Automatic design of reinforced concrete structures: Cross sections subjected to axial force and bi-axial moments

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Abstract

The main focus of this paper is to develop a precise foundation for the design of reinforced concrete structures subjected to both axial force and bi-axial moments. There are many developments around this subject, specially when it comes to seismic actions, however a gap remains between the action and the resistance. With the purpose of diminishing this lacuna, several methods were computed based on an object-oriented programming, in this case programming in *Python*, trying simultaneously integrate various existing concepts and to create a solid foundation for the design of reinforced concrete sections, allowing to decrease some safety factors. Although this is a theoretical problem, even it is studied a lot in the scientific community, there are a few more questions that need to be answered. These might be possible when the increasing computational capabilities is almost exponential. Beyond the main goal of this dissertation, the program has sufficient versatility to treat many other problems, as design or security check of different kinds of stresses for several types of cross sections, returning the results in a numerical or graphical mode, allowing comparisons with existent tables or graphics, still giving the designer his critical sense.

Keywords: Reinforced concrete sections, Resistant capacity surfaces, palavra 3, Design of structures, Axial force and bi-axial moments, Stress-resultant interaction

1. Introduction

The design of a reinforced concrete section submitted to axial force and bi-axial moments is most of the times an additional effort for the designers due to its three-dimensional form. This paper takes this problem into account trying to provide some tools allowing an easier design or safety checks. On the side of the action, the seismic action is one of the inspirations for the paper. Many of the studies carried out contributed to a greater knowledge of the action, of which sets of efforts are submitted to the structure and which the correlations between directions of the earthquake may exist. However, when the main goal is to design the structure, there is an added difficulty in its optimization. Section 2 presents an overview of the developments between the determination of the interaction surfaces of the seismic action and the optimization of the resistant capacity surface of the cross section, reinforcing once again that there is still a gap between both themes. Section 3 presents the methodologies used to answer this question, after presenting the existing cross sections for use in the methods. Section 4 gives the some interaction resistant curves and the results obtained by the previous methods, for a rectangular section. Finally, some conclusions are drawn from the existing methodology.

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2. Related Works

With the technological advances in construction, caused by an exponential population increase, civil engineering has a fundamental role in terms of optimizing competitive solutions from an economic point of view without compromising safety. A relevant subject in this regard that has been studied over time is seismic action. The EC8 [1] recommends the use of response spectra for the design of the structures. However, an analysis that does not take all correlations into account may oversize them.
The first developments on this subject were explored by Gupta [2]. He was able to prove that the interaction diagram for an earthquake using a quadratic combination takes the form of an ellipsoid. Considering that the ellipsoid has a well-known analytical form, it was a great advance for the knowledge of this subject. In practice, it will only be necessary to obtain a resistant diagram of the cross section that completely envelops the ellipsoid. To optimize the section using the minimum amount of reinforcement, this clause will be sufficient, which is not the case in current situations for the design of structures, as already mentioned by EC8, which can lead to an over-design of the structure and, what for a relatively current construction and with lower volumes it may not have a significant preponderance, it will certainly have for complex and/or larger structures. For this reason, it makes sense to approach the topic more thoroughly, especially on the resistance side. Menun and Kiureghian [3] state that in the

