Semiclassical approximation for non-Hermitian operators - Application to stochastic optimization

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Abstract

A connection is made between the formalism of stochastic optimization of a certain class of control systems and the semiclassical evolution generated by a quantum non Hermitian Hamiltonian. The non-Hermitian dynamics of Gaussian wave packets is described by a system of equations for the motion of the center and metric associated with the wave packet. We propose a new method of solving the system of equations that is able to simplify the calculations. This method is based on the inherent Kähler geometry of coherent states in phase space. An example of a quadratic Hamiltonian is explored where we show the existence of the infinite time limit of the center and the metric of the wave packet. The optimal control systems studied in the present paper are used in reinforcement learning algorithms as the policy improvement with path integrals proposed in [19].

Keywords: *Non-Hermitian Hamiltonians, semiclassical approximation, stochastic optimal control.*

1 Introduction

In Physics, one is usually interested in analyzing a given subsystem of the Universe and such a system is never fully isolated. One often then has to turn to the theory of open quantum systems (OQS) [2]. Nowadays, this theory serves as the backbone to modern research in quantum mechanics and its applications. OQS are very important for example in the fields of quantum information and computation. Here, although results are usually derived using the regular closed system quantum mechanics, one needs models that take into account that a quantum computer is an OQS and may have unwanted interactions that may change the dynamics.

One possible way to mimic the behavior of OQS is to consider an effective Hamiltonian that contains an imaginary term used to model the exchange of energy with the environment. This approach is called **non-Hermitian quantum mechanics**, which is a field that has been attracting a lot of visibility in recent years. Non-Hermitian Hamiltonians (NHH) naturally appear in several contexts like in the field of quantum optics and topological photonics [4,10,12,16].

One particular area of interest is the semiclassical limit of non-Hermitian dynamics [7–9]. Starting first by studying how NHH behave in the coherent state approximation it has been shown that, for both the state [9], and for the Wigner function of the state [7], one gets a generalized system of equations that describes the phase space evolution of the center of the state and of the associated metric [8]. In a recent work [11,14], a new method of solving this coupled system of equations has been proposed.

In the present work an equivalence between a class of stochastic optimal control systems and a class of non-Hermitian Hamiltonians has been found. Stochastic control theory has a wide range of applications nowadays, be it in robotics to accurately calculate a robot's course of actions [20] or even in finance, to model the dynamics of financial markets [1]. In particular, for a certain class of control systems it is possible to describe the dynamics of the value function using the formalism developed in this work. This study thus shows new methods to study these controlled systems under a new perspective, that of the non-Hermitian quantum formalism.

1.1 Symplectic and Kähler geometry

Symplectic manifolds [22] provide the natural setting for the Hamiltonian formulation of classical mechanics. They also provide the appropriate setup to study the semiclassical approximation of quantum dynamics. A **symplectic manifold** is a smooth manifold, M, with a closed non-degenerate 2-form ω, called the symplectic form. For the case of classical dynamics on $M = \mathbb{R}^{2n}$ this symplectic form is defined as:

$$\omega = dq \wedge dp = \sum_{j=1}^{n} dq_j \wedge dp_j \ . \tag{1}$$

Before moving forward we introduce the concept of Hamiltonian vector fields. Given a function from the

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manifold to the real numbers, $f : M \to \mathbb{R}$, we can obtain a unique vector field X_f, called the Hamiltonian vector field corresponding to f. In canonical coordinates, (p_i, q_i) , this vector field is represented by:

$$X_f = -\sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} \ . \tag{2}$$

We can also consider **complex structures** J on the symplectic manifold (M, ω)[15]. At every point, J is a linear transformation of the tangent space such that $J^2 = -\mathbb{1}_p$, i.e. the identity at point $p \in M$. Algebraically, this complex structure $J : T_pM \to T_pM$ plays the role of the imaginary unit, where T_pM is the tangent space at p. Note that, for a manifold to have a complex structure, it is required that the manifold is even dimensional. As symplectic manifolds are always even dimensional, that restriction does not pose a problem. Since $J^2 = -\mathbb{1}_p$, it has eigenvalues $\pm i$ that define conjugate eigenspaces of holomorphic and anti-holomorphic directions.

The symplectic form and the complex structure are said to be compatible if we require that the complex structure is a symplectic transformation, i.e. iff:

$$\omega\left(Ju, Jv\right) = \omega\left(u, v\right), \forall u, v \in T_pM. \tag{3}$$

For the matrix representation of the complex structure on canonical symplectic coordinates this means that $J^T \Omega J = \Omega$ with Ω being the matrix of the symplectic form, defined as:

$$\Omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{4}$$

in the basis (p_j, q_j). For the remainder of the present work we will, unless otherwise stated, always assume we are in the ordered basis (p_j, q_j).

Note that Eq. (3) implies that the tensor g defined as:

$$g(u, v) = \omega(u, Jv), \forall u, v \in T_pM, \tag{5}$$

is a symmetric tensor, defined on T_pM. If g is positive definite, i.e a metric, then (M, ω, J, g) is called a **Kähler Manifold**[13]. In a **Kähler Manifold**, the above three structures exist in such a way that they are mutually compatible with each other. Thus, knowing any pair of them allows one to determine the third.

Equivalently, in the matrix representation we have that:

$$G = \Omega J, \tag{6}$$

with G being the matrix representation of the metric g, J the complex structure and Ω the symplectic matrix defined above.

1.2 Wigner Distribution

In this work we will study non-unitary quantum dynamics in the semiclassical limit. For an harmonic oscillator, a coherent state is defined to be the unique eigenstate of the annihilation operator $\hat{a}$ with associated eigenvalue $\alpha \in \mathbb{C}$. It is a state with minimum uncertainty, that is, it saturates Heisenberg's uncertainty principle. Thus, they are often used as a good bridge between classical and quantum physics.

Since their proposal[6] coherent states have been generalized for arbitrary Lie groups. A good review can be found in Ref.[17]. Here, however, we will only consider coherent states for harmonic oscillators. An n-dimensional normalized Gaussian coherent state in the Schrödinger representation is defined as:

$$\psi(q) = \frac{(\det \operatorname{Im} B)^{\frac{1}{4}}}{(\pi\hbar)^{\frac{n}{4}}} \exp\left[\frac{i}{\hbar}\left(P \cdot (q - Q) + \frac{1}{2}(q - Q) \cdot B(q - Q)\right)\right], \tag{7}$$

where the vector $Y = (P, Q) \in \mathbb{R}^n \times \mathbb{R}^n$ is the center of the state, and the matrix $B \in M_n(\mathbb{C})$ is symmetric, with $\operatorname{Im} B > 0$ to ensure the state is normalizable.

