Control of a Robotic Leg for Walking on Irregular Surfaces

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Abstract

We present an articulated walking model based on the inverted pendulum, with massless rigid legs, capable of walking on irregular surfaces. The step of the biped robot is composed by the pendular motion of the leg that supports the robot (standing leg), and, an articulated motion associated to the knee and the hip joints of the other leg, divided in 3 phases, that enables the robot to overcome obstacles (trailing leg). The adaptation to irregular surfaces was possible by controlling the time of the step so that both legs made always the same angle between them when the step starts. Additionally, we conserved the mechanical energy of the robot in order to continue walking successfully with the same characteristics as the step before. We formulated two failing criteria for the robot to walk on irregular surfaces, one for the local maximums and other for the local minimums. The junction of both failing criteria allowed to formulate a global sufficient condition in order for the robot to walk without failing the step due to the standing leg. For the articulated leg the failure conditions were defined but no sufficient conditions were formulated. We also establish a relationship between the energy needed to maintain a limit cycle trajectory, the angle of the slope, and the aperture between the legs when the robot is walking down a ramp without active control present to conserve the mechanical energy at the start of each step.
Keywords: standing leg, trailing leg, walking models, irregular surfaces, limit cycle

1. Introduction

One of the key factors that distinguish humans from other primates is their capability of having a bipedal walking/running gait. This aspect served as an evolutionary mark regarding our movement in the sense that we could access a much higher mobility and a bigger vision range than before against our preys and our predators. Since the way we walk is unique and very complex [7] regarding the rest the animal kingdom, an effort has been made to reproduce the same type of walking and running in robots and other types of machines.

One of the first robots to be implemented in the context of the inverted pendulum model was developed by McGeer, in which the concept of passive dynamic walking was introduced [6]. When a step transition occurs, the heel strike applies an impulse which partially stops the motion from the previous step, however, the larger is the velocity from the step before, the bigger will be the velocity of the next step. If we have a descending floor with a fixed slope, the ending velocity when step ends is bigger than in the case of a flat floor. Therefore, an equilibrium is possible, and in this situation, the gravitational energy added, is compensated by the heel strike, thus providing a stable walking gait with no external control. Passive dynamic walking corresponds to the case when the robot walks continually, with no external control to maintain the stability of its movement. With access to the Poincaré map associated to a robot going down a surface, *i.e.* by recording the velocity and the angle at the start of each step for various initial conditions, and by determining the eigenvalues, λ_1, λ_2, of the Jacobian matrix associated to the Poincaré section from step to step, the existence of a limit cycle can be proven [5], which confirms that for a given descending floor with constant slope, there can be a walking stable gait powered by gravity. Another passive dynamic walker with knees was also implemented [2]. In this work, the step of the robot was divided in two phases, one where the knees were locked and another where the knees were unlocked. Besides the walking model developed, the region associated to the initial conditions of the attractor was determined, which is associated with the stable gaits of the passive dynamic walker going down a slope. An active dynamic walker [1] was also developed, meaning there is an active control in each step, in which mechanical energy of the inverted pendulum model was fixed for each step, allowing the robot, also in two phases, using the articulation of the knee, to successfully walk along a straight line on a flat surface.

2. Movement Along a Flat Horizontal Surface

In this section we develop a walking model composed of a robot with a mass m located in the hip, and two massless rigid legs of length l. These two legs have distinct functions. One of the legs supports the mass of the robot throughout the step and propagates this mass forward. This leg will be called the standing leg. The other leg, the trailing leg, resets the orientation of the standing leg from the previous step so that it has the proper initial conditions for the next step. This reset motion is composed of an articulated 2-link controlled movement via the hip joint that rotates the thigh and the knee joint that rotates the calf. This articulated motion allows the robot to overcome obstacles.

