Calculus of variations and Optimal Control

Dynamic Systems and Optimization (SD09)

This course is entitled **Calculus of variations and Optimal Control**, and is offered to the Ph. D. Course on Electrical and Computer Engineering of Instituto Superior Técnico (IST) of Universidade de Lisboa (Portugal) within the Course with official name *Dynamic Systems and Optimization (SD09)*.

**Responsible**

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**Students with potential interest**

The course has potential interest to the following groups of students

- Ph. D. students in the areas of Systems, Decision and Control, including and Robotics
- Ph. D. students with an interest on Control and/or Dynamic Optimization of deterministic systems in the areas of
  - Electrical and Computer Engineering
  - Aerospace Engineering
  - Mechanical Engineering
  - Applied Mathematics
  - Applied Physics
  - Biomedical Engineering
  - Informatics Engineering
  - Economics and Management
  - Chemical Engineering
- Advanced M. Sc. Students with a solid mathematical background and an interest on Control and/or Dynamic Optimization of deterministic systems in the same areas listed above.

**Required background**

The background required corresponds to the basic curricula of Mathematics taught during the first three years of all the Engineering Courses of IST, including integral and differential calculus in \( \mathbb{R}^n \), basic notions of differential equations and linear algebra.

Previous knowledge of control and dynamical systems is useful but not essential.
Objectives

The main overall objective consists of an introduction to infinite dimensional optimization problems in relation to Calculus of Variations and Optimal Control, that balances a rigorous, but not fully general, proof of the main results with their application to specific problems.

After completing the course, the students will be able to pursue on their own further studies of advanced topics. The course is in line with similar courses presented at the same level in reference universities around the world.

Syllabus

The following topics will be addressed in the course:

1. **Application problems leading to Dynamic Optimization**, in the Calculus of variation and Optimal Control. Introduction of a new class of optimization problems through examples addressed in the course:
   a. Calculus of variations
      i. The brachistochrone problem
      ii. Dido’s isoperimetric problem
      iii. Variational Mechanics
      iv. The Principle of Maximum Entropy
   b. Optimal Control
      i. Robotics
      ii. Shape optimization
      iii. Cancer therapy optimization
      iv. Optimal management

2. **Background on finite and infinite dimensional optimization**
   a. Finite dimensional optimization
   b. Infinite dimensional optimization

3. **Calculus of variations**
   a. Basic calculus of variations problem
   b. Weak and strong Extrema
   c. First order necessary conditions for weak extrema; Euler Lagrange (EL) equation
   d. Special cases of the EL equation.
   e. Variable-endpoint problems
   f. Free endpoint
   g. Hamilton’s canonical equations.
   h. Extremal condition on the Hamiltonian
   i. Legendre transformation
   j. Application to Mechanics and the Principle of Least Action
   k. Variational problems with constraints. Isoperimetric problems
   l. Second order conditions. Legendre 2nd order condition.
   m. Sufficient conditions for a weak minimum. Conjugate points and the Jacobi and Riccati equations.
n. Preparing the ground to Pontryagin’s Maximum Principle: Problems with corners.

4. Pontryagin’s Maximum Principle (PMP)
   b. Special cases.
   c. Proof of the PMP using Calculus of Variations methods
   d. Discussion of the limitations of the proof of PMP based on CV.
   e. A more general proof for piecewise continuous functions with derivative
   f. Time optimal control
   g. Singular arcs.

5. The Hamilton-Jacobi-Bellman equation

6. The Linear Quadratic problem
   a. Finite horizon problems
   b. Infinite horizon problem.
   c. Output regulation and the root-square locus.

7. Optimal control problems in discrete time

8. Introduction to numerical methods
   a. Indirect methods.
   b. Direct methods

Elements for study

- Slides
- Problems, including problems for the classroom, for self study and series of problems.

Bibliography

Main reference

The course tightly follows the reference


A preliminary (although unrevised) version of this book is available at [http://liberzon.csl.illinois.edu/teaching/cvoc/cvoc.html](http://liberzon.csl.illinois.edu/teaching/cvoc/cvoc.html)

The students are encouraged to follow the published book.

Complementary book references


**Complementary journal and report references**


**Grading**

Final grade is made of:
• Active participation at the classes during the semester (10%)
• 6 series of problems (50%)
• Final exam (40%)

**Format**

2 lectures of 1,5 hour each per week during 12 weeks.

The students are expected to attend the theoretical lectures. During most of the classes the students will be invited to solve problems related to the concepts being presented.