design of a structure subject to seismic actions, the simultaneous effects of forces acting on the structure must be considered. The use, in its dimensioning, of the response spectrum, identifies the maximum values of the actions to act separately, without taking into account that they may never occur simultaneously. Menun and Kiureghian state that, for a known orientation of the main axes, the interaction surface has the shape of an ellipsoid. When this orientation is unknown, a supreme envelope is defined, built from the critical orientation of the main axes. By superimposing a resistant surface that completely surrounds the interaction diagram, very effective results are obtained for the optimization of the structure. Exploring previous developments, and in order to correlate seismic action with the cross section resistance in a more expeditious manner, Rosati and colleagues [4] develop several algorithms to check if the seismic action envelope, the ellipsoid, is completely contained in the domain of the resistant section. In order to achieve the ultimate strength of the section without the need to create the entire surface, it is based on a convergence of the stiffness matrix secant, in order to reduce the number of iterations, since one of the main problems is the number of required equations for the efficient creation of an optimized resistance surface. Erlicher [5] reinforces the fact that the design is, on many occasions, based on a conservative response, since the response spectrum provides the maximum value of the action for one direction independently, without taking into account the probability of simultaneous responses in the different directions of the earthquake. In this case, the interaction diagram would correspond to a parallelepiped. Considering that these responses do not usually occur simultaneously, Erlicher believes that the so-called *hyper-ellipsoid*, considering the probabilities of the occurrence of different responses over time, is the way forward. To this end, he formulates equations for the creation of a polyhedron that approaches the interaction surface. It also proposes a new approach to the method of equivalent forces, in which the main problem, he considers, is the difficulty in defining a field of static forces that are representative of non-linear behaviour. Sessa and colleagues [6] continue to develop methods for the design of resistant cross sections, with an approach in which he calls *Seismic critical multiplier*. This concept takes into account new research in the field of seismic actions, with a *Supreme Envelope* [3, 7] with the combination CQC3 [8], which involves the various seismic actions regardless of their angle of occurrence or its changes in time. The objective will be to increase the surface of the envelope, until it finds a point tangent to the resistant surface and it completely surrounds the envelope. This multiplier, already mentioned, is the safety factor of the resistant section in relation to the action performed. An approach to the construction of the interaction diagrams, in [9], studies different modal combinations of the seismic action for its construction. They conclude that the interaction diagram can be a polyhedron, an ellipsoid or something in between depending on the rules adopted. However, this approach does not focus on the resistance of the cross section, emphasizing once again the research around the action.

3. Methods

The behaviour of the cross sections is discretized in a several group of elementary points. Each point receives some fundamental attributes for the creation of the cross section. Supported with the Bernoully hipotheses, the tension in each point can be calculated by (1), being **a** and **b** the vectors defined in (2).

$$\varepsilon = a \cdot b \tag{1}$$

$$a = \begin{bmatrix} \varepsilon_G \\ \chi_y \\ \chi_z \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ z \\ -y \end{bmatrix} \tag{2}$$

The stresses in each point are calculated by (3).

$$d\sigma = E_t d\varepsilon \tag{3}$$

Being $r = \begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix}$, axial force and bi-axial moments are defined by (4),

$$dr = K\, da \tag{4}$$

with **K** in (5), the secant stiffness matrix.

$$K = \begin{bmatrix} \int_A E_t\, dA & \int_A zE_t\, dA & \int_A -yE_t\, dA \\ \int_A zE_t\, dA & \int_A z^2E_t\, dA & \int_A -zyE_t\, dA \\ \int_A -yE_t\, dA & \int_A -zyE_t\, dA & \int_A y^2E_t\, dA \end{bmatrix} \tag{5}$$

To determine the resistance of the cross section to the ultimate limit state, the limit strains imposed by the regulations must be respected, which are generally of the type in (6).

$$\varepsilon_{min} < \varepsilon < \varepsilon_{max} \tag{6}$$

Methods 3 and 4

Methods 1 and 2 were the base of the next methods, so we will make some reference when they need to be called. It is important to note that conceptually, method 3 gives the safety factor of a cross section subjected to some kind of stresses and, on the other hand, method 4 gives the minimum rate of reinforcement that needs to resist to those stresses. In order to demonstrate the equations below, take note that all of this methods (1 to 4) only give one point of the resistant capacity surface.

Assuming that the applied forces are defined by an initial position (0) and a direction (1), a linear function of the type (7) can be considered.

$$N(\lambda) = N^0 + \lambda N^{(1)} \tag{7}$$

$$M_y(\lambda) = M_y^0 + \lambda {M_y}^{(1)} \tag{8}$$

$$M_z(\lambda) = M_z^0 + \lambda {M_z}^{(1)} \tag{9}$$

Be s the acting vector, in (10),

$$s = \begin{bmatrix} N(\lambda) \\ M_y(\lambda) \\ M_z(\lambda) \end{bmatrix} \tag{10}$$

it is possible to compact the equation, for the case of the linear function, in (11).

