \]  

\[ + g^2 \phi_i T_S^{a\mu} T_S^{b\nu} A^a_{\mu} A^b_{\nu} \]  

(7.174)

onde \( T_F^a \) e \( T_S^a \) são os geradores na representação em que se encontram os fermiões e os escalares respectivamente. Para se encontrar a contribuição destas partículas para a função \( \beta \) temos que calcular a contribuição delas para \( Z_g \). O mais fácil é usar os resultados de QED e electrodinâmica escalar que dizem que

\[ Z_g = Z_A^{-1/2} \]  

(7.175)

e calcular a contribuição para \( Z_A \) dos fermiões e escalares devido aos diagramas da figura 7.7. O resultado é

\[ Z_g(\text{fermiões + escalares}) = 1 + \frac{g^2}{16\pi^2} \left[ \frac{4}{3} T(R_F) + \frac{1}{3} T(R_S) \right] \frac{1}{\varepsilon} + \cdots \]  

(7.176)

pelo que

\[ \beta(\text{fermiões}) = \frac{g^3}{16\pi^2} \frac{4}{3} T(R_F) \]  

(7.177)
CHAPTER 7. RENORMALIZATION GROUP

Figure 7.7: Contribution from fermions and scalars to vacuum polarization.

\[ \beta(\text{escalares}) = \frac{g^3}{16\pi^2} \frac{1}{3} T(R_S) \] (7.178)

Pondo tudo junto obtemos

\[ \beta = \frac{g^3}{16\pi^2} \left[ \frac{11}{3} C_2(V) + \frac{4}{3} T(R_F) + \frac{1}{3} T(R_S) \right] \] (7.179)

onde as quantidades \( T(R) \) são definidas para uma dada representação por

\[ \text{Tr}(T^a T^b) = T(R)\delta^{ab} \] (7.180)

Se a teoria contém fermiões de Majorana (ou spinores de Weyl) ou campos escalares, os coeficientes em frente de \( T(R_F) \) e \( T(R_S) \) são multiplicados por um factor adicional de 1/2. Consideremos agora um exemplo simples:

**Exemplo 3.1:**

\( QCD \ (SU(3)) \) com as três famílias de quarks.

Para \( SU(N) \) temos

\[ C_2(V) = N \] (7.181)

e como os quarks se encontram na representação fundamental

\[ T(R_F) = \frac{1}{2} \] (7.182)

Então

\[ \beta = \frac{g}{16\pi^2} \left[ -\frac{33}{3} + \frac{4}{3} \times \frac{1}{2} \times 2N_g \right] \] (7.183)

ou seja \( (N_g = \text{número de gerações ou famílias}) \)

\[ \beta = \frac{g^3}{16\pi^2} \left[ -\frac{33 - 4N_g}{3} \right] \] (7.184)
Portanto $SU(3)$ será assimptoticamente livre se

$$33 - 4N_g > 0$$  \hspace{1cm} (7.185)

ou ainda

$$N_g < \frac{33}{4} \rightarrow N_g \leq 8$$  \hspace{1cm} (7.186)

São portanto permitidas 8 famílias ou seja 16 tripletos de $SU(3)$.

### 7.3.3 The vacuum of a NAGT as a paramagnetic medium ($\mu > 1$)

Um argumento recente (Nielsen 1981, Hughes 1981) permite compreender melhor o que se passa de diferente nas teorias de gauge não abelianas para que elas tenham liberdade assimptotica. Primeiro o facto de a carga diminuir a curta distância pode ser interpretado como um *anti-shielding* do vácuo, isto é

$$\varepsilon < 1$$  \hspace{1cm} (7.187)

O problema em compreender o que se passa resulta do facto de não conhecermos substâncias$^3$ com $\varepsilon < 1$. Contudo o vácuo deve ser invariante relativista e portanto deve ter uma permeabilidade $\mu$ tal que (estamos a fazer $c = 1$)

$$\mu \varepsilon = 1$$  \hspace{1cm} (7.188)

Assim o *antiscreening* corresponde a $\mu > 1$. Portanto o vácuo duma teoria de gauge não abeliana é um *paramagnético* e este conceito pode ser compreendido mais facilmente.

A permeabilidade magnética pode ser calculada calculando a densidade de energia do vácuo num campo exterior

$$u_0 = \frac{1}{2\mu} B_{ext}^2$$  \hspace{1cm} (7.189)

Nielsen e Hughes mostraram que $\mu = 1 + \chi$ onde a susceptibilidade $\chi$ é dada por

$$\chi \sim (-1)^2 q^2 \sum_{s_3} \left( -\frac{1}{3} + \gamma^2 s_3^3 \right)$$  \hspace{1cm} (7.190)

onde $s$ é o spin, $q$ a carga, $\gamma$ a razão giromagnética e $s_3$ a projecção de spin na direcção do campo magnético externo. Assim para escalares, fermiões e campos de gauge obtemos

*Escalares*

$$\chi_S \sim -\frac{1}{3} q^2 < 0$$ \hspace{1cm} (diamagnético)  \hspace{1cm} (7.191)

$^3$Em QED a carga aumenta a curta distância e portanto o vácuo é um dieléctrico normal $\varepsilon > 1$. 
Fermiões \((\gamma_F = 2)\)

\[ \chi_F \sim (-1)q_F^2 2 \left( -\frac{1}{3} + 1 \right) = -\frac{4}{3} q_F^2 \quad \text{(diamagnético)} \]  
(7.192)

Bosões de gauge \((\gamma_V = 2)\)

\[ \chi_V \sim q_V^2 2 \left( -\frac{1}{3} + 4 \right) = \frac{22}{3} q_V^2 \quad \text{(paramagnético)} \]  
(7.193)

e portanto

\[ \chi_{\text{Total}} \sim \frac{22}{3} q_V^2 - \frac{4}{3} q_F^2 - \frac{1}{3} q_S^2 \]  
(7.194)

Comparando com a função \(\beta\) podemos fazer a correspondência

\[ q_V^2 \rightarrow \frac{1}{2} C_2(V) \]
\[ q_F^2 \rightarrow T(R_F) \]
\[ q_S^2 \rightarrow T(R_S) \]  
(7.195)

o que permite compreender o vácuo das teorias de gauge não abelianas como um meio paramagnético.

### 7.4 Renormalization group applications

We consider the Grand Unified Theory (GUT) with the gauge group \(SU(5)\), that is

\[ SU(5) \supset SU_c(3) \times SU_L(2) \times U_Y(1) . \]  
(7.196)

The unification takes place at the GUT scale \(M_X\). Using the renormalization group equations and the low energy data on the coupling constants, it is possible to determine the scale \(M_X\) as well as other predictions for the theory at the low scale, which we take to be the scale \(M_Z\). For this we need to know how the different coupling constants evolve with the scale.

#### 7.4.1 Scale \(M_X\)

We start by writing the covariant derivatives for the unified theory and for the theory with the broken symmetry.

\[ SU(5) : D_\mu = \partial_\mu + iq_5 \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2} \]  
(7.197)
7.4. RENORMALIZATION GROUP APPLICATIONS

\[ SU(3) \times SU(2) \times U(1) : D_\mu = \partial_\mu + ig_3 \sum_\alpha G^{a}_\mu \lambda^a + i g_2 \sum_\alpha A^{a}_\mu \sigma^a + ig' Y B_\mu \]  \hspace{1cm} (7.198)

At the scale \(M_X\) where the unification takes place we have

\[ g_5 = g_3 = g_2 = g_1 \] \hspace{1cm} (7.199)

where \(g_1\) is the coupling constant of the abelian subgroup of \(SU(5)\). However for the abelian groups there are no constraints in the normalization of the generators, and therefore the the generator \(\lambda^0\) of that \(U(1)\) can be normalized in a different way from the hypercharge. We must have

\[ g_1 \lambda^0 = g' Y \] \hspace{1cm} (7.200)

As \(\lambda^0\) is a generator of \(SU(5)\) it is normalized according to

\[ T_F(\lambda^a \lambda^b) = 2 \delta^{ab} \] \hspace{1cm} (7.201)

that is, for the fundamental representation we must have

\[ \lambda^0 = \begin{pmatrix} 2 & \hspace{1cm} 2 \hspace{1cm} -3 \\ \hspace{1cm} 2 \hspace{1cm} & \hspace{1cm} -3 \hspace{1cm} & \hspace{1cm} -3 \end{pmatrix} \] \hspace{1cm} (7.202)

Now, for the fundamental representation, we have

\[ 5 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ \nu^c_R \end{pmatrix} \hspace{1cm} (7.203) \]

and the hypercharge\(^4\) can be read directly. We obtain,

\[ Y = \begin{pmatrix} -2/3 \\ -2/3 \\ -2/3 \hspace{1cm} 1 \end{pmatrix} \] \hspace{1cm} (7.204)

\(^4\)Remember that our convention is such that \(Q = T_3 + \frac{Y}{2} \).
Therefore \( Y = -\sqrt{\frac{4}{3}} \lambda^0 \) and \( g' = -\sqrt{\frac{4}{3}} g_1 \). This allows to determine \( \sin^2 \theta_W \) at the GUT scale \( M_X \),

\[
\sin^2 \theta_W (M_X) = \frac{g'^2}{g^2 + g'^2} = \frac{\frac{3}{5} g_1}{g_2 + \frac{3}{5} g_1} = \frac{3}{8} \quad (7.205)
\]

Also, for future reference, we note that

\[
g'^2 = \frac{3}{5} g_1^2. \quad (7.206)
\]

### 7.4.2 Scale \( M_Z \)

Let us look now at what happens at the scale \( M_Z \). The evolution of the coupling constants is governed by the RGE equations for the three gauge groups in the broken phase

\[
\frac{dg_i}{dt} = \beta_i \quad (7.207)
\]

These \( \beta \) functions are given by

\[
\beta_i = \frac{g_i^3}{16\pi^2} \left[ -\frac{11}{3} C_2(V) + \sum_j \frac{4}{3} T(R_{F_j}) + \sum_k \frac{1}{3} T(R_{S_k}) \right] \quad (7.208)
\]

where the sums are over all the fermion and scalar physical states of the theory at a given scale. Given the form of Eq. (7.208), it is usual to define

\[
\beta_i \equiv \frac{1}{16\pi^2} b_i g_i^3 \quad (7.209)
\]

and therefore the \( b_i \) are defined by the bracket in Eq. (7.208). Before we evaluate them let us introduce Eq. (7.209) into Eq. (7.207). We get

\[
\frac{dg_i}{dt} = \frac{b_i}{16\pi^2} g_i^3 \quad (7.210)
\]

Let us solve this equations before we evaluate the beta function coefficients \( b_i \). For that it is usual to introduce the generalization of the fine structure constant, that is, we define

\[
\alpha_i \equiv \frac{g_i^2}{4\pi} \quad (7.211)
\]

Multiplying both sides of Eq. (7.210) by \( g_i \) and doing some trivial algebra we get,

\[
\frac{d\alpha_i}{dt} = \frac{b_i}{2\pi} \alpha_i^2 \quad (7.212)
\]

Rearranging and integrating between some initial \( (\mu_i) \), and final scale \( (\mu_f) \), we get

\[
\frac{1}{\alpha_i^2} \frac{d\alpha_i}{dt} = \frac{b_i}{2\pi} \int_{\mu_i}^{\mu_f} dt \quad (7.213)
\]
or\(^5\)

\[
\left[ -\frac{1}{\alpha_i} \right]_i^f = \frac{b_i}{2\pi} (t_f - t_i) = \frac{b_i}{2\pi} \ln \left( \frac{\mu_f}{\mu_i} \right)
\]  

(7.214)

and finally

\[
\alpha_i^{-1}(\mu_f) = \alpha_i^{-1}(\mu_i) - \frac{b_i}{4\pi} \ln \left( \frac{\mu_f^2}{\mu_i^2} \right)
\]

(7.215)

As at the unification scale \(M_X\) we have, by definition (see Eq. (7.199)), that

\[
\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5
\]

(7.216)

where \(\alpha_5\) is the \(SU(5)\) unified value, and we can write the final solution

\[
\alpha_i^{-1}(\mu) = \alpha_5^{-1} + \frac{b_i}{4\pi} \ln \left( \frac{M_X^2}{\mu^2} \right), \quad i = 1, 2, 3
\]

(7.217)

We can rewrite these equations in terms of electromagnetic fine structure constant \(\alpha(\mu)\) and of the strong coupling equivalent \(\alpha_s(\mu)\), that are measured at the weak scale, to obtain

\[
\begin{align*}
\alpha_s^{-1}(\mu) &= \alpha_5^{-1} + \frac{b_3}{4\pi} \ln \left( \frac{M_X^2}{\mu^2} \right) \\
\alpha^{-1}(\mu) \sin^2 \theta_W(\mu) &= \alpha_5^{-1} + \frac{b_2}{4\pi} \ln \left( \frac{M_X^2}{\mu^2} \right) \\
\frac{3}{5} \cos^2 \theta_W(\mu) \alpha^{-1}(\mu) &= \alpha_5^{-1} + \frac{b_1}{4\pi} \ln \left( \frac{M_X^2}{\mu^2} \right)
\end{align*}
\]

(7.218)

From these equations we obtain,

\[
\ln \frac{M_X^2}{\mu^2} = \frac{12\pi}{-8b_3 + 3b_2 + 5b_1} \left[ \frac{1}{\alpha(\mu)} - \frac{8}{3} \frac{1}{\alpha_s(\mu)} \right]
\]

(7.219)

That allows to determine \(M_X\), once \(\alpha(\mu)\) and \(\alpha_s(\mu)\) are known, at a given scale \(\mu\), and

\[
\sin^2 \theta_W(\mu) = \frac{3(b_2 - b_3)}{5b_1 + 3b_2 - 8b_3} + \frac{5(b_1 - b_2)}{5b_1 + 3b_2 - 8b_3} \frac{\alpha(\mu)}{\alpha_s(\mu)}
\]

(7.220)

which allows to determine \(\sin^2 \theta_W\) at the scale \(\mu = M_Z\), once \(\alpha(M_Z)\) and \(\alpha_s(M_Z)\) are known. Finally we can also solve for the value of \(\alpha_5^{-1}\). We get

\[
\alpha_5^{-1} = \alpha^{-1}(\mu) \frac{1}{-3b_3 + (5b_1 + 3b_2) \frac{\alpha(\mu)}{\alpha_s(\mu)}}
\]

(7.221)

Now we turn to the evaluation of the coefficients \(b_i\) first in the Standard Model (SM) and the in the Minimal Supersymmetric Standard Model (MSSM).

\(^5\)Remember that \(t = \ln(\mu)\).
Standard Model

In the SM we have the gauge fields, \( N_g = 3 \) families of leptons, \( N_F = 2N_g = 6 \) quark flavours and one Higgs. With this information we can find the coefficients \( b_i \) for the SM using the definition

\[
b_i = -\frac{11}{3} C_2(V_i) + \sum_j \frac{2}{3} T(R_{F_j}) + \sum_k \frac{1}{3} T(R_{S_k})
\]  

(7.222)

where we have modified Eq. (7.208), as the sum in the fermions is done separately for each quarkality. This is important for the SM as the model is described in terms of left and right-handed fermions.

- **SU(3)**

  For \( SU(3) \), we have \( C_2(V_3) = 3 \) and the quarks are in the fundamental representation, therefore \( T(R_{F_j}) = 1/2 \). Then the counting goes as follows,

\[
b_3 = -\frac{11}{3} \times 3 + N_g \times \left[ \frac{2}{3} \times \frac{1}{2} \times (2 + 1 + 1) \right] = -7
\]

(7.223)

where the meaning of \( (2 + 1 + 1) \) is that we count the up and down components of each \((SU(2)_L)\) doublet and then the corresponding right-handed quarks for each generation.

- **SU(2)**

  For the \( SU(2) \) we get

\[
b_2 = -\frac{11}{3} \times 2 + N_g \times \left( \frac{N_c \times 2}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \times (2 + 1 + 1) \right) + \frac{1}{3} \times \frac{1}{2} = -\frac{19}{6}
\]

(7.224)

where \( N_c = 3 \) is the number of colours.

- **U(1)**

  Finally for the \( U(1) \) part, with the correct normalization, we have

\[
b_1 = \frac{3}{5} \times \left[ \frac{2}{3} \times \sum_{f_L,f_R} \frac{Y_f}{2}^2 + \frac{1}{3} \times \sum_{s_{scalars}} \frac{Y_f}{2}^2 \right]
\]

(7.225)
and therefore,
\[
\begin{align*}
\frac{3}{5} \times N_g & \times \left( \frac{2}{3} \times \left( -\frac{1}{2} \right)^2 \times 2 + \frac{2}{3} \times (-1)^2 + N_c \times \frac{2}{3} \times \left( \frac{1}{6} \right)^2 \times 2 + N_c \times \frac{2}{3} \times \left( \frac{2}{3} \right)^2 \right. \\
& \left. + N_c \times \frac{2}{3} \times \left( \frac{1}{3} \right)^2 \right) + \frac{1}{3} \times \left( \frac{1}{2} \right)^2 \times 2 \\
& = 4 + \frac{1}{10} \frac{41}{10} \\
& = \frac{41}{10}
\end{align*}
\] (7.226)

So in summary we have for the SM,
\[
\begin{align*}
b_1 &= \frac{41}{10}, \quad b_2 = -\frac{19}{6}, \quad b_3 = -7
\end{align*}
\] (7.227)

Now let us look to see what are the results for \( M_X, \sin^2 \theta_W(M_Z) \) and \( \alpha_5^{-1} \). We will use the current values from the Particle Data Group. These are (without worrying about errors)\(^6\)
\[
\begin{align*}
\alpha^{-1}(M_Z) &= 127.916, \quad \alpha_s(M_Z) = 0.118, \quad M_Z = 91.1896 \text{ GeV}
\end{align*}
\] (7.228)
we get
\[
\begin{align*}
M_X &= 6.7 \times 10^{14} \text{ GeV}, \quad \sin^2 \theta_W(M_Z) = 0.208, \quad \alpha_5^{-1} = 41.48
\end{align*}
\] (7.229)

At the time that this GUT model was proposed by the first time, the constants were not known so precisely as today. Also the bound on the lifetime of the proton was much lower than today. So at that time the model was completely consistent. However after many years of dedicated experiments for find the decay of the proton, the lower limit was substantially improved and also after LEP the coupling constants are known with greater precision. So today the value for \( M_X \) is too low, the same being true for the value of \( \sin^2 \theta_W(M_Z) \) (the best value today is around \( \sin^2 \theta_W(M_Z) = 0.230 \) \(^7\)).

This can be seen very clearly if we use Eq. (7.215), with \( \mu_i = M_Z \) and plot the \( \alpha_i^{-1} \) as a function of \( \ln(\mu^2/M_Z^2) \). This is shown in Fig. 7.8. We clearly see that the agreement is quite poor with today’s values.

**Minimal Supersymmetric Standard Model**

Let us now turn to the MSSM. Below the GUT scale the gauge group is the same as in the SM, but the particle content is larger, more than duplicated in relation to the SM. We summarize in the Table 7.1 the particle content and their quantum numbers under \( G = SU_c(3) \otimes SU_L(2) \otimes U_Y(1) \).

---

\(^6\)Not only errors but also the difference between different renormalization schemes. Also this discussion is at one-loop level.

\(^7\)Again, without discussing the very small errors and the dependence on the renormalization scheme.
Figure 7.8: Evolution of $\alpha_i$ as function of the scale $\mu$ in the Standard Model for a $SU(5)$ minimal GUT theory.

With the values in Table 7.1 we can calculate the contribution of the various particles to the $b_i$ coefficients. We will do it in succession for the three groups and for the different supermultiplets.

- $SU(3)$
  - Gauge Supermultiplet

We first do it in general for any gauge group and then apply it to the cases of interest. The gauge multiplet has a gauge boson contributing with

$$b_{\text{gauge boson}} = \frac{-11}{3} C_2(V)$$

(7.230)

and the left-handed gauginos in the adjoint representation of the gauge group. These therefore contribute

$$b_{\text{gauginos}} = \frac{2}{3} C_2(V)$$

(7.231)

and therefore

$$b_{\text{gauge SM}} = -3 C_2(V)$$

(7.232)

where SM stands here for super-multiplet. Applying now to $SU(3)$ we get

$$b_{3_{\text{gauge SM}}} = -9$$

(7.233)

- Left-handed Lepton Supermultiplet

$$b_{3_{\text{Leptons}_L, SM}} = 0$$

(7.234)
7.4. RENORMALIZATION GROUP APPLICATIONS

Supermultiplet $\quad SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$

<table>
<thead>
<tr>
<th>Quantum Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{V}_1 \equiv (\lambda, W^{\mu}_1)$</td>
</tr>
<tr>
<td>$\hat{V}_2 \equiv (\lambda_a, W^{\mu a}_2)$</td>
</tr>
<tr>
<td>$V_3 \equiv (g^b, W^{zb}_3)$</td>
</tr>
<tr>
<td>$\hat{L}_i \equiv (\bar{L}, L)_i$</td>
</tr>
<tr>
<td>$\hat{R}_i \equiv (\bar{R}, \ell^c)_i$</td>
</tr>
<tr>
<td>$\hat{Q}_i \equiv (\bar{Q}, Q)_i$</td>
</tr>
<tr>
<td>$\hat{D}_i \equiv (\bar{d}_R, d^c)_i$</td>
</tr>
<tr>
<td>$\hat{U}_i \equiv (\bar{u}_R, u^c)_i$</td>
</tr>
<tr>
<td>$\hat{H}_u \equiv (H_u, \tilde{H}_u)$</td>
</tr>
<tr>
<td>$\hat{H}_d \equiv (H_d, \tilde{H}_d)$</td>
</tr>
</tbody>
</table>

Table 7.1: Particle content of the MSSM. Note that $Q = T_3 + Y/2$.

- **Right-handed Lepton Supermultiplet**

  $$b^{\text{Leptons}_{R}}_{3, \text{SM}} = 0 \quad (7.235)$$

- **Left-handed Quark Supermultiplet**

  $$b^{\text{Quarks}_{L,SM}}_{3} = \frac{2}{3} \times \frac{1}{2} \times 2 + \frac{1}{3} \times \frac{1}{2} \times 2 = 1 \quad (7.236)$$

- **Right-handed Up-Quark Supermultiplet**

  $$b^{\text{Up-Quarks}_{R,SM}}_{3} = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2} \quad (7.237)$$

- **Right-handed Down-Quark Supermultiplet**

  $$b^{\text{Down-Quarks}_{R,SM}}_{3} = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2} \quad (7.238)$$

- **Up type Higgs Supermultiplet**
- Down type Higgs Supermultiplet

\[ b_3^{\text{Down-Higgs SM}} = 0 \] (7.240)

- SU(2)

  Gauge Supermultiplet

  We get

  \[ b_2^{\text{gauge SM}} = -6 \] (7.241)

- Left-handed Lepton Supermultiplet

  \[ b_2^{\text{Leptons}_L \ SM} = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2} \] (7.242)

- Right-handed Lepton Supermultiplet

  \[ b_2^{\text{Leptons}_R \ SM} = 0 \] (7.243)

- Left-handed Quark Supermultiplet

  \[ b_2^{\text{Quarks}_L \ SM} = N_c \left( \frac{2}{3} \times \frac{1}{2} + N_c \frac{1}{3} \times \frac{1}{2} \right) = N_c \frac{1}{2} = \frac{3}{2} \] (7.244)

- Right-handed Up-Quark Supermultiplet

  \[ b_2^{\text{Up-Quarks}_R \ SM} = 0 \] (7.245)

- Right-handed Down-Quark Supermultiplet

  \[ b_2^{\text{Down-Quarks}_R \ SM} = 0 \] (7.246)

- Up type Higgs Supermultiplet
7.4. RENORMALIZATION GROUP APPLICATIONS

\[ b_2^{\text{Up-Higgs SM}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = \frac{1}{2} \]  \hspace{1cm} (7.247)

– Down type Higgs Supermultiplet

\[ b_2^{\text{Down-Higgs SM}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = \frac{1}{2} \]  \hspace{1cm} (7.248)

• \( U(1) \)

Gauge Supermultiplet

We get

\[ b_1^{\text{gauge SM}} = 0 \]  \hspace{1cm} (7.249)

– Left-handed Lepton Supermultiplet

\[ b_1^{\text{Leptons}_L, \text{SM}} = \frac{3}{5} \times \left[ \frac{2}{3} \times \left( -\frac{1}{2} \right)^2 \times 2 + \frac{1}{3} \times \left( -\frac{1}{2} \right)^2 \times 2 \right] = \frac{3}{10} \]  \hspace{1cm} (7.250)

– Right-handed Lepton Supermultiplet

\[ b_1^{\text{Leptons}_R, \text{SM}} = \frac{3}{5} \times \left[ \frac{2}{3} \times (-1)^2 + \frac{1}{3} \times (-1)^2 \right] = \frac{3}{5} \]  \hspace{1cm} (7.251)

– Left-handed Quark Supermultiplet

\[ b_1^{\text{Quarks}_L, \text{SM}} = \frac{3}{5} \times N_c \times \left[ \frac{2}{3} \times \left( \frac{1}{6} \right)^2 \times 2 + \frac{1}{3} \times \left( \frac{1}{6} \right)^2 \times 2 \right] = N_c \times \frac{3}{5} \times \frac{1}{18} = \frac{1}{10} \]  \hspace{1cm} (7.252)

– Right-handed Up-Quark Supermultiplet
\[ b_1^{\text{Up-Quarks}_{\text{SM}}} = \frac{3}{5} N_c \times \left[ \frac{2}{3} \times \left( \frac{2}{3} \right)^2 + \frac{1}{3} \times \left( \frac{2}{3} \right)^2 \right] = N_c \times \frac{3}{5} \times \frac{4}{9} = \frac{4}{5} \quad (7.253) \]

– Right-handed Down-Quark Supermultiplet

\[ b_1^{\text{Down-Quarks}_{\text{SM}}} = \frac{3}{5} N_c \times \left[ \frac{2}{3} \times \left( \frac{-1}{3} \right)^2 + \frac{1}{3} \times \left( \frac{-1}{3} \right)^2 \right] = N_c \times \frac{3}{5} \times \frac{1}{9} = \frac{1}{5} \quad (7.254) \]

– Up type Higgs Supermultiplet

\[ b_1^{\text{Higgs}_u_{\text{SM}}} = \frac{3}{5} \times \left[ \frac{1}{3} \times \left( \frac{1}{2} \right)^2 \times 2 + \frac{2}{3} \times \left( \frac{1}{2} \right)^2 \times 2 \right] = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \quad (7.255) \]

– Down type Higgs Supermultiplet

\[ b_1^{\text{Higgs}_d_{\text{SM}}} = \frac{3}{5} \times \left[ \frac{1}{3} \times \left( \frac{-1}{2} \right)^2 \times 2 + \frac{2}{3} \times \left( \frac{-1}{2} \right)^2 \times 2 \right] = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \quad (7.256) \]

Now we put everything together to obtain for the MSSM,

\[ b_1 = N_g \times \left( \frac{3}{10} + \frac{3}{5} + \frac{1}{10} + \frac{4}{5} + \frac{1}{5} + \frac{3}{10} + \frac{3}{10} \right) = 3 \times 2 + \frac{3}{5} = \frac{33}{5} \]
\[ b_2 = -6 + N_g \times \left( \frac{1}{2} + \frac{3}{2} \right) + \frac{1}{2} + \frac{1}{2} = 1 \]
\[ b_3 = -9 + N_g \times \left( 1 + \frac{1}{2} + \frac{1}{2} \right) = -3 \quad (7.257) \]
Now let us look to see what are the results for $M_X$, $\sin^2 \theta_W(M_Z)$ and $\alpha^{-1}_5$ in the MSSM. Using the same inputs as for the SM, Eq. (7.228) $^8$, we get

$$M_X = 2.1 \times 10^{16} \text{ GeV}, \quad \sin^2 \theta_W(M_Z) = 0.231, \quad \alpha^{-1}_5 = 24.27$$ (7.258)

we immediately see that these values are quite good. This can be seen very clearly if we use Eq. (7.215), with $\mu_i = M_Z$ and plot the $\alpha_i^{-1}$ as a function of $\ln(\mu^2/M_Z^2)$. This is shown in Fig. 7.9 and the agreement is excellent.

![Figure 7.9: Evolution of $\alpha_i$ as function of the scale $\mu$ in the MSSM for a $SU(5)$ minimal GUT theory.](image)

We can still go a step further. We know that supersymmetry must be broken above the electroweak scale, so what we have done in Fig. 7.9 is not quite correct because we are running with the MSSM content down to the weak scale. Of course each particle will decouple at its mass, but assuming that their masses are not much different we can assume that there will a scale $M_{\text{SUSY}}$, below which we will have the SM RGEs. We can redo the calculation taking now the evolved SM values at $M_{\text{SUSY}}$ as the boundary conditions for the MSSM evolution. In Fig. and Fig. the results are shown for various values of the SUSY scale. We see from these results that if the SUSY scale is much higher than, say 1 TeV, the good agreement starts do disappear. Before we end we should emphasize that these are one loop results, without many fine details, like thresholds (talking in account that not all the supersymmetric particles decouple at the same scale) and the important two-loop effects.