We consider that a step begins when the standing leg connects with the floor, and a step ends when the standing leg leaves the floor. By transitioning to another step, the robot will instantly swap between standing and trailing legs in order to keep walking continually. An important characteristic about this model is that when a new step starts, the aperture between the standing and trailing legs is always the same. This means that the foot associated to the step before, the foot associated to the next step and the mass always create an isosceles triangle with the inner angle associated to the mass always being the same, when there is a step transition. Figure (1) illustrates two important angles, α and β, that express the aperture between the legs between step transitions. The angle β will be a parameter of the model which expresses how open the legs are when a step transition occurs. The angle α which corresponds to the interior angle of the triangle centered in one of the foots is given by,

$$\alpha = \frac{\pi - \beta}{2}. \tag{1}$$

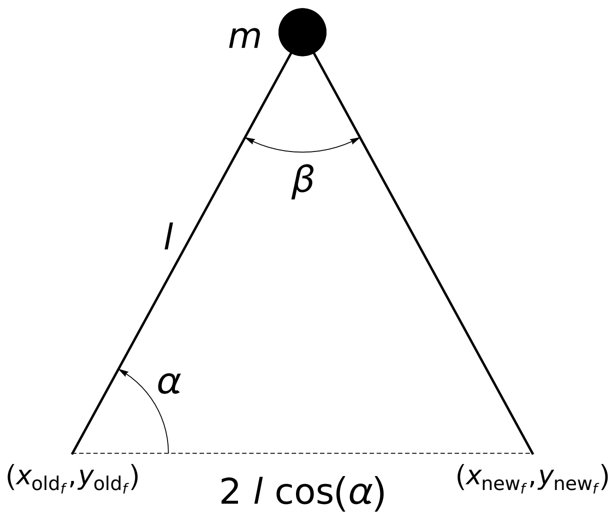

Figure 1: Transition instant of the two legs between steps. The mass m, the foot associated to the step before and the foot associated to the next step always create an isosceles triangle with the aperture between the legs corresponding to the angle β.

2.1. Standing Leg

The robot is transported by the inverted pendulum associated to the foot, the standing leg and the mass. The equation of motion associated to the mass of the robot is given by,

$$m\ddot{\phi} + m\frac{g}{l}\cos\phi = 0, \tag{2}$$

where g is the gravity acceleration. The angle ϕ is the angle of the standing leg measured from an horizontal line passing through the foot, to the mass. The pendulum in the phase space $(\phi, \dot{\phi})$ has the Hamiltonian function,

$$H = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\sin\phi. \tag{3}$$

We are interested in the domain where $0 < \phi < \pi$ and $\dot{\phi} \in \mathbb{R}^{0-}$, since the propagation of the mass m is done in the positive direction of the xx axis. In the open interval of ϕ considered, Eq. (2) has an unique unstable fixed point at $(\phi, \dot{\phi}) = (\pi/2, 0)$.

Since the inverted pendulum is an Hamiltonian system, the energy E is conserved for the step duration, which implies,

$$\dot{\phi} = -\sqrt{2\left(\frac{E}{ml^2} - \frac{g}{l}\sin\phi\right)}, \tag{4}$$

where $H = E$, and the negative sign comes from direction of motion of the mass. If $E > mgl$, the phase curves are defined for $\phi \in [0, \pi]$. If $0 < E \leq mgl$, the phase curves are defined for $\phi \in [0, \arcsin(E/mlg)] \cup [\pi - \arcsin(E/mlg), \pi]$. We are not interested in the later situation since the pendulum can fall before the step ends, colliding with the floor. If the pendulum has enough energy to achieve it's vertical position, $E > mgl$, which is where the potential energy is maximum, there will be no constraints regarding the conclusion of the next step. Taking into account Eq. (4), the sufficient condition that assures the standing leg doesn't fall is

$$\dot{\phi}(t_i) \leq -\sqrt{2\frac{g}{l}\left(1 - \sin\phi(t_i)\right)}, \tag{5}$$

where t_i corresponds to the initial time measured for the respective step. By determining the phase curves associated to the pendulum, a geometric criteria can be set that filters the solutions that respect the sufficient energy condition, $E > mgl$. Modifying the energy changes the outcome of the