$$s = s^{(0)} + \lambda s^{(1)} \tag{11}$$

To solve the equation, it is necessary to determine the value of λ in the ultimate limit state and the corresponding strains. Start by considering an initial estimative, for example $\lambda = 1$, $\varepsilon_G = 0$, $\chi_y = 0$ e $\chi_z = 0$. Be a^A the point defined by the estimative. The residual vector [10] will have the form presented in(12) or, defining the vector $r = \begin{bmatrix} N(\varepsilon_G, \chi_y, \chi_z) \\ M_y(\varepsilon_G, \chi_y, \chi_z) \\ M_z(\varepsilon_G, \chi_y, \chi_z) \end{bmatrix}$, the compacted form is written in (13).

$$R^A = \begin{bmatrix} N(\varepsilon_G, \chi_y, \chi_z) - N(\lambda) \\ M_y(\varepsilon_G, \chi_y, \chi_z) - M_y(\lambda) \\ M_z(\varepsilon_G, \chi_y, \chi_z) - M_z(\lambda \end{bmatrix} \tag{12}$$

$$R = r - s \tag{13}$$

It is necessary to find the solution a that satisfies R(a)=0. To use Newton´s method, be (14) its linearised form.

$$R^A + \frac{dR}{da}\Delta a + \frac{dR}{d\lambda}\Delta\lambda ... = 0 \tag{14}$$

Sensibilities are given by the matrix $\frac{dR}{da}$, that depends on the point A where it is applied, and on the fixed vector $\frac{dR}{d\lambda}$, presented in (15),

$$\left[\tfrac{dR}{da}\right] = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \quad \left[\tfrac{dR}{d\lambda}\right] = \begin{bmatrix} -N^{(1)} \\ -{M_y}^{(1)} \\ -{M_z}^{(1)} \end{bmatrix} = -Q \tag{15}$$

so the solution is given in (16),

$$\Delta a = -\left[\tfrac{dR}{da}\right]^{-1} (R^A - \Delta\lambda Q) = a^R + \Delta\lambda a^Q \tag{16}$$

where a^R, in (17), corresponds to the correction that needs to be done for $\lambda = constant$ and a^Q, in (18), corresponding to the correction when λ varies, keeping the error.

$$a^R = -\left[\tfrac{dR}{da}\right]^{-1} R^A = \begin{bmatrix} \varepsilon_G^R \\ \chi_y^R \\ \chi_z^R \end{bmatrix} \tag{17}$$

$$a^Q = \left[\tfrac{dR}{da}\right]^{-1} Q = \begin{bmatrix} \varepsilon_G^Q \\ \chi_y^Q \\ \chi_z^Q \end{bmatrix} \tag{18}$$

The new estimates of a and λ are presented in (19).

$$a^B = a^A + a^R + \Delta\lambda a^Q \qquad \lambda^B = \lambda^A + \Delta\lambda \tag{19}$$

$\Delta\lambda$ is calculated with the limit conditions in (6). For ach point of the section, the condition (20) must be respected.

$$\varepsilon_{min} < b \cdot a < \varepsilon_{max} \tag{20}$$

Replacing the value of the new estimate, it obtains (21).

$$\varepsilon_{min} < b \cdot \left(a^A + a^R + \Delta\lambda a^Q\right) < \varepsilon_{max} \tag{21}$$

If $b \cdot a^Q > 0$, which means increasing strains with λ, the equation (22) is called.

$$\frac{\varepsilon_{min} - b \cdot \left(a^A + a^R\right)}{b \cdot a^Q} < \Delta\lambda < \frac{\varepsilon_{max} - b \cdot \left(a^A + a^R\right)}{b \cdot a^Q} \tag{22}$$

On the other hand, if $b \cdot a^Q < 0$, is (23).

$$\frac{\varepsilon_{max} - b \cdot \left(a^A + a^R\right)}{b \cdot a^Q} < \Delta\lambda < \frac{\varepsilon_{min} - b \cdot \left(a^A + a^R\right)}{b \cdot a^Q} \tag{23}$$

$$a^{(0)} = a^A + a^R \qquad a^{(1)} = a^Q \qquad p = \Delta\lambda \tag{24}$$

The $\Delta\lambda$ to be adopted will be the minimum of the maximum values obtained for each material point. With $\Delta\lambda$ obtained, it is now possible considerar in the iterative process, in (19). The methodology for method 4 was almost the same as in method 3, but this time iterating f_ω instead.