It is possible to pass to a phase space representation of the state using the Wigner representation[3], originally proposed by Wigner[21]:

$$W(q, p) := \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy \psi^*(q + y)\psi(q - y)e^{2ipy/\hbar}. \tag{8}$$

The Wigner representation is fundamental to study phase space quantum mechanics. The role of the Wigner representation is analogous to a probability density function, as expectation values for functions of position and momentum can be calculated from it. However the Wigner function cannot really be a probability density function as it can be negative-valued. It worthy of note that, for Gaussian states, the Wigner distribution is always positive and Gaussian.

This approach to quantum mechanics is interesting because it makes quantum mechanics as similar as possible to classical Hamiltonian mechanics. As we will see, in the semiclassical limit we will obtain behavior very similar to the classical Hamiltonian dynamics.

It can be shown that the Wigner function of a Gaussian coherent state of the form of Eq. (7) is[7]:

$$W(y) = (\pi\hbar)^{-n} \exp\left[-\frac{1}{\hbar}(y - Y) \cdot G(y - Y)\right], \tag{9}$$

where $y = (p, q)$ and $Y = (P, Q)$. The matrix G is obtained from B by:

$$G = \begin{pmatrix} \mathbb{1} & 0 \\ -\operatorname{Re} B & \mathbb{1} \end{pmatrix} \begin{pmatrix} (\operatorname{Im} B)^{-1} & 0 \\ 0 & (\operatorname{Im} B) \end{pmatrix} \begin{pmatrix} \mathbb{1} & -\operatorname{Re} B \\ 0 & \mathbb{1} \end{pmatrix}. \tag{10}$$

We note that G is nondegenerate, positive and symmetric and can be seen as describing a constant metric on phase space[7]. Furthermore, it can be seen that it is also a symplectic metric, i.e., $G\Omega G = \Omega$ where Ω is the symplectic structure on phase space defined in Eq. (4). Due to this, using Eq. (5) it is possible to define a complex structure J, with $J^2 = -\mathbb{1}$ and $\Omega J = G$. Consequently, every coherent state (Eq. 7) endows the phase space with a flat **Kähler structure**.

1.3 Stochastic Optimal Control

The idea behind optimal control is that, for a given system that can be controlled through some variables, we want to find the optimal choice of control in order to maximize or minimize a given cost function. These systems can range from robotic arms trying to open doors to airplanes trying to lower their fuel consumption by controlling the thrust of the engines.

Stochastic optimal control is the extension of optimal control theory where we take into consideration the fact that both the measurements as well as the control can be affected by noise, which usually takes form of a Gaussian random variable.

In this work we will study systems whose evolution is described by a stochastic differential equation of the form:

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x_t}, \boldsymbol{t}) + \boldsymbol{G}(\boldsymbol{x}_t)\left(\boldsymbol{u}_t + \boldsymbol{\xi}_t\right) \ , \tag{11}$$

where $\boldsymbol{x_t} \in \mathbb{R}^n$ is the state of the system, $\boldsymbol{f}(\boldsymbol{x_t}, \boldsymbol{t}) \in \mathbb{R}^n$ describes the passive dynamics, $\boldsymbol{G}(\boldsymbol{x}_t) \in \mathbb{R}^{n\times p}$ is the control matrix, $\boldsymbol{u}_t \in \mathbb{R}^p$ the control variables and finally $\boldsymbol{\xi}_t \in \mathbb{R}^p$ is Gaussian white noise with variance matrix Σ_ξ.

The differential form of the above equation reads:

$$d\boldsymbol{x}_t = \left(\boldsymbol{f}(\boldsymbol{x_t}, \boldsymbol{t}) + \boldsymbol{G}(\boldsymbol{x}_t)\boldsymbol{u}_t\right) dt + \boldsymbol{G}(\boldsymbol{x}_t) d\boldsymbol{W}_t \ , \tag{12}$$

where $d\boldsymbol{W}_t$ is a Wiener process. A Wiener process, $\boldsymbol{W}_t$, is a continuous random process such that $\boldsymbol{W}_0 = 0$ and its increments, $\boldsymbol{W}_{t+u} - \boldsymbol{W}_t$, are Gaussian and independent, that is, they are independent of past values of $\boldsymbol{W}_s, s < t$ and they follow a Gaussian distribution with mean 0 and variance u.

For simplification, on the remainder of this section the index t indicates a dependency on the time and state of the system, i.e. $\boldsymbol{u}_t = \boldsymbol{u}(\boldsymbol{x}_t, t)$. Our objective is to optimize a given predefined cost function of the trajectory T_i of the system that at time t_i is in the state $\boldsymbol{x}_{t_i}$ and ends at time t_N in the state $\boldsymbol{x}_{t_N}$. This cost function is defined as:

$$R(T_i) = \phi_{t_N} + \int_{t_i}^{t_N} r_t dt \ , \tag{13}$$

where $\phi_{t_N} = \phi(\boldsymbol{x}_{t_N})$ represents a final cost/reward at time t_N and where r_t denotes the instantaneous cost at time t.

The idea of stochastic optimal control is to minimize the above cost function through the control variables $\boldsymbol{u}_t$. We define the value function as the minimum of the expected value of the cost function:

$$V(\boldsymbol{x}_{t_i}) = V_{t_i} = \min_{\boldsymbol{u}_{t_i:t_N}} E_{T_i}[R(T_i)] \ , \tag{14}$$

where this expected value is taken over all trajectories that start at $\boldsymbol{x}_{t_i}$.

It can be shown that the value function satisfies the so called Hamilton-Jacobi-Bellman equation (HJB)[5,18]:

$$-\partial_t V = \min_{\boldsymbol{u}} \left[r_t + \left(\partial_{\boldsymbol{x}} V\right)^T \boldsymbol{F}_t + \tfrac{1}{2} \operatorname{tr}\left(\left(\partial^2_{\boldsymbol{xx}} V\right)\left(\boldsymbol{G}_t \boldsymbol{\Sigma}_t \boldsymbol{G}_t^T\right)\right)\right] \ , \tag{15}$$

where we have the boundary terminal condition $V(t_N) = E\left[\phi(t_N)\right]$, $\partial_{\boldsymbol{x}} V$ is the gradient vector of V, $\partial^2_{xx} V$ is the Hessian matrix of V and $\boldsymbol{F}_t = \boldsymbol{f}(t, \boldsymbol{x}_t) + \boldsymbol{G}(\boldsymbol{x}_t)\boldsymbol{u}_t$.

As we will see, from the HJB equation (Eq. (15)), it is possible to derive an equation similar to the Schrödinger equation for a given class of controlled systems. For a given Hamiltonian the solution of the Schrödinger equation for a Gaussian initial states can be found, in the semiclassical approximation, by solving a system of equations which we present in the next section, which yield the motion for the center and the metric associated with the Gaussian state.

