INTRODUCTION TO DECISION ANALYSIS

Introduction

- Decision often must be made in uncertain environments.
- Examples:
  - Manufacturer introducing a new product in the marketplace.
  - Government contractor bidding on a new contract.
  - Oil company deciding to drill for oil in a particular location.
- Type of decisions that decision analysis is designed to address.
- Making decisions with or without experimentation.
Goferbroke Company owns a tract of land that can contain oil.

- Contracted geologist reports that chance of oil is 1 in 4.
- Another oil company offers 90,000€ for land.
- Cost of drilling is 100,000€. If oil is found, revenue is 800,000€ (expected profit is 700,000€).

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Status of land</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Oil</td>
<td>Dry</td>
</tr>
<tr>
<td>Drill for oil</td>
<td>700,000€</td>
<td>-100,000€</td>
</tr>
<tr>
<td>Sell the land</td>
<td>90,000€</td>
<td>90,000€</td>
</tr>
<tr>
<td>Chance of status</td>
<td>1 in 4</td>
<td>3 in 4</td>
</tr>
</tbody>
</table>

Analogy to game theory:

- **Players**: decision maker (player 1) and nature (player 2).
- **Available strategies for 2 players**: alternative actions and possible states of nature, respectively.
- Combination of strategies results in some payoff to player 1 (decision maker).

- But, are both players rational?
Decision making without experimentation

- The **decision maker** needs to choose one of the **alternative actions**.
- **Nature** choose one of the possible **states of nature**.
- Each combination of an action and state of nature results in a **payoff**, which is one entry of a **payoff table**.
- Probabilities for states of nature provided by the prior distribution are **prior probabilities**.
- Payoff table is used to find an **optimal action** for the decision making according to an appropriate **criterion**.

### Payoff table for Goferbroke Co. problem

<table>
<thead>
<tr>
<th>Alternative</th>
<th>State of nature</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>1. Drill for oil</td>
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<td>-100</td>
<td></td>
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<tr>
<td>2. Sell the land</td>
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<td></td>
</tr>
<tr>
<td>Prior probability</td>
<td>0.25</td>
<td>0.75</td>
<td></td>
</tr>
</tbody>
</table>
Maximin payoff criterion

- **Game against nature.**
- **Maximin payoff criterion:** For each possible decision alternative, find the *minimum payoff* over all states. Next, find the *maximum* of these minimum payoffs.
  - *Best guarantee* of payoff: pessimistic viewpoint.

<table>
<thead>
<tr>
<th>Alternative</th>
<th>State of nature</th>
<th>Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Oil</td>
<td>Dry</td>
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<tr>
<td>1. Drill for oil</td>
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Maximum likelihood criterion

- **Maximum likelihood criterion:** Identify *most likely* state. For this state, find decision alternative with the maximum payoff. Choose this action.
  - *Most likely* state: ignores important information and low-probability big payoff.
Bayes’ decision rule

Bayes’ decision rule: Using prior probabilities, calculate the expected value of payoff for each decision alternative. Choose the action with the maximum expected payoff.

- For the prototype example:
  - E[Payoff (drill)] = 0.25(700) + 0.75(–100) = 100.
  - E[Payoff (sell)] = 0.25(90) + 0.75(90) = 90.
- Incorporates all available information (payoffs and prior probabilities).

What happens when probabilities are inaccurate?

Sensitivity analysis with Bayes’ rules

- Prior probabilities can be questionable.
- True probabilities of having oil are 0.15 to 0.35 (so, probabilities for dry land are from 0.65 to 0.85).
- \( p = \) prior probability of oil.
- Example: expected payoff from drilling for any \( p \):
  - \( E[\text{Payoff (drill)}] = 700p – 100(1 – p) = 800p – 100 \).
- In figure, the crossover point is where the decision changes from one alternative to another:
  - \( E[\text{Payoff (drill)}] = E[\text{Payoff (sell)}] \)
  - \( 800p – 100 = 90 \rightarrow p = 0.2375 \)
The decision is very sensitive to \( p \)!

**Expected payoff for alternative changes**

**Decision making with experimentation**

- Improved estimates are called **posterior probabilities**.
- **Example**: a detailed seismic survey costs 30 000€.
  - USS: unfavorable seismic soundings: oil is fairly unlikely.
  - FSS: favorable seismic soundings: oil is fairly likely.
- Based on past experience, the following probabilities are given:
  - \( P(\text{USS} \mid \text{State=Oil}) = 0.4 \); \( P(\text{FSS} \mid \text{State=Oil}) = 1 - 0.4 = 0.6 \)
  - \( P(\text{USS} \mid \text{State=Dry}) = 0.8 \); \( P(\text{FSS} \mid \text{State=Dry}) = 1 - 0.8 = 0.2 \)
### Posterior probabilities

- $n =$ number of possible states.
- $P(\text{State} = \text{state } i) =$ prior probability that true state is state $i$.
- Finding = finding from experimentation (random var.)
- Finding $j =$ one possible value of finding.
- $P(\text{State} = \text{state } i \mid \text{Finding} = \text{finding } j) =$ posterior probability that true state of nature is state $i$, given Finding = finding $j$.
- Given $P(\text{State} = \text{state } i)$ and $P(\text{Finding} = \text{find } j \mid \text{State} = \text{state } i)$, what is $P(\text{State} = \text{state } i \mid \text{Finding} = \text{finding } j)$?