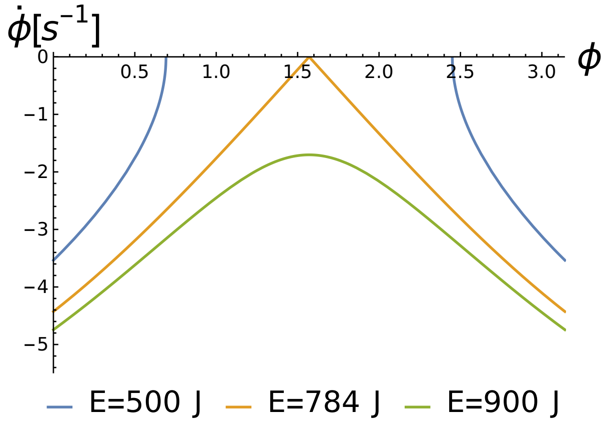

Figure 2: Phase curves for the inverted pendulum in Eq. (4) for, $m = 80$ Kg, $l = 1$ m $g = 9.8$ m/s^2, and for various values of energy $H = E$, namely, $E = 500$ J$< mlg$ (blue), $E = 784$ J$= mlg$ (orange) and $E = 900$ J$> mlg$ (green). The level of energy of $E = mlg = 784$ J, is the phase curve associated boundary of the energetic sufficient condition $E = mgl$. Only solutions with $\dot{\phi} \leq 0$ are taken.

fall. Figure (2) is a representation of the phase curves for the different energies.

Suppose the robot is on the step number n in a set course. The initial instant associated to the step n is designated by $t_{i_{(n)}}$, and the angle of the standing leg at this time, $\phi\left(t = t_{i_{(n)}}\right)$, is called the angle of attack associated to step n. Likewise, the final instant associated to the step n is designated by $t_{f_{(n)}}$, and the angle of the standing leg at this time, $\phi\left(t = t_{f_{(n)}}\right)$, is the exit angle associated to step n. Three parameters can be associated to the standing leg, namely, $\phi_0 = \phi\left(t_{i_{(1)}}\right)$, the angle of attack associated to the first step, which is also the initial condition of the system. The second parameter is the aperture angle between the legs measured at the time there is a transition of steps, β. The third parameter of the standing leg is the energy E associated with the pendular movement of the step in Eq. (3). In a flat surface, the angle α from Fig. (1) and Eq. (1) is the same as the exit angle associated to the first step since both measurement references coincide with

$$\phi\left(t_{f_{(1)}}\right) = \alpha = \frac{\pi - \beta}{2}. \tag{6}$$

Equation (2) can be integrated since the energy of the system is constant and we obtain an implicit expression for the time elapsed on the step n as a function of the initial and final angle of the standing leg

$$T_{(n)} = \sqrt{\frac{2\,ml^2}{E - mgl}}\left[F\left(\frac{1}{4}(\pi - 2\phi)\middle| - \frac{2\,mgl}{E - mgl}\right)\right]_{\phi=\phi_{i_{(n)}}}^{\phi=\phi_{f_{(n)}}}, \tag{7}$$

where $F(\varphi|k)$ is the elliptical integral of the first kind [8], defined as

$$F(\varphi|k) = \int_0^{\varphi} \frac{1}{\sqrt{1 - k\sin^2\vartheta}} d\vartheta. \tag{8}$$

2.2. Trailing Leg

The trailing leg has two types of possible rotations. One of the rotations is associated to the hip joint, which rotates the thigh, and therefore the knee of this leg around the hip. The other rotation, associated to the knee joint, directly rotates the calf, and therefore, the foot around the knee. The trailing leg, trails the standing leg so that in the next step it is properly reset and ready to connect with the floor. The respective lengths of the thigh and calf, l_1 and l_2, are controlled by $\lambda \in [0, 1]$ a parameter of the trailing leg. The length of the thigh is

$$l_1 = \lambda l, \tag{9}$$

and the length of the calf that is expressed as

$$l_2 = (1 - \lambda)l. \tag{10}$$

Two angles describe the dynamics of the trailing leg. The angle θ is the angle measured from the standing leg to the thigh of the trailing leg in the interval $\theta \in]-\pi, +\pi[$. The angle μ is the angle measured from the thigh of the trailing leg to the calf in the interval $\mu \in]0, 2\pi[$. The positions of the knee and the foot can be obtained by linked rotations associated to the angles μ and θ from the position of the mass, which is supported by the standing leg. Figure (3) is a representation of the angles of the standing leg, ϕ, as well of the thigh, θ, and the calf, μ of the trailing leg.