---

$^8$Again we do not take into account errors and the difference between different renormalization schemes. Also this discussion is at one-loop level and the effects of the supersymmetric particles not decoupling at the same scale (thresholds) are not taken in account.
Figure 7.10: Evolution of $\alpha_i$ as function of the scale in the MSSM for a $SU(5)$ minimal GUT theory. On the left panel $M_{\text{SUSY}} = 500$ GeV and on the right panel $M_{\text{SUSY}} = 1000$ GeV.

Figure 7.11: Evolution of $\alpha_i$ as function of the scale in the MSSM for a $SU(5)$ minimal GUT theory. On the left panel $M_{\text{SUSY}} = 10^4$ GeV and on the right panel $M_{\text{SUSY}} = 10^8$ GeV.
Problems for Chapter 7

3.1 Verify the equation 7.161. For this note that $\beta^i = \frac{dg^i}{dt}$ where $\beta^i$ is calculated from the following diagrams:

![Diagrams](image)

3.2 Calculate by subtraction minimum (MS) the renormalization constant $Z_3$ for QED, equation 7.166.

3.3 Calculate in MS the renormalization constant $Z_3$ in scalar electrodynamics, equation 7.168.

3.4 Consider a non-abelian theory with group symmetry $G$ and no matter. Calculate the renormalization constants of the propagator $Z_A$ and of the triple vertex $Z_1$.

3.5 Consider a non-abelian theory in interaction with scalar and fermionic fields. Calculate the contribution of these fields to $Z_A$ and $Z_1$. Utilize these results together with the results of problem 3.4 to determine the function $\beta$ of the group of renormalization for this theory.

3.6 Consider the standard model of weak and strong interactions. Considering all fields of the model (including the Higgs) calculate the coefficients $b_1$, $b_2$, and $b_3$ defined in equation (tc2-3.201).

3.7 Consider now the minimal supersymmetric standard model (MSSM). Calculate the coefficients $b_1$, $b_2$, and $b_3$ defined in equation (tc2-3.201). Repeat the analysis of convergence.
das constantes de acoplamento no quadro duma teoria de grande unificação supersimétrica com o grupo $SU(5)$.
Appendix A

Path Integral in Quantum Mechanics

A.1 Introduction

A formulation usual is given by the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |a(t)\rangle = H |a(t)\rangle \]  

(A.1)

where

\[ H = \frac{P^2}{2m} + V(Q) \]  

(A.2)

and

\[ [Q, P] = i\hbar \]  

(A.3)

This formulation is equivalent to another defined in terms of path integrals. For this we base on the observation that in quantum mechanics we know how to answer any question about the system if we know how to calculate the transition amplitudes

\[ \langle b(t') | a(t) \rangle = \langle b | e^{-iH(t'-t)} | a \rangle \]  

(A.4)

These transition amplitudes are defined in terms of path integrals. For this we base on the observation that in quantum mechanics we know how to answer any question about the system if we know how to calculate the transition amplitudes.

A.2 Configuration space

Introduce the states \( |q\rangle \) and \( |p\rangle \)

293
\[ Q |q⟩ = q |q⟩ \quad ; \quad P |p⟩ = p |p⟩ \]

\[ ⟨q'|q⟩ = δ(q' - q) \quad ; \quad ⟨p'|p⟩ = δ(p' - p) \]

\[ ⟨q|p⟩ = ⟨p|q⟩^∗ = \frac{1}{\sqrt{2\pi}} e^{ipq} \]  \hspace{1cm} (A.5)

Então

\[ ⟨q_f,t_f|q_i,t_i⟩ = \int D(q)e^{±\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(q,\dot{q})} \]  \hspace{1cm} (A.6)

onde \( D \) é uma medida de integração definida pelo limite

\[ D(q) = \lim_{n \to \infty} \prod_{1}^{n-1} dq_p \left[ \frac{nme^{-i\pi/2}}{2\pi(t_f-t_i)} \right]^{\frac{n}{2}} \]  \hspace{1cm} (A.7)

sendo \( n \) o número de intervalos em que se fez a partição do intervalo \((t_i,t_f)\). O limite \( n \to \infty \) é bastante complicado e só existe prova matemática para certas classes de potenciais. A Eq. (A.6) permite uma interpretação da mecânica clássica como limite da mecânica quântica. De facto quando \( \hbar \to 0 \) a maior contribuição para a amplitude vem das trajetórias que minimizam a acção, isto é, as trajetórias clássicas. A mecânica quântica é então vista como o estudo das flutuações à volta da trajetória clássica.

\subsection*{A.2.1 Matrix elements of operators}

Usando a propriedade dos integrais de caminho

\[ \int D(q)e^{iS(f,i)} = \int dq(t) \int D(q)e^{iS(f,t)} \int D(q)e^{iS(t,i)} \]  \hspace{1cm} (A.8)

onde \( t_i < t < t_f \), é fácil mostrar que

\[ ⟨q_f,t_f|O(t)|q_i,t_i⟩ = \int dq′dq″ \int D(q)e^{iS(q_f,t_f;q″,t)} \]

\[ \langle q″|O|q′⟩ \int D(q)e^{iS(q″,t_q,t_i)} \]  \hspace{1cm} (A.9)

Então se \( O \) for diagonal no espaço das coordenadas, isto é, se

\[ \langle q″|O|q′⟩ = O(q′)δ(q′ - q″) \]  \hspace{1cm} (A.10)

obtemos

\[ ⟨q_f,t_f|O|q_i,t_i⟩ = \int D(q) e^{iS(f,t)}O(q(t)) \]  \hspace{1cm} (A.11)
A.2.2 Time ordered product of operators

Seja \( \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n) \) com \( t_1 \geq t_2 \geq \cdots \geq t_n \). Então é fácil de mostrar que a ordenação no tempo é automática no integral de caminho, isto é,

\[
\langle q_f, t_f | \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n) | q_i, t_i \rangle = \int D(q)e^{iS(f,i)}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n
\]

(A.12)

Este resultado é particularmente importante, pois permitirá escrever as funções de Green de produtos de operadores ordenados no tempo como simples integrais de caminho de produtos dos equivalentes clássicos desses operadores.

A.2.3 Exact results I: harmonic oscillator

Para alguns potenciais é possível calcular exactamente o limite introduzido em (A.5). Para esses casos o integral de caminho é portanto perfeitamente bem definido. Esses potenciais não são muitos, mas são particularmente importantes. Para o seguimento interessa-nos discutir dois deles. O primeiro é o oscilador harmónico definido pelo potencial

\[
V(Q) = \frac{m\omega^2}{2}Q^2
\]

(A.13)

Para este caso obtém-se, (os integrais são gaussianos e por isso podem ser explicitamente calculados)

\[
\langle f|i \rangle = \left( \frac{m\omega e^{-i\pi/2}}{2\pi \sin \omega t} \right)^{\frac{1}{2}} \exp \left\{ \frac{i m \omega}{2} \left[ (q_f^2 + q_i^2) \cot \omega t - \frac{2q_f q_i}{\sin \omega t} \right] \right\}
\]

(A.14)

Este resultado vai ser útil adiante.

A.2.4 Exact results II: external force

Consideremos agora uma força exterior tal que o potencial é dado por

\[
V(Q) = -QF(t)
\]

(A.15)

Neste caso obtemos

\[
\langle f|i \rangle_F = \left[ \frac{me^{-i\pi/2}}{2\pi(t_f - t_i)} \right]^{\frac{1}{2}} e^{iS(f,i)}
\]

(A.16)

onde \( S(f,i) \) é a acção calculada ao longo da trajetória clássica,

\[
S(f,i) = \frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i} + \int_{t_i}^{t_f} dtF(t) \left( q_f \frac{t - t_i}{t_f - t_i} + q_i \frac{t_f - t}{t_f - t_i} \right)
\]
onde \( G(t', t'') = t' - t'' \) é a função de Green simétrica para o problema \( \dot{q} = F(t)/m \) com as condições na fronteira \( G(0, t'') = G(t', 0) = 0 \).

### A.2.5 Perturbation theory

A importância do resultado exacto para a força exterior deve-se ao facto que usando esse resultado podemos formalmente resolver o problema dum potencial qualquer. Para isso notemos que a derivação funcional em realação à fonte \( F(t) \) faz baixar \( Q(t) \). Mais explicitamente

\[
\langle f | Q(t) | i \rangle_F = \frac{\delta}{i \delta F(t)} \langle f | i \rangle_F
\]

onde \( \langle | \rangle_F \) significa calculado na presença da fonte exterior (i.e. para o hamiltoniano \( H = p^2/2m - QF(t) \)). Então para um potencial arbitrário \( V(q) \) temos

\[
\langle f | i \rangle = \int \mathcal{D}(q)e^{i \int_{t_i}^{t_f} dt \left[ \frac{1}{2} \dot{q}^2 + V(q) \right]} = \exp \left\{ -i \int_{t_i}^{t_f} dt V \left[ \frac{\delta}{\delta F(t)} \right] \right\} \langle f | i \rangle_F \bigg|_{F=0} \tag{A.19}
\]

Esta expressão formal torna-se muito útil quando a exponencial é expandida em série. Então todos os integrais são do tipo gaussiano e podem ser exactamente executados. Obtemos assim a teoria das perturações. Claro que só terá significado se houver um parâmetro pequeno no potencial. É importante notar que enquanto se faça teoria das perturbações não há qualquer problema com a indefinição matemática do integral de caminho, pois todas as integrações são gaussianas.

### A.3 Phase space formulation

Para este caso obtemos

\[
\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}(p, q)e^{i \int_{t_i}^{t_f} dt \left[ p \dot{q} + h(p, q) \right]} \tag{A.20}
\]

onde \( h(p, q) \) é o hamiltoniano clássico e a medida é dada pelo limite

\[
\mathcal{D}(p, q) = \lim_{n \to \infty} \frac{1}{(2\pi)^n} \prod_{i=1}^{n} dp_i \prod_{i=1}^{n-1} dq_i
\tag{A.21}
\]
A fase da exponencial é novamente a acção clássica expressa nas variáveis canónicas \( p \) e \( q \). Se \( h(p,q) \) depender quadraticamente de \( p \) como é usual, pode-se fazer a integração gaussiana em \( p \) e a expressão reduz-se à do caso anterior, equação (A.4).

### A.4 Bargmann-Fock space (coherent states)

Nesta representação usamos funções analíticas de variável complexa para descrevermos os operadores \( a \) e \( a^\dagger \) (\([a, a^\dagger] = 1\)). A correspondência é feita do modo seguinte. As funções analíticas geram um espaço de Hilbert com o produto interno definido por

\[
\langle g | f \rangle \equiv \int \frac{dz d\xi}{2\pi i} e^{-z\xi} \overline{g}(z) f(\xi)
\]  

Os operadores \( a \) e \( a^\dagger \) são representados neste espaço por

\[
a \rightarrow \frac{\partial}{\partial z}
\]

\[
a^\dagger \rightarrow \overline{\xi}
\]

Dado um estado \(| f \rangle\), representado pela função \( f(\overline{\xi}) \), a acção do operador \( A \) em \(| f \rangle\) produz outro estado que também pode ser representado por funções analíticas. Se designarmos por \( g(\overline{\xi}) \) essa função temos

\[
g(\overline{\xi}) \equiv \langle \overline{\xi} | A | f \rangle \equiv \int \frac{d\xi d\xi'}{2\pi i} e^{-\xi \xi'} A(\overline{\xi}, \xi) f(\xi)
\]

onde \( A(\overline{\xi}, \xi) \) é o kernel do operador \( A \). Uma representação explícita para o kernel é fácil de obter. Para isso introduzimos os estados \(| n \rangle\), definidos por

\[
|n\rangle = \frac{a^\dagger^n}{\sqrt{n!}} |0\rangle
\]

É fácil de verificar (ver Problema A.2) que com a definição de produto interno acima introduzida estes estados são ortonormados, isto é \( \langle f_m | f_n \rangle = \delta_{mn} \). Então

\[
\langle \overline{\xi} | A | f \rangle = \sum_{n,m} \frac{\overline{\xi}^n}{\sqrt{n!}} \langle n | A | m \rangle \langle m | f \rangle
\]

\[
= \sum_{n,m} \frac{\overline{\xi}^n}{\sqrt{n!}} A_{n,m} \int \frac{d\xi d\xi'}{2\pi i} e^{-\xi \xi'} \frac{\xi'^m}{\sqrt{m!}} f(\xi)
\]

\[
= \int \frac{d\xi d\xi'}{2\pi i} e^{-\xi \xi'} \left[ \sum_{n,m} \frac{\overline{\xi}^n}{\sqrt{n!}} A_{n,m} \frac{\xi'^m}{\sqrt{m!}} \right] f(\xi)
\]
Portanto

\[ A(\tau, \xi) \equiv \sum_{n,m} \frac{\tau^n m \xi^m}{\sqrt{n!} \sqrt{m!}} \]  

(A.27)

O kernel de qualquer operador é assim obtido desde que se conheçam os seus elementos de matriz na base \( |n\rangle \).

Já vimos como se representam estados e operadores. Vamos ver como representar produtos de operadores. Sejam dois operadores \( A_1 \) e \( A_2 \) e um estado \( |f\rangle \). Seja ainda

\[ g(\eta) = \langle \eta | A_2 | f \rangle = \int \frac{d\xi d\bar{\xi}}{2\pi i} e^{-\xi \bar{\xi}} A_2(\eta, \xi) f(\bar{\xi}) \]  

(A.28)

Então

\[ \langle \eta | A_1 | g \rangle = \int \frac{d\xi d\bar{\xi}}{2\pi i} e^{-\xi \bar{\xi}} A_1(\tau, \eta) g(\eta) \]

\[ = \int \frac{d\eta d\bar{\eta}}{2\pi i} \frac{d\xi d\bar{\xi}}{2\pi i} e^{-\xi \bar{\xi}} e^{-\eta \bar{\eta}} A_1(\tau, \eta) A_2(\bar{\eta}, \xi) f(\bar{\xi}) \]

\[ = \int \frac{d\xi d\bar{\xi}}{2\pi i} e^{-\xi \bar{\xi}} \left[ \int \frac{d\eta d\bar{\eta}}{2\pi i} e^{-\eta \bar{\eta}} A_1(\tau, \eta) A_2(\bar{\eta}, \xi) \right] f(\bar{\xi}) \]

\[ = \int \frac{d\xi d\bar{\xi}}{2\pi i} e^{-\xi \bar{\xi}} A_3(\tau, \xi) f(\bar{\xi}) \]  

(A.29)

Portanto o kernel do operador \( A_3 = A_1 A_2 \) é obtido por convolução dos kernéis de \( A_1 \) e \( A_2 \), isto é

\[ A_3(\tau, \eta) = \int \frac{d\eta d\bar{\eta}}{2\pi i} e^{-\eta \bar{\eta}} A_1(\tau, \eta) A_2(\bar{\eta}, \xi) \]  

(A.30)

### A.4.1 Normal form for an operator

Como já sabemos representar estados, operadores e produtos de operadores já temos todos os ingredientes para fazer mecânica quântica neste espaço. Há contudo um outro assunto que é importante tendo em atenção que pretendemos aplicar este formalismo em teoria quântica dos campos. Trata-se da forma normal dum operador\(^1\). O operador \( A \) na sua forma normal é definido por

\[ A = \sum_{n,m} A^{N}_{n,m} \frac{a^n \dagger a^m}{\sqrt{n!} \sqrt{m!}} \]  

(A.31)

isto é, os operadores de destruição estão à direita dos operadores de criação. O kernel normal é definido por

\(^1\)Notar que em teoria quântica dos campos tem que se proceder ao ordenamento normal do hamiltoniano para definir o zero da energia.
A.4. BARGMANN-FOCK SPACE (COHERENT STATES)

\[ A^N(z, z) \equiv \sum_{n,m} \frac{z^n}{\sqrt{n!}} A^N_{n,m} \frac{z^m}{\sqrt{m!}} \quad (A.32) \]

Isto é, é obtido por substituição directa dos operadores de destruição por \( z \) e dos de criação por \( \overline{z} \). Para um operador dado na sua forma normal este é o kernel imediato de obter. É contudo diferente do kernel atrás definido. Para ver a relação entre eles notemos a seguinte relação

\[
\begin{align*}
  f(\overline{z}) &= \sum_n \frac{\overline{z}}{\sqrt{n!}} \langle n|f angle \\
  &= \sum_n \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\overline{\xi} \xi} \xi^n f(\overline{\xi}) \\
  &= \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\overline{\xi} \xi} e^{z\xi} f(\overline{\xi}) \tag{A.33}
\end{align*}
\]

O kernel \( e^{z\xi} \) é portanto uma função delta neste espaço. Usando este resultado obtemos

\[
\begin{align*}
  \langle \overline{z} | A | f \rangle &= \sum_{n,m} \frac{A^N_{n,m}}{\sqrt{n!}\sqrt{m!}} \frac{z^n}{\sqrt{n!}} \frac{z^m}{\sqrt{m!}} f(\overline{z}) \\
  &= \sum_{n,m} \frac{A^N_{n,m}}{\sqrt{n!}\sqrt{m!}} \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\overline{\xi} \xi} e^{z\xi} \xi^n \xi^m f(\overline{\xi}) \\
  &= \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\overline{\xi} \xi} e^{z\xi} A^N(\overline{z}, \xi) f(\overline{\xi}) \tag{A.34}
\end{align*}
\]

onde resulta

\[ A(\overline{z}, \xi) = e^{z\xi} A^N(\overline{z}, \xi) \quad (A.35) \]

Esta relação é muito importante pois permite imediatamente escrever o kernel dum ope-
rador qualquer uma vez que seja conhecida a sua forma normal. Isto é particularmente úti-
le em teoria quântica dos campos onde o hamiltoniano é dado na sua forma normal.

A.4.2 Evolution operator

Podemos obter agora a expressão para o operador de evolução nesta representação. De
acordo com aquilo que acabámos de dizer, para um intervalo infinitesimal, devemos ter
para o kernel de \( U \)

\[ U(\overline{z}, \xi, \Delta t) = e^{z\xi} e^{-i\Delta t h(\overline{z}, \xi)} \tag{A.36} \]
onde $h(\Xi, \xi)$ é o kernel normal obtido por substituição directa dos operadores $a^\dagger$ e $a$ pelas variáveis complexas $\Xi$ e $\xi$. Notar que quando $\Delta t \to 0$ o kernel do operador de evolução se reduz ao kernel da identidade, $e^{\Xi}$, que como vimos é a função $\delta$ neste espaço.

Para um intervalo de tempo finito $t = t_f - t_i$, dividimos o intervalo em $n$ intervalos

$$
z_0 \quad z_1 \quad z_2 \quad \cdots \quad z_{n-1} \quad z_n$$

$$\Delta t = \frac{t}{n} \quad (A.37)$$

Então

$$U(\Xi_1, z_0) \simeq e^{\Xi_1 z_0 - i\Delta t h(\Xi_1, z_0)}$$
$$U(\Xi_2, z_1) \simeq e^{\Xi_2 z_1 - i\Delta t h(\Xi_2, z_1)}$$
$$\vdots \quad \vdots \quad \vdots$$
$$U(\Xi_n, z_{n-1}) \simeq e^{\Xi_n z_{n-1} - i\Delta t h(\Xi_n, z_{n-1})} \quad (A.38)$$

Aplicando agora a regra de multiplicação dos kernéis obtemos

$$U(\Xi_f, t_f; z_i, t_i) = \lim_{n \to \infty} \int \prod_{k=1}^{n-1} \frac{dz_k d\Xi_k}{2\pi i} \exp \left[ \sum_{k=1}^{n} \Xi_k z_{k-1} - \sum_{k=1}^{n-1} \Xi_k z_k - i \sum_{k=1}^{n} h(\Xi_k, z_{k-1}) \Delta t \right]$$

ou seja

$$U(\Xi_f, t_f; z_i, t_i) \equiv \int D(z, \Xi) e^{i \frac{1}{2} (z_f z_f + \Xi_f \Xi_f) + i \int^{t_f}_{t_i} [\frac{1}{2}(z z' - \Xi \Xi') - h(z, \xi)] dt} \quad (A.39)$$

Nesta expressão $\Xi_f(t_f)$ e $z_i(t_i)$ são fixados pelas condições fronteiras mas $\Xi_f(t_i)$ e $z_i(t_f)$ são arbitrários. A fase da exponencial é novamente a acção, agora escrita nas variáveis complexas $z$ e $\Xi$. Para ver isso basta lembrar que

$$\frac{1}{2}(pdq + qdp) = \frac{1}{2i}(zd\Xi - \Xi dz) \quad (A.41)$$

**A.4.3 Exact results I: harmonic oscillator**

Também aqui vamos analisar os casos importantes em que há resultados exactos, nomeadamente o oscilador hamónico e o caso das fontes externas. Começamos pelo oscilador hamónico. O hamiltoniano é dado por

$$H_0 = \omega a^\dagger a \quad (A.42)$$
Trata-se portanto dum caso em que o hamiltoniano é dado na forma normal. Este problema pode ser resolvido exactamente. Temos

\[
U(z_f, z_i, t) = \lim_{n \to \infty} \int d\zeta_1 d\zeta_{n-1} \frac{1}{2\pi i} \exp \left[ \sum_{k=1}^{n-1} \zeta_k z_{k-1} - \sum_{k=1}^{n} \bar{\zeta}_k \zeta_k - i\omega t \sum_{k=1}^{n} \zeta_k \bar{\zeta}_{k-1} \right]
\]

onde

\[
X = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{bmatrix} \ ; \quad \bar{X} = (\zeta_1, \zeta_2, \ldots, \zeta_{n-1}) \quad (A.44)
\]

e

\[
B = \begin{bmatrix} z_0 a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ ; \quad \bar{B} = (0, 0, \ldots, 0, z_n a) \quad (A.45)
\]

com \( z_0 = z_i \) e \( z_n = z_f \). A matriz \( A \) de dimensão \( (n-1) \times (n-1) \) é dada por

\[
A = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ -a & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & -a & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \cdots & \cdots & 0 & -a & 1 & 0 \\ \cdots & \cdots & \cdots & 0 & -a & 1 \end{bmatrix} \quad (A.46)
\]

onde se definiu

\[
a \equiv 1 - i\hbar \omega t \quad (A.47)
\]

As \( (n-1) \) integrações gaussianas podem ser facilmente feitas usando o resultado (ver Problema A.3),

\[
\int d\zeta_1 d\zeta_{n-1} \frac{1}{2\pi i} e^{-\bar{\zeta}A\zeta + \bar{\zeta}B + \bar{\zeta}A^{-1}u} = (\text{det} A)^{-1} e^\frac{1}{2} \bar{\zeta}A^{-1}u \quad (A.48)
\]

obtemos então
APPENDIX A. PATH INTEGRAL IN QUANTUM MECHANICS

\[ U_0(z_f, z_i; t) = \lim_{n \to \infty} \left[(\det A)^{-1} e^{\frac{B A^{-1}}{2}}\right] \]

\[ = \lim_{n \to \infty} \left[(\det A)^{-1} e^{\frac{1}{2}A^{-1}}\right] \]

(A.49)

É fácil de verificar que para a matriz \( A \) se tem

\[ (A^{-1})_{k,m} = \begin{cases} a^{k-m} & \text{se } k \geq m \\ 0 & \text{se } k < m \end{cases} \] (A.50)

e portanto

\[ \det A = 1 \] (A.51)

e

\[ A^{-1}_{n-1,1} = (-1)^n(-a)^{n-2} \] (A.52)

donde se conclui que

\[ \lim_{n \to \infty} a^2(A^{-1})_{n-1,1} = \lim_{n \to \infty} \left(1 - \frac{i \omega t}{n}\right)^n = e^{-i \omega t} \] (A.53)

Obtemos então finalmente

\[ U_0(z_f, z_i; t) = \exp \{z_f z_i e^{-i \omega t}\} \] (A.54)

Podemos verificar que este resultado é da forma \( e^{iS} \) onde \( S \) é a acção calculada ao longo da trajectória clássica. De facto a estacionaridade do expoente da exponencial dá

\[ \delta \left\{ \frac{1}{2}(z_f z(t_f) + z(t_i)z_i) + \int_{t_i}^{t_f} \left[ \frac{z_z z_z}{2} - i \omega z\bar{z}\right] dt \right\} \]

\[ = \frac{1}{2} \delta z_f z(t_f) + \frac{1}{2} z_i \delta z(t_i) \]

\[ + \int_{t_i}^{t_f} \left[ \delta z (\bar{z} - i \omega z) - \delta \bar{z} (z + i \omega z) \right] dt \] (A.55)

pois \( \delta z_f = \delta z_i = 0 \). As equações de movimento são portanto

\[ \begin{align*}
\bar{z} - i \omega \bar{z} &= 0 \\
\bar{z} + i \omega z &= 0
\end{align*} \]

com

\[ \begin{align*}
\bar{z}(t_f) &= z_f \\
z(t_i) &= z_i
\end{align*} \] (A.56)
que têm como solução

\[
\begin{align*}
  z(t) &= z_i e^{i\omega(t_i - t)} \\
  \bar{z}(t) &= \bar{z}_f e^{i\omega(t - t_f)}
\end{align*}
\]

Substituindo estas soluções no expoente obtemos

\[
\frac{1}{2} \left[ \bar{z}_f z(t_f) + z_i \bar{z}(t_i) \right] + \int_{t_i}^{t_f} \left[ \frac{1}{2}(\bar{z} \bar{z} - \bar{z} \bar{z}) - i\omega \bar{z} \right] dt = \bar{z}_f z_i e^{i\omega(t_i - t_f)}
\]

para \( t \equiv t_f - t_i \), como queríamos mostrar.

Um outro resultado importante do oscilador harmônico é que a evolução dum estado sob a acção de \( H_0 = \omega a^\dagger a \) é particularmente simples neste espaço das funções de variável complexa. Seja \( f(\bar{z}) \) a representação do estado \( | f \rangle \). A evolução debaixo de \( H_0 \) é dada por

\[
U_0(t) f(\bar{z}) = \int d\xi d\bar{\xi} \frac{2\pi i}{e^{-\xi \bar{\xi}}} e^{\xi \bar{\xi} - i\omega t} f(\xi) = f(\bar{\xi} e^{-i\omega t})
\]

isto é, é reduzida à multiplicação por \( e^{-i\omega t} \)

\[
\bar{z} \rightarrow \bar{z} e^{-i\omega t}
\]

Isto é importante para descrever a matriz \( S \), em que os estados assimptóticos evoluem de acordo com o hamiltoniano livre.