2 The semiclassical approximation for non-Hermitian Hamiltonians

In their paper on the evolution of wave packets[7] (described by Gaussian coherent states), Graefe and Schubert derive a system of coupled equations for the time evolution of the Wigner function of a NHH given by $\hat{\mathcal{H}} = \hat{H} - i\hat{\Gamma}$. They start by inserting a Gaussian ansatz for the time dependent Wigner function:

$$W(t, y) = \frac{\alpha(t)}{(\hbar\pi)^n} \exp\left(-\frac{1}{\hbar}(y - Y(t)) \cdot G(t)(y - Y(t))\right). \tag{16}$$

Arriving at the following system of equations for the time dependent parameters $\alpha(t), G(t), Y(t)$:

$$\begin{aligned} \dot{Y} &= \Omega \nabla H(Y) - G^{-1} \nabla \Gamma(Y) \\ \dot{G} &= H''(Y)\Omega G - G\Omega H''(Y) + \Gamma''(Y) - G\Gamma''_\Omega(Y)G, \\ \frac{\dot{\alpha}}{\alpha} &= -\frac{2}{\hbar}\Gamma(Y) - \frac{1}{2}\operatorname{Tr}\left[\Gamma''_\Omega(Y)G\right] \end{aligned} \tag{17}$$

where Ω is the usual symplectic matrix as defined in Eq. (4), the vector $Y(t)$ is the center of the Wigner wave packet, the metric $G(t)$ which is obtained from the complex matrix $B(t)$ (see Eq. (10)), $\Gamma''_\Omega := \Omega^t \Gamma'' \Omega$ and the notation A'' denotes the Hessian matrix of $A(y)$ at y.

Upon further inspection of Eqs. (17), we can see that, for the evolution of the center of the Wigner state, we have a regular Hamiltonian flow for the Hermitian part of the Hamiltonian but with an added contribution from the non-Hermitian part. This contribution from the non-Hermitian part is a gradient flow into the minima of Γ[7]. This implies that, were we to choose Γ with no minima, for example $\Gamma = -q^2 - p^2$, then the anti-Hermitian term would drive the motion of the localized state away from the origin and exponentially so. As for the metric, we see that its evolution equation depends on the center $Y(t)$, making this equation coupled if the Hamiltonian is not quadratic. Lastly, for the coefficient $\alpha(t)$, we can see that its magnitude will in general no longer be unitary and this corresponds either to an increase or decrease in the overall probability due to the non-Hermitian term of the NHH. This is further corroborated by the fact that the dynamics of $\alpha(t)$ only depend on Γ and has no dependence on the Hermitian term of $\mathcal{H}$.

Notice that the system of equations in Eqs. (17) is highly coupled and this makes finding its analytical solution difficult except for trivial cases. However, noticing that this evolution takes place in a Kähler manifold, we can rewrite the first dynamical equation for the center of the wave packet using Eq. (6):

$$\dot{Y}(t) = \Omega\nabla H(Y) - G^{-1}\nabla\Gamma(Y) = X_H(Y) - J(t)X_\Gamma(Y). \tag{18}$$

Notice that J has to depend on time because the metric depends on time and the symplectic structure does not. We see from Eq. (18) that, if the Hamiltonian was Hermitian, we would have obtained the classical Hamilton equations as the classical limit for the evolution of these quantum states. However, in the presence of a non-Hermitian term we can see that this evolution no longer corresponds to the flow of a simple Hamiltonian vector field. Instead, the evolution has a component which is mediated by the metric/complex structure. We note that this dependence on the metric for the evolution of the center is unprecedented in unitary quantum mechanics and is an effect only observable when studying NHH.

One important simplification is achieved by decoupling in Eq. (17) the equation for the center from the equation for the metric. One rewrites the above equation in terms of functions z that are $J(t)$ holomorphic[14], and for those functions z we have that:

$$\dot{z} = X_H(z) - J(t)X_\Gamma(z) = X_H(z) - iX_\Gamma(z) = X_{\mathcal{H}}(z). \tag{19}$$

That is, a equation solely dependent on the time dependent function z. We define ϕ_t from $Y_t = \phi_t(Y_0)$, where $Y(t)$ is the solution of Eq. (18). Then, solving equation (19) would allow one to obtain the pullback $\phi_t^*(z_0)$ that defines the time evolution for the center $Y(t)$. The pullback $\phi_t^*(z_0)$ is defined from:

$$z_t = z_0 \circ \phi_t = \phi_t^*(z_0)\ . \tag{20}$$

Then, using $\phi_t^*(z_0)$ one can calculate the time-dependent complex structure and thus the metric, $G(t)$. Doing so, we are able to obtain the Wigner function $W(p,q,t)$ except for the factor $\alpha(t)$ which can be found by solving the last equation in Eq. (17). In the next subsection 2.1, we go into more detail on how to obtain both the motion of the center as well as the motion of the complex structure, in the two dimensional phase space.

2.1 Alternative method of decoupling the equations for the center from the equations for the intrinsic geometry of the state

One option of solving the system of Eqs. (17) is by directly solving all the coupled equations for the intrinsic metric and the center of the Wigner function. Another option, which we propose, is to make use of the Kähler structure underlying the system to rewrite the equation for the evolution of the center into Eq. (18).

It follows from[11,14] that a solution to the coupled system of equations for the center and the intrinsic geometry of the Gaussian coherent states can be found as follows: First, we calculate the flow in $\mathbb{C}^2$ of the complex vector field $-X_{\mathcal{H}} : (p_t, q_t)$. Then, we find the unique, if it exists, diffeomorphism $\tilde{\varphi}_t^{-X_{\mathcal{H}}} : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$\left(\tilde{\varphi}_t^{-X_{\mathcal{H}}}\right)^* (z_0(p,q)) = e^{-tX_{\mathcal{H}}} z_0(p,q) = z_t(p,q) = z_0(p_t,q_t)\ , \tag{21}$$

where z_0 denotes the global complex coordinate defining J_0. After that, the solution of Eqs. (17) is given by $\varphi_t = \left(\tilde{\varphi}_t^{-X_{\mathcal{H}}}\right)^{-1}$, i.e.:

$$Y_t = \varphi_t^*\left(Y_0\right) \tag{22}$$

$$G_t = \Omega\left(\varphi_t^{-1}\right)^*\left(J_0\right)\ . \tag{23}$$

The intuition behind this method is that we evolve the state using the complex vector field $-X_{\mathcal{H}}$. However, for the centers of the state, the opposite is true i.e., they should evolve not with $-X_{\mathcal{H}}$ but in the opposite direction of the evolution of the state, which corresponds to $X_{\mathcal{H}}$, as seen in Eq. (19).

We remark that, for real Hamiltonians we have:

$$\left(\varphi_t^{-X_{\mathcal{H}}}\right)^{-1} = \varphi_t^{X_{\mathcal{H}}}\ , \tag{24}$$

however, for complex Hamiltonians the same cannot be said. Arguments from geometric quantization[11,14] seem to indicate that, for non quadratic Hamiltonians, the equations for the motion of the center as described by this methodology might give a better approximation than the ones described in Eqs. (17).