The positions of the foot of the standing leg have coordinates $(x_{SF_{(n)}}, y_{SF_{(n)}})$, where n is n$^{\text{th}}$ step of the robot. Given Fig. (3), the coordinates of the position of the hip, H, knee of the trailing leg, TK, and foot of the trailing leg, TF, are expressed as rotations applied by the angles θ, μ on respectively the hip join and the knee joint.

2.2.1 Control of the Trailing Leg

The articulated movement of the trailing leg is divided into 3 phases, phases a), b) and c). The start of the step, $t_{i_{(n)}}$, corresponds to the start of phase a), $t_{a_{(n)}}$. The start of phase b), $t_{b_{(n)}}$, is controlled by an independent parameter, τ_b. This parameter

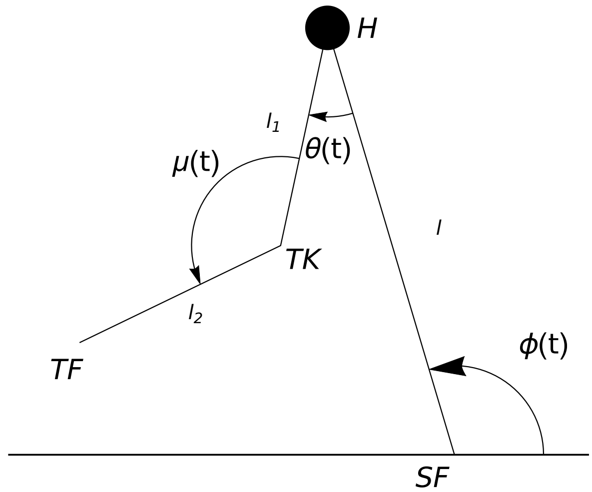

Figure 3: Angles of the thigh (θ) and the calf (μ) of the trailing leg. The thigh of the trailing leg is segmented by two points, the hip, H, and the knee, TK, and its length is l_1. The calf of the trailing leg is segmented by the foot, TF and the knee, TK, and its length is l_2.

corresponds to the portion of the time elapsed in a step until phase b) starts

$$t_{b_{(n)}} = t_{i_{(n)}} + \tau_b T_{(n)}, \tag{11}$$

where $T_{(n)}$ is the total time elapsed in step n defined in Eq. (7). The time phase c) starts, $t_{c_{(n)}}$, is also controlled by an independent parameter, τ_c. This parameter corresponds to a portion of the time elapsed in a step until phase c) starts

$$t_{c_{(n)}} = t_{i_{(n)}} + \tau_c T_{(n)}. \tag{12}$$

The parameters τ_b and τ_c satisfy a precedence requirement

$$0 < \tau_{b_{(n)}} < \tau_{c_{(n)}} < 1. \tag{13}$$

Introducing ϵ as an independent parameter that corresponds to an offset to the angle of the thigh on the final instant of phase b) that is within in the interval $\epsilon \in [0, \pi - \beta]$ and γ, also an independent parameter that corresponds to the minimum of $\mu(t)$ in the step,

$$\gamma = \min \mu(t), \quad \text{if } t \in \left[t_{i_{(n)}}, t_{f_{(n)}}\right], \tag{14}$$

we can broadly define the control functions θ and μ. Given Fig. (3), any continuous function, from the start to the end of a step n, that is non-decreasing [4] from $t_{a_{(n)}}$ to $t_{c_{(n)}}$, respecting

$$\theta\left(t_{a_{(n)}}\right) = -\beta, \tag{15}$$

$$\theta\left(t_{c_{(n)}}\right) = \beta + \epsilon, \tag{16}$$

and, is non-increasing [4] in the interval $t_{c_{(n)}}$ to $t_{f_{(n)}}$ from $\beta + \epsilon$ to β, is a suitable control law $\theta(t)$, the rotation associated to the hip joint. Likewise, for any continuous function $\mu(t)$, that is non-increasing from $t_{a_{(n)}}$ to $t_{b_{(n)}}$, goes in this interval from π to γ, is non-decreasing from $t_{b_{(n)}}$ to $t_{c_{(n)}}$ so that $\mu\left(t_{c_{(n)}}\right) = \pi$, and, is kept constant until the step ends, is a suitable control law for the rotation of the calf.