### A.4.4 Exact results II: external force

Seja o hamiltoniano

\[
H = \omega a^\dagger a - f(t) a^\dagger - \bar{f}(t) a
\]

Este hamiltoniano também conduz a um resultado exacto. Usando os mesmos métodos que foram utilizados para o oscilador harmônico pode-se mostrar que neste caso também temos

\[
U(\bar{z}_f, z_i; t) = e^{iS(f, i)}
\]

onde \( S(f, i) \) é a acção calculada ao longo das trajectórias clássicas (ver Problema A.1).
A.5 Fermion systems

Vamos generalizar os resultados anteriores ao caso de sistemas de fermiões. Começamos com sistemas com dois níveis com os operadores \( a^\dagger \) e \( a \) tais que

\[
\{ a^\dagger, a \} = 1 ; \quad a^2 = a^{2\dagger} = 0 \quad (A.63)
\]

Para efectuar a construção anterior vamos tentar representar estes operadores num espaço de Hilbert de funções analíticas. Isto é possível se considerarmos funções (de facto polinómios) com coeficientes complexos em duas variáveis que anticomutam \( \eta \) e \( \overline{\eta} \), designadas por variáveis de Grassmann e que obedecem a

\[
\eta \overline{\eta} + \overline{\eta} \eta = 0 ; \quad \eta^2 = \overline{\eta}^2 = 0 \quad (A.64)
\]

Então qualquer função \( P(\eta, \overline{\eta}) \) terá a forma

\[
P(\eta, \overline{\eta}) = p_0 + p_1 \eta + \tilde{p}_1 \eta + p_{12} \eta \overline{\eta} \quad (A.65)
\]

### A.5.1 Derivatives

Neste espaço a derivação é definida por (as derivadas são esquerdas)

\[
\frac{\partial P}{\partial \eta} = \tilde{p}_1 + p_{12} \overline{\eta}
\]

\[
\frac{\partial P}{\partial \overline{\eta}} = p_1 - p_{12} \eta \quad (A.66)
\]

De entre todas as funções nas variáveis \( \eta \) e \( \overline{\eta} \) definimos o subconjunto das funções analíticas tais que

\[
\frac{\partial}{\partial \eta} f = 0 \quad (A.67)
\]

isto é as funções analíticas têm a forma

\[
f = f_0 + f_1 \overline{\eta} \quad (A.68)
\]

### A.5.2 Dot product

No espaço das funções analíticas define-se o produto interno

\[
(g, f) = \overline{g}_0 \, f_0 + \overline{g}_1 \, f_1 \quad (A.69)
\]

este produto interno pode ser representado por um integral desde que definamos a integração convenientemente (ver equação (A.74)) .
A.5.3 Integration

A integração nas variáveis de Grassmann é definida pelas relações

\[ \int d\eta \eta = \int d\bar{\eta} \bar{\eta} = 1 \]
\[ \int d\eta 1 = \int d\bar{\eta} 1 = 0 \]  \hspace{1cm} (A.70)

Notar que a integração assim definida é semelhante à derivação. De facto²

\[ \int d\eta P = \partial P \quad ; \quad \int d\bar{\eta} P = \overline{\partial} P \]
\[ \int d\bar{\eta} d\eta P = \overline{\partial} \partial P \]  \hspace{1cm} (A.72)

Devido à forma da equação A.62 é claro que se tem

\[ \overline{\partial}^2 = \partial^2 = 0 \]  \hspace{1cm} (A.73)

e que portanto o integral dum derivada é zero. Consideremos agora a mudança de variáveis nos integrais. Seja

\[ \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = A \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \]  \hspace{1cm} (A.74)

Então obtemos

\[ \eta \bar{\eta} = (A_{11} \xi + A_{12} \bar{\xi})(A_{21} \xi + A_{22} \bar{\xi}) \]
\[ = (A_{11} A_{22} - A_{12} A_{21}) \xi \bar{\xi} \]
\[ = \det A \xi \bar{\xi} \]  \hspace{1cm} (A.75)

Pelo que

\[ \int d\bar{\eta} d\eta P(\eta, \bar{\eta}) = \int d\bar{\xi} d\xi (\det A)^{-1} Q(\xi, \bar{\xi}) \]  \hspace{1cm} (A.76)

onde \( Q(\xi, \bar{\xi}) \) é o polinómio que se obtém de \( P(\eta, \bar{\eta}) \) por substituição de \( \eta \) e \( \bar{\eta} \) por \( \xi \) e \( \bar{\xi} \).

²Estamos a usar a notação compacta

\[ \partial = \frac{\partial}{\partial \eta} \quad ; \quad \overline{\partial} = \frac{\partial}{\partial \bar{\eta}} \]  \hspace{1cm} (A.71)
Finalmente notemos que se definirmos a conjugação complexa de $f$ por

$$
\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1 \eta
$$

então podemos encontrar uma representação integral para o produto interno dada por

$$
(g, f) = \int d\eta d\eta' e^{-\eta' \eta} g f
$$

Para vermos isso calculemos o integral. Obtemos

$$
\int d\eta d\eta' e^{-\eta_0 \eta} f
$$

$$
= \int d\eta d\eta' (1 - \eta \eta')(\mathcal{T}_0 + \mathcal{T}_1 \eta)(f_0 + f_1 \eta)
$$

$$
= \mathcal{T}_0 f_0 + \mathcal{T}_1 f_1
$$

$$
= (g, f)
$$

(A.79)

### A.5.4 Representation of operators

Os operadores $a$ e $a^\dagger$ podem ser representados por

$$
a \rightarrow \mathcal{O}
$$

$$
a^\dagger \rightarrow \mathcal{\bar{O}}
$$

É fácil de ver que com estas definições temos $a^2 = a^\dagger a^\dagger = 0$ e $aa^\dagger + a^\dagger a = 1$.

Consideremos agora os estados $|0\rangle$ e $|1\rangle = a^\dagger |0\rangle$ a que correspondem as funções $1$ e $\mathcal{\bar{O}}$.

Então podemos encontrar o kernel de qualquer operador

$$
A = \sum_{n,m} |n\rangle A_{n,m} \langle m|
$$

(A.81)

De facto

$$
(Af)\mathcal{\bar{O}} = \sum_{n,m} \mathcal{\bar{O}}^n A_{n,m} \langle m|f\rangle
$$

$$
= \int d\xi d\xi' e^{\xi \xi'} \sum_{n,m} \mathcal{\bar{O}}^n A_{n,m} \xi^m f(\xi)
$$

$$
= \int d\xi d\xi' e^{-\xi \xi'} A(\mathcal{\bar{O}}, \xi) f(\xi)
$$

(A.82)
A.5. FERMION SYSTEMS

onde

\[ A(\eta, \xi) \equiv \sum_{n,m} \eta^m A_{n,m} \xi^n \quad ; \quad n, m = 0, 1 \quad (A.83) \]

Para o produto de operadores é fácil de ver que temos como anteriormente

\[ A_1 A_2(\eta, \eta) = \int d\xi d\xi e^{-\xi \xi} A_1(\eta, \xi) A_2(\xi, \eta) \quad (A.84) \]

A.5.5 Normal form for operators

Seja um operador definido por

\[ A = \sum_{n,m} |n\rangle A_{n,m} \langle m| = \sum_{n,m} a^n |0\rangle \langle 0| a^m A_{n,m} \quad (A.85) \]

O projector do estado base é

\[ |0\rangle \langle 0| = : e^{-a^\dagger a} : \quad (A.86) \]

logo

\[ A = \sum_{n,m} A_{n,m} : a^n e^{-a^\dagger a} a^m : \]

\[ = \sum_{n,m} A^N_{n,m} a^n a^m \quad (A.87) \]

O kernel normal é então definido pela substituição \( a^\dagger \rightarrow \eta \) e \( a \rightarrow \eta \), isto é

\[ A^N(\eta, \eta) = \sum_{n,m} A^N_{n,m} \eta^n \eta^m \quad (A.88) \]

O kernel da identidade é \( e^{\eta^m} \), isto é

\[ f(\eta) = \int d\xi d\xi e^{-\xi \xi} e^{\eta \xi} f(\xi) \quad (A.89) \]

o que permite obter a relação entre o kernel usual e kernel normal. De facto

\[ [a^n a^m f] \eta = \eta^n \frac{\partial f}{\partial \eta^m} \]

\[ = \int d\xi d\xi e^{-\xi \xi} e^{\eta \xi} \eta^n \xi^m f(\xi) \quad (A.90) \]
o que permite escrever a relação procurada

\[ A(\eta, \eta) = e^{\mathbf{\eta} t} \ A^N(\eta, \eta) \]  

(A.91)

Finalmente seguindo um raciocínio análogo ao do sistema de bosões é fácil obter o kernel do operador de evolução

\[ U(\eta_f, t_f; \eta_i, t_i) = \int D(\eta, \eta) \ e^{\frac{i}{\hbar} \left( \mathbf{p}_f - \mathbf{p}_i \right)} \ e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ \frac{1}{2} \left( \mathbf{\eta}_f - \mathbf{\eta}_i \right)^2 - h(\mathbf{\eta}, \eta) \right]} \]  

(A.92)
Problemas do Apêndice A

Problems for Appendix A

A.1 Mostre o resultado da equação A.59. Em particular mostre que

\[ iS(f,i) = \zeta_f e^{-i\omega(t_f-t_i)} z_i + i \int_{t_i}^{t_f} dt \left[ \zeta_f e^{-i\omega(t_f-t)} f(t) + \bar{f}(t) e^{-i\omega(t-t_i)} z_i \right] \]

\[ - \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \bar{f}(t) e^{-i\omega(t-t')} f(t') \theta(t-t') \]  

(A.93)

Este resultado é útil em muitas aplicações (ver equação B.9).

A.2 Mostre que os representantes dos estados \( |n \rangle \) e \( |m \rangle \), \( \frac{\zeta_n}{\sqrt{n!}} \) e \( \frac{\zeta_m}{\sqrt{m!}} \), respectivamente, são ortonormados, isto é, \( \langle f_n | f_m \rangle = \delta_{n,m} \).

A.3 Mostrar que para integrais gaussianos se tem

\[ \int \prod dz_k \frac{d\zeta_k}{2\pi i} e^{-\pi A_z + \pi z + \pi u} = (\det A)^{-1} e^{\pi A^{-1} u} \]  

(A.94)

Notar que o expoente é o valor no ponto de estacionaridade.

A.4 Mostrar que para integrais gaussianos se obtém

\[ \int \prod d\eta_k d\eta_k e^{\sum \eta_k \Lambda_k \eta_k + \sum (\bar{\eta}_k \xi_k + \bar{\xi}_k \eta_k)} \]

\[ = \det A e^{\sum \xi_k (A^{-1})_{kl} \xi_l} \]  

(A.95)

Comparar com o resultado do problema A.3.
Appendix B

Path Integral in Quantum Field Theory

B.1 Path integral quantization

Vamos aqui generalizar os resultados do apêndice A para o caso de sistemas com um número infinito de graus de liberdade que são os que interessam em teoria quântica dos campos. Para evitar complicações com índices e com problemas decorrentes da invariância de gauge vamos estudar o caso do campo escalar cuja acção clássica em presença duma fonte exterior é

\[ S(\phi, J) = S_0(\phi, J) + \int d^4x \, V(x) \]  \hfill (B.1)

onde

\[ S_0(\phi, J) = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J \phi \right] \]  \hfill (B.2)

é a acção do campo escalar livre acoplada a uma fonte exterior. Vamos primeiro estudar este caso, isto é, supor que \( V = 0 \). O caso geral é fácil, de obter a partir deste, como veremos mais à frente. O hamiltoniano é dado por

\[ H = \int d^3x \left[ \frac{1}{2} \pi^{\text{op}}^2 + \frac{1}{2} (\nabla \phi^{\text{op}})^2 + \frac{1}{2} m^2 \phi^{\text{op}}^2 - J \phi^{\text{op}} \right] \]  \hfill (B.3)

e podemos introduzir os operadores \( a(k) \) e \( a^\dagger(k) \) tais que num certo instante

\[ \phi^{\text{op}} = \int \tilde{d}k \left[ a(k) \, e^{i\vec{k} \cdot \vec{x}} + a^\dagger(k) \, e^{-i\vec{k} \cdot \vec{x}} \right] \]  \hfill (B.4)

e

\[ \pi^{\text{op}} = -i \int \tilde{d}k \left[ a(k) \, e^{i\vec{k} \cdot \vec{x}} - a^\dagger(k) \, e^{-i\vec{k} \cdot \vec{x}} \right] \omega(k) \]  \hfill (B.5)
então

\[ H = \int \tilde{d}k \left[ \omega(k) a(k) a(k) - f(t, \vec{k}) a(k) - \overline{f}(t, \vec{k}) a(k) \right] \]  \hspace{1cm} (B.6)

onde introduzimos a transformada de Fourier espacial da fonte,

\[ f(t, \vec{k}) = \int d^3x e^{-i\vec{k} \cdot \vec{x}} j(t, \vec{x}) \]  \hspace{1cm} (B.7)

e onde definimos

\[ \tilde{d}k \equiv \frac{d^3k}{(2\pi)^3} \frac{2\pi\omega(k)}{2} \]

usando os resultados do problema A.1 podemos escrever imediatamente o kernel do operador de evolução

\[ U(\overline{\xi}_f, t_f; \xi_i, t_i) = \exp \left\{ \int \tilde{d}k \left[ \overline{\xi}_f(k) e^{-i\omega(k)(t_f-t_i)} \xi_i(k) \right. \right. \]

\[ \left. + i \int_{t_i}^{t_f} dt \left[ \overline{\xi}_f(k) e^{-i\omega(k)(t_f-t)} f(t, \vec{k}) + \overline{f}(t, \vec{k}) e^{-i\omega(k)(t-t_i)} \xi_i(k) \right] \right. \]

\[ \left. - \frac{1}{2} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \overline{f}(t, \vec{k}) e^{-i\omega(k)(t-t')} f(t', \vec{k}) \right\} \]  \hspace{1cm} (B.9)

A matriz \( S \) é então definida como o limite

\[ \lim_{-t_i, t_f \to \infty} e^{it_f H_0} U(t_f, t_i) e^{-it_i H_0} \]  \hspace{1cm} (B.10)

onde \( H_0 \) é obtido a partir de \( H \) fazendo \( J = 0 \). Na representação que estamos a usar a acção de \( e^{-itH_0} \) é uma simples multiplicação (ver eq. A.60).

\[ \overline{\xi} \rightarrow \overline{\xi} e^{-i\omega t} \]  \hspace{1cm} (B.11)

Portanto o kernel da matriz \( S \) é

\[ S(\overline{\xi}_f, \xi_i) = \lim_{-t_i, t_f \to \infty} \exp \left\{ \int \tilde{d}k \overline{\xi}_f(k) \xi_i(k) \right\} \exp \left\{ \int \tilde{d}k \left[ \right. \right. \right. \]

\[ \left. i \int_{t_i}^{t_f} \frac{dt}{t_i} \left[ \overline{\xi}_f(k) e^{i\omega(k)t} f(t, \vec{k}) + \overline{f}(t, \vec{k}) e^{-i\omega(k)t} \xi_i(k) \right] \right. \]

\[ \left. - \frac{1}{2} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \overline{f}(t, \vec{k}) e^{-i\omega(k)(t-t')} f(t', \vec{k}) \right\} \]  \hspace{1cm} (B.12)

O primeiro factor é aquilo que é necessário para passar do kernel usual para o kernel normal. O restante pode ser interpretado se definirmos
Como \( \overline{z_f} \) não é o complexo conjugado de \( z_i \) então \( \phi_{as} \) é dado em termos de condições na fronteira com frequências positivas para \( t \to -\infty \) e frequências negativas para \( t \to \infty \). Estas são precisamente as condições na fronteira de Feynman. Com estas convenções e notações obtemos para o primeiro termo

\[
\int \tilde{d}k \int_{t}^{t_f} dt \left[ \overline{z_f}(k) e^{i\omega(k)t} f(t, \tilde{k}) + \overline{f}(t, \tilde{k}) e^{-i\omega(k)t} z_i(k) \right]
\]

\[
= \int d^4x \int \tilde{d}kJ(x) \left[ \overline{z_f} e^{i\omega(k)t-i\vec{k} \cdot \vec{x}} + z_i(k) e^{-i\omega(k)t+i\vec{k} \cdot \vec{x}} \right]
\]

\[
= \int d^4xJ(x)\phi_{as}(x) \quad (B.14)
\]

e para o segundo

\[
\int \tilde{d}k \int dt \int dt' \overline{f}(t, \tilde{k}) e^{-i\omega(k)(t-t')} f(t', \tilde{k})
\]

\[
= \int d^4xd^4x' J(x)J(x') \int \tilde{d}k e^{-i\omega(k)(t-t')+i\vec{k} \cdot (\vec{x}-\vec{x}')}
\]

\[
= \int d^4xd^4x' J(x)G_F(x-x')J(x') \quad (B.15)
\]

pois

\[
\int \tilde{d}k e^{-i\omega(k)(t-t')+i\vec{k} \cdot (\vec{x}-\vec{x}')}
\]

\[
= \int \tilde{d}k e^{-i\omega(k)(t-t')+i\vec{k} \cdot (\vec{x}-\vec{x}') \theta(t-t')}
\]

\[
+ \int \tilde{d}k e^{i\omega(k)(t-t')+i\vec{k} \cdot (\vec{x}-\vec{x}') \theta(t'-t) \frac{1}{k^2 - m^2 + i\varepsilon}}
\]

\[
= G_F(x-x') \quad (B.16)
\]

Notar que as condições na fronteira mistas conduzem ao propagador de Feynman. Podemos portanto finalmente escrever o kernel normal da matriz \( S \) na presença da fonte \( J \),

\[
S^N(Z_f, z_i) \big| J = e^j \int d^4x j(x) \phi_{as}(x) e^{-\frac{1}{2} \int d^4xd^4x' J(x)G_F^*_{\mu}(x-x')J(x')} \quad (B.17)
\]
Para se obter o operador $S$ substituímos $\phi$ por $\phi_{op}$ e fazemos o ordenamento normal, isto é

$$S_0(J) =: e^{i \int d^4x \ J(x) \phi_{op}(x)} : e^{-\frac{1}{2} \int d^4x d^4x' \ J(x) G^0_{\mathcal{P}}(x-x') J(x')} \tag{B.18}$$

Como o funcional gerador das funções de green é $\langle 0 | S_0(J) | 0 \rangle$ obtemos imediatamente

$$Z_0(J) = e^{-\frac{1}{2} \int d^4x d^4x' \ J(x) G^0_{\mathcal{P}}(x-x') J(x')} \tag{B.19}$$

Este resultado permite resolver o problema de qualquer potencial $V(x)$. De facto é fácil de mostrar que no caso geral os kernéis estão relacionados por

$$S^N = \exp \left[ -i \int d^4x V \left( \frac{\delta}{i \delta J(x)} \right) \right] S_0(J) \bigg|_{J=0} \tag{B.20}$$

e o operador $S$ é

$$S =: e^{f \int d^4x J(x) \phi_{op}(x)} : \exp \left[ -i \int d^4x \ V \left( \frac{\delta}{i \delta J(x)} \right) \right] Z_0(J) \bigg|_{J=0} \tag{B.21}$$

ou seja

$$Z(J) = \exp \left[ -i \int d^4x \ V \left( \frac{\delta}{i \delta J(x)} \right) \right] Z_0(J) \tag{B.22}$$

com

$$Z_0(J) = e^{-\frac{1}{2} \int d^4x d^4x' \ J(x) G^0_{\mathcal{P}}(x-x') J(x')} \tag{B.23}$$

Estas expressões permitem calcular qualquer função de green com as regras usuais da teoria das perturbações. A quantificação usando os integrais de caminho conduziu aos mesmos resultados (em teoria das perturbações) que a quantificação canônica. As expressões para os funcionais geradores embora deem resultados perturbativos dum forma imediata não são as mais úteis quando estamos interessados em encontrar resultados válidos para além da teoria das perturbações. Para esses casos (identidades de Ward, etc) é mais útil ter uma expressão formal em termos dum integral de caminho. É isso que vamos agora estudar.

### B.2 Path integral for generating functional

O ponto de partida é a expressão para o kernel da matriz $S$,

$$S(\mathcal{F}_f, z_i) = \lim_{-t_i, t_f \to \infty} \int D(\mathcal{F}, z) e^{\frac{i}{2} \int d \mathcal{k} [\mathcal{F}(k,t_f) z(k,t_f) + \mathcal{F}(k,t_i) z(k,t_i)]} \exp \left\{ i \int_{t_i}^{t_f} dt \int d \mathcal{k} \left[ \frac{1}{2i} (\mathcal{F}(k,t) z(k,t) - \mathcal{F}(k,t) \dot{z}(k,t)) \right] \right\}$$
\[ -\omega(k)z(k, t) z(k, t) - V(z, z) \] \{(B.24) \]

com as condições na fronteira\(^1\)

\[ \begin{aligned}
  \xi(k, t_f) &= \xi_f(k) e^{i\omega t_f} \\
  z(k, t_i) &= z_i(k) e^{-i\omega t_i}
\end{aligned} \] \{(B.25) \]

Em vez das variáveis \(z(k, t)\) e \(\xi(k, t)\) vamos introduzir os campos clássicos \(\phi(\vec{x}, t)\) e \(\pi(\vec{x}, t)\) definidos por

\[ \phi(\vec{x}, t) = \int \tilde{d}k \left[ z(k, t) e^{i\vec{k} \cdot \vec{x}} + \xi(k, t) e^{-i\vec{k} \cdot \vec{x}} \right] \] \{(B.26) \]

e

\[ \pi(\vec{x}, t) = -i \int \tilde{d}k \omega(k) \left[ z(k, t) e^{i\vec{k} \cdot \vec{x}} - \xi(k, t) e^{-i\vec{k} \cdot \vec{x}} \right] \] \{(B.27) \]

Estas fórmulas são obviamente sugeridas pelas relações entre \(\phi_{\text{op}}, \pi_{\text{op}}\) e \(a(k), a^\dagger(k)\) expressas nas equações B.4 e B.5, só que aqui não se trata de operadores mas sim de campos clásicos. Começemos por escrever a acção em termos das novas variáveis,

\[ \int_{t_i}^{t_f} dt \int \tilde{d}k \left[ \frac{1}{2} \left( \xi(k, t) z(k, t) - \xi(k, t) z(k, t) \right) \right] - \omega(k)\xi(k, t) z(k, t) - V(\xi, z) \]

\[ = \int d^3 x \int_{t_i}^{t_f} dt \left[ \frac{1}{2} \left( \pi \partial_0 \phi - \partial_0 \pi \phi \right) - \frac{1}{2} \pi^2 \right. \]

\[ - \frac{1}{2} \left( \partial_k \phi \right)^2 - \frac{1}{2} m^2 \phi^2 - V(\phi) \] \{(B.28) \]

Introduzimos agora novas variáveis \(\phi_1(\vec{x}, t)\) e \(\pi_1(\vec{x}, t)\) definidas do modo seguinte

\[ \begin{aligned}
  \phi(\vec{x}, t) &= \phi_{\text{as, in}}(\vec{x}, t) + \phi_1(\vec{x}, t) \\
  \pi(\vec{x}, t) &= \partial_0 \phi(\vec{x}, t) + \phi_1(\vec{x}, t)
\end{aligned} \] \{(B.29) \]

onde

\[ \phi_{\text{as, in}}(\vec{x}, t) = \int \tilde{d}k \left[ \xi_{\text{in}}(k) e^{i\vec{k} \cdot \vec{x}} + z_{\text{in}}(k) e^{-i\vec{k} \cdot \vec{x}} \right] \] \{(B.30) \]

\(^1\)Não há restrições em \(\xi(k, t_i)\) e \(z(k, t_f)\).
\( \phi_{\text{as, out}}(\vec{x}, t) = \int dk \left[ z_{\text{out}}(k) e^{ik \cdot x} + z_{\text{in}}(k) e^{-ik \cdot x} \right] \)  \hspace{1cm} (B.31)

onde \( \text{in} \equiv t \to -\infty \) e \( \text{out} \equiv t \to +\infty \) com

\[
\begin{align*}
\left\{ \begin{array}{l}
z_{\text{out}}(k) \equiv z_f(k) \\
z_{\text{in}}(k) \equiv z_i(k)
\end{array} \right. \hspace{1cm} (B.32)
\]

O campo \( \phi_{\text{as}} \) tem portanto as condições fronteira apropriadas para o problema e satisfaz à equação de Klein-Gordon

\( (\Box + m^2) \phi_{\text{as}} = 0 \) \hspace{1cm} (B.33)

Escrevemos a acção nas novas variáveis

\[
\begin{align*}
\int d^3x \int_{t_i}^{t_f} dt \left[ \frac{1}{2} (\pi \partial_0 \phi - \partial_0 \pi \phi) - \frac{1}{2} \pi^2 - \frac{1}{2} (\partial_k \phi)^2 - \frac{1}{2} m^2 \phi^2 - V(\phi) \right] \\
= \int d^3x \left[ -\frac{1}{2} \pi \phi \right]_{t_i}^{t_f} \\
+ \int d^3x \int_{t_i}^{t_f} dt \left[ \pi \partial_0 \phi - \frac{1}{2} (\pi^2 + 2 \pi \partial_0 \phi - (\partial_0 \phi)^2) - \frac{1}{2} (\partial_k \phi_{\text{as}})^2 - \frac{1}{2} (\partial_k \phi_1)^2 \\
- \partial_k \phi_{\text{as}} \partial_k \phi_1 - \frac{1}{2} m^2 \phi_{\text{as}}^2 - \frac{1}{2} m^2 \phi_1^2 - m^2 \phi_{\text{as}} \phi_1 - V(\phi) \right] \\
= \int d^3x \left[ -\frac{1}{2} \pi \phi \right]_{t_i}^{t_f} \\
+ \int d^3x \int_{t_i}^{t_f} dt \left[ -\frac{1}{2} \pi^2 + \frac{1}{2} (\partial_0 \phi_{\text{as}})^2 + \partial_0 \phi_{\text{as}} \partial_0 \phi_1 + \frac{1}{2} (\partial_0 \phi_1)^2 - \frac{1}{2} (\partial_k \phi_{\text{as}})^2 \\
- \frac{1}{2} (\partial_k \phi_1)^2 - \partial_k \phi_{\text{as}} \partial_k \phi_1 - \frac{1}{2} m^2 \phi_{\text{as}}^2 - \frac{1}{2} m^2 \phi_1^2 - m^2 \phi_{\text{as}} \phi_1 - V(\phi) \right] \\
= \int d^3x \left[ \frac{1}{2} \partial_0 \phi_{\text{as}} \phi_{\text{as}} + \partial_0 \phi_{\text{as}} \phi_1 - \frac{1}{2} \pi \phi \right]_{t_i}^{t_f} \\
+ \int d^3x \int_{t_i}^{t_f} dt \left[ -\frac{1}{2} \pi^2 + \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1 - V(\phi) \right] \\
= \int d^3x \left[ \partial_0 \phi_{\text{as}} \phi - \frac{1}{2} \partial_0 \phi_{\text{as}} \phi_{\text{as}} - \frac{1}{2} \pi \phi \right]_{t_i}^{t_f} \\
+ \int d^3x \int_{t_i}^{t_f} dt \left[ -\frac{1}{2} \pi^2 + \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1 - V(\phi) \right] \hspace{1cm} (B.34)
\]
B.2. PATH INTEGRAL FOR GENERATING FUNCTIONALS

Vemos que no segundo termo as variáveis \( \phi_1 \) e \( \pi_1 \) estão separadas e \( \pi_1 \) aparece quadraticamente. Isto permitirá eliminar \( \pi_1 \) como veremos no seguimento. Analisemos contudo primeiro o termo que tem as condições na fronteira. Usando as definições de \( \phi_{as} \), \( \phi \) e \( \pi \) podemos escrever

\[
i \int d^3x \left[ \partial_0 \phi_{as} \phi - \frac{1}{2} \partial_0 \phi_{as} \phi - \frac{1}{2} \pi \phi \right]_{t_i}^{t_f}
\]

\[
= \int \hat{dk} \left\{ \bar{z}_f(k)z_i(k) - \frac{1}{2} [\bar{z}(k, t_f)z(k, t_f) + \bar{z}(k, t_i)z(k, t_i)] \right. \\
- \left. \frac{1}{4} [z(k, t_f) - z_i(k) e^{-i\omega t_f}]^2 - \frac{1}{4} [\bar{z}(k, t_i) - \bar{z}_f(k) e^{i\omega t_i}]^2 \right\} (B.35)
\]

Nesta expressão o primeiro termo dá a passagem do kernel usual para o kernel normal, o segundo cancela exactamente o termo na fronteira na definição inicial de \( S(z_f, z_i) \) e os últimos têm que ser estudados em detalhe. Reunindo tudo até este ponto a expressão do kernel normal da matriz \( S \) é

\[
S^N(\phi_{as}) = \lim_{-t_i, t_f \to \infty} \int \mathcal{D}(\phi, \pi) \exp \left\{ -\frac{1}{4} \int \hat{dk} \left[ (z(k, t_f) - z_i(k) e^{-i\omega t_f})^2 \\
+ (\bar{z}(k, t_i) - \bar{z}_f(k) e^{i\omega t_i})^2 \right] \right\} \\
\exp \left\{ \int d^3x \int_{t_i}^{t_f} dt \left[ -\pi_1^2 + \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 - V(\phi) \right] \right\} (B.36)
\]

Esta expressão já está próxima do resultado final. Falta só mostrar que os termos dentro da primeira exponencial tendem para zero quando \(-t_i, t_f \to \infty\). Esta é a parte mais delicada do argumento. Vamos expô-lo por passos:

i) Funções rapidamente decrescentes

Queremos que \( I(t) = \int d^3x \, \pi_1^2(\vec{x}, t) \) seja integrável. Dizemos então que funções como \( \pi_1(\vec{x}, t) \) são rapidamente decrescentes (RD) quando \(|t| \to \infty\).

ii) Informação sobre \( z_{1,\text{out}}(k, t) \) e \( z_{1,\text{in}}(k, t) \)

Da definição \( \phi = \phi_{as} + \phi_1 \) resultam as definições

\[
\begin{align*}
\bar{z}(k, t) &= \bar{z}_f(k) e^{i\omega t} + \bar{z}_1(k, t) \\
z(k, t) &= z_i(k) e^{-i\omega t} + z_1(k, t)
\end{align*}
\]

As condições na fronteira dizem-nos que \( z_{1,\text{out}}(k, t) \) e \( z_{1,\text{in}}(k, t) \) são funções RD quando \( t \to +\infty \) e \( t \to -\infty \) respectivamente, mas não nos dizem nada sobre \( \bar{z}_{1,\text{in}} \) e \( z_{1,\text{out}} \), que são precisamente os limites que precisamos.

iii) Informação sobre os limites \( z_{1,\text{out}} \) e \( \bar{z}_{1,\text{in}} \)

Informação sobre os limites \( z_{1,\text{out}} \) e \( \bar{z}_{1,\text{in}} \) obtém-se a partir do seguinte raciocínio,
\[ \pi_1 = \pi - \partial_0 \phi \]
\[ = \int \tilde{d}k \left\{ \left[ i\omega(k) \varphi(k, t) - \partial_0 \varphi(k, t) \right] e^{-ik \cdot \vec{x}} - \left[ i\omega(k) z(k, t) + \partial_0 z(k, t) \right] e^{ik \cdot \vec{x}} \right\} \]
\[ \equiv - \int \tilde{d}k \left[ z_2(k, t) e^{-ik \cdot \vec{x}} + z_2(k, t) e^{ik \cdot \vec{x}} \right] \]  

(B.38)

Para que \( \pi_1 \) seja do tipo RD quando \( |t| \to \infty \) também teremos que ter \( z_2(k, t) \) e \( \varphi_2(k, t) \) RD nesses limites. Vejamos qual a informação contida neste resultado.