2.2 From holomorphic coordinates to J and G

To apply this method we need to choose an appropriate complex coordinate such that we reobtain the initial complex structure and, consequently, the initial metric. We know that, if we were to choose the complex $J-$holomorphic coordinate $z = x + iy$ then, in the (x,y) basis, the complex structure is:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ . \tag{25}$$

The method to calculate J is by noticing that, from the definition of z and $\bar{z}$, we have:

$$\begin{cases} dz = dx + idy \\ d\bar{z} = dx - idy \end{cases} \Leftrightarrow \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} = M \begin{pmatrix} dx \\ dy \end{pmatrix}\ , \tag{26}$$

where M is the matrix encapsulating the transformation from $dz, d\bar{z}$ to dx, dy. And so, we can expand the definition of J:

$$J = i\frac{\partial}{\partial z} \otimes dz - i\frac{\partial}{\partial \bar{z}} \otimes d\bar{z}\ , \tag{27}$$

to obtain the standard complex structure. To do that, we note that:

$$\begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} = M \begin{pmatrix} dx \\ dy \end{pmatrix} \Leftrightarrow \begin{pmatrix} dx \\ dy \end{pmatrix} = M^{-1} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} \ . \tag{28}$$

However, in order to obtain the expressions for $\partial_z, \partial_{\bar{z}}$ we need to consider a test function, $F(x,y) = U(x,y) + iV(x,y)$. The differential of this test function is given by:

$$\begin{aligned} dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dz\left(M_{11}^{-1}\frac{\partial F}{\partial x} + M_{21}^{-1}\frac{\partial F}{\partial y}\right) + \\ + d\bar{z}\left(M_{12}^{-1}\frac{\partial F}{\partial x} + M_{22}^{-1}\frac{\partial F}{\partial y}\right) := \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial \bar{z}}d\bar{z} \ , \end{aligned} \tag{29}$$

and so we have that:

$$\begin{pmatrix} \partial_z \\ \partial_{\bar{z}} \end{pmatrix} = \left(M^{-1}\right)^T \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \ . \tag{30}$$

Using the above we can now, for example, see how J is written in the basis (p,q) when we have the complex coordinate $z = p + (-a - ib)q$. Substituting Eq. (28) and Eq. (30) into Eq. (27) thus gives:

$$J = \begin{pmatrix} -\frac{a}{b} & \frac{a^2+b^2}{b} \\ -\frac{1}{b} & \frac{a}{b} \end{pmatrix} \implies G = \begin{pmatrix} \frac{1}{b} & -\frac{a}{b} \\ -\frac{a}{b} & b + \frac{a^2}{b} \end{pmatrix} \ , \tag{31}$$

where in the last step we used Eq. (10). Notice that we obtain precisely the form for the metric matrix as in Eq. (6). With this calculation we are able to pass from a given complex structure to its corresponding complex coordinate and vice-versa:

$$z = p + (-a - ib)q \Leftrightarrow G = \begin{pmatrix} \frac{1}{b} & -\frac{a}{b} \\ -\frac{a}{b} & b + \frac{a^2}{b} \end{pmatrix} \ , \tag{32}$$

in the basis (p,q).

We remark that scaling the entire complex coordinate by a real factor does not affect the resulting metric nor the complex structure. Moreover, a global scale by a complex number also does not change the inherent complex structure. This in turn means that we can either start with a complex coordinate of the form $z = p + (-a - ib)q$ or with $w = -\frac{1}{a+ib}p + q$ as they correspond to the same metric and complex structure. This remark may be helpful in certain examples where a given choice of initial complex coordinate might simplify the process of obtaining the solution.

3 Connecting Stochastic Optimal Control with non-Hermitian Hamiltonians

3.1 Deriving a Schrödinger-like equation from the HJB equation

Following [19], we consider an instantaneous cost of the form:

$$r_t = q(\boldsymbol{x}_t, t) + \frac{1}{2}\boldsymbol{u}_t^T \boldsymbol{R}\boldsymbol{u}_t \ , \tag{33}$$

which has a state dependent part and a quadratic term on the control and where $\boldsymbol{R}$ is a positive definite matrix.

Substituting this cost function in (15), taking the gradient of the part we want to minimize with respect to $\boldsymbol{u}_t$ and setting it to zero enables us to get an expression for the optimal control:

$$\boldsymbol{u}(\boldsymbol{x}_t) = -\boldsymbol{R}^{-1}\boldsymbol{G}_t^T\left(\partial_{\boldsymbol{x}}V\right) \ . \tag{34}$$

Moreover we can substitute the expression for the optimal control in the HJB equation (15) to get:

$$\begin{aligned} -\partial_t V = & q_t - \frac{1}{2}\left(\partial_{\boldsymbol{x}}V\right)^T \boldsymbol{G}_t\boldsymbol{R}^{-1}\boldsymbol{G}_t^T\left(\partial_{\boldsymbol{x}}V\right) + \left(\partial_{\boldsymbol{x}}V\right)^T \boldsymbol{f}_t + \\ & + \frac{1}{2}\operatorname{tr}\left(\left(\partial_{\boldsymbol{x}\boldsymbol{x}}^2 V\right)\left(\boldsymbol{G}_t\boldsymbol{\Sigma}_t\boldsymbol{G}_t^T\right)\right) \ , \end{aligned} \tag{35}$$

which is a second order nonlinear partial differential equation.

By making a change of dependent variable [19]:

$$V_t = -\lambda \log \psi_t \ , \tag{36}$$

as well as using the assumption that $\lambda\boldsymbol{R}^{-1} = \boldsymbol{\Sigma}$ we obtain the following linear equation for ψ:

$$-\partial_t\psi = \tfrac{1}{2}\operatorname{Tr}\left(\partial_{\boldsymbol{x}\boldsymbol{x}}^2(\psi_t)\,\boldsymbol{G}_t\Sigma\boldsymbol{G}_t^T\right) + \boldsymbol{f}_t^T\left(\nabla_x\psi_t\right) - \tfrac{1}{\lambda}q_t\psi_t \ , \tag{37}$$

with the terminal boundary condition $\psi_{t_N} = \exp\left(-\phi(t_N)/\lambda\right)$. The assumption used is the statement that a high variance control input implies cheap control cost whereas small variance control yields high control cost [19]. By relabeling the parameter $\lambda \to \hbar$ and multiplying (37) by $\hbar$ we can see that the above equation has a Schrödinger-like form for a real wave function with a non-Hermitian Hamiltonian associated with it:

$$-\hbar\partial_t\psi = \tfrac{1}{2}\hbar^2\operatorname{Tr}\left(\partial_{\boldsymbol{x}\boldsymbol{x}}^2(\psi_t)\,\boldsymbol{G}_t\boldsymbol{R}^{-1}\boldsymbol{G}_t^T\right) + \boldsymbol{f}_t^T\hbar\left(\nabla_x\psi_t\right) - q_t\psi_t \ . \tag{38}$$

In the next subsection, we identify the Hamiltonian corresponding to Eq. (38) and we study several examples of controlled systems under the NHH formalism.