$$\begin{array}{lclcl} N(\varepsilon_G, \chi_y, \chi_z, f_\omega) & = & \int_A \sigma \, dA & = & N^b + f_\omega N^{a1} \\ M_y(\varepsilon_G, \chi_y, \chi_z, f_\omega) & = & \int_A z\sigma \, dA & = & M_y^b + f_\omega M_y^{a1} \\ M_z(\varepsilon_G, \chi_y, \chi_z, f_\omega) & = & -\int_A y\sigma \, dA & = & M_z^b + f_\omega M_z^{a1} \end{array} \tag{25}$$

$$R^A + \frac{dR}{da}\Delta a + \frac{dR}{df_\omega}\Delta f_\omega ... = 0 \tag{26}$$

The previous methods generally determine a point on the resistant surface, through a given direction. It is necessary to find the critical direction of the resistant surface. The following methods discuss the first steps towards achieving it.

In the event that the stresses result from a seismic action analysed by response spectra, it is necessary to obtain its interaction diagram [9]. Be $s^{(0)}$ the solicited stresses of a static analysis for permanent loads and $s_k^{(i)}$ the stresses of the responseof the structure for mode i due to the base component k accordind to modal dynamic analysis. Vetot s has three components, with $m = 3$.
The total dynamic response can be obtained by applying quadratic combination rules, such as *CQC* to combine modes and *SRSS* to combine base components. To consider the interaction between efforts, it is convenient to apply the latter to linear combinations of the type in (27),

$$s_t = s \cdot t \tag{27}$$

being t the vector that contains the linear combination factores.
The dynamic response in direction t in defined in (28),

$$t \cdot s \leq s_t^{(CQC+SRSS)} = \sqrt{\sum_{k=1}^{n_g}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}(t \cdot s_k^{(i)})(t \cdot s_k^{(j)})} \tag{28}$$

where n is the number of modes, n_g the number of base components, ρ_{ij} the modal correlation factors defined by CQC rule which depend fundamentally of the relation between the frequencies of each mode.
Evidencing t, the equation can be written as in (29).

$$s_t^{(CQC+SRSS)} = \sqrt{t \cdot \left(\sum_{k=1}^{n_g}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}(s_k^{(i)} \otimes s_k^{(j)})\right) t} \tag{29}$$

Defining the matrix S, in (30),

$$S = \sum_{k=1}^{n_g}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}(s_k^{(i)} \otimes s_k^{(j)}) \tag{30}$$

the combination rule can be compacted to (31).

$$t \cdot s \leq s_t^{(CQC+SRSS)} = \sqrt{t \cdot St} \tag{31}$$

Assume that the response space does not contain linearly independent components and that therefore the matrix S is invertible and $s_t^{(CQC+SRSS)} > 0$ for any direction t.
The set of conditions (31) written for all directions t allows to define an ellipsoidal response diagram. It can be shown that the critical point of this ellipsoid in the direction t is given by (32),

$$s^{(CQC+SRSS)} = s^{(0)} + \frac{St}{\sqrt{St \cdot t}} \tag{32}$$

where the static contribution is included. The ellipsoid is therefore tangent to a line/plane perpendicular to t. Note that although the lenght of the vector t is irrelevant, since it is affected by the numerator and the denominator, its orientation is important. It is also possible to obtain the point of the intersection of the ellipsoid with a given direction v using (33). Figure 1 illustrates the relationship between the vectors v and t and the seismic portion of s.

$$s^{(CQC+SRSS)} = s^{(0)} + \frac{v}{\sqrt{v \cdot S^{-1}v}} \tag{33}$$

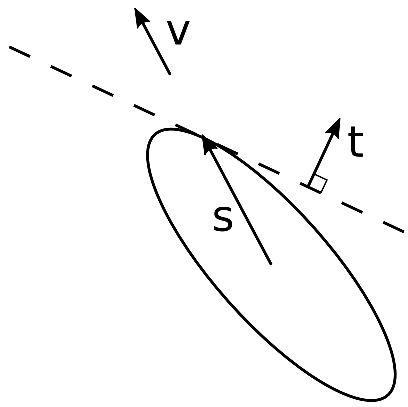


Figure 1: Response interaction surface of seismic action - relationship between v and t

To extrapolate the previous methods for the interaction surfaces, there are two existing approaches, either the cross section safety check, or their design, thus optimizing the section.