- \( t \to +\infty \)

Obtemos então que a função

\[ z_2(k, t) \equiv \partial_0 z(k, t) + i\omega(k) z(k, t) \]  

(B.39)

é RD quando \( t \to +\infty \). A informação sobre \( \varphi_2(k, t) \) não trás nada de novo já que está contida nas condições fronteiras. De facto

\[ \lim_{t \to +\infty} \varphi_2(k, t) \equiv \lim_{t \to +\infty} \left[ \partial_0 \varphi(k, t) - i\omega(k) \varphi(k, t) \right] \]
\[ = i\omega(k) \varphi_f(k) e^{i\omega t} + \partial_0 \varphi_{1,\text{out}}(k, t) \]
\[ -i\omega(k) \varphi_f(k) e^{i\omega t} - i\omega(k) \varphi_{1,\text{out}}(k, t) \]
\[ = \text{RD} \quad t \to +\infty \]  

(B.40)

- \( t \to -\infty \)

A informação contida nas condições na fronteira é

\[ \varphi_2(k, t) \equiv \partial_0 \varphi(k, t) - i\omega(k) \varphi(k, t) \]  

(B.41)

\( iv) \) Demonstração que \( z_{1,\text{out}} \) e \( \varphi_{1,\text{in}} \) são RD

Da definição

\[ \phi(\vec{x}, t) = \phi_{\text{as}} + \phi_1 \]  

(B.42)

resulta

\[ \begin{aligned} 
\phi(\vec{x}, t) &= \phi_{\text{as,\text{in}}}(\vec{x}, t) + \phi_{1,\text{in}}(\vec{x}, t) \quad t \to -\infty \\
\phi(\vec{x}, t) &= \phi_{\text{as,\text{out}}}(\vec{x}, t) + \phi_{1,\text{out}}(\vec{x}, t) \quad t \to +\infty 
\end{aligned} \]  

(B.43)
ou seja

\[
\begin{align*}
\xi(k, t) &= \xi_{in}(k) e^{i\omega t} + \xi_{1,in}(k, t) \quad t \to -\infty \\
z(k, t) &= z_{out}(k) e^{-i\omega t} + z_{1,out}(k, t) \quad t \to +\infty
\end{align*}
\]

(B.44)

Mas usando os resultados anteriores

\[
\begin{align*}
\partial_0 \xi(k, t) - i\omega(k) \xi(k, t) &= RD \quad t \to -\infty \\
&= i\omega(k) \xi_{in}(k) e^{i\omega t} + \partial_0 \xi_{1,in}(k, t) \\
&\quad - i\omega(k) \xi_{in}(k) e^{i\omega t} - i\omega(k) \xi_{1,in}(k, t)
\end{align*}
\]

ou seja

\[
\xi_{1,in}(k, t) = RD \quad t \to -\infty
\]

(B.46)

e igualmente

\[
z_{1,out}(k, t) = RD \quad t \to +\infty
\]

(B.47)

Isto quer dizer que

\[
\begin{align*}
\phi_{1,in} &= RD \quad t \to -\infty \\
\phi_{1,out} &= RD \quad t \to +\infty
\end{align*}
\]

(B.48)

isto é, assimptoticamente

\[
\phi = \phi_{as} + RD
\]

(B.49)

v) Resultado final

Estamos agora em condições de atacar o nosso problema. Temos

\[
\lim_{t_i \to -\infty} \int \tilde{dk} \left[ \left( \xi(k, t_i) - z_f(k) e^{i\omega t_i} \right)^2 \right]
\]

\[
= \lim_{t_i \to -\infty} \int \tilde{dk} \left[ \left( \xi_{in}(k) - z_f(k) \right) e^{i\omega t_i} + \xi_{1,in}(k, t_i) \right]^2
\]

\[
= \lim_{t_i \to -\infty} \int \tilde{dk} \left[ \left( \xi_{in}(k) - z_f(k) \right)^2 e^{2i\omega t_i} + 2 \left( \xi_{in}(k) - z_f(k) \right) e^{i\omega t_i} \xi_{1,in}(k, t_i) \\
+ \xi_{1,in}(k, t_i) \right]
\]
Para o outro termo obter-se-ia o mesmo resultado. Chegamos portanto ao resultado

\[ S_N(\phi_{as}) = \int D(\phi, \pi) \exp \left\{ -\frac{i}{2} \int d^4 x \pi_1^2 \right\} \times \exp \left\{ i \int d^4 x \left[ \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 - V(\phi) \right] \right\} \]

(B.51)

onde a integração é feita sobre os campos \( \phi = \phi_{as} + \phi_1 \) com as condições fronteiras apropriadas. Fazendo a integração sobre \( \pi_1 \) obtemos (a menos duma normalização)

\[ S_N(\phi_{as}) = \int D(\phi) e^{i \int d^4 x \left[ \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 - V(\phi) \right]} \]

(B.52)

Na presença de fontes exteriores obtemos

\[ S_N(\phi_{as}, J) = \int_{\phi=\phi_{as}+\phi_1} D(\phi) e^{i \int d^4 x \left[ \mathcal{L}(\phi_1) - (V(\phi) - V(\phi_1)) + J\phi \right]} \]

(B.53)

Normalmente não estamos interessados na matriz \( S \) mas no funcional gerador das funções de Green. Por definição

\[ Z(J) \equiv S(\phi_{as}, J) \Big|_{\phi_{as}=0} \]

(B.54)

Obtemos portanto a expressão fundamental

\[ Z(J) = \int D(\phi) e^{i \int d^4 x \left[ \mathcal{L}(\phi) + J\phi \right]} \]

(B.55)

### B.3 Fermion systems

O uso de variáveis de Grassmann permite escrever expressões de integrais de caminho para a matriz \( S \) e para o funcional gerador das funções de Green \( Z \) para este caso. Não vamos aqui repetir os cálculos que fizemos para os sistemas de bosões, mas antes apresentar somente os resultados deixando as demonstrações para os problemas.

O ponto de partida é a definição do funcional gerador das funções de Green em presença das fontes exteriores fermiônicas. Este é dado por\(^4\)

\(^4\)Comparar com a definição do caso bosónico, equação 5.15.
B.3. FERMION SYSTEMS

\[ Z[\eta, \overline{\eta}] = \langle 0 | T \exp \left[ i \int d^4x \left( \overline{\eta}(x) \psi(x) + \overline{\psi}(x) \eta(x) \right) \right] | 0 \rangle \]  \hspace{1cm} (B.56)

Então as funções de Green

\[ G^{2n}(x_1, \ldots, y_n) \equiv \langle 0 | T \psi(x_1) \cdots \psi(x_n) \overline{\psi}(y_1) \cdots \overline{\psi}(y_n) | 0 \rangle \]  \hspace{1cm} (B.57)

são dadas por

\[ G^{2n}(x_1, \ldots, y_n) = \frac{\delta^{2n} Z}{i \delta \eta(y_n) \cdots i \delta \eta(y_1) i \delta \overline{\eta}(x_n) \cdots i \delta \overline{\eta}(x_1)} \]  \hspace{1cm} (B.58)

onde as derivadas são esquerdas, isto é

\[ \frac{\delta}{\delta \overline{\eta}(x)} \int d^4y \overline{\psi}(y) \psi(y) = \psi(x) \]

\[ \frac{\delta}{\delta \eta(x)} \int d^4y \overline{\psi}(y) \eta(y) = -\overline{\psi}(x) \]  \hspace{1cm} (B.59)

e por convenção a ordem da derivação é a indicada, isto é

\[ \frac{\delta}{i \delta \eta(y_n)} \cdots \frac{\delta}{i \delta \overline{\eta}(x_1)} \]  \hspace{1cm} (B.60)

Consideramos agora o lagrangeano de Dirac livre

\[ \mathcal{L} = \overline{\psi} (i \gamma^\mu - m) \psi \]  \hspace{1cm} (B.61)

Pode-se mostrar (ver problema B.2) que o funcional gerador é neste caso dado por

\[ Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y \overline{\eta}(x) S_F^0(x-y) \eta(y)} \]  \hspace{1cm} (B.62)

onde \( S_F^0(x-y) \) é o propagador de Feynman para a teoria de Dirac livre, dado por

\[ S_F^0(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x-y)}}{p - m + i\varepsilon} \]  \hspace{1cm} (B.63)

Seguindo métodos semelhantes ao do caso bosónico podemos também mostrar que este funcional gerador pode ser representado pelo integral de caminho,

\[ Z_0[\eta, \overline{\eta}] = \int \mathcal{D}(\psi, \overline{\psi}) \ e^{i \int d^4x \left[ \mathcal{L}(x) + \overline{\eta} \psi + \overline{\psi} \eta \right]} \]  \hspace{1cm} (B.64)

Tendo o funcional gerador para a teoria livre podemos formalmente escrever o funcional gerador para qualquer teoria fermiônica com interações. Um exemplo é dado no Problema B.4.
Problems Appendix B

B.1 Mostre que as funções de Green

\[ G^{2n}(x_1, \ldots, y_n) \equiv \langle 0 | T\psi(x_1) \cdots \psi(x_n) \overline{\psi}(y_1) \cdots \overline{\psi}(y_n) | 0 \rangle \]  

\[ (B.65) \]

são dadas por

\[ G^{2n}(x_1, \ldots, y_n) = \frac{\delta^{2n}Z}{i\delta\eta(y_n)} \cdots i\delta\eta(y_1) i\delta\overline{\eta}(x_n) \cdots i\delta\overline{\eta}(x_1) \]  

\[ (B.66) \]

B.2 Mostre que o funcional gerador das funções de Green para a teoria de Dirac livre é dado por

\[ Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y \overline{\eta}(x) S^\mu_0(x-y) \eta(y)} \]

\[ (B.67) \]

B.3 Mostre que o funcional gerador das funções de Green para a teoria de Dirac livre se pode representar pelo seguinte integral de caminho

\[ Z_0[\eta, \overline{\eta}] = \int \mathcal{D}(\psi, \overline{\psi}) \ e^{i\int d^4x \left[ \mathcal{L}(x) + \overline{\psi}\psi + \overline{\psi}\eta \right]} \]

\[ (B.68) \]

B.4 Considere o lagrangiano seguinte,

\[ \mathcal{L}(x) = -\overline{\psi}(i\partial - m)\psi + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - g \overline{\psi}(x) \psi(x) \phi(x) \]

\[ (B.69) \]
que descreve a interacção dum campo de Dirac com um campo escalar.

a) Mostre que

\[ Z[\eta, \overline{\eta}, J] = \exp \left\{ -ig \int d^4x \left( \frac{\delta}{i\delta \eta} \right) \left( \frac{\delta}{i\delta \overline{\eta}} \right) \left( \frac{\delta}{i\delta J} \right) \right\} \] (B.70)

onde

\[ Z_0[\eta, \overline{\eta}, J] = e^{-\int d^4xd^4y [\overline{\eta}(x)S_\phi^0(x-y)\eta(y)+\frac{1}{2}J(x)\Delta_F(x-y)J(y)]} \] (B.71)

\(\Delta_F\) é o propagador livre do campo escalar.

b) Mostre que \(Z[\eta, \overline{\eta}, J]\) se pode exprimir por meio do integral de caminho

\[ Z[\eta, \overline{\eta}, J] = \int \mathcal{D}(\psi, \overline{\psi}, \phi) e^{i \int d^4x [L(x)+J\phi+\overline{\psi}\gamma^\mu\eta\gamma^\mu\psi]} \] (B.72)
Appendix C

Useful techniques for renormalization

C.1 $\mu$ parameter

The reason for the $\mu$ parameter introduced in section 4.1.1 is the following. In dimension $d = 4 - \epsilon$, the fields $A_\mu$ and $\psi$ have dimensions given by the kinetic terms in the action,

\[
\int d^dx \left[ -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \overline{\psi} \gamma \cdot \partial \psi \right]
\]

We have therefore

\[
0 = -d + 2 + 2[A_\mu] \Rightarrow [A_\mu] = \frac{1}{2} (d - 2) = 1 - \frac{\epsilon}{2}
\]

\[
0 = -d + 1 + 2[\psi] \Rightarrow [\psi] = \frac{1}{2} (d - 1) = \frac{3}{2} - \frac{\epsilon}{2}
\]

Using these dimensions in the interaction term

\[
S_I = \int d^dx \, e \overline{\psi} \gamma_\mu \psi A^\mu
\]

we get

\[
[S_I] = -d + [e] + 2[\psi] + [A]
\]

\[
= -4 + \epsilon + [e] + 3 - \epsilon + 1 - \frac{\epsilon}{2}
\]

\[
= [e] - \frac{\epsilon}{2}
\]

Therefore, if we want the action to be dimensionless (remember that we use the system where $\hbar = c = 1$), we have to set

\[
\frac{\epsilon}{2} = 1
\]

\[
\Rightarrow \epsilon = 2
\]
\[ |e| = \frac{\epsilon}{2} \]  

(C.5)

We see then that in dimensions \( d \neq 4 \) the coupling constant has dimensions. As it is more convenient to work with a dimensionless coupling constant we introduce a parameter \( \mu \) with dimensions of a mass and in \( d \neq 4 \) we will make the substitution

\[ e \to e \mu^\epsilon \quad (\epsilon = 4 - d) \]  

(C.6)

while keeping \( e \) dimensionless.

### C.2 Feynman parametrization

The most general form for a 1-loop is \(^1\)

\[ \hat{T}_{n}^{\mu_{1} \cdots \mu_{p}} = \int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_{1}} \cdots k^{\mu_{p}}}{D_{0}D_{1} \cdots D_{n-1}} \]  

(C.7)

where

\[ D_{i} = (k + r_{i})^{2} - m_{i}^{2} + i\epsilon \]  

(C.8)

and the momenta \( r_{i} \) are related with the external momenta (all taken to be incoming) through the relations,

\[ r_{j} = \sum_{i=1}^{j} p_{i} \quad ; \quad j = 1, \ldots, n - 1 \]

\[ r_{0} = \sum_{i=1}^{n} p_{i} = 0 \]  

(C.9)

as indicated in Fig. (C.1). In these expressions there appear in the denominators products of the denominators of the propagators of the particles in the loop. It is convenient to

\(^1\)We introduce here the notation \( \hat{T} \) to distinguish from a more standard notation that will be explained in subsection C.9.
C.2. FEYNMAN PARAMETRIZATION

combine these products in just one common denominator. This is achieved by a technique
due to Feynman. Let us exemplify with two denominators.

\[ \frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1 - x)]^2} \quad (C.10) \]

The proof is trivial. In fact

\[ \int dx \frac{1}{[ax + b(1 - x)]^2} = \frac{x}{b[(a - b)x + b]} \quad (C.11) \]

and therefore Eq. (C.10) immediately follows. Taking successive derivatives with respect
to \( a \) and \( b \) we get

\[ \frac{1}{a^p b^q} = \frac{\Gamma(p + q)}{\Gamma(p)\Gamma(q)} \int_0^1 \frac{dx}{[ax + b(1 - x)]^{p+q}} \quad (C.12) \]

and using induction we obtain a general formula

\[ \frac{1}{a_1 a_2 \cdots a_n} = \Gamma(n) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \frac{dx_{n-1}}{[a_1 x_1 + a_2 x_2 + \cdots + a_n (1 - x_1 - \cdots - x_{n-1})]^n} \quad (C.13) \]

Before closing the section let us give an example that will be useful in the self-energy case.
Consider the situation with the kinematics described in Fig. (C.2).

We get

\[ I = \int \frac{d^4k}{(2\pi)^d} \frac{1}{[(k + p)^2 - m_1^2 + i\epsilon] \left[ k^2 - m_2^2 + i\epsilon \right]} \]

\[ = \int_0^1 dx \int \frac{d^4k}{(2\pi)^d} \frac{1}{\left[ k^2 + 2p \cdot k \cdot x + p^2 x - m_1^2 x - m_2^2 (1 - x) + i\epsilon \right]^2} \]

\[ = \int_0^1 dx \int \frac{d^4k}{(2\pi)^d} \frac{1}{\left[ k^2 + 2P \cdot k - M^2 + i\epsilon \right]^2} \]

**Figure C.2:**
\[ = \int_{0}^{1} dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k + P)^2 - P^2 - M^2 + i\epsilon]^2} \quad (C.14) \]

where in the last line we have completed the square in the term with the loop momenta \( k \). The quantities \( P \) and \( M^2 \) are, in this case, defined by

\[ P = xp \quad (C.15) \]

and

\[ M^2 = -x p^2 + m_1^2 x + m_2^2 (1 - x) \quad (C.16) \]

They depend on the masses, external momenta and Feynman parameters, but not in the loop momenta. Now changing variables \( k \rightarrow k - P \) we get rid of the linear terms in \( k \) and finally obtain

\[ I = \int_{0}^{1} dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - C + i\epsilon]^2} \quad (C.17) \]

where \( C \) is independent of the loop momenta \( k \) and it is given by

\[ C = P^2 + M^2 \quad (C.18) \]

Notice that the \( i\epsilon \) factors will add correctly and can all be put as in Eq. (C.17).

## C.3 Wick Rotation

From the example of the last section we can conclude that all the scalar integrals can be reduced to the form

\[ I_{r,m} = \int \frac{d^d k}{(2\pi)^d} \frac{k^{2r}}{[k^2 - C + i\epsilon]^m} \quad (C.19) \]

It is also easy to realize that also all the tensor integrals can be obtained from the scalar integrals. For instance

\[ \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{[k^2 - C + i\epsilon]^m} = 0 \]

\[ \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2 - C + i\epsilon]^m} = \frac{1}{d} \ g^{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{[k^2 - C + i\epsilon]^m} \quad (C.20) \]

and so on. Therefore the integrals \( I_{r,m} \) are the important quantities to evaluate. We will consider that \( C > 0 \). The case \( C < 0 \) can be done by analytical continuation of the final formula for \( C > 0 \).

To evaluate the integral \( I_{r,m} \) we will use integration in the complex plane of the variable \( k^0 \) as described in Fig. C.3. We can then write

\[ I_{r,m} = \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \int d k^0 \frac{k^{2r}}{[k^0 - |\vec{k}|^2 - C + i\epsilon]^m} \quad (C.21) \]
C.4 SCALAR INTEGRALS IN DIMENSIONAL REGULARIZATION

The function under the integral has poles for

\[ k^0 = \pm \left( \sqrt{|\mathbf{k}|^2 + C - i\epsilon} \right) \]  

as shown in Fig. C.3. Using the properties of functions of complex variables (Cauchy theorem) we can deform the contour, changing the integration from the real to the imaginary axis plus the two arcs at infinity. This can be done because in deforming the contour we do not cross any pole. Notice the importance of the \( i\epsilon \) prescription to be able to do this. The contribution from the arcs at infinity vanishes in dimension sufficiently low for the integral to converge, as we assume in dimensional regularization. We have then changed the integration along the real axis into an integration along the imaginary axis in the plane of the complex variable \( k^0 \). If we write

\[ k^0 = i k_0^E \quad \text{com} \quad \int_{-\infty}^{+\infty} dk^0 \rightarrow i \int_{-\infty}^{+\infty} dk_0^E \]  

and

\[ k^2 = (k^0)^2 - |\mathbf{k}|^2 = -(k_0^E)^2 - |\mathbf{k}|^2 \equiv -k_E^2, \]  

where \( k_E = (k_0^E, \mathbf{k}) \) is an euclidean vector. By this we mean that we calculate the scalar product using the euclidean metric \( \text{diag}(+,+,+,+) \),

\[ k_E^2 = (k_0^E)^2 + |\mathbf{k}|^2 \]  

We can then write

\[ I_{r,m} = i(-1)^{r-m} \int \frac{d^dk_E}{(2\pi)^d} \frac{k_E^{2r}}{[k_E^2 + C]^m} \]  

where we do not need the \( i\epsilon \) because the denominator is strictly positive \( (C > 0) \). This procedure is known as Wick Rotation. We note that the Feynman prescription for the propagators that originated the \( i\epsilon \) rule for the denominators is crucial for the Wick rotation to be possible.

C.4 Scalar integrals in dimensional regularization

We have seen in the last section that the scalar integrals to be calculated with dimensional regularization had the general form of Eq. (C.25). We are now going to find a general
formula for $I_{r,m}$. We begin by writing

$$\int d^d k_E = \int d\vec{k} \, \vec{k}^{-d-1} \, d\Omega_{d-1} \quad (C.26)$$

where $\vec{k} = \sqrt{(k_0^2 + \vec{k}^2)}$ is the length of the vector $k_E$ in the euclidean space in $d$ dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates in that euclidean space. The angles are defined by

$$k_E = \vec{k}(\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \ldots, \sin \theta_1 \cdots \sin \theta_{d-1}) \quad (C.27)$$

We can then write

$$\int d\Omega_{d-1} = \int_0^\pi \sin^{d-2} \theta_1 \, d\theta_1 \cdots \int_0^{2\pi} d\theta_{d-1} \quad (C.28)$$

Using now

$$\int_0^\pi \sin^m \theta \, d\theta = \sqrt{\pi} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})} \quad (C.29)$$

where $\Gamma(z)$ is the gamma function (see section C.6) we get

$$\int d\Omega_{d-1} = 2 \pi^{\frac{d}{2}} \frac{\pi^{d}}{\Gamma(\frac{d}{2})} \quad (C.30)$$

The integration in $\vec{k}$ is done using the result

$$\int_0^\infty dx \, \frac{x^p}{(x^n + a^n)^q} = \pi (-1)^{q-1} a^{p+1-nq} \frac{\Gamma(\frac{p+1}{n})}{n \sin(\pi \frac{p+1}{n})} \frac{\Gamma(\frac{p+1}{2} - q + 1)}{\Gamma(\frac{p+1}{2})} \quad (C.31)$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}} \left( -1 \right)^{r-m} \frac{\Gamma(r + \frac{d}{2})}{\left( 4\pi \right)^\frac{d}{2}} \frac{\Gamma(2 + r - \frac{d}{2})}{\Gamma(2 - \frac{d}{2})} \frac{\Gamma(m - r - 2 + \frac{d}{2})}{\Gamma(m)} \quad (C.32)$$

Before ending the section we note that the integral representation for $I_{r,m}$, Eq. (C.19), is valid only for $d < 2(m - r)$ to ensure convergence when $\vec{k} \to \infty$. However the final form in Eq. (C.32) can be analytically continued for all values of $d$ except for those where the function $\Gamma(m - r - d/2)$ has poles, that is for (see section C.6),

$$m - r - \frac{d}{2} \neq 0, -1, -2, \ldots \quad (C.33)$$

For the application in dimensional regularization it is convenient to rewrite Eq. (C.32) using the relation $d = 4 - \epsilon$. we get

$$I_{r,m} = iC^{r-m+\frac{d}{2}} \left( -1 \right)^{r-m} \frac{\Gamma(r + \frac{d}{2})}{\left( 4\pi \right)^\frac{d}{2}} \frac{\Gamma(2 + r - \frac{d}{2})}{\Gamma(2 - \frac{d}{2})} \frac{\Gamma(m - r - 2 + \frac{d}{2})}{\Gamma(m)} \quad (C.34)$$
C.5 Tensor integrals in dimensional regularization

We are frequently faced with the task of evaluating the tensor integrals of the form of Eq. (C.7),

\[ \hat{T}^{\mu_1 \cdots \mu_p}_{\mu_1 \cdots \mu_p} = \int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \cdots k^{\mu_p}}{D_0 D_1 \cdots D_{n-1}} \]  

(C.35)

The first step is to reduce to one common denominator using the Feynman parameterization technique. The result is,

\[ \hat{T}^{\mu_1 \cdots \mu_p}_{\mu_1 \cdots \mu_p} = \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \cdots k^{\mu_p}}{[k^2 + 2k \cdot P - M^2 + i\epsilon]^n} \]  

(C.36)

where we have defined

\[ I^{\mu_1 \cdots \mu_p}_{\mu_1 \cdots \mu_p} = \int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \cdots k^{\mu_p}}{[k^2 + 2k \cdot P - M^2 + i\epsilon]^n} \]  

(C.37)

that we call, from now on, the tensor integral. In principle all these integrals can be written in terms of scalar integrals. It is however convenient to have a general formula for them. This formula can be obtained by noticing that

\[ \frac{\partial}{\partial P^\mu} \frac{1}{[k^2 + 2k \cdot P - M^2 + i\epsilon]^n} = -n \frac{2k_\mu}{[k^2 + 2k \cdot P - M^2 + i\epsilon]^{n+1}} \]  

(C.38)

Using the last equation one can write the final result

\[ I^{\mu_1 \cdots \mu_p}_{\mu_1 \cdots \mu_p} = \frac{i}{16\pi^2} \frac{(4\pi)^{\epsilon/2}}{\Gamma(n)} \int_0^\infty \frac{dt}{(2t)^\mu} t^{n-\epsilon/2} \frac{\partial}{\partial P^\mu_1} \cdots \frac{\partial}{\partial P^\mu_p} e^{-tC} \]  

(C.39)

where \( C = P^2 + M^2 \). After doing the derivatives the remaining integrals can be done using the properties of the \( \Gamma \) function (see section C.6). Notice that \( P, M^2 \) and therefore also \( C \) depend not only in the Feynman parameters but also in the exterior momenta. The advantage of having a general formula is that it can be programmed [9] and all the integrals can then be obtained automatically.

C.6 \( \Gamma \) function and useful relations

The \( \Gamma \) function is defined by the integral

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \]  

(C.40)

or equivalently

\[ \int_0^\infty t^{z-1} e^{-t} dt = \mu^{-z} \Gamma(z) \]  

(C.41)
The function $\Gamma(z)$ has the following important properties

\[
\begin{align*}
\Gamma(z + 1) &= z \Gamma(z) \\
\Gamma(n + 1) &= n!
\end{align*}
\] (C.42)

Another related function is the logarithmic derivative of the $\Gamma$ function, with the properties,

\[
\begin{align*}
\psi(z) &= \frac{d}{dz} \ln \Gamma(z) \quad \text{(C.43)} \\
\psi(1) &= -\gamma \quad \text{(C.44)} \\
\psi(z + 1) &= \psi(z) + \frac{1}{z} \quad \text{(C.45)}
\end{align*}
\]

where $\gamma$ is the Euler constant. The function $\Gamma(z)$ has poles for $z = 0, -1, -2, \ldots$. Near the pole $z = -m$ we have

\[
\Gamma(z) = \frac{(-1)^m}{m!} \frac{1}{m + z} + \frac{(-1)^m}{m!} \psi(m + 1) + O(z + m) \quad \text{(C.46)}
\]

From this we conclude that when $\epsilon \to 0$

\[
\begin{align*}
\Gamma \left( \frac{\epsilon}{2} \right) &= \frac{2}{\epsilon} + \psi(1) + O(\epsilon) \\
\Gamma(-n + \epsilon) &= \frac{(-1)^n}{n!} \left[ \frac{2}{\epsilon} + \psi(n + 1) + 1 \right] \quad \text{(C.47)}
\end{align*}
\]

and

\[
\Gamma(1 + \epsilon) = 1 - \gamma \epsilon + \left( \gamma^2 + \frac{\pi^2}{6} \right) \frac{\epsilon^2}{2!} + \cdots, \quad \epsilon \to 0 \quad \text{(C.48)}
\]

Using these results we can expand our integrals in powers of $\epsilon$ and separate the divergent and finite parts. For instance for the one of the integrals of the self-energy,

\[
I_{0,2} = \frac{i}{(4\pi)^2} \left( \frac{4\pi}{C} \right) \left[ \frac{1}{\epsilon} - \gamma + \ln \frac{4\pi}{\epsilon} \right]
\]

\[
= \frac{i}{16\pi^2} \left[ \frac{2}{\epsilon} - \gamma \ln 4\pi - \ln C + O(\epsilon) \right]
\]

\[
= \frac{i}{16\pi^2} \left[ \Delta + \ln C + O(\epsilon) \right] \quad \text{(C.49)}
\]

where we have introduced the notation

\[
\Delta = \frac{2}{\epsilon} - \gamma + \ln 4\pi \quad \text{(C.50)}
\]

for a combination that will appear in all expressions.
C.7 Explicit formulæ for the 1–loop integrals

Although we have presented in the previous sections the general formulæ for all the integrals that appear in 1–loop, Eqs. (C.34) and (C.39), in practice it is convenient to have expressions for the most important cases with the expansion on the $\epsilon$ already done. The results presented below were generated with the Mathematica package OneLoop [9] from the general expressions. In these results the integration on the Feynman parameters has still to be done. This is in general a difficult problem and we will present in section C.9 an alternative way of expressing these integrals more convenient for a numerical evaluation.