3.2 Deriving a Schrödinger-like equation from the HJB equation - 1D case

In one dimension, for both the state and the control, equation (38) reads:

$$-\hbar\frac{\partial}{\partial t}\psi = \frac{1}{2}\hbar^2 A_t\Delta\left(\psi_t\right) + \hbar f_t\left(\frac{\partial}{\partial x}\psi_t\right) - q(x)\psi_t \ , \tag{39}$$

where $A_t = G_t R^{-1} G_t$ and $\Delta = \frac{\partial^2}{\partial x^2}$. We remark that, in this section, the index in order to time means that a variable may depend on the state x_t and/or explicitly on time. The above equation can be rewritten with the change $t \to -s$ as:

$$\hbar\frac{\partial}{\partial s}\psi_s = \hbar^2 A(x_s)\frac{\Delta}{2}\psi_s + \hbar f(x_s)\left(\frac{\partial}{\partial x}\psi_s\right) - q(x_s)\psi_s \ , \tag{40}$$

implying that the wave function at an instant s is given by:

$$\psi_s = \exp\left[\frac{s}{\hbar}\left(\hbar^2\frac{A(x)\Delta}{2} + \hbar f_s\partial_x - q(x)\right)\right]\psi_0 \tag{41}$$

We can identify a non-Hermitian Hamiltonian in the above expression by rewriting it as:

$$\psi_s = \exp\left[-\frac{i}{\hbar}s\left(i\hbar f\partial_x + i\left(\hbar^2 A(x)\frac{\Delta}{2} - q(x)\right)\right)\right]\psi_0 \tag{42}$$

Thus the non-Hermitian Hamiltonian is given by:

$$\hat{\mathcal{H}} = -\left(f(\hat{x})\hat{p}\right) - i\left(\frac{A(\hat{x})(\hat{p})^2}{2} + q(\hat{x})\right) \ , \tag{43}$$

in the next section we will see how to obtain a classical Hamiltonian from the above NHH.

3.3 Obtaining the classical Hamiltonian

To be able to apply the formalism presented in the previous section, we must be able to obtain the classical Hamiltonian from the quantum one to solve the semi-classical dynamics. In order to do that, we will assume that the quantization of a classical observable is given by the symmetric quantization scheme i.e., we assume that the quantization of a classical function $f(x)g(p)$ is $f(x)g(p) \to \frac{1}{2}\left(f(\hat{x})g(\hat{p}) + g(\hat{p})f(\hat{x})\right)$. Under this choice of quantization we get that functions of solely the position or momentum variable correspond to operators that solely depend on the position or momentum operator.

In the Schrödinger representation, we assume that the quantization of the function $F = f(x)p$ is thus given by the symmetric quantization, i.e.:

$$\hat{F} = -\frac{1}{2}f(x)i\hbar\partial_x - \frac{i\hbar}{2}\partial_x \circ f(x) = f(\hat{x})\hat{p} - \frac{i\hbar}{2}f'(\hat{x}) \tag{44}$$

Thus the quantization of the classical observable $G = f(x)p + \frac{i\hbar}{2}f'(x)$ is given by:

$$\hat{G} = f(\hat{x})\hat{p} \tag{45}$$

It remains to be shown which classical observable gives us the operator $\widehat{A(x)p^2}$. We note that the quantization of the classical observable $f(x)p^2$ under our choice of quantization scheme is:

$$\begin{aligned}(\hat{f(x)p^2}) &= -\frac{1}{2}\hbar^2 f(x)\partial_x^2 - \frac{1}{2}\hbar^2\partial_x^2 \circ f(x) = \\ &= f(\hat{x})\hat{p}^2 - i\hbar f'(\hat{x})\hat{p} - \frac{1}{2}\hbar^2 f''(\hat{x})\end{aligned} \tag{46}$$

Thus joining the previous result it follows that the quantization of the following classical observable is:

$$O = f(x)p^2 + i\hbar f'(x)p \implies \hat{O} = f(\hat{x})\hat{p}^2 \tag{47}$$

And so our quantum Hamiltonian arises from the quantization of the classical observable:

$$\mathcal{H} = -f(x)p - \frac{i\hbar}{2}f'(x) - \frac{i}{2}\left(A(x)p^2 + i\hbar A'(x)p\right) - iq(x) \ , \tag{48}$$

from where we identify:

$$\begin{cases} H = \left(\hbar\frac{A'(x)}{2} - f(x)\right)p \\ \Gamma = \frac{A(x)(p)^2}{2} + \left(\hbar\frac{f'(x)}{2} + q(x)\right) \end{cases} \tag{49}$$

The evolution of the above system of equations can be thus evaluated using the formalism developed in the previous sections. In the next section we explore a simple example, with a complex quadratic Hamiltonian.

4 Example - Quadratic Hamiltonian

In this section we will work with following stochastic differential equation:

$$\dot{x} = -\alpha x + (u_t + \xi) \ , \tag{50}$$

where ξ has variance δ so that we have $A_t = \delta$. The parameter α controls the linear drift and the parameter δ controls the noise. We can think of Eq. (50) as an exponentially decaying system that can be influenced via the control u_t and its associated noise ξ.

Moreover, we also take as a cost function a quadratic function of the following form:

$$r_t = q(x) + \frac{1}{2}Ru^2 = \frac{\beta}{2}x^2 + \frac{1}{2}\frac{\hbar}{\delta}u^2 \tag{51}$$

With the above assumptions, and following Eq. (49), we get the following classical Hamiltonian function:

$$\begin{cases} H = \alpha x p \\ \Gamma = \frac{\delta}{2}p^2 + \frac{\beta}{2}x^2 - \frac{\alpha\hbar}{2} \end{cases} \ . \tag{52}$$

Looking at Eq. (36), if the value function at $t = 0$ is quadratic then we must have an initial state that is localized and purely real, i.e., a Gaussian which is centered at a point $Y = (0, x_0)$:

$$\psi_s = \left(\frac{b}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{b}{2\hbar}(x - x_0)^2\right] \ . \tag{53}$$

Furthermore, by evolving this Wigner function using Eqs. (17), we are able to obtain the evolution for the value function as well, which will be given by the logarithm of our wave function.

Looking at the Hamiltonian in Eqs. (52) and at the second equation in (17), assuming an initial diagonal metric of the form:

$$G(t=0) = \begin{pmatrix} g_1(t=0) & 0 \\ 0 & g_2(t=0) \end{pmatrix} \ ,$$

then we obtain the following equation for the metric components:

$$\begin{aligned} \dot{g_1}(t) &= \delta + 2\alpha g_1(t) - \beta g_1(t)^2 \ , \\ \dot{g_2}(t) &= \beta - 2\alpha g_2(t) - \delta g_2(t)^2 \ . \end{aligned}$$

One can then verify that if we define $g_2(t) = 1/g_1(t)$, we obtain the second equation from the first one and vice-versa. This implies that if the initial conditions for the metric are such that its determinant is equal to one, then, for all time t, the same is true, and the components are inverse of each other. This also implies that we only need to solve one differential equation to get the complete time evolution of the metric.