Method 5

Making sense to consider a permanent load, but the seismic response affected by a parameter λ, we have (34),

$$s^{(CQC+SRSS)} = s^{(0)} + \lambda\frac{St}{\sqrt{St \cdot t}} \tag{34}$$

completely analogous to (11), being $s^{(1)} = \frac{St}{\sqrt{St \cdot t}}$, as long as the vector t is fixed.
Verifying the security of a section involves the application of method 3 to a representative set of t directions, so it is important to verify that the smallest of the λ values is still greater than 1. Essentially, this method is governed by the SED strategy considered by Sessa et al. [6], although they consider the supreme envelope of the CQC3 rule and use the secant strategy to obtain λ. In practice, the interaction surface will be increased by a λ factor until it intersects the resistant surface.

Method 6

If the design of the cross section is the main objective of the theme, we have (35),

$$s^{(CQC+SRSS)} = s^{(0)} + 1 \times \frac{St}{\sqrt{St \cdot t}} \tag{35}$$

thus, the parameter λ receives a fixed value of 1. In this case, the interaction surface does not change, unlike the resistant one. Therefore, method 4 should be considered for each t choosing the largest of the minimum values obtained. Subsequently, method 3 will be called to draw the resistant surface with $f_\omega = f_\omega^{min}$.

4. Results

4.1. Resistant interaction surfaces

This section shows the results for three different kinds of cross sections: rectangular, circular and T, respectively. Before showing the results, the properties of each cross section are shown by its corresponding table and is discretization is illustrated on the figure below.

Rectangular section

Table 1: Rectangular section - properties

Concrete	Steel	b $[m]$	h $[m]$	c $[m]$	R_{inf}	R_{sup}
C30/37	A500	0.3	0.5	0.042	5ϕ16	2ϕ16

Circular section

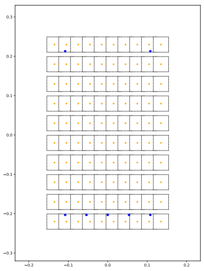
Table 2: Circular section - properties

Concrete	Steel	d $[m]$	c $[m]$	R_{tot}
C20/25	A400	0.45	0.032	25ϕ16

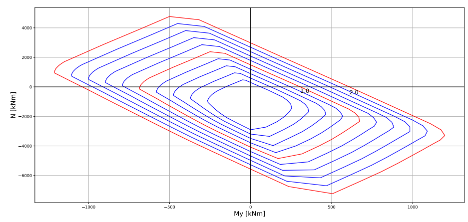
T section

Table 3: T section - properties

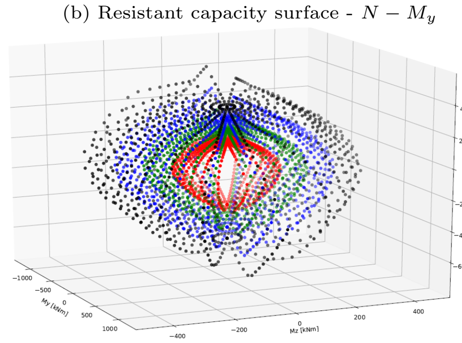
Concrete	Steel	b_f $[m]$	t_f $[m]$	b_w $[m]$	t_w $[m]$	c $[m]$	R_{inf}	R_{sup}
C35/45	A500	0.6	0.3	0.6	0.2	0.030	3ϕ20	9 ϕ16



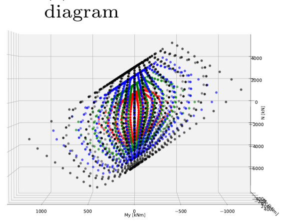
(a) Rectangular section - Discretization



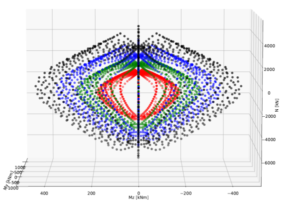
(b) Resistant capacity surface - $N - M_y$



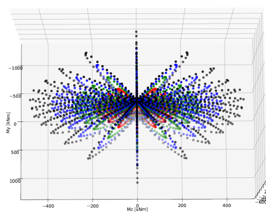
(c) Resistant capacity surface - Three-dimensional diagram