C.7.1 Tadpole integrals

With the definitions of Eqs. (C.34) and (C.39) we get

$$I_{0,1} = \frac{i}{16\pi^2} C(1 + \Delta_\epsilon - \ln C)$$

$$I_1^\mu = 0$$

$$I_1^{\mu\nu} = \frac{i}{16\pi^2} \frac{1}{8} C^2 g^{\mu\nu}(3 + 2\Delta_\epsilon - 2\ln C)$$

where for the tadpole integrals

$$P = 0 \quad ; \quad C = m^2$$

because there are no Feynman parameters and there is only one mass. In this case the above results are final.

C.7.2 Self–Energy integrals

For the integrals with two denominators we get,

$$I_{0,2} = \frac{i}{16\pi^2} (\Delta_\epsilon - \ln C)$$

$$I_2^\mu = \frac{i}{16\pi^2} (-\Delta_\epsilon + \ln C)P^\mu$$

$$I_2^{\mu\nu} = \frac{i}{16\pi^2} \frac{1}{2} \left[ C g^{\mu\nu}(1 + \Delta_\epsilon - \ln C) + 2(\Delta_\epsilon - \ln C)P^\mu P^\nu \right]$$

$$I_2^{\mu\nu\alpha} = \frac{i}{16\pi^2} \frac{1}{2} \left[ - C g^{\mu\nu}(1 + \Delta_\epsilon - \ln C)P^\alpha - C g^{\nu\alpha}(1 + \Delta_\epsilon - \ln C)P^\mu 
- C g^{\mu\alpha}(1 + \Delta_\epsilon - \ln C)P^\nu - (2\Delta_\epsilon P^\alpha P^\mu - 2\ln C P^\alpha P^\mu)P^\nu \right]$$

where, with the notation and conventions of Fig. (C.1), we have

$$P^\mu = x r_1^\mu \quad ; \quad C = x^2 r_1^2 + (1 - x) m_2^2 + x m_1^2 - x r_1^2$$
C.7.3 Triangle integrals

For the integrals with three denominators we get,

\[ I_{0,3} = \frac{i}{16\pi^2} \frac{-1}{2C} \]

\[ I_3^\mu = \frac{i}{16\pi^2} \frac{1}{2C} P^\mu \]

\[ I_3^{\mu \nu} = \frac{i}{16\pi^2} \frac{1}{4C} \left[ C g^{\mu \nu} (\Delta_\epsilon - \ln C) - 2 P^\mu P^\nu \right] \]

\[ I_3^{\mu \nu \alpha} = \frac{i}{16\pi^2} \frac{1}{4C} \left[ C g^{\mu \nu} (-\Delta_\epsilon + \ln C) P^\alpha + C g^{\nu \alpha} (-\Delta_\epsilon + \ln C) P^\mu \right. \]
\[ \left. \quad + C g^{\mu \alpha} (-\Delta_\epsilon + \ln C) P^\nu + 2 P^\alpha P^\mu P^\nu \right] \]

\[ I_3^{\mu \nu \alpha \beta} = \frac{i}{16\pi^2} \frac{1}{8C} \left[ C^2 (1 + \Delta_\epsilon - \ln C) \left( g^{\mu \alpha} g^{\nu \beta} + g^{\mu \beta} g^{\nu \alpha} + g^{\alpha \beta} g^{\mu \nu} \right) \right. \]
\[ \quad + 2C (\Delta_\epsilon - \ln C) \left( g^{\mu \nu} P^\alpha P^\beta + g^{\nu \beta} P^\alpha P^\mu + g^{\nu \alpha} P^\beta P^\mu + g^{\mu \alpha} P^\beta P^\nu \right. \]
\[ \left. \quad + g^{\mu \beta} P^\alpha P^\nu \right) - 4 P^\alpha P^\beta P^\mu P^\nu \] (C.55)

where

\[ P^\mu = x_1 r_1^\mu + x_2 r_2^\mu \]

\[ C = x_1^2 r_1^2 + x_2^2 r_2^2 + 2x_1 x_2 r_1 \cdot r_2 + x_1 m_1^2 + x_2 m_2^2 \]
\[ + (1 - x_1 - x_2) m_3^2 - x_1 r_1^2 - x_2 r_2^2 \] (C.56)

C.7.4 Box integrals

\[ I_{0,4} = \frac{i}{16\pi^2} \frac{1}{6C^2} \]

\[ I_4^\mu = \frac{i}{16\pi^2} \frac{-1}{6C^2} P^\mu \]

\[ I_4^{\mu \nu} = \frac{i}{16\pi^2} \frac{-1}{12C^2} \left[ C g^{\mu \nu} - 2 P^\mu P^\nu \right] \]

\[ I_4^{\mu \nu \alpha} = \frac{i}{16\pi^2} \frac{1}{12C^2} \left[ C (g^{\mu \nu} P^\alpha + g^{\nu \alpha} P^\mu + g^{\mu \alpha} P^\nu) - 2 P^\alpha P^\mu P^\nu \right] \]
\[ I_4^{\mu\nu\alpha\beta} = \frac{i}{16\pi^2} \frac{1}{24C^2} \left[ C^2 (\Delta_\epsilon - \ln C) \left( g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + g^{\alpha\beta} g^{\mu\nu} \right) \right. \\
\left. - 2C \left( g^{\mu\nu} P^\alpha P^\beta + g^{\mu\beta} P^\alpha P^\nu + g^{\nu\alpha} P^\beta P^\mu + g^{\alpha\beta} P^\mu P^\nu \right) \right. \\
\left. + g^{\mu\beta} P^\alpha P^\nu + g^{\alpha\beta} P^\mu P^\nu \right) + 4P^\alpha P^\beta P^\mu P^\nu \right] \] (C.57)

where

\[ P^\mu = x_1 r_1^\mu + x_2 r_2^\mu + x_3 r_3^\mu \]

\[ C = x_1^2 r_1^2 + x_2^2 r_2^2 + x_3^2 r_3^2 + 2x_1 x_2 r_1 \cdot r_2 + 2x_1 x_3 r_1 \cdot r_3 + 2x_2 x_3 r_2 \cdot r_3 \]

\[ + x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2 + (1 - x_1 - x_2 - x_3) m_4^2 \]

\[ - x_1 r_1^2 - x_2 r_2^2 - x_3 r_3^2 \] (C.58)

\section*{C.8 Divergent part of 1–loop integrals}

When we want to study the renomalization of a given theory it is often convenient to have expressions for the divergent part of the one-loop integrals, with the integration on the Feynman parameters already done. We present here the results for the most important cases. These divergent parts were calculated with the help of the package \texttt{OneLoop} \cite{9}. The results are for the functions \( \hat{T}_n^{\mu,\mu_2,\ldots,\mu_n} \) defined in Eq. (C.35).

\subsection*{C.8.1 Tadpole integrals}

\[ \text{Div} \left[ \hat{T}_1 \right] = \frac{i}{16\pi^2} \Delta_\epsilon m^2 \]

\[ \text{Div} \left[ \hat{T}_1^\mu \right] = 0 \]

\[ \text{Div} \left[ \hat{T}_1^{\mu\nu} \right] = \frac{i}{16\pi^2} \frac{1}{4} \Delta_\epsilon m^4 g^{\mu\nu} \] (C.59)

\subsection*{C.8.2 Self–Energy integrals}

\[ \text{Div} \left[ \hat{T}_2 \right] = \frac{i}{16\pi^2} \Delta_\epsilon r_1^\mu \]

\[ \text{Div} \left[ \hat{T}_2^\mu \right] = \frac{i}{16\pi^2} \left( -\frac{1}{2} \right) \Delta_\epsilon r_1^\mu \]

\[ \text{Div} \left[ \hat{T}_2^{\mu\nu} \right] = \frac{i}{16\pi^2} \frac{1}{12} \Delta_\epsilon \left( 3m_1^2 + 3m_2^2 - r_1^2 \right) g^{\mu\nu} + 4r_1^\mu r_1^\nu \]
\[
\text{Div} \left[ \hat{T}_2^{\mu \alpha} \right] = \frac{i}{16\pi^2} \left( -\frac{1}{24} \right) \Delta_{\epsilon} \left[ (4m_1^2 + 2m_2^2 - r_1^2) (g^{\mu \nu} r_1^\alpha + g^{\nu \alpha} r_1^\mu + g^{\mu \alpha} r_1^\nu) + 6 r_1^\alpha r_1^\mu r_1^\nu \right] 
\]

\[
\text{Div} \left[ \hat{T}_3^{\mu \nu} \right] = 0 
\]

\[
\text{Div} \left[ \hat{T}_4^{\mu \nu} \right] = 0 
\]

\[
\text{Div} \left[ \hat{T}_4^{\mu \alpha \beta} \right] = \frac{i}{16\pi^2} \frac{1}{48} \Delta_{\epsilon} \left[ (2m_1^2 + 2m_2^2 + 2m_3^2) (g^{\mu \alpha} g^{\nu \beta} + g^{\alpha \beta} g^{\mu \nu} + g^{\beta \mu} g^{\alpha \nu}) 
\]
\[
+ g^{\alpha \beta} [2r_1^\alpha r_1^\nu + r_1^\mu r_1^\nu + (r_1 \leftrightarrow r_2)] + g^{\mu \beta} [2r_1^\alpha r_1^\nu + r_1^\mu r_1^\nu + (r_1 \leftrightarrow r_2)] 
\]
\[
+ g^{\nu \beta} [2r_1^\alpha r_1^\nu + r_1^\mu r_1^\nu + (r_1 \leftrightarrow r_2)] + g^{\mu \nu} [2r_1^\alpha r_1^\nu + r_1^\mu r_1^\nu + (r_1 \leftrightarrow r_2)] 
\]
\[
+ g^{\mu \alpha} [2r_1^\beta r_1^\nu + r_1^\beta r_1^\nu + (r_1 \leftrightarrow r_2)] + g^{\nu \alpha} [2r_1^\beta r_1^\nu + r_1^\beta r_1^\nu + (r_1 \leftrightarrow r_2)] 
\]
\[
+ (\frac{1}{24} r_1^2 + r_1 \cdot r_2 - r_2^2) (g^{\mu \alpha} g^{\nu \beta} + g^{\alpha \beta} g^{\mu \nu} + g^{\beta \mu} g^{\alpha \nu}) \right] 
\]

\[
\text{Div} \left[ \hat{T}_4^{\mu \nu \alpha \beta} \right] = \frac{i}{16\pi^2} \frac{1}{24} \Delta_{\epsilon} \left[ g^{\mu \nu} g^{\alpha \beta} + g^{\alpha \beta} g^{\mu \nu} + g^{\mu \alpha} g^{\nu \beta} \right] 
\]

### C.9 Passarino-Veltman Integrals

#### C.9.1 The general definition

The description of the previous sections works well if one just wants to calculate the divergent part of a diagram or to show the cancellation of divergences in a set of diagrams. If one actually wants to numerically calculate the integrals the task is normally quite...
complicated. Except for the \textit{self-energy} type of diagrams the integration over the Feynman parameters is normally quite difficult.

To overcome this problem a scheme was first proposed by Passarino and Veltman [10]. These scheme with the conventions of [11, 12] was latter implemented in the \texttt{Mathematica} package \texttt{FeynCalc} [13] and, for numerical evaluation, in the \texttt{LoopTools} package [14, 15]. The numerical evaluation follows the code developed earlier by van Oldenborgh [16].

We will now describe this scheme. We will write the generic one-loop tensor integral as
\[ T_{n}^{\mu_{1} \cdots \mu_{p}} = \frac{(2\pi\mu)^{4-d}}{i\pi^{2}} \int d^{d}k \frac{k^{\mu_{1}} \cdots k^{\mu_{p}}}{D_{0}D_{1} \cdots D_{n-1}} \] (C.63)
where we follow for the momenta the conventions of section C.2 and Fig. C.1 and defined \( D_{0} \equiv D_{n} \) and \( m_{n} = m_{0} \) so that \( D_{0} = k^{2} - m_{0}^{2} \) (remember that \( r_{n} \equiv r_{0} = 0 \). The main difference between this definition and the previous one Eq. (C.7) is that a factor of \( \frac{i}{16\pi^{2}} \) is taken out. This is because, as we have seen in section C.3 these integrals always give that prefactor. So with our new convention that prefactor \textit{has} to included in the end. Factoring out the \( i \) has also the convenience of dealing with real functions in many cases.\footnote{The one loop functions are in general complex, but in some cases they can be real. These cases correspond to the situation where cutting the diagram does not corresponding to a kinematically allowed process.}

From all those integrals in Eq. (C.63) the scalar integrals are, has we have seen, of particular importance and deserve a special notation. It can be shown that there are only four independent such integrals, namely \((4 - d = \epsilon)\)
\[ A_{0}(m_{0}^{2}) = \frac{(2\pi\mu)^{\epsilon}}{i\pi^{2}} \int d^{d}k \frac{1}{k^{2} - m_{0}^{2}} \] (C.64)
\[ B_{0}(r_{10}^{2}, m_{1}^{2}, m_{2}^{2}) = \frac{(2\pi\mu)^{\epsilon}}{i\pi^{2}} \int d^{d}k \prod_{i=0}^{1} \frac{1}{(k + r_{i})^{2} - m_{i}^{2}} \] (C.65)
\[ C_{0}(r_{10}^{2}, r_{12}^{2}, r_{20}^{2}, m_{1}^{2}, m_{2}^{2}, m_{3}^{2}) = \frac{(2\pi\mu)^{\epsilon}}{i\pi^{2}} \int d^{d}k \prod_{i=0}^{2} \frac{1}{(k + r_{i})^{2} - m_{i}^{2}} \] (C.66)
\[ D_{0}(r_{10}^{2}, r_{12}^{2}, r_{23}^{2}, r_{30}^{2}, r_{20}^{2}, r_{13}^{2}, m_{1}^{2}, \ldots, m_{4}^{2}) = \frac{(2\pi\mu)^{\epsilon}}{i\pi^{2}} \int d^{d}k \prod_{i=0}^{3} \frac{1}{(k + r_{i})^{2} - m_{i}^{2}} \] (C.67)
where
\[ r_{ij}^{2} = (r_{i} - r_{j})^{2} ; \quad \forall \ i, j = (0, n - 1) \] (C.68)
Remember that with our conventions \( r_{0} = 0 \) so \( r_{10}^{2} = r_{i}^{2} \). In all these expressions the \( i\epsilon \) part of the denominator factors is supressed. The general one-loop tensor integrals are not independent. Their decomposition is not unique. We follow the conventions of [13, 15] to write
\[ B^{\mu} = \frac{(2\pi\mu)^{4-d}}{i\pi^{2}} \int d^{d}k k^{\mu} \prod_{i=0}^{1} \frac{1}{(k + r_{i})^{2} - m_{i}^{2}} \] (C.69)
\[ B^{\mu\nu} = \frac{(2\pi\mu)^{4-d}}{i\pi^{2}} \int d^{d}k k^{\mu} k^{\nu} \prod_{i=0}^{1} \frac{1}{(k + r_{i})^{2} - m_{i}^{2}} \] (C.70)
These integrals can be decomposed in terms of (reducible) functions in the following way:

\[ C^{\mu} = \frac{(2\pi)^{4-d}}{i^2} \int d^d k \, k^{\mu} \prod_{i=0}^{2} \frac{1}{(k + r_i)^2 - m_i^2} \]  
(C.71)

\[ C^{\mu \nu} = \frac{(2\pi)^{4-d}}{i^2} \int d^d k \, k^{\mu} k^{\nu} \prod_{i=0}^{2} \frac{1}{(k + r_i)^2 - m_i^2} \]  
(C.72)

\[ C^{\mu \nu \rho} = \frac{(2\pi)^{4-d}}{i^2} \int d^d k \, k^{\mu} k^{\nu} k^{\rho} \prod_{i=0}^{2} \frac{1}{(k + r_i)^2 - m_i^2} \]  
(C.73)

\[ D^{\mu} = \frac{(2\pi)^{4-d}}{i^2} \int d^d k \, k^{\mu} \prod_{i=0}^{3} \frac{1}{(k + r_i)^2 - m_i^2} \]  
(C.74)

\[ D^{\mu \nu} = \frac{(2\pi)^{4-d}}{i^2} \int d^d k \, k^{\mu} k^{\nu} \prod_{i=0}^{3} \frac{1}{(k + r_i)^2 - m_i^2} \]  
(C.75)

\[ D^{\mu \nu \rho} = \frac{(2\pi)^{4-d}}{i^2} \int d^d k \, k^{\mu} k^{\nu} k^{\rho} \prod_{i=0}^{3} \frac{1}{(k + r_i)^2 - m_i^2} \]  
(C.76)

\[ D^{\mu \nu \rho \sigma} = \frac{(2\pi)^{4-d}}{i^2} \int d^d k \, k^{\mu} k^{\nu} k^{\rho} k^{\sigma} \prod_{i=0}^{3} \frac{1}{(k + r_i)^2 - m_i^2} \]  
(C.77)

These integrals can be decomposed in terms of (reducible) functions in the following way:

\[ B^{\mu} = r_1^{\mu} B_1 \]  
(C.78)

\[ B^{\mu \nu} = g^{\mu \nu} B_{00} + r_1^{\mu} r_1^{\nu} B_{11} \]  
(C.79)

\[ C^{\mu} = r_1^{\mu} C_1 + r_2^{\mu} C_2 \]  
(C.80)

\[ C^{\mu \nu} = g^{\mu \nu} C_{00} + \sum_{i=1}^{2} r_i^{\mu} r_i^{\nu} C_{ij} \]  
(C.81)

\[ C^{\mu \nu \rho} = \sum_{i=1}^{2} (g^{\mu \nu \rho} r_i^{\mu} + g^{\mu \rho \nu} r_i^{\rho} + g^{\nu \mu \rho} r_i^{\nu}) \, C_{00} + \sum_{i,j,k=1}^{2} r_i^{\mu} r_j^{\nu} r_k^{\rho} C_{ijk} \]  
(C.82)

\[ D^{\mu} = \sum_{i=1}^{3} r_i^{\mu} D_i \]  
(C.83)

\[ D^{\mu \nu} = g^{\mu \nu} D_{00} + \sum_{i=1}^{3} r_i^{\mu} r_i^{\nu} D_{ij} \]  
(C.84)

\[ D^{\mu \nu \rho} = \sum_{i=1}^{3} (g^{\mu \nu \rho} r_i^{\mu} + g^{\mu \rho \nu} r_i^{\rho} + g^{\nu \mu \rho} r_i^{\nu}) \, D_{00} + \sum_{i,j,k=1}^{3} r_i^{\mu} r_j^{\nu} r_k^{\rho} D_{ijk} \]  
(C.85)

\[ D^{\mu \nu \rho \sigma} = (g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho} + g_{\mu \rho} g_{\sigma \nu}) \, D_{0000} + \sum_{i,j=1}^{3} \left( g^{\mu \nu \rho \sigma} r_i^{\mu} r_j^{\nu} + g^{\mu \rho \nu \sigma} r_i^{\rho} r_j^{\sigma} + g^{\nu \mu \rho \sigma} r_i^{\nu} r_j^{\rho} + g^{\nu \rho \mu \sigma} r_i^{\nu} r_j^{\rho} + g^{\rho \mu \nu \sigma} r_i^{\rho} r_j^{\nu} ight) D_{00ij} \]  
(C.86)
\[ + \sum_{i,j,k,l=1}^{3} r_i^\mu r_j^\nu r_k^\rho r_l^\sigma C_{ijkl} \]  

(C.87)

All coefficient functions have the same arguments as the corresponding scalar functions and are totally symmetric in their indices. In the FeynCalc [13] package one generic notation is used,

\[ \text{PaVe} [i,j,\ldots,\{r_{10}^2, r_{12}^2, \ldots\}, \{m_0^2, m_1^2, \ldots\}] \]  

(C.88)

for instance

\[ B_{11}(r_{10}^2, m_0^2, m_1^2) = \text{PaVe} [1, 1, \{r_{10}^2\}, \{m_0^2, m_1^2\}] \]  

(C.89)

All these coefficient functions are not independent and can be reduced to the scalar functions. FeynCalc provides the command \texttt{PaVeReduce} \ldots to accomplish that. This is very useful if one wants to check for cancellation of divergences or for gauge invariance where a number of diagrams have to cancel.

### C.9.2 The divergences

The package \texttt{LoopTools} provides ways to numerically check for the cancellation of divergences. However it is useful to know the divergent part of the Passarino-Veltman integrals. Only a small number of these integrals are divergent. They are

\[
\begin{align*}
\text{Div} [A_0(m_0^2)] &= \Delta \epsilon m_0^2 \\
\text{Div} [B_0(r_{10}^2, m_0^2, m_1^2)] &= \Delta \epsilon \\
\text{Div} [B_1(r_{10}^2, m_0^2, m_1^2)] &= -\frac{1}{2} \Delta \epsilon \\
\text{Div} [B_{00}(r_{10}^2, m_0^2, m_1^2)] &= \frac{1}{12} \Delta \epsilon (3m_0^2 + 3m_1^2 - r_{10}^2) \\
\text{Div} [B_{11}(r_{10}^2, m_0^2, m_1^2)] &= \frac{1}{3} \Delta \epsilon \\
\text{Div} [C_{00}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)] &= \frac{1}{4} \Delta \epsilon \\
\text{Div} [C_{001}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)] &= -\frac{1}{12} \Delta \epsilon \\
\text{Div} [C_{002}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)] &= -\frac{1}{12} \Delta \epsilon \\
\text{Div} [D_{0000}(r_{10}^2, \ldots, m_0^2, \ldots)] &= \frac{1}{24} \Delta \epsilon
\end{align*}
\]

(C.90) - (C.99)

These results were obtained with the package \texttt{LoopTools}, after reducing to the scalar integrals with the command \texttt{PaVeReduce}, but they can be verified by comparing with our results of section C.8, after factoring out the \(i/(16\pi^2)\).

### C.9.3 Useful results for PV integrals

Although the PV approach is intended primarily to be used numerically there are situations where one wants to have explicit results. These can be useful to check cancellation of
APPENDIX C. USEFUL TECHNIQUES FOR RENORMALIZATION

divergences or because in some simple cases the integrals can be done analytically. We note that as our conventions for the momenta are the same in sections C.9 and C.7 one can read immediately the integral representation of the PV in terms of the Feynman parameters just by comparing both expressions, not forgetting to take out the \( i/(16\pi^2) \) factor. For instance, from Eq. (C.81) for \( C^{\mu\nu} \) and Eq. (C.55) for \( I^{\mu\nu}_3 \) we get

\[
C_{12}(r_{11}^2, r_{12}^2, r_{22}^2, m_0^2, m_1^2, m_2^2) = -\Gamma(3) \frac{2}{4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{C} \quad (C.100)
\]

with

\[
C = x_1^2 r_1^2 + x_2^2 r_2^2 + x_1 x_2 (r_1^2 + r_2^2 - r_{12}^2) + x_1 m_1^2 + x_2 m_2^2 \\
+ (1 - x_1 - x_2) m_0^2 - x_1 r_1^2 - x_2 r_2^2 \quad (C.101)
\]

Explicit expression for \( A_0 \)

This integral is trivial. There is no Feynman parameter and the integral can be read from Eq. (C.51). We get, after factoring out the \( i/(16\pi^2) \),

\[
A_0(m^2) = m^2 \left( \Delta_\epsilon + 1 - \ln \frac{m^2}{\mu^2} \right) \quad (C.102)
\]

Explicit expressions for the \( B \) functions

Function \( B_0 \)

The general form of the integral \( B_0(p^2, m_1^2, m_2^2) \) can be read from Eq. (C.53). We obtain

\[
B_0(p^2, m_0^2, m_1^2) = \Delta_\epsilon - \int_0^1 dx \ln \left[ \frac{-x(1-x)p^2 + x m_1^2 + (1-x)m_0^2}{\mu^2} \right] \quad (C.103)
\]

From this expression one can easily get the following results,

\[
B_0(0, m_0^2, m_1^2) = \Delta_\epsilon + 1 - \frac{m_0^2 \ln \frac{m_0^2}{\mu^2} - m_1^2 \ln \frac{m_1^2}{\mu^2}}{m_0^2 - m_1^2} \quad (C.104)
\]

\[
B_0(0, m_0^2, m_1^2) = \frac{A_0(m_0^2) - A_0(m_1^2)}{m_0^2 - m_1^2} \quad (C.105)
\]

\[
B_0(0, m_0^2, m_0^2) = \Delta_\epsilon - \ln \frac{m_0^2}{\mu^2} = \frac{A_0(m_0^2)}{m_0^2} - 1 \quad (C.106)
\]

\[
B_0(m_0^2, 0, m_0^2) = \Delta_\epsilon + 2 - \ln \frac{m_0^2}{\mu^2} = \frac{A_0(m_0^2)}{m_0^2} + 1 \quad (C.107)
\]

\[
B_0(0, 0, m_0^2) = \Delta_\epsilon + 1 - \ln \frac{m_0^2}{\mu^2} \quad (C.108)
\]
Function $B'_0$

The derivative of the $B_0$ function with respect to $p^2$ appears many times. From Eq. (C.103) one can derive an integral representation,

$$B'_0(p^2, m_0^2, m_1^2) = -\int_0^1 dx \frac{x(1-x)}{-p^2 x(1-x) + x m_1^2 + (1-x)m_0^2} \quad (C.109)$$

An important particular case corresponds to $B'_0(m^2, m_0^2, m^2)$ that appears in the self-energy of the electron. In this case $m$ is the electron mass and $m_0 = \lambda$ is the photon mass that one has to introduce to regularize the IR divergent integral. The integral in this case reduces to

$$B'_0(m^2, \lambda^2, m^2) = -\int_0^1 dx \frac{x(1-x)}{m^2 x^2 + (1-x)\lambda^2}$$

$$= \frac{1}{m^2} + \frac{1}{2m^2} \ln \frac{\lambda^2}{m^2} \quad (C.110)$$

It is clear that in the limit $\lambda \to 0$ this integral diverges.

Function $B_1$

The explicit expression can be read from Eq. (C.53). We have

$$B_1(p^2, m_0^2, m_1^2) = -\frac{1}{2} \Delta_\epsilon + \int_0^1 dx \ln \left[ -\frac{x(1-x)p^2 + x m_1^2 + (1-x)m_0^2}{\mu^2} \right] \quad (C.111)$$

For $p^2 = 0$ this integral can be easily evaluated to give

$$B_1(0, m_0^2, m_1^2) = -\frac{1}{2} \Delta_\epsilon + \frac{1}{2} \ln \left( \frac{m_0^2}{\mu^2} \right) - \frac{3 + 4t - t^2 - 4t \ln t + 2t^2 \ln t}{4(-1+t)^2} \quad (C.112)$$

where we defined

$$t = \frac{m_1^2}{m_0^2} \quad (C.113)$$

From Eq. (C.112) one can shown that even for $p^2 = 0$ $B_1$ is not a symmetric function of the masses,

$$B_1(p^2, m_0^2, m_1^2) \neq B_1(p^2, m_1^2, m_0^2) \quad (C.114)$$

As this might appear strange let us show with one example how the coefficient functions are tied to our conventions about the order of the momenta and Feynman parameters. Let us consider the contribution to the self-energy of a fermion of mass $m_f$ of the exchange of a scalar with mass $m_s$. We can consider the two choices in Fig. C.4,

Now with the first choice (diagram on the left of Fig. C.4) we have
while with the second choice we have

\[ -i\Sigma_2 = \frac{i}{16\pi^2} \left[ -\hat{p}B_1(p^2, m_f^2, m_s^2) + m_f B_0(p^2, m_1^2, m_0^2) \right] \] (C.116)

How can these two expressions be equal? The reason has precisely to do with the non

\[ B_1(p^2, m_0^2, m_1^2) = -\left( B_0(p^2, m_1^2, m_0^2) + B_1(p^2, m_1^2, m_0^2) \right) \] (C.117)

where we have changed variables \((x \rightarrow 1 - x)\) in the integral and used the definitions of

\[ B_0(p^2, m_1^2, m_0^2) = B_0(p^2, m_s^2, m_f^2), \]

and therefore Eqs. (C.115) and (C.116) are equivalent.