The above remark is valid only in the case of quadratic Hamiltonians of the form of (52). For example, if we had a real component quadratic in position, then we would not only lose the above property but metric would no longer be diagonal for a generic time t .

For the rest of this section we will analyze the Hamiltonian in (52) in three different cases:

- First the case $\alpha = \beta = 0$ with $\delta \neq 0$;
- Secondly, the case $\alpha = 0$ and $\beta \neq 0, \delta \neq 0$;
- Lastly, the case where $\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$,

and for the last case we shall see what is the role of the several parameters on the behavior of the metric and position of the center of the wave packet. To avoid confusion with the parameter α of the Hamiltonian, for the rest of this section we will denote the multiplicative factor, described by the third equation in Eqs. (17), by $N(t)$.

4.1 Quadratic only in momentum

We consider the system of (52) where we put $\alpha = \beta = 0$ and $\delta \neq 0$. Thus, our classical Hamiltonian is given by:

$$\begin{cases} H = 0 \\ \Gamma = \frac{\delta p^2}{2} \end{cases} \tag{54}$$

This is a purely imaginary quadratic Hamiltonian that only depends on momentum. We take an initial Gaussian state which is centered at a point $Y = (0, x_0)$:

$$\psi_s = \left(\frac{b}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{b}{2\hbar}(x - x_0)^2\right] , \tag{55}$$

from where we see that $B = ib \in i\mathbb{R}$. Thus, the Wigner function for this wave function has initial center at the origin $Y = (0, x_0)$ and with $G = \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & b \end{pmatrix}$.

In this case the equation for G is independent of Y and is an equation of the following form:

$$\dot{G} = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} - G \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix} G , \tag{56}$$

where, as the Hamiltonian is quadratic, this implies that one of the components of the metric is the inverse of the other. It thus suffices to calculate the momentum related component:

$$\dot{g}(t) = \delta \implies \delta t + \frac{1}{b} , \tag{57}$$

where we already took into account the initial conditions for the metric. This result implies that the metric is given by:

$$G(t) = \begin{pmatrix} \delta t + \frac{1}{b} & 0 \\ 0 & \left(\delta t + \frac{1}{b}\right)^{-1} \end{pmatrix} . \tag{58}$$

Furthermore, the center's equation of motion is given by:

$$\dot{Y} = -G(t)^{-1} \nabla\Gamma(Y) , \tag{59}$$

which can be written as:

$$\frac{\dot{p}}{p} = -\frac{1}{t + \frac{1}{\delta b}} \tag{60}$$

$$\dot{x} = 0 . \tag{61}$$

Integrating the above equations thus give us:

$$p(t) = \frac{p(0)}{1 + \delta b t} \tag{62}$$

$$x(t) = x(0) , \tag{63}$$

where the initial conditions for the position and momentum are taken into account. As we have that the initial center is $Y = (0, x_0)$, we see that the center will not move, remaining constant for all time.

Moreover, for large time we observe that the metric tends to the matrix:

$$G_\infty = \begin{pmatrix} +\infty & 0 \\ 0 & 0 \end{pmatrix} . \tag{64}$$

Furthermore, knowing the evolution of the metric and the center it is thus possible to calculate the evolution of the multiplicative factor of the wave packet using (17). In this case it is of the form:

$$N(t) = \frac{N_0}{\sqrt{b\delta t + 1}} , \tag{65}$$

where N_0 is the initial multiplicative factor (normalization constant).

4.2 Quadratic in position and momentum

We now consider the system of (52) in the case where $\alpha = 0$ and $\beta \neq 0, \delta \neq 0$. Thus, our classical Hamiltonian is given by:

$$\begin{cases} H = 0 \\ \Gamma = \frac{\delta p^2}{2} + \frac{\beta x^2}{2} \end{cases} \tag{66}$$

Again, this is a purely imaginary Hamiltonian and its dynamics can be seen as the purely imaginary time evolution of a regular Hermitian Hamiltonian. We take as the initial state the following Gaussian which is centered at a given point:

$$\psi_s = \left(\frac{b}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{b}{2\hbar}(x - x_0)^2\right] , \tag{67}$$

from where we see that $B = ib \in i\mathbb{R}$. The Wigner function for this wave function has initial center at the point $Y = (0, x_0)$ and with $G = \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & b \end{pmatrix}$.

In this case the equation for G is independent of Y and is a Ricatti equation:

$$\dot{G} = \begin{pmatrix} \delta & 0 \\ 0 & \beta \end{pmatrix} - G \begin{pmatrix} \beta & 0 \\ 0 & \delta \end{pmatrix} G \ . \tag{68}$$

Once more it suffices to solve the differential equation for the momentum component of the metric:

$$\dot{g}(t) = \delta - \beta g(t)^2 \ , \tag{69}$$

whose solution is:

$$g(t) = \frac{b\delta \sinh\left(t\sqrt{\beta\delta}\right) + \sqrt{\beta\delta}\cosh\left(t\sqrt{\beta\delta}\right)}{b\sqrt{\beta\delta}\cosh\left(t\sqrt{\beta\delta}\right) + \beta\sinh\left(t\sqrt{\beta\delta}\right)} \ , \tag{70}$$

where the initial conditions were already taken into account.

The metric is thus given by:

$$G(t) = \begin{pmatrix} g(t) & 0 \\ 0 & (g(t))^{-1} \end{pmatrix} \ . \tag{71}$$

Furthermore, the center's equation of motion is given by:

$$\frac{\dot{p}}{p} = -\delta\,(g(t))^{-1} \tag{72}$$

$$\frac{\dot{x}}{x} = -\beta g(t) \tag{73}$$

Integrating the above equations and using the initial condition that $Y = (0, x_0)$ thus give us:

$$p(t) = 0 \tag{74}$$

$$x(t) = \frac{bx_0\sqrt{\beta\delta}}{b\sqrt{\beta\delta}\cosh\left(t\sqrt{\beta\delta}\right) + \beta\sinh\left(t\sqrt{\beta\delta}\right)}, \tag{75}$$

For large time we observe that the center Y tends to the origin $Y_\infty = (0, 0)$. Moreover, the metric tends to the matrix:

$$G_\infty = \begin{pmatrix} \sqrt{\frac{\delta}{\beta}} & 0 \\ 0 & \sqrt{\frac{\beta}{\delta}} \end{pmatrix} \tag{76}$$

4.3 Quadratic with linear mixed term

We now consider the system of (52) in the case where $\alpha = 0$ and $\beta \neq 0, \delta \neq 0$. Thus, our classical Hamiltonian is given by:

$$\begin{cases} H = \alpha xp \\ \Gamma = \frac{\delta p^2}{2} + \frac{\beta x^2}{2} - \frac{\alpha\hbar}{2} \end{cases} \tag{77}$$

Where the Hamiltonian is now no longer purely imaginary being instead complex. We again take as the initial state the following Gaussian:

$$\psi_s = \left(\frac{b}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{b}{2\hbar}(x - x_0)^2\right] \ , \tag{78}$$

from where we see that $B = ib \in i\mathbb{R}$. The Wigner function for this wave function has initial center at the origin $Y = (0, x_0)$ and with $G = \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & b \end{pmatrix}$.