### Posterior probabilities

From probability theory the Bayes’ theorem can be obtained:

$$P(\text{State} = \text{state } i \mid \text{Finding} = \text{finding } j) =$$

$$= \frac{P(\text{Finding} = \text{finding } j \mid \text{State} = \text{state } i)P(\text{State} = \text{state } i)}{\sum_{k=1}^{n} P(\text{Finding} = \text{finding } j \mid \text{State} = \text{state } k)P(\text{State} = \text{state } k)}$$
Bayes’ theorem in prototype example

If seismic survey in unfavorable (USS):

\[ P(\text{State} = \text{Oil} \mid \text{Finding} = \text{USS}) = \frac{0.4(0.25)}{0.4(0.25) + 0.8(0.75)} = \frac{1}{7}, \]

\[ P(\text{State} = \text{Dry} \mid \text{Finding} = \text{USS}) = 1 - \frac{1}{7} = \frac{6}{7}. \]

If seismic survey in favorable (FSS):

\[ P(\text{State} = \text{Oil} \mid \text{Finding} = \text{FSS}) = \frac{0.6(0.25)}{0.6(0.25) + 0.2(0.75)} = \frac{1}{2}, \]

\[ P(\text{State} = \text{Dry} \mid \text{Finding} = \text{FSS}) = 1 - \frac{1}{2} = \frac{1}{2}. \]


Probability tree diagram

Unconditional probabilities: 

\[ P(\text{FSS}) = 0.15 + 0.15 = 0.3 \]

\[ P(\text{finding}) \]

\[ P(\text{USS}) = 0.1 + 0.6 = 0.7 \]
Expected payoffs

Expected payoffs can be found using again Bayes’ decision rule for the prototype example, with posterior probabilities replacing prior probabilities:

- **Expected payoffs if finding is USS:**
  \[ E[\text{Payoff (drill | Finding = USS)}] = \frac{1}{7}(700) + \frac{6}{7}(-100) - 30 = -15.7 \]
  \[ E[\text{Payoff (sell | Finding = USS)}] = \frac{1}{7}(90) + \frac{6}{7}(90) - 30 = 60 \]

- **Expected payoffs if finding is FSS:**
  \[ E[\text{Payoff (drill | Finding = FSS)}] = \frac{1}{2}(700) + \frac{1}{2}(-100) - 30 = 270 \]
  \[ E[\text{Payoff (sell | Finding = FSS)}] = \frac{1}{2}(90) + \frac{1}{2}(90) - 30 = 60 \]

Optimal policy

Using Bayes’ decision rule, the optimal policy of optimizing payoff is given by:

<table>
<thead>
<tr>
<th>Finding from seismic survey</th>
<th>Optimal alternative</th>
<th>Expected payoff excluding cost of survey</th>
<th>Expected payoff including cost of survey</th>
</tr>
</thead>
<tbody>
<tr>
<td>USS</td>
<td>Sell the land</td>
<td>90</td>
<td>60</td>
</tr>
<tr>
<td>FSS</td>
<td>Drill for oil</td>
<td>300</td>
<td>270</td>
</tr>
</tbody>
</table>

Is it worth spending 30.000€ to conduct the experimentation?
### Value of experimentation

- Before performing an experimentation, determine its potential value.
- Two methods:
  1. **Expected value of perfect information** – it is assumed that all uncertainty is removed. Provides an *upper bound* of potential value of experiment.
  2. **Expected value of information** – is the expected increase in payoff, not just its upper bound.

### Expected value of perfect information

<table>
<thead>
<tr>
<th>Alternative</th>
<th>State of nature</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Oil</td>
</tr>
<tr>
<td>1. Drill for oil</td>
<td>700</td>
</tr>
<tr>
<td>2. Sell the land</td>
<td>90</td>
</tr>
<tr>
<td>Maximum payoff</td>
<td>700</td>
</tr>
<tr>
<td>Prior probability</td>
<td>0.25</td>
</tr>
</tbody>
</table>

*Expected payoff with perfect information = 0.25(700) + 0.75(90) = 242.5*

- **Expected value of perfect information (EVPI)** is:
  \[
  \text{EVPI} = \text{expected payoff with perfect information} - \text{expected payoff without experimentation}
  \]
- **Example:** \(\text{EVPI} = 242.5 - 100 = 142.5\). This value is > 30
Expected value of information

- Requires *expected payoff with experimentation*:

  \[ \sum_j P(\text{Finding} = j)E[\text{payoff} | \text{Finding} = j] \]

- **Example**: see *probability tree diagram*, where:
  \[ P(\text{USS}) = 0.7, \quad P(\text{FSS}) = 0.3. \]

  - Expected payoff (excluding cost of survey) was obtained in *optimal policy*:
    - \( E(\text{payoff} | \text{Finding} = \text{USS}) = 90 \),
    - \( E(\text{payoff} | \text{Finding} = \text{FSS}) = 300 \).