**Explicit expressions for the \( C \) functions**

In Eq. (C.100) we have already given the general form of \( C_{12} \). The other functions are very similar. In the following we just present the results for the particular case of \( p^2 = 0 \). This case is important in many situations where it is a good approximation to neglect the
external momenta in comparison with the masses of the particles in the loop. We also warn the reader that the coefficient functions $C_i, C_{ij}$ obtained from LoopTools are not well defined in this limit. Hence there is some utility in given them here.

**Function $C_0$**

$$
C_0(0, 0, 0, m_0^2, m_1^2, m_2^2) = -\frac{1}{2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 m_1^2 + x_2 m_2^2 + (1 - x_1 - x_2) m_0^2}
$$

$$
= -\frac{1}{m_0^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 t_1 + x_2 t_2 + (1 - x_1 - x_2)}
$$

$$
= -\frac{1}{m_0^2} t_1 \ln t_1 + t_1 t_2 \ln t_1 + t_2 \ln t_2 - t_1 t_2 \ln t_2
$$

where

$$
t_1 = \frac{m_1^2}{m_0^2} ; \quad t_2 = \frac{m_2^2}{m_0^2}
$$

Using the properties of the logarithms one can show that in this limit $C_0$ is a symmetric function of the masses. This expression is further simplified when two of the masses are equal, as it happens in the $\mu \rightarrow e\gamma$ problem. Then $t_1 = t_2$,

$$
C_0(0, 0, 0, m_0^2, m_1^2, m_1^2) = \frac{-1 + t - \ln t}{(-1 + t)^2}
$$

in agreement with Eq.(20) of [17]. In the case of equal masses for all the loop particles we have

$$
C_0(0, 0, 0, m_0^2, m_0^2, m_0^2) = \frac{-1}{2m_0^2}
$$

Before we close this section on $C_0$ there is another particular case when it is useful to have an explicit case for it. This in the case when it is IR divergent as in the QED vertex. The functions needed is $C_0(m^2, m^2, 0, m_2^2, \lambda^2, m^2)$. Using the definition we have

$$
C_0(m^2, m^2, 0, m_2^2, \lambda^2, m^2) = -\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{m^2(1 - 2x_1 + x_2^2) + x_1 \lambda^2}
$$

$$
= -\int_0^1 dx_1 \frac{1 - x_1}{m^2(1 - x_1)^2 + (1 - x_1) \lambda^2}
$$

$$
= -\int_0^1 dx \frac{x}{m^2 x^2 + x \lambda^2}
$$

$$
= \frac{1}{2m^2} \ln \frac{\lambda^2}{m^2} = B'_0(m^2, \lambda^2, m^2) - \frac{1}{m^2}
$$

**Function $C_{00}$**

$$
C_{00}(0, 0, 0, m_0^2, m_1^2, m_2^2) = \Gamma(3) \frac{1}{4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[ \Delta_\epsilon - \ln \left( \frac{C}{\mu^2} \right) \right]
$$
\[ \frac{1}{4} \Delta_x - \frac{1}{2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln \left[ \frac{x_1m_0^2 + x_2m_0^2 + (1 - x_1 - x_2)m_0^2}{\mu^2} \right] \]

\[ = \frac{1}{4} \left( \Delta_x - \ln \frac{m_0^2}{\mu^2} \right) + \frac{3}{8} \cdot \frac{t_1^2}{4(t_1 - 1)(t_1 - t_2)} \ln t_1 + \frac{t_2^2}{4(t_2 - 1)(t_1 - t_2)} \ln t_2 \]

where, as before

\[ t_1 = \frac{m_1^2}{m_0^2} ; \quad t_2 = \frac{m_2^2}{m_0^2} \]

Using the properties of the logarithms one can show that in this limit \( C_{00} \) is a symmetric function of the masses. This expression is further simplified when two of the masses are equal. Then \( t = t_1 = t_2 \),

\[ C_{00}(0,0,0,m_0^2,m_1^2,m_2^2) = \frac{1}{4} \left( \Delta_x - \ln \frac{m_0^2}{\mu^2} \right) - \frac{3}{8} \cdot \frac{4t - t^2 - 4t \ln t + 2t^2 \ln t}{8(t - 1)^2} \]

\[ = -\frac{1}{2} B_1(0,m_0^2,m_1^2) \]

Functions \( C_i \) and \( C_{ij} \)

We recall that the definition of the coefficient functions is not unique, it is tied to a particular convention for assigning the loop momenta and Feynman parameters, as shown in Fig. C.1. For the particular case of the \( C \) functions we show our conventions in Fig. C.5.

With the same techniques we obtain,

\[ C_1(0,0,0,m_0^2,m_1^2,m_2^2) = \frac{1}{m_0^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1}{x_1t_1 + x_2t_2 + (1 - x_1 - x_2)} \]

\[ = -\frac{1}{m_0^2} \left[ \frac{t_1}{2(-1 + t_1)(t_1 - t_2)} - \frac{t_1(t_1 - 2t_2 + t_1t_2)}{2(-1 + t_1)^2(t_1 - t_2)^2} \right] \ln t_1 \]
In the limit

\[
\frac{t_2^2 - 2t_1t_2^2 + t_1^2t_2^2}{2(-1 + t_1)^2(t_1 - t_2)^2(-1 + t_2)} \ln t_2
\]
(C.127)

\[
C_2(0, 0, m_0^2, m_1^2, m_2^2) = \frac{1}{m_0^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_2}{x_1t_1 + x_2t_2 + (1 - x_1 - x_2)}
\]
\[
= -\frac{1}{m_0^2} \left[ -\frac{t_2}{2(t_1 - t_2)(-1 + t_2)} + \frac{\ln t_1}{2(-1 + t_1)(-1 + t_2)^2}
\right. \\
- \frac{2t_1t_2 - 2t_1^2t_2 - t_2^2 + t_1^2t_2^2}{2(-1 + t_1)(t_1 - t_2)^2(-1 + t_2)^2} \ln \left( \frac{t_1}{t_2} \right)
\right]
(C.128)

\[
C_{ij}(0, 0, m_0^2, m_1^2, m_2^2) = -\frac{1}{m_0^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_i x_j}{x_1t_1 + x_2t_2 + (1 - x_1 - x_2)}
(C.129)
\]

where we have not written explicitly the \( C_{ij} \) for \( i, j = 1, 2 \) because they are rather lengthy. However a simple Fortran program can be developed [9] to calculate all the three point functions in the zero external limit case. This is useful because in this case some of the functions from LoopTools will fail. Notice that the \( C_i \) and \( C_{ij} \) functions are not symmetric in their arguments. This a consequence of their non-uniqueness, they are tied to a particular convention. This is very important when ones compares with other results. However using their definition one can get some relations. For instance we can show

\[
C_1(0, 0, m_0^2, m_1^2, m_2^2) = C_1(0, 0, m_0^2, m_2^2, m_1^2)
(C.130)
\]
\[
C_2(0, 0, m_0^2, m_1^2, m_2^2) = -C_0(0, 0, m_0^2, m_2^2, m_1^2) - C_1(0, 0, m_0^2, m_2^2, m_1^2) \\
- C_2(0, 0, m_0^2, m_2^2, m_1^2)
(C.131)
\]

In the limit \( m_1 = m_2 \) we get the simple expressions,

\[
C_1(0, 0, m_0^2, m_1^2, m_1^2) = C_2(0, 0, m_0^2, m_1^2, m_1^2)
\]
\[
= -\frac{1}{m_0^2} \frac{3 - 4t + t^2 + 2 \ln t}{4(-1 + t)^4}
(C.132)
\]
\[
C_{11}(0, 0, m_0^2, m_1^2, m_1^2) = C_{22}(0, 0, m_0^2, m_1^2, m_1^2) = 2C_{12}(0, 0, m_0^2, m_1^2, m_1^2)
\]
\[
= -\frac{1}{m_0^2} \frac{-11 + 18t - 9t^2 + 2t^3 - 6 \ln t}{18(-1 + t)^4}
(C.133)
\]
in agreement with Eqs. (21-22) of [17]. The case of masses equal gives

\[
C_1(0, 0, m_0^2, m_0^2, m_0^2) = C_2(0, 0, m_0^2, m_0^2, m_0^2) = \frac{1}{6m_0^2}
(C.134)
\]
\[
C_{11}(0, 0, m_0^2, m_0^2, m_0^2) = C_{22}(0, 0, m_0^2, m_0^2, m_0^2) = -\frac{1}{12m_0^2}
(C.135)
\]
$$C_{12}(0, 0, 0, m_{0}^2, m_{1}^2, m_{2}^2) = -\frac{1}{24 m_{0}^2} \quad (C.136)$$

The package PVzem

As we said before, in many situations it is a good approximation to neglect the external momenta. In this case, the loop functions are easier to evaluate and one approach is for each problem to evaluate them. However our approach here is more in the direction of automatically evaluating the one-loop amplitudes. If one does that with the use of **FeynCalc**, has we have been doing, then the result is given in terms of standard functions that can be numerically evaluated with the package **LoopTools**. However this package has problems with this limit. This is because this limit is unphysical. Let us illustrate this point calculating the functions $C_1(m^2, 0, 0, m_F^2, m_r^2, m_2^2)$ and $C_2(m^2, 0, 0, m_F^2, m_r^2, m_2^2)$ for $m_B = 100$ GeV, $m_F = 80$ GeV and $m_2$ ranging from $10^{-6}$ to 100 GeV. To better illustrate our point we show two plots with different scales on the axis.

In these plots, $C_1^{\text{Ex}}$ are the exact $C_i$ functions calculated with **LoopTools** and $C_i^{\text{Ap}}$ are the $C_i$ calculated in the zero momenta limit. We can see that only for external momenta (in this case corresponding to the mass $m_2$) close enough to the masses of the particles in the loop, the exact result deviates from the approximate one. However for very small values of the external momenta, **LoopTools** has numerical problems as shown in the right panel of Fig. C.6. To overcome this problem I have developed a **Fortran** package that evaluates all the $C$ functions in the zero external momenta limit. There are no restrictions on the masses being equal or different and the conventions are the same as in **FeynCalc** and **LoopTools**, for instance,

$$c_{12zem}(m_{02}, m_{12}, m_{22}) = c_{0i}(cc_{12}, 0, 0, m_{02}, m_{12}, m_{22}) \quad (C.137)$$

where $c_{0i}(cc_{12}, \cdots)$ is the **LoopTools** notation and $c_{12zem}(\cdots)$ is the notation of my package, called **PVzem**. It can be obtained from the address indicated in Ref.[9].
approximate functions shown in Fig. C.6 were calculated using that package. We include here the Fortran code used to produce that figure.
APPENDIX C. USEFUL TECHNIQUES FOR RENORMALIZATION

program LoopToolsExample
implicit none

* LoopTools has to be used with FORTRAN programs with the
* extension .F in order to have the header file "looptools.h"
* preprocessed. This file includes all the definitions used
* by LoopTools.
* Functions c1zem and c2zem are provided by the package PVzem.
*
#include "looptools.h"

integer i
real*8 m2,mF2,mS2,m
real*8 lgmmin,lgmmax,lgm,step
real*8 rc1,rc2
real*8 c1zem,c2zem

mS2=100.d0**2
mF2=80.d0**2

* Initialize LoopTools. See the LoopTools manual for further
details. There you can also learn how to set the scale MU
* and how to handle the UR and IR divergences.
*
call ltini

lgmmax=log10(100.d0)
lgmmin=log10(1.d-6)
step=(lgmmax-lgmmin)/100.d0
lgm=lgmmin-step

open(10,file='plot.dat',status='unknown')
do i=1,101
   lgm=lgm+step
   m=10.d0**lgm
   m2=m**2

end do
In LoopTools the $c_{0i}(\ldots)$ are complex functions. For the
kinematics chosen here they are real, so we take the real
part for comparison.

\[
\begin{align*}
rc_1 &= \text{dble}(c_{0i}(cc_1, m_2, 0.d0, 0.d0, m_{S2}, m_{F2}, m_{F2})) \\
rc_2 &= \text{dble}(c_{0i}(cc_2, m_2, 0.d0, 0.d0, m_{S2}, m_{F2}, m_{F2})) \\
\text{write}(10, 100)m, rc_1*m_{S2}, rc_2*m_{S2}, c_{1zem}(m_{S2}, m_{F2}, m_{F2})*m_{S2}, \\
& \text{& } c_{2zem}(m_{S2}, m_{F2}, m_{F2})*m_{S2}
\end{align*}
\]

enddo

100 format(5(e22.14))
call ltexi
end

************** End of Program LoopToolsExample.F **************

When the above program is compiled, the location of the header file \texttt{looptools.h}
must be known by the compiler. This is best achieved by using a \texttt{Makefile}. We give
below, as an example, the one that was used with the above program. Depending on the
installation details of \texttt{LoopTools} the paths might be different.

\begin{verbatim}
FC =
LT = /usr/local/lib/LoopTools
FFLAGS = -c -O -I$(LT)/include
LDFLAGS =
LINKER = $(FC)

LIB = -L$(LT)/lib
LIBS = -looptools

.f.o:
   $(FC) $(FFLAGS) $*.F

files = LoopToolsExample.o PVzem.o

all:
   $(files)
   $(LINKER) $(LDFLAGS) -o Example $(files) $(LIB) $(LIBS)
\end{verbatim}

Explicit expressions for the $D$ functions

\textbf{Function $D_0$}

The various $D$ functions can be calculated in a similar way. However they are rather
lengthy and have to handled numerically \cite{9}. Here we just give $D_0$ for the equal masses
C.10 Examples of 1-loop calculations with PV functions

In this section we will work out in detail a few examples of one-loop calculations using the FeynCalc package and the Passarino-Veltman scheme.

C.10.1 Vacuum Polarization in QED

We have done this example in section 4.1.1 using the techniques described in sections C.3, C.4 and C.5. Now we will use FeynCalc. The first step is to write the Mathematica program. We list it below:

(* ************ Program VacPol.m *************)

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)

dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(* Now write the numerator of the Feynman diagram. We define the constant

C=alpha/(4 pi)
C.10. EXAMPLES OF 1-LOOP CALCULATIONS WITH PV FUNCTIONS

*)


(* Tell FeynCalc to evaluate the integral in dimension D *)

SetOptions[OneLoop, Dimension -> D]

(* Define the amplitude *)

amp := num * FeynAmpDenominator[PropagatorDenominator[q + k, m], \n    PropagatorDenominator[q, m]]

(* Calculate the result *)

res := (-I / Pi^2) OneLoop[q, amp]

(***************** End of Program VacPol.m *********************)

The output from Mathematica is:

```
Out[4]= (4 C (k + 6 m B0[0, m, m] - 3 (k + 2 m) B0[k, m, m])
2 2 2 2 2 2 2 2 2
2
(k g[mu, nu] - k[mu] k[nu])) / (9 k)
```

Now remembering that,

\[
C = \frac{\alpha}{4\pi}
\]

and

\[
i \Pi_{\mu\nu}(k, \varepsilon) = -i k^2 P^T_{\mu\nu} \Pi(k, \varepsilon)
\]

we get

\[
\Pi(k, \varepsilon) = \frac{\alpha}{4\pi} \left[ -\frac{4}{9} - \frac{8 m^2}{3 k^2} B_0(0, m^2, m^2) + \frac{4}{3} \left( 1 + \frac{2m^2}{k^2} \right) B_0(k^2, m^2, m^2) \right]
\]

To obtain the renormalized vacuum polarization one needs to know the value of \( \Pi(0, \varepsilon) \).

To do that one has to take the limit \( k \to 0 \) in Eq. (C.141). For that one uses the derivative of the \( B_0 \) function

\[
B_0'(p^2, m_1^2, m_2^2) \equiv \frac{\partial}{\partial p^2} B_0(p^2, m_1^2, m_2^2)
\]
to obtain

$$\Pi(0, \varepsilon) = \frac{\alpha}{4\pi} \left[ -\frac{4}{9} + \frac{4}{3} B_0(0, m^2, m^2) + \frac{8}{3} m^2 B'_0(0, m^2, m^2) \right]$$

(C.143)

Using

$$B'_0(0, m^2, m^2) = \frac{1}{6m^2}$$

(C.144)

we finally get

$$\Pi(0, \varepsilon) = -\delta Z_3 = \frac{\alpha}{4\pi} \left[ \frac{4}{3} B_0(0, m^2, m^2) \right]$$

(C.145)

and the final result for the renormalized vertex is:

$$\Pi^R(k) = \frac{\alpha}{3\pi} \left[ -\frac{1}{3} + \left( 1 + \frac{2m^2}{k^2} \right) (B_0(k^2, m^2, m^2) - B_0(0, m^2, m^2)) \right]$$

(C.146)

If we want to compare with our earlier analytical results we need to know that

$$B_0(0, m^2, m^2) = \Delta_\varepsilon - \ln \frac{m^2}{\mu^2}$$

(C.147)

Then Eq. (C.146) reproduces the result of Eq. (4.54). The comparison between Eq. (C.146) and Eq. (4.56) can be done numerically using the package LoopTools[15].

C.10.2 Electron Self-Energy in QED

In this section we repeat the calculation of section 4.1.2 using the Passarino-Veltman scheme. We start with the Mathematica program,

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)

dm[\mu_\_]:=*DiracMatrix[\mu,Dimension->D]
dm[5]:=*DiracMatrix[5]
ds[p_\_]:=*DiracSlash[p]
m[\mu_,\nu_\_]:=*MetricTensor[\mu,\nu]
f[v[p_\_,\mu_\_]:=*FourVector[p,\mu]
epsilon[a_\_,b_\_,c_\_,d_\_]:=*LeviCivita[a,b,c,d]
id[n_\_]:=*IdentityMatrix[n]
s[p_\_,q_\_]:=*ScalarProduct[p,q]
l[mu_\_]:=*LorentzIndex[mu]

L:=dm[7]
R:=dm[6]

(* Tell FeynCalc to reduce the result to scalar functions *)
C.10. EXAMPLES OF 1-LOOP CALCULATIONS WITH PV FUNCTIONS

SetOptions[{B0,B1,B00,B11},BReduce->True]

(* Now write the numerator of the Feynman diagram. We define the constant

   C = - \alpha/(4 \pi)

The minus sign comes from the photon propagator. The factor i/(16 \pi^2) is already included in this definition. *)

num:= C dm\[mu\] . (ds[p]+ds[k]+m) . dm\[mu\]

(* Tell FeynCalc to evaluate the one-loop integral in dimension D *)

SetOptions[OneLoop,Dimension->D]

(* Define the amplitude *)

amp:= num \[\backslash\]
FeynAmpDenominator[PropagatorDenominator[p+k,m], \[\backslash\]
    PropagatorDenominator[k]]

(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[k,amp]

ans=-res;

(* The minus sign in ans comes from the fact that -i \Sigma = diagram *)

(* Calculate the functions A(p^2) and B(p^2) *)

A=Coefficient[ans,DiracSlash[p],0];
B=Coefficient[ans,DiracSlash[p],1];

(* Calculate deltm *)
APPENDIX C. USEFUL TECHNIQUES FOR RENORMALIZATION

\[ \text{delm} = A + m B \quad \text{/. \ } p \rightarrow m \]

\[ \text{(* Calculate delZ2 *)} \]

\[ \text{Ap2} = A \quad \text{/. \ } \text{ScalarProduct}[p,p] \rightarrow p^2 \]
\[ \text{Bp2} = B \quad \text{/. \ } \text{ScalarProduct}[p,p] \rightarrow p^2 \]

\[ \text{aux} = 2 m D[\text{Ap2},p^2] + Bp2 \]
\[ + 2 m^2 D[Bp2,p^2] \quad \text{/. \ } D[B0[p^2,0,m^2],p^2] \rightarrow D[B0[p^2,0,m^2]] \]

\[ \text{aux2} = \text{aux} \quad \text{/. \ } p^2 \rightarrow m^2 \]

\[ \text{aux3} = \text{aux2} \quad \text{/. \ } A0[m^2] \rightarrow m^2 (B0[m^2,0,m^2] - 1) \]

\[ \text{delZ2} = \text{Simplify}[\text{aux3}] \]

\[ \text{(* *************** End of Program SelfEnergy.m *************** *)} \]

The output from Mathematica is:

\[ A = -(C (-2 m + 4 m B0[p^2,0,m^2])) \]

\[ B = -(\frac{C (p + A0[m^2] - (m + p) B0[p^2,0,m^2])}{p}) \]

\[ \text{delm} = \frac{C (m - A0[m^2] - 2 m B0[m^2,0,m^2])}{m} \]

\[ \text{delZ2} = C (-2 + B0[m^2,0,m^2] - 4 m DB0[m^2,0,m^2]) \]

We therefore get

\[ A = \frac{\alpha m}{\pi} \left[ -\frac{1}{2} + B_0(p^2,0,m^2) \right] \quad \text{(C.148)} \]
$$B = \frac{\alpha}{4\pi} \left[ 1 + \frac{1}{p^2} A_0(m^2) - \left( 1 + \frac{m^2}{p^2} \right) B_0(p^2, 0, m^2) \right]$$  \hspace{1cm} (C.149)

$$\delta_m = \frac{3\alpha m}{4\pi} \left[ -\frac{1}{3} + \frac{1}{3m^2} A_0(m^2) + \frac{2}{3} B_0(m^2, 0, m^2) \right]$$  \hspace{1cm} (C.150)

One can check that Eq. (C.150) is in agreement with Eq. (4.80). For that one needs the following relations,

$$A_0(m^2) = m^2 \left( B_0(m^2, 0, m^2) - 1 \right)$$  \hspace{1cm} (C.151)

$$B_0(m^2, 0, m^2) = \Delta_\varepsilon + 2 - \ln \frac{m^2}{\mu^2}$$  \hspace{1cm} (C.152)

$$\int_0^1 dx (1 + x) \ln \frac{m^2 x^2}{\mu^2} = - \frac{5}{2} + \frac{3}{2} \ln \frac{m^2}{\mu^2}$$  \hspace{1cm} (C.153)

For $\delta Z_2$ we get

$$\delta Z_2 = \frac{\alpha}{4\pi} \left[ 2 - B_0(m^2, 0, m^2) - 4m^2 B'_0(m^2, \lambda^2, m^2) \right]$$  \hspace{1cm} (C.154)

This expression can be shown to be equal to Eq. (4.83) although this is not trivial. The reason is that $B'_0$ is IR divergent, hence the parameter $\lambda$ that controls the divergence.

### C.10.3 QED Vertex

In this section we repeat the calculation of section 4.1.3 for the QED vertex using the Passarino-Veltman scheme. The Mathematica program should by now be easy to understand. We just list it here,

```plaintext
(* First input FeynCalc *)
<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)
dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]
```
(* Tell FeynCalc to reduce the result to scalar functions *)

SetOptions[{B1,B00,B11},BReduce->True]

(* Now write the numerator of the Feynman diagram. We define the constant

\[ C = \frac{\alpha}{4\pi} \]

The kinematics is: \( q = p_1 - p_2 \) and the internal momenta is \( k \). *)

num:=Spinor[p1,m] . dm[ro] . (ds[p1]-ds[k]+m) . ds[a] \n   . (ds[p2]-ds[k]+m) . dm[ro] . Spinor[p2,m]

SetOptions[OneLoop,Dimension->D]

amp:=C num \nFeynAmpDenominator[PropagatorDenominator[k,lbd], \n   PropagatorDenominator[k-p1,m], \n   PropagatorDenominator[k-p2,m]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m^2,ScalarProduct[p2,p2]->m^2, \n   ScalarProduct[p1,p2]->m^2-q2/2}

(* Define the divergent part of the relevant PV functions*)

div={B0[0,0,m^2]->Div,B0[0,m^2,m^2]->Div, \n   B0[m^2,0,m^2]->Div,B0[m^2,lbd^2,m^2]->Div, \n   B0[q2,m^2,m^2]->Div,B0[0,lbd^2,m^2]->Div}

res1:=(-I / Pi^2) OneLoop[k,amp]

res:=res1 /. onshell

auxV1:= res /. onshell
auxV2:= PaVeReduce[auxV1]
auxV3:= auxV2 /. div

divV:=Simplify[Div*Coefficient[auxV3,Div]]
(* Check that the divergences do not cancel *)

testdiv := Simplify[divV]

ans1 = res;
var = Select[Variables[ans1], (Head[#] === StandardMatrixElement) &]
Set @@ {var, {ME[1], ME[2]}}

(* Extract the different Matrix Elements *

Mathematica writes the result in terms of 2 Standard Matrix Elements. To have a simpler result we substitute these elements by simpler expressions (ME[1], ME[2]).

{StandardMatrixElement[u[p1, m1] . u[p2, m2]], StandardMatrixElement[u[p1, m1] . ga[μ] . u[p2, m2]]}
*)

ans2 = Simplify[PaVeReduce[ans1]]

CE11 = Coefficient[ans2, ScalarProduct[a, p1] ME[1]]
CE12 = Coefficient[ans2, ScalarProduct[a, p2] ME[1]]
CE2 = Coefficient[ans2, ME[2]]


test1 := Simplify[CE11 - CE12]
test2 := Simplify[ans2 - ans3]

(* ME[2] is \(\overline{u}(p')\gamma_{\mu}u(p)\) and ME[3] is \(\frac{i}{2m}\overline{u}(p')\sigma_{\mu\nu}q^{\nu}\) u(p) *)

ans4 = ans3 /.

CGamma := Coefficient[ans4, ME[2]]
CSigmaAux := Coefficient[ans4, ME[3]]

From this program we can obtain first the value of $\delta Z_1$. We get

$$\delta Z_1 = C(1 - B_0[0, 0, m^2] + 2 B_0[0, m^2, m^2] - 2 B_0[m^2, 0, m^2] - 4 m^2 C_0[m^2, 2 m^2, 0, m^2, \lambda^2, m^2])$$

which can be written as

$$\delta Z_1 = \frac{\alpha}{2\pi} \left[ 1 - B_0(0, 0, m^2) + 2B_0(0, m^2, m^2) - 2B_0(m^2, m^2) - 4m^2C_0(m^2, m^2, 0, m^2, \lambda^2, m^2) \right]$$

(C.155)

where we have introduced a small mass for the photon in the function $C_0(m^2, m^2, 0, m^2, \lambda^2, m^2)$ because it is IR divergent when $\lambda \to 0$ (see Eq. (C.123)). Using the results of Eqs. (C.106), (C.107), (C.108) and Eq. (C.123) we can show the important result

$$\delta Z_1 = \delta Z_2$$

(C.156)

where $\delta Z_2$ was defined in Eq. (C.154). After performing the renormalization the coefficient $F_1(k^2)$ is finite and given by

$$F_1 = C \left( \frac{-q^2}{2} \right) \left[ \frac{-q^2 B_0[0, 0, m^2]}{2} + \frac{2 q^2 B_0[0, m^2, m^2]}{2} - \frac{2 q^2 B_0[m^2, 0, m^2]}{2} - \frac{8 m B_0[0, m^2, m^2]}{2} + \frac{3 q^2 B_0[q^2, m^2, m^2]}{2} - \frac{8 m B_0[q^2, m^2, m^2]}{2} + \frac{q^2 - 4 m}{2} \right]$$
C.10. EXAMPLES OF 1-LOOP CALCULATIONS WITH PV FUNCTIONS  

\[ \begin{align*}
\text{In[5]} & = F1 \, / \, . \, q2\rightarrow 0 \\
\text{Out[5]} & = 0
\end{align*} \]

while the coefficient \( F_2(q^2) \) does not need renormalization and it is given by,

\[ F_2 = C \frac{2}{q_2 - 4m} \]

and for \( F_2(0) \) we get

\[ F_2[0] = -C \frac{2}{q_2 - 4m} \]

Using the results of the Appendix we can show that,

\[ F_2(0) = \frac{\alpha}{2\pi} \]  

(C.157)

a well known result, first obtained by Schwinger even before the renormalization program was fully understood (\( F_2(q^2) \) is finite).
C.11 Modern techniques in a real problem: $\mu \rightarrow e\gamma$

In the previous sections we have redone most of the QED standard textbook examples using the PV decomposition and automatic tools. Here we want to present a more complex example, the calculation of the partial width $\mu \rightarrow e\gamma$ in an arbitrary theory where the charged leptons couple to scalars and fermions, charged or neutral. This has been done in Ref.[17] for fermions and bosons of arbitrary charge $Q_F$ and $Q_B$, but for simplicity I will consider here separately the cases of neutral and charged scalars.