Let us now solve this problem using the method described in subsection 2.1 instead of directly solving Eqs. (17). We have that:

$$-X_{\mathcal{H}} = (\alpha p - i\beta x)\frac{\partial}{\partial p} + (i\delta p - \alpha x)\frac{\partial}{\partial x} \ . \tag{79}$$

And so, we have:

$$\begin{cases} p_t = e^{-X_{\mathcal{H}}t}p \implies \dot{p}_t = e^{-X_{\mathcal{H}}t}\left(-X_{\mathcal{H}}p\right) = \alpha p_t - i\beta x_t \\ x_t = e^{-X_{\mathcal{H}}t}x \implies \dot{x}_t = e^{-X_{\mathcal{H}}t}\left(-X_{\mathcal{H}}x\right) = -\alpha x_t + i\delta p_t \end{cases} , \tag{80}$$

which can be written in matrix form as:

$$\begin{pmatrix} \dot{p}_t \\ \dot{x}_t \end{pmatrix} = \begin{pmatrix} \alpha & -i\beta \\ i\delta & -\alpha \end{pmatrix} \begin{pmatrix} p_t \\ x_t \end{pmatrix} \ . \tag{81}$$

This is a matrix differential equation of the form $\dot{\boldsymbol{v}}_t = M\boldsymbol{v}_t$. The solution is thus given by the matrix exponential, $\boldsymbol{v}_t = e^{Mt}\boldsymbol{v}_0$. The matrix exponential can be obtained by diagonalizing the matrix $M = UDU^{-1}$ and calculating $e^{Mt} = Ue^{Dt}U^{-1}$. In this case, the matrix has eigenvalues:

$$\lambda_\pm = \pm\sqrt{\alpha^2 + \beta\delta} \ , \tag{82}$$

and so, denoting by λ the positive eigenvalue, we obtain the following matrix exponential:

$$e^{Mt} = \begin{pmatrix} \cosh(\lambda t) + \frac{a}{\lambda}\sinh(\lambda t) & -i\frac{b}{\lambda}\sinh(\lambda t) \\ i\frac{d}{\lambda}\sinh(\lambda t) & \cosh(\lambda t) - \frac{a}{\lambda}\sinh(\lambda t) \end{pmatrix} . \tag{83}$$

Taking into account the starting metric $G = \mathrm{diag}(\frac{1}{b}, b)$ we have the following initial holomorphic coordinate:

$$z_0 = p - ibx \ , \tag{84}$$

and so applying the flow of the Hamiltonian vector field we get:

$$z_t = p\left(\cosh(\lambda t) + \tfrac{\alpha+b\delta}{\lambda}\sinh(\lambda t)\right) - ibx\left(\cosh(\lambda t) + \tfrac{\beta/b-\alpha}{\lambda}\sinh(\lambda t)\right) \ , \tag{85}$$

which, when we note that $z_t := p_t - ibx_t$, gives:

$$\tilde{\varphi}_t^{-X_{\mathcal{H}}} = \begin{pmatrix} \cosh(\lambda t) + \frac{\alpha+b\delta}{\lambda}\sinh(\lambda t) & 0 \\ 0 & \cosh(\lambda t) + \frac{\beta/b-\alpha}{\lambda}\sinh(\lambda t) \end{pmatrix} . \tag{86}$$

From the inverse of the above we thus conclude that the center motion is given by:

$$\begin{cases} P(t) = \frac{P(0)}{\cosh(\lambda t) + \frac{\alpha+b\delta}{\lambda}\sinh(\lambda t)} = 0 \\ X(t) = \frac{X(0)}{\cosh(\lambda t) + \frac{\beta/b-\alpha}{\lambda}\sinh(\lambda t)} = \frac{\lambda b X(0)}{\lambda b\cosh(\lambda t) + (\beta - b\alpha)\sinh(\lambda t)} \end{cases} \tag{87}$$

Finally, we can obtain the metric by first obtaining the complex structure directly from Eq. (85). And so, as mentioned in subsection 2.2, a scaling of the holomorphic does not alter the inherent complex structure. In this case we can divide the entire equation by the real factor of the x coordinate to directly obtain $g(t)$ as the coefficient of the p coordinate. We thus have:

$$g(t) = \frac{\lambda\cosh(\lambda t) + (\alpha + b\delta)\sinh(\lambda t)}{b\lambda\cosh(\lambda t) + (\beta - b\alpha)\sinh(\lambda t)} \ , \tag{88}$$

where $\lambda = \sqrt{\alpha^2 + \beta\delta}$. We note that we can reobtain the previous equations by simply setting any of the α, β, δ to zero.

For large time we observe that the center Y tends to the origin $Y_\infty = (0, 0)$. And for large time the metric tends to the matrix:

$$G_\infty = \begin{pmatrix} \frac{\sqrt{\alpha^2+\beta\delta}+\alpha}{\beta} & 0 \\ 0 & \left(\frac{\sqrt{\alpha^2+\beta\delta}+\alpha}{\beta}\right)^{-1} \end{pmatrix}, \tag{89}$$

where the dependence in b is not present. Thus, in the infinite time limit, the system forgets its initial geometry, which is related to the b parameter.

4.4 Center and metric behavior with change of parameters

In this subsection we shall see the role of the parameters α, β, δ on the time evolution of the momentum component of the metric and of the center's position. In all the below images the darker the coloring of the curve the smaller the value of the parameter being used to plot said curve.

4.4.1 Influence of parameter α

The parameter α is associated with the linear drag on equation (50). In the next image we can see its influence on the momentum component of the metric, where we see that it controls the limit to which this components tends to, while maintaining all the other parameters fixed. The bigger the parameter the larger the limit this metric component tends to.