  - Expected payoff with experimentation =
    \[ \sum_j P(\text{Finding} = j)E[\text{payoff} | \text{Finding} = j] \]

  - So, *expected payoff with experimentation* is
    - Expected payoff with experim. = \( 0.7(90) + 0.3(300) = 153 \).

  - **Expected value of experimentation (EVE)** is:
    \[ \text{EVE} = \text{expected payoff with experimentation} - \text{expected payoff without experimentation} \]

  - **Example**: \( \text{EVE} = 153 - 100 = 53 \).
    - As 53 exceeds 30, the seismic survey should be done.
Prototype example has a sequence of two questions:

1. Should a seismic survey be conducted before an action is chosen?
2. Which action (drill for oil or sell the land) should be chosen?

These questions have a corresponding decision tree.

Junction points are nodes, and lines are branches.

A decision node, represented by a square, indicates that a decision needs to be made at that point.

An event node, represented by a circle, indicates that a random event occurs at that point.
Decision tree with probabilities

Performing the analysis

1. Start at right side of decision tree and move *one column* at a time. For each column, perform step 2 or step 3, depending if nodes are *event* or *decision* nodes.

2. For each *event node*, calculate its *expected payoff*, by multiplying expected payoff of each branch by probability of that branch and summing these products.

3. For each *decision node*, compare the expected payoffs of its branches, and choose alternative with largest *expected payoff*. Record the choice by inserting a double dash in each rejected branch.
The decision tree results in the following decisions:

1. Do the seismic survey.
2. If the result is unfavorable, sell the land.
3. If the result is favorable, drill for oil.
4. The expected payoff (including the cost of the seismic survey) is 123 (123 000€).

- Same result as obtained with experimentation.
- For any decision tree, the backward induction procedure always will lead to the optimal policy.
Utility theory

- You are offered the choice of:
  1. Accepting a 50:50 chance of winning $100,000 or nothing;
  2. Receiving $40,000 with certainty.
- What do you choose?

- A company may be unwilling to invest a large sum of money in a new product even when the expected profit is substantial if there is a risk of losing its investment and thereby becoming bankrupt.

- People buy insurance even though it is a poor investment from the viewpoint of the expected payoff.

Utility theory

- **Utility functions** $u(M)$ for money $M$: usually there is a decreasing marginal utility for money (individual is *risk-averse*).
Utility function for money

- It is also possible to exhibit a mixture of these kinds of behavior (*risk-averse, risk seeker, risk-neutral*)
- An individual’s attitude toward risk may be different when dealing with one’s personal finances than when making decisions on behalf of an organization.
- When a *utility function for money* is incorporated into a decision analysis approach to a problem, this utility function must be constructed to fit the preferences and values of the decision maker involved. (The decision maker can be either a single individual or a group of people.)

Utility theory

- **Fundamental property:** the decision maker’s *utility function for money* has the property that the decision maker is *indifferent* between two alternatives if they have the *same expected utility*.
  - **Example.** Offer: an opportunity to obtain either $100000 (utility = 4) with probability $p$ or nothing (utility = 0) with probability $1 - p$. Thus, $E(utility) = 4p$.
    - Decision maker is indifferent for e.g.:
      - Offer with $p = 0.25$ ($E(utility) = 1$) or definitely obtaining $10000$ (utility = 1).
      - Offer with $p = 0.75$ ($E(utility) = 3$) or definitely obtaining $60000$ (utility = 3).
Role of utility theory

- If utility function is used to measure worth of possible monetary outcomes, Bayes’ decision rule replaces monetary payoffs by corresponding utilities.
- Thus, optimal action is the one that maximizes the expected utility.

- Note that utility functions may not be monetary.
  - Example: doctor’s decision alternatives in treating a patient involves the future health of the patient.

Applying utility theory to example

- The Goferbroke Co. does not have much capital, so a loss of 100 000€ would be quite serious.

- The complete utility function can be found using the following values:

<table>
<thead>
<tr>
<th>Monetary payoff</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>-130</td>
<td>-150</td>
</tr>
<tr>
<td>-100</td>
<td>-105</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>90</td>
<td>90</td>
</tr>
<tr>
<td>670</td>
<td>580</td>
</tr>
<tr>
<td>700</td>
<td>600</td>
</tr>
</tbody>
</table>
A popular form is the exponential utility function:

\[ u(M) = R \left(1 - e^{-\frac{M}{R}}\right) \]

- \( R \) = decision maker’s risk tolerance.
- This is designing a risk-averse individual.
  - For prototype example, \( R = 2250 \) for \( u(670) \), and \( R = 465 \) for \( u(-130) \).
- Note that, in general, it is not possible to have different values of \( R \).
The solution is exactly the same as before, except for substituting utilities for monetary payoffs.

Thus, the value obtained to evaluate each fork of the tree is now the expected utility rather than the expected monetary payoff.

- **Optimal decisions selected by Bayes’ decision rule maximize the expected utility for the overall problem.**