(d) Projection $N - M_y$

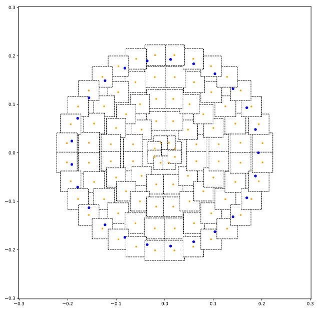


(e) Projection $N - M_z$

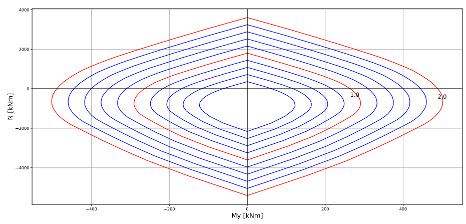


(f) Projection $M_y - M_z$

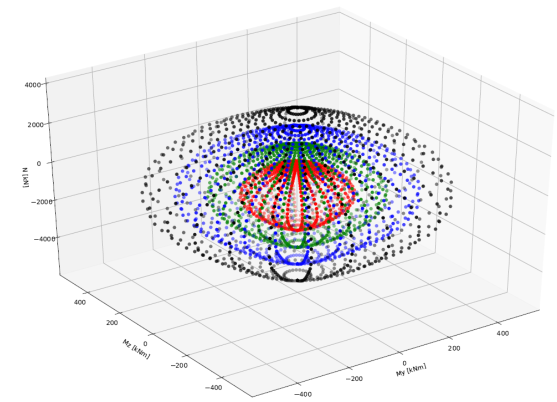
Figure 2: Definition of the resistant capacity surfaces according to the reinforcement rates ω of the reinforced concrete rectangular section



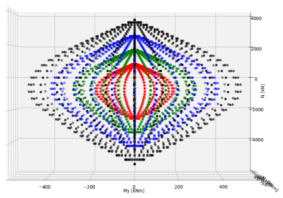
(a) Circular section - Discretization



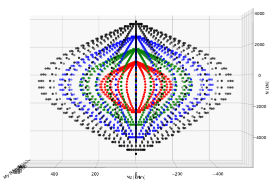
(b) Resistant capacity surface - $N - M_y$



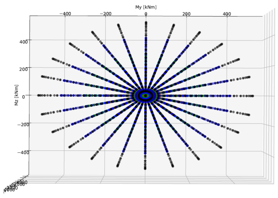
(c) Resistant capacity surface - Three-dimensional diagram



(d) Projection $N - M_y$

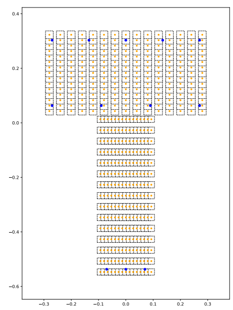


(e) Projection $N - M_z$

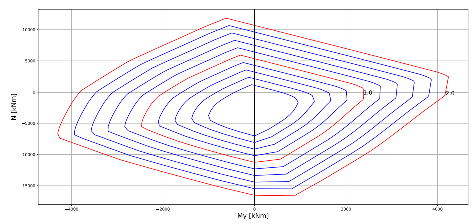


(f) Projection $M_y - M_z$

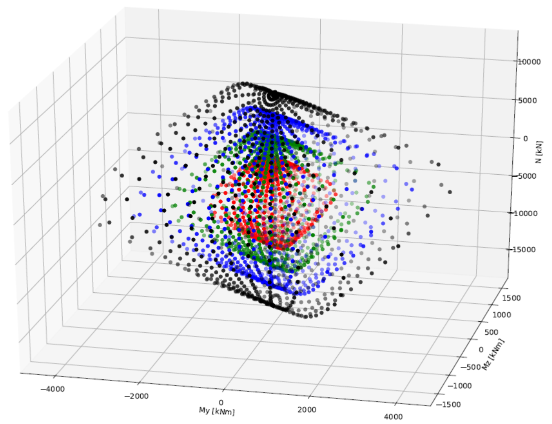
Figure 3: Definition of the resistant capacity surfaces according to the reinforcement rates ω of the reinforced concrete circular section