C.11.1 Neutral scalar charged fermion loop

We will consider a theory with the following interactions,

$$\begin{align*}
    & l^- \quad S^0 \quad l^- \quad (A_L P_L + A_R P_R) \\
    & F^- \quad S^0 \quad F^- \quad (B_L P_L + B_R P_R)
\end{align*}$$

where $F^-$ is a fermion with mass $m_F$ and $S^0$ a neutral scalar with mass $m_S$. In fact $B_{L,R}$ are not independent of $A_{L,R}$ but it is easier for our programming to consider them completely general. The Feynman rule for the coupling of the photon with the lepton is $-i e Q_{\ell} \gamma^\mu$ where $e$ is the positron charge (for an electron $Q_{\ell} = -1$). $\ell^-_1$ can be any of the leptons but we will omit all indices in the program, the lepton being identified by its mass and from the assumed kinematics

$$\ell_2(p_2) \rightarrow \ell_1(p_1) + \gamma(k) \quad \text{(C.158)}$$

The diagrams contributing to the process are given in Fig. C.7,

![Diagram](image)

Figure C.7:

where

$$D_1 = q^2 - m_S^2; \quad D_2 = (p_2 + q)^2 - m_F^2; \quad D_3 = (q + p_2 - k)^2 - m_F^2 \quad \text{(C.159)}$$
The amplitudes are

\[ D_4 = D_3 \quad ; \quad D_6 = D_2 \quad ; \quad D_5 = (p_2 - k)^2 - m_2^2 = -2p_2 \cdot k \] (C.160)

\[ D_7 = (p_1 + k)^2 - m_1^2 = 2p_1 \cdot k = -D_5 \] (C.161)

The amplitudes are

\[
iM_1 = \frac{e Q_\ell}{D_1 D_2 D_3} \bar{\pi}(p_1) (A_L P_L + A_R P_R) (\not{q} + \not{p}_2 - \not{k} + m_F) \gamma^\mu (\not{q} + \not{p}_2 + m_F)
(B_L P_L + B_R P_R) u(p_2) \varepsilon_\mu(k)
\] (C.162)

\[
iM_2 = \frac{e Q_\ell}{D_1 D_4 D_5} \bar{\pi}(p_1) (A_L P_L + A_R P_R) (\not{q} + \not{p}_2 - \not{k} + m_F) (B_L P_L + B_R P_R)
(p_\perp - \not{k}_2 + m_2) \gamma^\mu u(p_2) \varepsilon_\mu(k)
\] (C.163)

\[
iM_3 = \frac{e Q_\ell}{D_1 D_5 D_7} \bar{\pi}(p_1) \gamma^\mu (\not{q} + \not{k} + m_F) (A_L P_L + A_R P_R) (\not{q} + \not{p}_2 + m_1)
(B_L P_L + B_R P_R) u(p_2) \varepsilon_\mu(k)
\] (C.164)

On-shell the amplitude will take the form (we have \( p_1 \cdot k = p_2 \cdot k \))

\[
iM = 2p_2 \cdot \varepsilon(k) \left[ C_L \bar{u}(p_1) P_L u(p_2) + C_R \bar{u}(p_1) P_R u(p_2) \right]
+ D_L \bar{u}(p_1) \not{\varepsilon} P_L u(p_2) + D_R \bar{u}(p_1) \not{\varepsilon} P_R u(p_2)
\] (C.165)

If we write the amplitude as

\[ M = M_\mu \varepsilon^\mu(k) \] (C.166)

then gauge invariance implies

\[ M_\mu k^\mu = 0 \] (C.167)

Imposing this condition on Eq. (C.165) we get the relations

\[ D_L = -m_2 C_R - m_1 C_L \] (C.168)

\[ D_R = -m_1 C_R - m_2 C_L \] (C.169)

Assuming these relations the amplitude can be written as

\[
iM = C_L [2p_2 \cdot \varepsilon(k) \bar{u}(p_1) P_L u(p_2) - m_1 \bar{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_2 \bar{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)]
+ C_R [2p_2 \cdot \varepsilon(k) \bar{u}(p_1) P_R u(p_2) - m_1 \bar{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_2 \bar{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)]
\] (C.170)

and the decay width will be
\[ \Gamma = \frac{1}{16\pi m_2^2} (m_2^2 - m_1^2)^3 \left( |C_L|^2 + |C_R|^2 \right) \]  

(C.171)

As the coefficient of \( p_2 \cdot \varepsilon(k) \) only comes from the 3-point function (amplitude \( M_1 \)) this justifies the usual procedure of just calculating that coefficient and forgetting about the self-energies (amplitudes \( M_2 \) and \( M_3 \)). However these amplitudes are crucial for the cancellation of divergences and for gauge invariance. Now we will show the power of the automatic \texttt{FeynCalc} [13] program and calculate both the coefficients \( C_{L,R} \) and \( D_{L,R} \), showing the cancellation of the divergences and that the relations, Eqs. (C.168) and (C.169) needed for gauge invariance are satisfied. We start by writing the \texttt{mathematica} program:

(* Program mueg-ns.m *)

This program calculates the COMPLETE (both the 3 point amplitude and the two self energy type on each external line) amplitudes for \( \mu \rightarrow e \gamma \) when the fermion line in the loop is charged and the neutral line is a scalar. The \( \mu \) has momentum \( p_2 \) and mass \( m_2 \), the electron \( (p_1,m_1) \) and the photon momentum \( k \). The momentum in the loop is \( q \).

The assumed vertices are,

1) Electron-Scalar-Fermion:

\[ \text{Spinor}[p_1,m_1] \ (A L P_L + A R P_R) \text{Spinor} \ [p_2,m_2] \]

2) Fermion-Scalar-Muon:

\[ \text{Spinor}[p_2,m_2] \ (B L P_L + B R P_R) \text{Spinor} \ [p_1,m_1] \]

(*

\[
\begin{align*}
\text{dm}[\mu_] &= \text{DiracMatrix}[\mu, \text{Dimension}\rightarrow D] \\
\text{dm}[5] &= \text{DiracMatrix}[5] \\
\text{ds}[p_] &= \text{DiracSlash}[p] \\
\text{mt}[\mu_, \nu_] &= \text{MetricTensor}[\mu, \nu] \\
\text{fv}[p_, \mu_] &= \text{FourVector}[p, \mu] \\
\text{epsilon}[a_, b_, c_, d_] &= \text{LeviCivita}[a, b, c, d] \\
\text{id}[n_] &= \text{IdentityMatrix}[n] \\
\text{sp}[p_, q_] &= \text{ScalarProduct}[p, q] \\
\text{li}[\mu_] &= \text{LorentzIndex}[\mu] \\
L &= \text{dm}[7] \\
R &= \text{dm}[6] \\
\end{align*}
\]

(*

\[\text{SetOptions}[[B0, B1, B00, B11], \text{BReduce}\rightarrow \text{True}]\]

*)
C.11. MODERN TECHNIQUES IN A REAL PROBLEM: $\mu \rightarrow E\gamma$

\[ gA := AL \text{DiracMatrix}[7] + AR \text{DiracMatrix}[6] \]
\[ gB := BL \text{DiracMatrix}[7] + BR \text{DiracMatrix}[6] \]

\[ \text{num1} := \text{Spinor}[p1,m1] \cdot gA \cdot (ds[q]+ds[p2]-ds[k]+mf) \cdot ds[\text{Polarization}[k]] \cdot \]
\[ (ds[q]+ds[p2]+mf) \cdot gB \cdot \text{Spinor}[p2,m2] \]

\[ \text{num2} := \text{Spinor}[p1,m1] \cdot gA \cdot (ds[q]+ds[p1]+mf) \cdot gB \cdot (ds[p1]+m2) \cdot \\
\text{ds}[\text{Polarization}[k]] \cdot \text{Spinor}[p2,m2] \]

\[ \text{num3} := \text{Spinor}[p1,m1] \cdot \text{ds}[\text{Polarization}[k]] \cdot (ds[p2]+m1) \cdot gA \cdot \\
(ds[q]+ds[p2]+mf) \cdot gB \cdot \text{Spinor}[p2,m2] \]

SetOptions[OneLoop, Dimension -> D]

\[ \text{amp1} := \text{num1} \]
\[ \text{FeynAmpDenominator}[\text{PropagatorDenominator}[q+p2-k,mf], \\
\text{PropagatorDenominator}[q+p2,mf], \\
\text{PropagatorDenominator}[q,ms]] \]

\[ \text{amp2} := \text{num2} \]
\[ \text{FeynAmpDenominator}[\text{PropagatorDenominator}[q+p1,mf], \\
\text{PropagatorDenominator}[p2-k,m2], \\
\text{PropagatorDenominator}[q,ms]] \]

\[ \text{amp3} := \text{num3} \]
\[ \text{FeynAmpDenominator}[\text{PropagatorDenominator}[p1+k,m1], \\
\text{PropagatorDenominator}[q+p2,mf], \\
\text{PropagatorDenominator}[q,ms]] \]

(* Define the on-shell kinematics *)

\[ \text{onshell} = \{ \text{ScalarProduct}[p1,p1] \rightarrow m1^2, \text{ScalarProduct}[p2,p2] \rightarrow m2^2, \ \\
\text{ScalarProduct}[k,k] \rightarrow 0, \text{ScalarProduct}[p1,k] \rightarrow (m2^2-m1^2)/2, \ \\
\text{ScalarProduct}[p2,k] \rightarrow (m2^2-m1^2)/2, \ \\
\text{ScalarProduct}[p2,\text{Polarization}[k]] \rightarrow p2epk, \ \\
\text{ScalarProduct}[p1,\text{Polarization}[k]] \rightarrow p2epk \} \]
APPENDIX C. USEFUL TECHNIQUES FOR RENORMALIZATION

(* Define the divergent part of the relevant PV functions*)

\[
div = \{ B0[m1^2, mf^2, ms^2] -> Div, B0[m2^2, mf^2, ms^2] -> Div, \\
        B0[0, mf^2, ms^2] -> Div, B0[0, mf^2, mf^2] -> Div, B0[0, ms^2, ms^2] -> Div \}
\]

res1 := (-I / Pi^2) OneLoop[q, amp1]
res2 := (-I / Pi^2) OneLoop[q, amp2]
res3 := (-I / Pi^2) OneLoop[q, amp3]
res := res1 + res2 + res3 /. onshell

auxT1 := res1 /. onshell
auxT2 := PaVeReduce[auxT1]
auxT3 := auxT2 /. div
divT := Simplify[Div*Coefficient[auxT3, Div]]

auxS1 := res2 + res3 /. onshell
auxS2 := PaVeReduce[auxS1]
auxS3 := auxS2 /. div
divS := Simplify[Div*Coefficient[auxS3, Div]]

(* Check cancellation of divergences
   testdiv should be zero because divT = -divS *)
testdiv := Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 8 Standard Matrix Elements. To have a simpler result we substitute these elements by simpler expressions (ME[1], ..., ME[8]). But they are not all independent. The final result can just be written in terms of 4 Matrix Elements. *

\{
    StandardMatrixElement[p2epk u[p1,m1] . ga[7] . u[p2,m2]],
    StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[6] . u[p2,m2]],
    StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[7] . u[p2,m2]],
    StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[6] . u[p2,m2]],
    StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[7] . u[p2,m2]]
\} *\)
C.11. MODERN TECHNIQUES IN A REAL PROBLEM: $\mu \to E\gamma$

We first do the tests. The output of mathematica is

```mathematica
ans1 = res;
var = Select[Variables[ans1], (Head[#] == StandardMatrixElement) &]
Set @@ {var, {ME[1], ME[2], ME[3], ME[4], ME[5], ME[6], ME[7], ME[8]}}

ans2 = ans1 /. identities;
ans = Simplify[ans2];

CR = Coefficient[ans, ME[1]]/2;
CL = Coefficient[ans, ME[2]]/2;
DR = Coefficient[ans, ME[5]];
DL = Coefficient[ans, ME[6]];

(* Test to see if we did not forget any term *)

(* Test that the divergences cancel term by term *)
auxCL = PaVeReduce[CL] /. div;
testdivCL := Simplify[Coefficient[auxCL, Div]]
auxCR = PaVeReduce[CR] /. div;
testdivCR := Simplify[Coefficient[auxCR, Div]]
auxDL = PaVeReduce[DL] /. div;
testdivDL := Simplify[Coefficient[auxDL, Div]]
auxDR = PaVeReduce[DR] /. div;
testdivDR := Simplify[Coefficient[auxDR, Div]]

(* Test the gauge invariance relations *)
testGI1 := Simplify[PaVeReduce[(m2^2 - m1^2)*CR - DR*m1 + DL*m2]]
testGI2 := Simplify[PaVeReduce[(m2^2 - m1^2)*CL + DR*m2 - DL*m1]]
```

********************** End Program mueg-ns.m ******************
Now we obtain the results for $C_L$:
C.11. MODERN TECHNIQUES IN A REAL PROBLEM: $\mu \to E\gamma$

\[
\begin{align*}
C_{11}. \text{ MODERN TECHNIQUES IN A REAL PROBLEM: } & \mu \to E\gamma \\
\text{\bf Mathematica output:} & \\
\text{In[13]}:= \text{CL} \\
\text{Out[13]}= (-4 \text{ AL BL mf CO[0, m2, m1, mf, mf, ms]} + \\
2 \text{ 2 2 2 2 2} & ) / 4 \\
\text{In[15]}:= \text{CR} \\
\text{Out[15]}= (-4 \text{ AR BR mf CO[0, m2, m1, mf, mf, ms]} + \\
2 \text{ 2 2 2 2 2} & ) / 4 \\
\end{align*}
\]

The expressions for $D_{L,R}$ are quite complicated. They are not normally calculated because they can be related to $C_{L,R}$ by gauge invariance. However the power of this automatic program can be illustrated by asking for these functions. As they are very long we calculate them by pieces. We just calculate $D_L$ because one can easily check that $D_R = D_L(L \leftrightarrow R)$. 
Mathematica output

In[12]:= Coefficient[PaVeReduce[DL],AL BL]

\[\frac{2}{m_1 - m_2} \frac{m_1 m_2 B_0[m_1, m_f, m_s]}{m_1 - m_2} - \frac{m_1 m_2 C_0[m_1, m_2, 0, m_f, m_s, m_f]}{m_1 - m_2}\]

Out[12]= \[
\frac{2}{m_1 m_2} \left( \frac{m_1 m_2 - m_2 m_f + m_2 m_s}{m_1 - m_2} \right) B_0[m_1, m_f, m_s] - \frac{2}{m_1 (m_1 - m_2)} \left( \frac{m_1 m_2 - m_1 m_f + m_1 m_s}{m_1 - m_2} \right) B_0[m_2, m_f, m_s] + \frac{2}{m_2 (m_1 - m_2)} \]

In[13]:= Coefficient[PaVeReduce[DL],AL BR]

\[\frac{2}{m_1 m_2} \left( \frac{m_f - m_s}{m_1 - m_2} \right) B_0[0, m_f, m_s]\]

Out[13]= \[
\frac{2}{m_1 m_2} \left( \frac{-2 m_1 m_f + 2 m_1 m_s}{m_1 - m_2} \right) B_0[m_1, m_f, m_s] - \frac{2}{m_2 (m_1 - m_2)} \left( \frac{-2 m_2 m_f + 2 m_2 m_s}{m_1 - m_2} \right) B_0[m_2, m_f, m_s]\]

In[14]:= Coefficient[PaVeReduce[DL],AR BL]

\[\frac{1}{2} \left( \frac{-2 m_1 m_f + 2 m_1 m_s}{m_1 - m_2} \right) B_0[m_1, m_f, m_s]\]

Out[14]= \[
\frac{2}{m_1 (m_1 - m_2)} \left( \frac{-2 m_1 m_f + 2 m_1 m_s}{m_1 - m_2} \right) B_0[m_1, m_f, m_s] - \frac{2}{m_2 (m_1 - m_2)} \left( \frac{-2 m_2 m_f + 2 m_2 m_s}{m_1 - m_2} \right) B_0[m_2, m_f, m_s]\]
C.11. MODERN TECHNIQUES IN A REAL PROBLEM: $\mu \rightarrow E\gamma$

\[ C_{11} \] 

From these expressions one can immediately verify that the divergences cancel in $D_{L,R}$ and that they are not present in $C_{L,R}$. To finish this section we just rewrite the $C_{L,R}$ in our usual notation. We get

\[ C_L = \frac{e Q_L}{16\pi^2} \left[ A_L B_L m_F \left( -C_0(0,m_2^2,m_1^2,m_2^2,m_2^2,m_5^2) - C_2(0,m_1^2,m_2^2,m_2^2,m_2^2,m_5^2) \right) + A_L B_R m_2 \left( C_2(0,m_1^2,m_2^2,m_2^2,m_2^2) + C_{12}(0,m_1^2,m_2^2,m_2^2,m_2^2) + C_{22}(0,m_1^2,m_2^2,m_2^2,m_2^2) \right) \right. 
\[ + \left. A_R B_L m_1 C_{12}(0,m_1^2,m_2^2,m_2^2,m_2^2,m_5^2) \right] \]  
(C.172)

\[ C_R = C_L(L \leftrightarrow R) \]  
(C.173)

These equations are in agreement with Eqs. (32-34) and Eqs. (38-39) of Ref. [17], although some work has to be done in order to verify that\(^3\). This has to do with the fact that the PV decomposition functions are not independent (see the Appendix for further details on this point). We can however use the power of FeynCalc to verify this. We list below a simple program to accomplish that.

\[ (*************** End of Mathematica output **************) \]

\[^3\]An important difference between our conventions and those of Ref.[17] is that $p_1$ and $p_2$ (and obviously $m_1$ and $m_2$) are interchanged.
APPENDIX C. USEFUL TECHNIQUES FOR RENORMALIZATION

<< FeynCalc.m
<< mueg-ns.m

(*
Now write Lavoura integrals in the notation of FeynCalc. Be careful with the order of the entries.
*)

c1:=PaVe[1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
c2:=PaVe[2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
d1:=PaVe[1,1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
d2:=PaVe[2,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
f:=PaVe[1,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

(* Write Eqs. (32)-(34) of hepph/0302221 in our notation *)

k1:=PaVeReduce[m2*(c1+d1+f)]
k2:=PaVeReduce[m1*(c2+d2+f)]
k3:=PaVeReduce[mf*(c1+c2)]

(*
Now test the results. For this we should use the equivalences:
\rho -> AL BR
\lambda -> AR BL
\xi -> AR BR
\nu -> AL BL
*)

testCLALBR:=Simplify[PaVeReduce[Coefficient[CL, AL BR]-k1]]
testCLARBL:=Simplify[PaVeReduce[Coefficient[CL, AR BL]-k2]]
testCLALBL:=Simplify[PaVeReduce[Coefficient[CL, AL BL]-k3]]
testCRALBR:=Simplify[PaVeReduce[Coefficient[CR, AL BR]-k2]]
testCRARBL:=Simplify[PaVeReduce[Coefficient[CR, AR BL]-k1]]
testCRARBR:=Simplify[PaVeReduce[Coefficient[CR, AR BR]-k3]]

(*************** End of Program lavoura-ns.m ***************)

One can easily check that the output of the six tests is zero, showing the equivalence between our results. And all this is done in a few seconds.
C.11. MODERN TECHNIQUES IN A REAL PROBLEM: $\mu \to E\gamma$

C.11.2 Charged scalar neutral fermion loop

We consider now the case of the scalar being charged and the scalar neutral. The general case of both charged [17] can also be easily implemented, but for simplicity we do not consider it here. The couplings are now

$$\mu \to E\gamma$$

and the diagrams contributing to the process are given in Fig. C.8, where all the denominators are as in Eqs. (C.159)-(C.161) except that

$$D'_1 = q^2 - m_F^2 ; \quad D'_2 = (q - p_1)^2 - m_S^2 ; \quad D'_3 = (q - p_1 - k)^2 - m_S^2$$  \hspace{1cm} (C.174)

Also the coupling of the photon to the charged scalar is, in our notation,

$$-ieQ_l (-2q + p_1 + p_2)^\mu$$  \hspace{1cm} (C.175)

The procedure is very similar to the neutral scalar case and we just present here the *mathematica* program and the final result. All the checks of finiteness and gauge invariance can be done as before.

```
(* Program mueg-cs.m *)

This program calculates the COMPLETE (both the 3 point amplitude and the two self energy type on each external line) amplitudes for 
$\mu \to e \gamma$ when the fermion line in the loop is neutral and the charged line is a scalar. The $\mu$ has momentum p2 and mass m2, the 
electron (p1,m1) and the photon momentum k. The momentum in the loop 
is q.
```
The assumed vertices are,

1) Electron-Scalar-Fermion:

\[ \text{Spinor}[p_1,m_1] \text{ (AL} \ P_L + \text{ AR} \ P_R) \text{ Spinor} \ [p_f,m_f] \]

2) Fermion-Scalar-Muon:

\[ \text{Spinor}[p_f,m_f] \text{ (BL} \ P_L + \text{ BR} \ P_R) \text{ Spinor} \ [p_2,m_2] \]

(*
\[ \text{dm}[\mu_\_] := \text{DiracMatrix}[\mu, \text{Dimension} \to 4] \]
\[ \text{dm}[5] := \text{DiracMatrix}[5] \]
\[ \text{ds}[p_\_] := \text{DiracSlash}[p] \]
\[ \text{mt}[\nu_\_, \mu_\_] := \text{MetricTensor}[\nu, \mu] \]
\[ \text{fv}[\nu_\_, \mu_\_] := \text{FourVector}[\nu, \mu] \]
\[ \text{epsilon}[a_\_, b_\_, c_\_, d_\_] := \text{LeviCivita}[a, b, c, d] \]
\[ \text{id}[n_\_] := \text{IdentityMatrix}[n] \]
\[ \text{sp}[p_\_, q_\_] := \text{ScalarProduct}[p, q] \]
\[ \text{li}[\mu_\_] := \text{LorentzIndex}[\mu] \]
\[ L := \text{dm}[7] \]
\[ R := \text{dm}[6] \]

(*
\text{SetOptions}[\{B_0, B_1, B0_0, B1_1\}, \text{BReduce} \to \text{True}] \]
*)

\[ gA := \text{AL} \ \text{DiracMatrix}[7] + \text{AR} \ \text{DiracMatrix}[6] \]
\[ gB := \text{BL} \ \text{DiracMatrix}[7] + \text{BR} \ \text{DiracMatrix}[6] \]

\[ \text{num1} := \text{Spinor}[p_1,m_1] \cdot gA \cdot (\text{ds}[q] + mf) \cdot gB \cdot \text{Spinor}[p_2,m_2] \]
\[ \text{PolarizationVector}[k, \mu_\_] \left( - 2 \text{fv}[q, \mu] + \text{fv}[p_1, \mu] + \text{fv}[p_2, \mu] \right) \]

\[ \text{num11} := \text{DiracSimplify}[\text{num1}] \]

\[ \text{num2} := \text{Spinor}[p_1,m_1] \cdot gA \cdot (\text{ds}[q] + \text{ds}[p_1] + mf) \cdot gB \cdot (\text{ds}[p_1] + mf) \cdot \]
\[ \text{ds}[\text{Polarization}[k]] \cdot \text{Spinor}[p_2,m_2] \]

\[ \text{num3} := \text{Spinor}[p_1,m_1] \cdot \text{ds}[\text{Polarization}[k]] \cdot (\text{ds}[p_2] + mf) \cdot gA \cdot \]
\[ (\text{ds}[q] + \text{ds}[p_2] + mf) \cdot gB \cdot \text{Spinor}[p_2,m_2] \]

\text{SetOptions}[\text{OneLoop}, \text{Dimension} \to 4] \]
C.11. MODERN TECHNIQUES IN A REAL PROBLEM: $\mu \to E\gamma$

\[
\begin{align*}
\text{amp1} &:= \text{num1} \ F\text{eynAmpDenominator}[\text{PropagatorDenominator}[q, mf], \nonumber \\
& \quad \quad \quad \quad \quad \quad \quad \text{PropagatorDenominator}[q-p1, ms], \
& \quad \quad \quad \quad \quad \quad \quad \text{PropagatorDenominator}[q-p1-k, ms]] \\
\text{amp2} &:= \text{num2} \ F\text{eynAmpDenominator}[\text{PropagatorDenominator}[q+p1, mf], \
& \quad \quad \quad \quad \quad \quad \quad \text{PropagatorDenominator}[p2-k, m2], \
& \quad \quad \quad \quad \quad \quad \quad \text{PropagatorDenominator}[q, ms]] \\
\text{amp3} &:= \text{num3} \ F\text{eynAmpDenominator}[\text{PropagatorDenominator}[p1+k, m1], \
& \quad \quad \quad \quad \quad \quad \quad \text{PropagatorDenominator}[q+p2, mf], \
& \quad \quad \quad \quad \quad \quad \quad \text{PropagatorDenominator}[q, ms]] \\

(* Define the on-shell kinematics *)

\text{onshell} = \{\text{ScalarProduct}[p1, p1] \to m1^2, \text{ScalarProduct}[p2, p2] \to m2^2, \quad \nonumber \\
& \text{ScalarProduct}[k, k] \to 0, \text{ScalarProduct}[p1, k] \to (m2^2-m1^2)/2, \
& \text{ScalarProduct}[p2, k] \to (m2^2-m1^2)/2, \
& \text{ScalarProduct}[p2, \text{Polarization}[k]] \to p2epk, \
& \text{ScalarProduct}[p1, \text{Polarization}[k]] \to p2epk\}

(* Define the divergent part of the relevant PV functions*)

\text{div} = \{B0[m1^2, mf^2, ms^2] \to \text{Div}, B0[m2^2, mf^2, ms^2] \to \text{Div}, \nonumber \\
& B0[0, mf^2, ms^2] \to \text{Div}, B0[0, mf^2, mf^2] \to \text{Div}, B0[0, ms^2, ms^2] \to \text{Div}\}

\text{res1} := (-I / \pi^2) \text{OneLoop}[q, \text{amp1}] \\
\text{res2} := (-I / \pi^2) \text{OneLoop}[q, \text{amp2}] \\
\text{res3} := (-I / \pi^2) \text{OneLoop}[q, \text{amp3}] \\
\text{res} := \text{res1} + \text{res2} + \text{res3} /. \text{onshell}

\text{auxT1} := \text{res1} /. \text{onshell} \\
\text{auxT2} := \text{PaVeReduce}[\text{auxT1}] \\
\text{auxT3} := \text{auxT2} /. \text{div} \\
\text{divT} := \text{Simplify}[\text{Div}*\text{Coefficient}[	ext{auxT3}, \text{Div}]]

\text{auxS1} := \text{res2} + \text{res3} /. \text{onshell} \\
\text{auxS2} := \text{PaVeReduce}[\text{auxS1}] \\
\text{auxS3} := \text{auxS2} /. \text{div} \\
\text{divS} := \text{Simplify}[\text{Div}*\text{Coefficient}[	ext{auxS3}, \text{Div}]]
(* Check cancellation of divergences

    testdiv should be zero because divT=-divS
*)

    testdiv := Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 6 Standard Matrix Elements. To have a simpler result we substitute these elements by simpler expressions (ME[1],...ME[6]). Not all are independent.