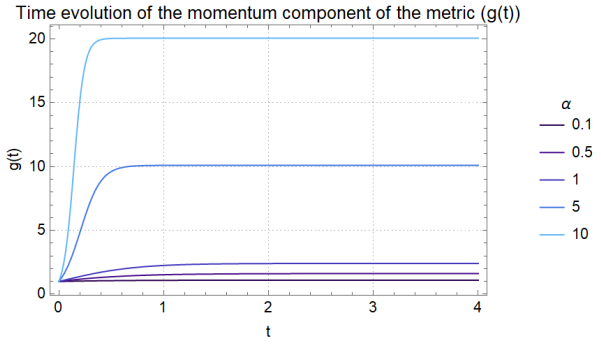
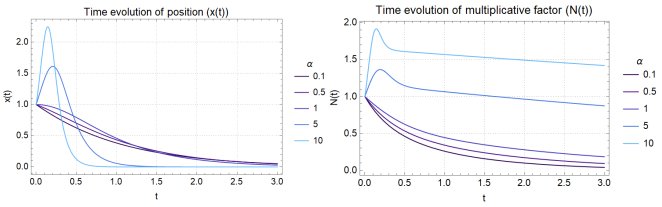


Figure 1: Momentum metric component g(t) for different values of α where $\beta = \delta = b = 1$

Whereas on the center's position, displayed in the next image on the left side, we can see that the parameter α controls the height of the peak of maximum distance from the origin that the center has before tending to the origin. As for the role of the parameter α on the evolution of the multiplicative factor we see that it is similar to the one in the position case, being responsible for a deviation from the origin, for big enough values of α.



(a) Center's position x(t) for different values of α where $\beta = \delta = b = x_0 = 1$

(b) Multiplicative factor N(t) for different values of α where $\beta = \delta = b = x_0 = N_0 = \hbar = 1$

Figure 2: Influence of the parameter α on the time evolution for position of the center of the wave packet (left image) and for the multiplicative factor (right image)

4.4.2 Influence of parameter β

The parameter β is associated with the quadratic term on the position, that appears due to our choice of the cost function in (51). In the next image we can see its influence on the momentum component of the metric, where we see that it controls the limit to which this components tends to. This limit is smaller for large values of this β.

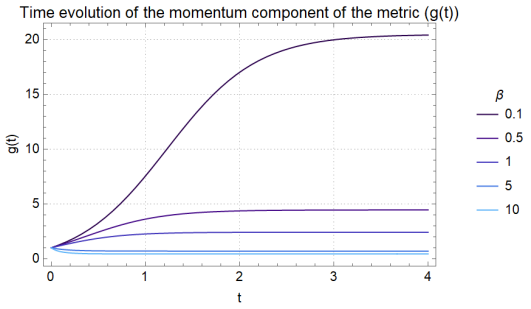
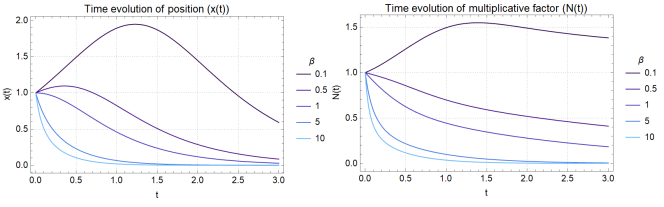


Figure 3: Momentum metric component g(t) for different values of β where $\alpha = \delta = b = 1$

Whereas on the center's position, displayed in the next image on the left side, we can see that the parameter β also controls the height of the peak of maximum distance from the origin that the center has before tending to the origin but in a manner opposite to the parameter α. The smaller β is, the bigger this peak becomes. The same can be said for the influence of β on the evolution of the multiplicative factor.



(a) Center's position x(t) for different values of β where $\alpha = \delta = b = x_0 = 1$

(b) Multiplicative factor N(t) for different values of β where $\alpha = \delta = b = x_0 = N_0 = \hbar = 1$

Figure 4: Influence of the parameter β on the time evolution for position of the center of the wave packet (left image) and for the multiplicative factor (right image)

4.4.3 Influence of parameter δ

The parameter δ is associated with the quadratic term on the momentum. It is associated with the stochastic noise in equation (50). In the next image we can see its influence on the momentum component of the metric, where we see that it controls the limit to which this components tends to. This limit is larger for large values of δ although it only grows with the square root of the parameter.

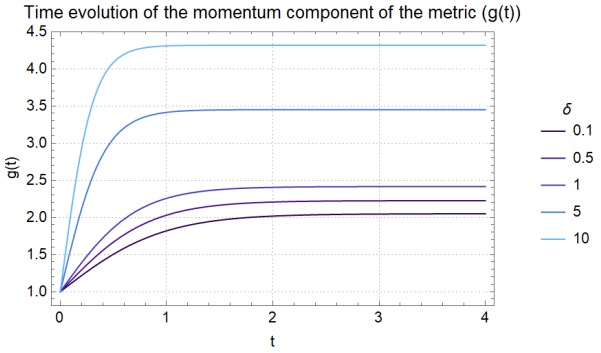
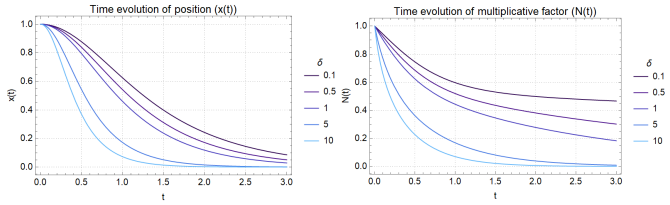


Figure 5: Momentum metric component g(t) for different values of δ where $\alpha = \beta = b = 1$

The role of δ on both the evolution of the center position and of the multiplicative factor is to regulate the decay present on the curve, the larger the value of δ the faster both curves tend to zero, as one can see on the below figure.



(a) Center's position x(t) for different values of δ where $\alpha = \beta = b = x_0 = 1$

(b) Multiplicative factor N(t) for different values of δ where $\alpha = \beta = b = x_0 = N_0 = \hbar = 1$

Figure 6: Influence of the parameter δ on the time evolution for position of the center of the wave packet (left image) and for the multiplicative factor (right image)

5 Conclusions and future work

In this work, we introduced a new method of handling the system of equations (17), which describes the motion of the center, the motion of the intrinsic metric of an initial coherent state and of the multiplicative normalization factor. The proposed method allows one to decouple the equation for the center from the equation for the intrinsic metric. This turns a potentially complicated system of equations into a simpler one, where after solving the complexified motion of the center, we can algebraically extract both the motion of the real center as well as the motion of the intrinsic metric of the state.

We also saw that it is possible to relate the time evolution of the value function for a given class of stochastic control systems to the time evolution with respect to non-Hermitian Hamiltonians. We studied the dynamics of initial Gaussian wave packets, which corresponds to having a given final cost that is a quadratic function of the state.

For the specific case of quadratic Hamiltonians we found the exact solution for the motion of the center and the metric of the wave packet and showed that both tend to finite values in the infinite time limit.

One interesting aspect to be further explored is the fact that the Schrödinger like equation describes the evolution of a wave function that is real valued with respect to a complex valued classical Hamiltonian. Furthermore, more work needs to be done in order to assess the usefulness of the NHH paradigm on the context of stochastic optimal control.

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