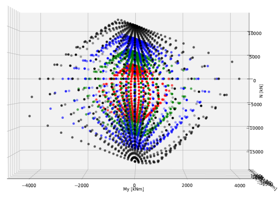
(a) T section - Discretization



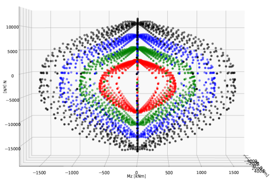
(b) Resistant capacity surface - $N - M_y$



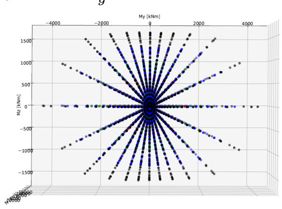
(c) Resistant capacity surface - Three-dimensional diagram



(d) Projection $N - M_y$



(e) Projection $N - M_z$



(f) Projection $M_y - M_z$

Figure 4: Definition of the resistant capacity surfaces according to the reinforcement rates ω of the reinforced concrete T section

4.2. Resistant interaction surfaces subjected to a seismic action

The results that follows consider the two possible situations for the occurrence of a seismic action: security check and its design. The seismic action is defined by its permanent components (**s**0) and dynamics(**S**)as shown in (36).

$$s^{(0)} = \begin{bmatrix} -150 & 50 & 0 \end{bmatrix} \quad S = \begin{bmatrix} 640100 & -24000 & 500 \\ -24000 & 23400 & 0 \\ 500 & 0 & 2500 \end{bmatrix} \tag{36}$$

The cross section is rectangular and it is defined in Table 4 and illustrated in Figure 5. The safety fator λ is equal to 0.29, which means that the dynamic component of the earthquake would have to decrease until 0.29 times in order to verify the safety of the cross section.

Table 4: Rectangular section - properties

Concrete	Steel	b $[m]$	h $[m]$	c $[m]$	R_{inf}	R_{sup}
C25/30	A500	0.3	0.3	0.04	$4\phi16$	$4\phi16$

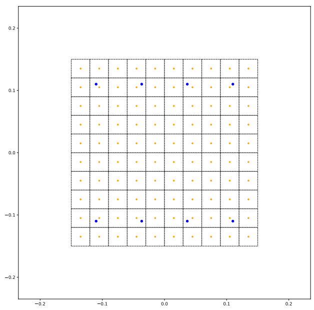


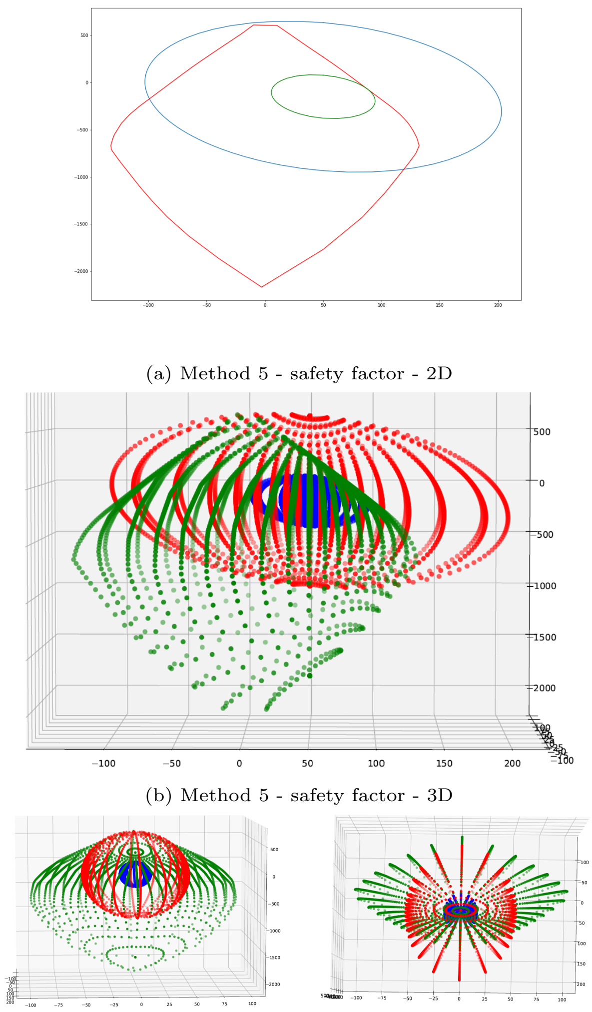
Figure 5: Rectangular section - discretization

Let's consider the design. Here, the main purpose is to find the minimum rate of reinforcement ω so the cross section resists to the earthquake.