 StandardMatrixElement[p2epk u[p1, m1] . ga[7] . u[p2, m2]],
 StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[6] . u[p2, m2]],
 StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[7] . u[p2, m2]]}

*)

    ans1 = res;
    var = Select[Variables[ans1], (Head[#] === StandardMatrixElement) &]


    ans2 = ans1 /. identities;
    ans = Simplify[ans2];

    CR = Coefficient[ans, ME[1]]/2;
    CL = Coefficient[ans, ME[2]]/2;
    DR = Coefficient[ans, ME[5]];
    DL = Coefficient[ans, ME[6]];

    (* Test to see if we did not forget any term *)

C.11. MODERN TECHNIQUES IN A REAL PROBLEM: $\mu \to E\gamma$

(* Test that the divergences cancel term by term *)

auxCL := PaVeReduce[CL] /. div;
testdivCL := Simplify[Coefficient[auxCL, Div]]

auxCR := PaVeReduce[CR] /. div;
testdivCR := Simplify[Coefficient[auxCR, Div]]

auxDL := PaVeReduce[DL] /. div;
testdivDL := Simplify[Coefficient[auxDL, Div]]

auxDR := PaVeReduce[DR] /. div;
testdivDR := Simplify[Coefficient[auxDR, Div]]

(* Test the gauge invariance relations *)
testGI1 := PaVeReduce[(m2^2 - m1^2)*CR - DR*m1 + DL*m2]
testGI2 := PaVeReduce[(m2^2 - m1^2)*CL + DR*m2 - DL*m1]

(********************** End Program mueg-cs.m ***********************)

Note that although these programs look large, in fact they are very simple. Most of it are comments and tests. The output of this program gives,

(* Mathematica output *********************** *)

In[3]:= CL

Out[3]= (-2 AR BL m1 C0[0, m1, m2, ms, ms, mf] -

2 2 2 2
2 AR BL m1 PaVe[1, {m1, 0, m2}, {mf, ms, ms}] -

2 2 2 2
4 AR BL m1 PaVe[1, {m1, m2, 0}, {ms, mf, ms}] -

2 2 2 2
2 AL BL mf PaVe[1, {m1, m2, 0}, {ms, mf, ms}] -

2 2 2 2
2 AL BR m2 PaVe[2, {m1, 0, m2}, {mf, ms, ms}] -

2 2 2 2
2 AR BL m1 PaVe[2, {m1, m2, 0}, {ms, mf, ms}] +
To finish this section we just rewrite the \( C_{L,R} \) in our usual notation. We get

\[
C_L = \frac{e Q_L}{16\pi^2} \left[ A_{LB} m_F \left( -C_1(m_1^2, m_2^2, 0, m_S^2, m_F^2, m_S^2) \right) + A_{LB} m_2 \left( -C_2(m_1^2, 0, m_2^2, m_F^2, m_S^2, m_S^2) + C_2(m_1^2, m_2^2, 0, m_S^2, m_F^2, m_S^2) \right) + C_{12}(m_1^2, m_2^2, 0, m_S^2, m_F^2, m_S^2) \right]
\]

\[
+ A_{RB} m_1 \left( -C_0(0, m_1^2, m_2^2, m_S^2, m_F^2, m_S^2) - C_1(m_1^2, 0, m_2^2, m_F^2, m_S^2, m_S^2) + 2C_1(m_1^2, m_2^2, 0, m_S^2, m_F^2, m_S^2) - C_2(m_1^2, m_2^2, 0, m_S^2, m_F^2, m_S^2) \right)
\]

\[
- C_{11}(m_1^2, m_2^2, 0, m_S^2, m_F^2, m_S^2) - C_{12}(m_1^2, m_2^2, 0, m_S^2, m_F^2, m_S^2)) \right]
\]

\[
C_R = C_L(L \leftrightarrow R) \quad \text{(C.176)}
\]

It is left as an exercise to write a \texttt{mathematica} program that proves that these equations are in agreement with Eqs. (35-37) and Eqs. (38-39) of Ref. [17].
Appendix D

Feynman Rules for the Standard Model

D.1 Introduction

To do actual calculations it is very important to have all the Feynman rules with consistent conventions. In this Appendix we will give the complete Feynman rules for the Standard Model in the general $R_\xi$ gauge.

D.2 The Standard Model

One of the most difficult problems in having a consistent set of Feynman rules are the conventions. We give here those that are important for building the SM. We will separate them by gauge group.

D.2.1 Gauge Group $SU(3)_c$

Here the important conventions are for the field strengths and the covariant derivatives. We have

\[ G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + gf^{abc} G^b_\mu G^c_\nu, \quad a = 1, \ldots, 8 \quad (D.1) \]

where $f^{abc}$ are the group structure constants, satisfying

\[ [T^a, T^b] = if^{abc}T^c \quad (D.2) \]

and $T^a$ are the generators of the group. The covariant derivative of a (quark) field $q$ in some representation $T^a$ of the gauge group is given by

\[ D_\mu q = (\partial_\mu - ig G^a_\mu T^a) q \quad (D.3) \]

In QCD the quarks are in the fundamental representation and $T^a = \lambda^a/2$ where $\lambda^a$ are the Gell-Mann matrices. A gauge transformation is given by a matrix

\[ U = e^{-iT^a\alpha^a} \quad (D.4) \]
and the fields transform as
\[ q \rightarrow e^{-iT^a\alpha^a} q \quad \quad \delta q = -iT^a\alpha^a q \]
\[ G^a T^a \rightarrow U G^a T^a U^{-1} - \frac{i}{g} \partial_{\mu} U U^{-1} \quad \delta G^a = -\frac{1}{g} \partial_{\mu} \alpha^a + f^{abc} \alpha^b G^c \] (D.5)
where the second column is for infinitesimal transformations. With these definitions one can verify that the covariant derivative transforms like the field itself,
\[ \delta(D_{\mu} q) = -i T^a\alpha^a (D_{\mu} q) \] (D.6)
ensuring the gauge invariance of the Lagrangian.

**D.2.2 Gauge Group SU(2)_L**

This is similar to the previous case. We have
\[ W^a_{\mu\nu} = \partial_{\mu} W^a_{\nu} - \partial_{\nu} W^a_{\mu} + g\epsilon^{abc} W^b_{\mu} W^c_{\nu}, \quad a = 1, \ldots, 3 \] (D.7)
where, for the fundamental representation of SU(2)_L we have \( T^a = \sigma^a/2 \) and \( \epsilon^{abc} \) is the completely anti-symmetric tensor in 3 dimensions. The covariant derivative for any field \( \psi_L \) transforming non-trivially under this group is,
\[ D_{\mu} \psi_L = (\partial_{\mu} - i g W^a_{\mu} T^a) \psi_L \] (D.8)

**D.2.3 Gauge Group U(1)_Y**

In this case the group is abelian and we have
\[ B_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} \] (D.9)
with the covariant derivative given by
\[ D_{\mu} \psi_R = (\partial_{\mu} + i g' Y B_{\mu}) \psi_R \] (D.10)
where \( Y \) is the hypercharge of the field. Notice the different sign convention between Eq. (D.8) and Eq. (D.9). This is to have the usual definition\(^1\)
\[ Q = T_3 + Y. \] (D.12)

It is useful to write the covariant derivative in terms of the mass eigenstates \( A_{\mu} \) and \( Z_{\mu} \). These are defined by the relations,
\[ \begin{align*}
W^3_{\mu} &= Z_{\mu} \cos \theta_W - A_{\mu} \sin \theta_W \\
B_{\mu} &= Z_{\mu} \sin \theta_W + A_{\mu} \cos \theta_W
\end{align*} \quad \quad \begin{align*}
Z_{\mu} &= W^3_{\mu} \cos \theta_W + B_{\mu} \sin \theta_W \\
A_{\mu} &= -W^3_{\mu} \sin \theta_W + B_{\mu} \cos \theta_W
\end{align*} \] (D.13)
\(^1\)For this to be consistent one must also have, under hypercharge transformations, for a field of hypercharge \( Y \),
\[ \psi' = e^{i\gamma^a Y_{\mu}} \psi, \quad B'_{\mu} = B_{\mu} - \frac{1}{g'} \partial_{\mu} \alpha_Y. \] (D.11)
This is important when finding the ghost interactions. It would have been possible to have a minus sign in Eq. (D.10), with a definition \( \theta_W \to \theta_W + \pi \). This would also mean reversing the sign in the exponent of the hypercharge transformation in Eq. (D.11) maintaining the similarity with Eq. (D.5).
D.2. THE STANDARD MODEL

<table>
<thead>
<tr>
<th>Field</th>
<th>$\ell_L$</th>
<th>$\ell_R$</th>
<th>$\nu_L$</th>
<th>$u_L$</th>
<th>$d_L$</th>
<th>$u_R$</th>
<th>$d_R$</th>
<th>$\phi^+$</th>
<th>$\phi^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_3$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>$Y$</td>
<td>$-\frac{1}{2}$</td>
<td>$-1$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{3}{2}$</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>0</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{3}{2}$</td>
<td>$-\frac{1}{3}$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table D.1: Values of $T_3$, $Y$, and $Q$ for the SM particles.

For a field $\psi_L$, with hypercharge $Y$, we get,

$$D_\mu \psi_L = \left[ \partial_\mu - i \frac{g}{\sqrt{2}} (\tau^+ W^+ \tau^- W^-) - i \frac{g}{2} \tau_3 W_3^\mu + ig' Y B_\mu \right] \psi_L$$

$$= \left[ \partial_\mu - i \frac{g}{\sqrt{2}} (\tau^+ W^+ \tau^- W^-) + ie Q A_\mu - i \frac{g}{\cos \theta_W} \left( \frac{\tau_3}{2} - Q \sin^2 \theta_W \right) Z_\mu \right] \psi_L$$

where, as usual, $\tau^\pm = (\tau_1 \pm i \tau_2)/2$ and the charge operator is defined by

$$Q = \begin{bmatrix} \frac{1}{2} + Y & 0 \\ 0 & -\frac{1}{2} + Y \end{bmatrix},$$

and we have used the relations,

$$e = g \sin \theta_W = g' \cos \theta_W,$$

and the usual definition,

$$W^\pm_\mu = \frac{W_1^\mu \mp i W_2^\mu}{\sqrt{2}}.$$ (D.17)

For a singlet of $SU(2)_L$, $\psi_R$ we have,

$$D_\mu \psi_R = \left[ \partial_\mu + ig' Y B_\mu \right] \psi_R$$

$$= \left[ \partial_\mu + ie Q A_\mu + i \frac{g}{\cos \theta_W} Q \sin^2 \theta_W Z_\mu \right] \psi_R.$$ (D.18)

We collect in Table D.1 the quantum number of the SM particles.

D.2.4 The Gauge Field Lagrangian

For completeness we write the gauge field Lagrangian. We have

$$L_{gauge} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{4} W_{\mu
u}^a W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$ (D.19)

where the field strengths are given in Eqs. (D.1), and (D.9).
D.2.5 The Fermion Fields Lagrangian

Here we give the kinetic part and gauge interaction, leaving the Yukawa interaction for a next section. We have

\[ \mathcal{L}_{\text{Fermion}} = \sum_{\text{quarks}} i \bar{q}_i \gamma^\mu D_\mu q + \sum_{\psi_L} i \bar{\psi}_L \gamma^\mu D_\mu \psi_L + \sum_{\psi_R} i \bar{\psi}_R \gamma^\mu D_\mu \psi_R \]  

(D.20)

where the covariant derivatives are obtained with the rules in Eqs. (D.3), (D.14) and (D.18).

D.2.6 The Higgs Lagrangian

In the SM we use an Higgs doublet with the following assignments,

\[ \Phi = \begin{bmatrix} \phi^+ \\ v + H + i\varphi_Z \end{bmatrix} \]  

(D.21)

The hypercharge of this doublet is \(1/2\) and therefore the covariant derivative reads

\[ D_\mu \Phi = \left[ \partial_\mu - i \frac{g}{\sqrt{2}} \left( \tau^+ W^+_\mu + \tau^- W^-_\mu \right) - i \frac{g'}{2} \tau^3 W^3_\mu + i g' B_\mu \right] \Phi \]  

(D.22)

\[ = \left[ \partial_\mu - i \frac{g}{\sqrt{2}} \left( \tau^+ W^+_\mu - \tau^- W^-_\mu \right) + i e Q A_\mu - \frac{g}{\cos \theta_W} \left( \frac{\tau^3}{2} - Q \sin^2 \theta_W \right) Z_\mu \right] \Phi \]

The Higgs Lagrangian is then

\[ \mathcal{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger D_\mu \Phi + \mu^2 \Phi^\dagger \Phi - \lambda \left( \Phi^\dagger \Phi \right)^2 \]  

(D.23)

If we expand this Lagrangian we find the following terms

\[ \mathcal{L}_{\text{Higgs}} = \cdots + \frac{1}{8} g' v^2 W^3_\mu W^{\mu 3} + \frac{1}{8} g' v^2 B_\mu B^\mu + \frac{1}{4} g g' v^2 W^3_\mu B^\mu + \frac{1}{4} g' v^2 W^*_\mu W^-_\mu \]

\[ + \frac{1}{2} v \partial^\mu \varphi_Z \left( g' B_\mu + g W^3_\mu \right) + \frac{i}{2} g v W^*_\mu \partial^\mu \varphi^+ - \frac{i}{2} g' v W^*_\mu \partial^\mu \varphi^- \]  

(D.24)

The first three terms give, after diagonalization, a massless field, the photon, and a massive one, the \(Z\), with the relations given in Eq. (D.13), while the fourth gives the mass to the charged \(W^\pm_\mu\) boson. Using Eq. (D.13) we get,

\[ \mathcal{L}_{\text{Higgs}} = \cdots + \frac{1}{2} M_Z^2 Z^\mu Z_\mu + M^2 W^*_\mu W^-_\mu \]

\[ + M_Z \partial^\mu \varphi_Z + i M_W \left( W^-_\mu \partial^\mu \varphi^+ - W^*_\mu \partial^\mu \varphi^- \right) \]  

(D.25)

where

\[ M_W = \frac{1}{2} g v, \quad M_Z = \frac{1}{\cos \theta_W} \frac{1}{2} g v = \frac{1}{\cos \theta_W} M_W \]  

(D.26)

By looking at Eq. (D.25) we realize that besides finding a realistic spectra for the gauge bosons, we also got a problem. In fact the terms in the last line are quadratic in the fields and complicate the definition of the propagators. We now see how one can use the needed gauge fixing to solve also this problem.
D.2. THE STANDARD MODEL

D.2.7 The Yukawa Lagrangian

Now we have to spell out the interaction between the fermions and the Higgs doublet that after spontaneous symmetry breaking gives masses to the elementary fermions. We have,

\[ \mathcal{L}_{\text{Yukawa}} = -Y_l \bar{T}_\ell R - Y_d \bar{Q}_d R - Y_u \bar{Q}_u \tilde{\Phi} u_R + \text{h.c.} \] (D.27)

where sum is implied over generations, \( L (Q) \) are the lepton (quark) doublets and,

\[ \tilde{\Phi} = i \sigma_2 \Phi^* = \begin{bmatrix} v + H - i \phi Z \sqrt{2} \\ -\phi - \phi Z \end{bmatrix} \] (D.28)

D.2.8 The Gauge Fixing

As it is well known, we have to gauge fix the gauge part of the Lagrangian to be able to define the propagators. We will use a generalization of the class of Lorenz gauges, the so-called \( R_\xi \) gauges. With this choice the gauge fixing Lagrangian reads

\[ \mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} F^2_G - \frac{1}{2\xi} F^2_A - \frac{1}{2\xi} F^2_Z - \frac{\xi}{F_+ F_-} \] (D.29)

where

\[ F^a_G = \partial_\mu G^a_\mu, \quad F^a_A = \partial_\mu A^a_\mu, \quad F^a_Z = \partial_\mu Z^a_\mu - \xi M Z^a \phi Z \]

\[ F_+ = \partial_\mu W^+_\mu - i \xi M W^+_\phi, \quad F_- = \partial_\mu W^-_\mu + i \xi M W^- \phi \] (D.30)

One can easily verify that with these definitions we cancel the quadratic terms in Eq. (D.25).

D.2.9 The Ghost Lagrangian

The last piece in writing the SM Lagrangian is the ghost Lagrangian. As it is well known, this is given by the Fadeev-Popov prescription,

\[ \mathcal{L}_{\text{Ghost}} = \sum_{i=1}^4 \left[ \tau^+_i \frac{\partial (\delta F^+_\alpha)}{\partial \alpha^i} + \tau^-_i \frac{\partial (\delta F^-_\alpha)}{\partial \alpha^i} + \tau^Z_i \frac{\partial (\delta F^Z_\alpha)}{\partial \alpha^i} + \tau^A_i \frac{\partial (\delta F^A_\alpha)}{\partial \alpha^i} \right] c_i \]

\[ + \sum_{a,b=1}^8 \omega^a \frac{\partial (\delta F^a_G)}{\partial \beta^b} \omega^b \] (D.31)

where we have denoted by \( \omega^a \) the ghosts associated with the \( SU(3)_c \) transformations defined by,

\[ U = e^{-iT^a_\alpha \beta^a}, \quad a = 1, \ldots, 8 \] (D.32)

and by \( c_\pm, c_A, c_Z \) the electroweak ghosts associated with the gauge transformations,

\[ U = e^{-iT^a_\alpha \alpha^a}, \quad a = 1, \ldots, 3, \quad U = e^{iY \alpha^4} \] (D.33)
For completeness we write here the gauge transformations of the gauge fixing terms needed to find the Lagrangian in Eq. (D.31). It is convenient to redefine the parameters as

\[\alpha^\pm = \alpha^1 \mp \alpha^2 / \sqrt{2}\]

\[\alpha_Z = \alpha^3 \cos \theta_W + \alpha^4 \sin \theta_W\]

\[\alpha_A = -\alpha^3 \sin \theta_W + \alpha^4 \cos \theta_W\]  \hfill (D.34)

We then get

\[\delta F^a_G = -\partial_\mu \beta^a + g_s f^{abc} \beta^b G^c_{\mu}\]

\[\delta F_A = -\partial_\mu \alpha_A\]

\[\delta F_Z = \partial_\mu (\delta Z^{\mu}) - M_Z \delta \varphi_Z\]

\[\delta F_+ = \partial_\mu (\delta W^{\mu}_+ ) - i M_W \delta \varphi^+\]

\[\delta F_- = \partial_\mu (\delta W^{\mu}_- ) + i M_W \delta \varphi^-\]  \hfill (D.35)

Using the explicit form of the gauge transformations we can finally find the missing pieces,

\[\delta Z_\mu = -\partial_\mu \alpha_Z + ig \cos \theta_W (W^{\mu}_+ \alpha^- - W^{\mu}_- \alpha^+)\]  \hfill (D.36)

\[\delta W^{\mu}_+ = -\partial_\mu \alpha^+ + ig \left[ \alpha^+(Z_\mu \cos \theta_W - A_\mu \sin \theta_W) - (\alpha_Z \cos \theta_W - \alpha_A \sin \theta_W) W^{\mu}_+ \right]\]

\[\delta W^{\mu}_- = -\partial_\mu \alpha^- - ig \left[ \alpha^- (Z_\mu \cos \theta_W - A_\mu \sin \theta_W) - (\alpha_Z \cos \theta_W - \alpha_A \sin \theta_W) W^{\mu}_- \right]\]

and

\[\delta \varphi_Z = -\frac{1}{2} g \left( \alpha^- \varphi^+ + \alpha^+ \varphi^- \right) + \frac{g}{2 \cos \theta_W} \alpha_Z (v + H)\]

\[\delta \varphi^+ = -i \frac{g}{2} (v + H + i \varphi_Z) \alpha^+ - i \frac{g \cos 2 \theta_W}{2 \cos \theta_W} \varphi^+ \alpha_Z + i e \varphi^+ \alpha_A\]

\[\delta \varphi^- = i \frac{g}{2} (v + H - i \varphi_Z) \alpha^- + i \frac{g \cos 2 \theta_W}{2 \cos \theta_W} \varphi^- \alpha_Z - i e \varphi^- \alpha_A\]  \hfill (D.37)

D.2.10 The Complete SM Lagrangian

Finally the complete Lagrangian for the Standard Model is obtained putting together all the pieces. We have,

\[\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}}\]  \hfill (D.38)

where the different terms were given in Eqs. (D.19), (D.20), (D.23), (D.27), (D.29), (D.31).
D.3 The Feynman Rules for QCD

We give separately the Feynman Rules for QCD and the electroweak part of the Standard Model.

D.3.1 Propagators

\[ g_{\mu a} \begin{array}{c} \mu, a \\ g \\ \nu, b \end{array} - i\delta_{ab} \left[ \frac{g_{\mu \nu}}{k^2 + i\epsilon} - (1 - \xi) \frac{k_\mu k_\nu}{(k^2)^2} \right] \]  
\[a \quad \omega \quad b \quad \delta_{ab} \frac{i}{k^2 + i\epsilon} \]  
\[ \frac{g_{\rho c}}{p_3} \begin{array}{c} \rho, c \\ p_3 \end{array} \]

D.3.2 Triple Gauge Interactions

\[ g f^{abc} \left[ g^{\mu \nu} (p_1 - p_2)^\rho + g^{\rho \mu} (p_2 - p_3)^\nu ight. 
\left. + g^{\rho \nu} (p_3 - p_1)^\mu \right] \]
\[ p_1 + p_2 + p_3 = 0 \]  
\[ \frac{p_1}{p_1} + \frac{p_2}{p_2} + \frac{p_3}{p_3} = 0 \]  
\[ \frac{p_4}{p_4} \]

D.3.3 Quartic Gauge Interactions

\[ -ig^2 \left[ f_{eab} f_{ecd} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \right. 
\left. + f_{eac} f_{ebd} (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma}) \right. 
\left. + f_{ead} f_{ebc} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) \right] \]
\[ p_1 + p_2 + p_3 + p_4 = 0 \]
D.3.4 Fermion Gauge Interactions

\[ ig(\gamma^\mu)_{\beta\alpha} T_{ij}^a \] (D.43)

D.3.5 Ghost Interactions

\[ g C^{abc} p_1^\mu \] (D.44)

\[ p_1 + p_2 + p_3 = 0 \]

D.4 The Feynman Rules for the Electroweak Theory

D.4.1 Propagators

\[ \gamma \]

\[ -i \left[ \frac{g_{\mu\nu}}{k^2 + i\epsilon} - (1 - \xi) \frac{k_\mu k_\nu}{(k^2)^2} \right] \] (D.45)

\[ W \]

\[ \frac{-ig_{\mu\nu}}{k^2 - M_W^2 + i\epsilon} \] (D.46)

\[ Z \]

\[ \frac{-ig_{\mu\nu}}{k^2 - M_Z^2 + i\epsilon} \] (D.47)

\[ p \]

\[ \frac{i(\not p + m_f)}{p^2 - m_f^2 + i\epsilon} \] (D.48)

\[ h \]

\[ \frac{i}{p^2 - M_h^2 + i\epsilon} \] (D.49)

\[ \varphi \]

\[ \frac{i}{p^2 - \xi m_Z^2 + i\epsilon} \] (D.50)
D.4. THE FEYNMAN RULES FOR THE ELECTROWEAK THEORY

\[ i \frac{\varphi^\pm}{p^2 - \xi m_W^2 + i\epsilon} \] (D.51)

D.4.2 Triple Gauge Interactions

\[ -ie [g_{\alpha\beta}(p - k)_{\mu} + g_{\beta\mu}(k - q)_{\alpha} + g_{\mu\alpha}(q - p)_{\beta}] \] (D.52)

\[ ig \cos \theta_W [g_{\alpha\beta}(p - k)_{\mu} + g_{\beta\mu}(k - q)_{\alpha} + g_{\mu\alpha}(q - p)_{\beta}] \] (D.53)

D.4.3 Quartic Gauge Interactions

\[ -ie^2 [2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}] \] (D.54)

\[ -ig^2 \cos^2 \theta_W [2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}] \] (D.55)

\[ ieg \cos \theta_W [2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}] \] (D.56)

\[ ig^2 [2g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}] \] (D.57)
D.4.4 Charged Current Interaction

\[ i \frac{g}{\sqrt{2}} \gamma_\mu \frac{1 - \gamma_5}{2} \]  

\[ \hat{W}^\pm \]

D.4.5 Neutral Current Interaction

\[ \psi_f \]

\[ \psi_f \]

\[ Z^\mu \]

\[ A^\mu \]

\[ i \cos \theta_W \gamma_\mu (g_V^f - g_A^f \gamma_5) \]

\[ -ieQ_f \gamma_\mu \]  

where

\[ g_V^f = \frac{1}{2} T^3_f - Q_f \sin^2 \theta_W, \quad g_A^f = \frac{1}{2} T^3_f. \]  

D.4.6 Fermion-Higgs and Fermion-Goldstone Interactions

\[ -i \frac{g}{2} \frac{m_f}{m_W} \]  

\[ gT^3_f \frac{m_f}{m_W} \gamma_5 \]  

\[ i \frac{g}{\sqrt{2}} \left( \frac{m_u}{m_W} P_{R,L} - \frac{m_d}{m_W} P_{L,R} \right) \]
D.4.7 Triple Higgs-Gauge and Goldstone-Gauge Interactions

\[ -ie \ (p_+ - p_-)_\mu \] \hspace{1cm} (D.64)

\[ ig \frac{\cos 2\theta_W}{2 \cos \theta_W} (p_+ - p_-)_\mu \] \hspace{1cm} (D.65)

\[ \mp \frac{i}{2} g (k - p)_\mu \] \hspace{1cm} (D.66)

\[ \frac{g}{2} (k - p)_\mu \] \hspace{1cm} (D.67)

\[ \frac{g}{2 \cos \theta} (k - p)_\mu \] \hspace{1cm} (D.68)

\[ -ie m_W g_{\mu\nu} \] \hspace{1cm} (D.69)
D.4.8 Quartic Higgs-Gauge and Goldstone-Gauge Interactions

\[ -ig m_Z \sin^2 \theta_W g_{\mu\nu} \]  
(D.70)

\[ ig m_W g_{\mu\nu} \]  
(D.71)

\[ i \frac{g}{\cos \theta_W} m_Z g_{\mu\nu} \]  
(D.72)

\[ \frac{i}{2} g^2 g_{\mu\nu} \]  
(D.73)

\[ \frac{i}{2} g^2 g_{\mu\nu} \]  
(D.74)

\[ \frac{i}{2} \frac{g^2}{\cos^2 \theta_W} g_{\mu\nu} \]  
(D.75)
D.4. THE FEYNMAN RULES FOR THE ELECTROWEAK THEORY

\[ i \frac{g^2}{2 \cos^2 \theta_W} g_{\mu \nu} \]  
(D.76)

\[ 2i e^2 g_{\mu \nu} \]  
(D.77)

\[ i \frac{g}{2} \left( \frac{g \cos 2\theta_W}{\cos \theta_W} \right)^2 g_{\mu \nu} \]  
(D.78)

\[ i \frac{g^2}{2} g_{\mu \nu} \]  
(D.79)

\[ -i \frac{g^2}{2 \cos \theta_W} \sin^2 \theta_W g_{\mu \nu} \]  
(D.80)

\[ \mp g^2 \frac{\sin^2 \theta_W}{2 \cos \theta_W} g_{\mu \nu} \]  
(D.81)
D.4.9 Triple Higgs and Goldstone Interactions

\[ -i \frac{1}{2} e g g_{\mu\nu} \] (D.82)

\[ \pm \frac{1}{2} e g g_{\mu\nu} \] (D.83)

\[ -i e g \frac{\cos 2\theta_W}{\cos \theta_W} g_{\mu\nu} \] (D.84)

\[ -i \frac{1}{2} e g \frac{m_h^2}{m_W} \] (D.85)

\[ -\frac{3}{2} i e g \frac{m_h^2}{m_W} \] (D.86)
D.4. THE FEYNMAN RULES FOR THE ELECTROWEAK THEORY

\[ -\frac{i}{2} g^2 \frac{m_h^2}{m_W} \] (D.87)

D.4.10 Quartic Higgs and Goldstone Interactions

\[ -\frac{i}{2} g^2 \frac{m_h^2}{m_W} \] (D.88)

\[ -\frac{i}{4} g^2 \frac{m_h^2}{m_W} \] (D.89)

\[ -\frac{i}{4} g^2 \frac{m_h^2}{m_W} \] (D.90)

\[ -\frac{3}{4} i g^2 \frac{m_h^2}{m_W} \] (D.91)

\[ -\frac{i}{4} g^2 \frac{m_h^2}{m_W} \] (D.92)
APPENDIX D. FEYNMAN RULES FOR THE STANDARD MODEL

\[ -\frac{3}{4} \, ig^2 \frac{m_h^2}{m_W^2} \]  \hspace{1cm} (D.93)

D.4.11 Ghost Propagators

\[ \frac{i}{k^2 + i\epsilon} \]  \hspace{1cm} (D.94)

\[ \frac{i}{k^2 - \xi m_W^2 + i\epsilon} \]  \hspace{1cm} (D.95)

\[ \frac{i}{k^2 - \xi m_Z^2 + i\epsilon} \]  \hspace{1cm} (D.96)

D.4.12 Ghost Gauge Interactions

\[ \mp i e p_\mu \]  \hspace{1cm} (D.97)

\[ \pm ig \cos \theta_W p_\mu \]  \hspace{1cm} (D.98)
D.4. THE FEYNMAN RULES FOR THE ELECTROWEAK THEORY

\[ p \rightarrow W^\pm_\mu \mp ig \cos \theta_W p_\mu \]  
(D.99)

\[ c^\pm \rightarrow W^\pm_\mu \pm ie p_\mu \]  
(D.100)

\[ c^\pm \rightarrow W^\mp_\mu \mp ig \cos \theta_W p_\mu \]  
(D.101)

\[ c^\pm \rightarrow W^\mp_\mu \pm ie p_\mu \]  
(D.102)

D.4.13 Ghost Higgs and Ghost Goldstone Interactions

\[ c^\pm \rightarrow \varphi_Z \pm \frac{q}{2} \xi m_W \]  
(D.103)

\[ -\frac{i}{2} g \xi m_W \]  
(D.104)
\[ h \to c^+ c^- h \] 
\[ h \to c^0 Z c^0 Z \] 
\[ h \to \phi^\pm \phi^\pm \] 
\[ \phi^\pm \to c^\pm \phi^\pm \] 
\[ \phi^\pm \to c^0 A \phi^\pm \] 

\[ \frac{ig}{2 \cos \theta_W} \xi m_Z \] (D.105) 

\[ \frac{i}{2} g \xi m_Z \] (D.106) 

\[ -\frac{ig \cos 2\theta_W}{2 \cos \theta_W} \xi m_W \] (D.107) 

\[ ie \xi m_W \] (D.108)
Bibliography