In this case, the resistant capacity surface will be changed until it reaches the interaction surface. For that, the minimum reinforcement rate ω of the cross section in order to verify the safety of the structure is equal to 2.48.

5. Conclusion

Since the main goal of this paper is to define minimal resistance surfaces that resist the totality of the active efforts, several iterative methods were created. In general, the final objective of the present work is to build a resistant surface, as little as possible, that envelops the interaction surface. In other words, the interaction surface must



(a) Method 5 - safety factor - 2D

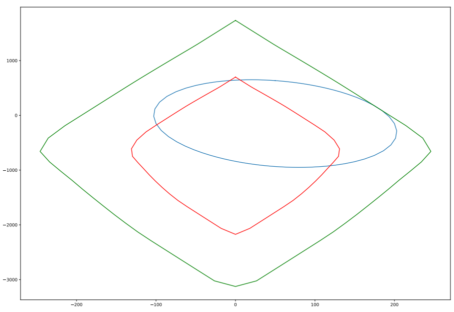
(b) Method 5 - safety factor - 3D

(c) Projection (1)

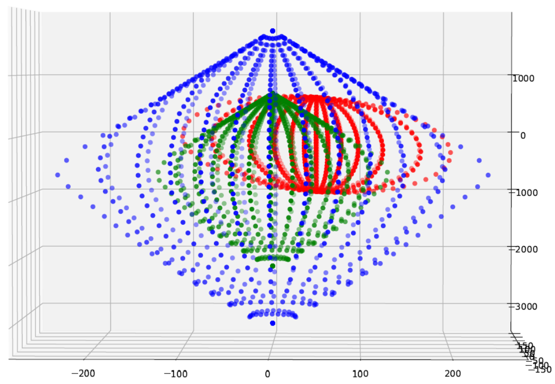
(d) Projection (2)

Figure 6: Results for method 5 - Find safety factor λ of the reinforced concrete T section

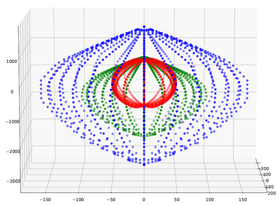
be completely contained in the domain of the resistance surface. The first methods, methods 1 and 2, the ones already known and most studied, served as the basis for the construction of the following methods. Method 3 can be considered for a safety check, since given a direction of efforts and a starting point, it returns the value by which the direction provided can be multiplied until the interaction surface is reached, which means that if the direction is known, it is only necessary to define a point of the resistant section. Method 4, on the other hand, must be considered for the design. Unlike method 3, method 4 defines the minimum reinforcement rate to withstand the combination of the stresses, in this case with a known direction. If this direction is totally unknown, it will be necessary to think about the resistant surface instead of a point. In this case, methods 3 and 4 are extrapolated to methods 5 and 6, creating routines for the recognition of the entire surface. The programming developed in this paper also allows the



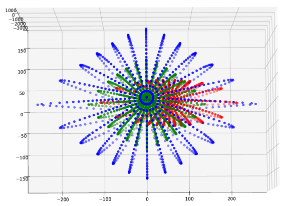
(a) Method 6 - Minimum reinforcement - 2D



(b) Method 6 - Minimum reinforcement - 3D



(c) Projection (1)



(d) Projection (2)

Figure 7: Results for method 6 - Minimum reinforcement factor f_ω of the reinforced concrete T section

definition of interaction curves for any type of cross section defined in the work, or any combination thereof, advantage over the abacuses of reinforced concrete tables, since they are limited to certain types of sections and coverings,and its accuracy is lower. The abacuses were defined with a reduction of 15 % in the breaking value of concrete under compression, and as already explained, the EC2 allows this reduction not to exist for current situations, since in the conditions of loading in time the concrete is, in general, required to lower stress levels. Object-oriented programming also allows that, after the main program is created, each method can be defined with a very small number of lines of code, making its creation accessible. Theoretically, another step has been taken in the development of this limitation, but it is important to create solid bases for the implementation of a design based on surfaces of interaction that, not being oversized, do not compromise structural safety.

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