2.2.2 Linearized Control of the Trailing Leg

Defining a normalized time variable $\eta(n, t)$,

$$\eta(n, t) = n + \frac{t - t_{i_{(n)}}}{T_{(n)}}, \quad \text{if } t_{i_{(n)}} \leq t \leq t_{f_{(n)}}, \tag{17}$$

we see that this continuous variable easily references the state of a given step and belongs in the interval $\eta \in [1, \max(n) + 1[$, where $\max(n)$ is the maximum step number achieved by the robot. The integer number corresponds to the step, and the decimal number to the state of completion of the current step. This way, if we say that the robot has made to time $\eta = 5.6$, this means that the robot is has completed 60% of the 5$^{\text{th}}$ step. The inversion of this variable is given by the following

$$t(\eta) = \sum_{j=0}^{\lfloor \eta \rfloor - 1} T_{(j)} + \{\eta\} T_{(\lfloor \eta \rfloor)}, \tag{18}$$

where $\lfloor \eta \rfloor$ and $\{\eta\}$ are respectively the extraction of the lowest integer and the fractional part [3] associated to η, and $T_{(0)}$ is the initial instant when the first step starts, $t_{i_{(1)}}$.

Using the variable η and setting up linear controls on the rotations associated to the hip and the knee so that $\dot{\theta}$ and $\dot{\mu}$ are constant, the expressions $\theta(\eta)$ and $\mu(\eta)$ become

$$\theta(\eta) = \begin{cases} -\beta + \dfrac{2\beta + \epsilon}{\tau_c}\{\eta\}, & \text{if } 0 \leq \{\eta\} < \tau_c, \\ \beta + \epsilon - \dfrac{\epsilon}{1 - \tau_c}\left(\{\eta\} - \tau_c\right), & \text{if } \tau_c \leq \{\eta\} < 1, \end{cases} \tag{19}$$

$$\mu(\eta) = \begin{cases} \pi + \dfrac{\gamma - \pi}{\tau_b}\{\eta\}, & \text{if } 0 \leq \{\eta\} < \tau_b, \\ \gamma + \dfrac{\pi - \gamma}{(\tau_c - \tau_b)}\left(\{\eta\} - \tau_b\right), & \text{if } \tau_b \leq \{\eta\} < \tau_c, \\ \pi, & \text{if } \tau_c \leq \{\eta\} < 1. \end{cases} \tag{20}$$

Attributing values to the parameters present in Eqs. (19) and (20), the motion of the trailing leg in the first step is known. The parameters chosen for the standing leg were the following

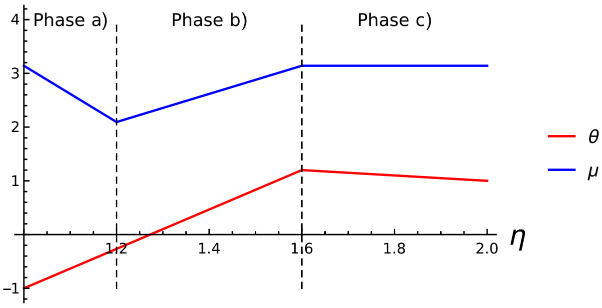

Figure 4: Control angles θ (red) and μ (blue) as a function of $\eta \in [1, 2]$. The parameters $\tau_b = 0.2$ and $\tau_c = 0.6$ have a direct relationship with the variable η and each phase since on the first step, phase $a)$ goes from $1 \leq \eta < 1 + \tau_b$, phase $b)$ goes from $1 + \tau_b \leq \eta < 1 + \tau_c$ and phase $c)$ goes from $1 + \tau_c \leq \eta < 2$.

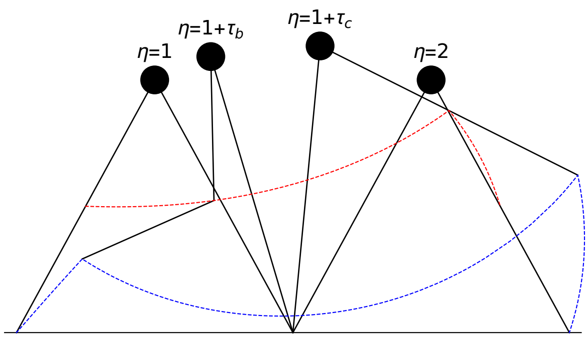

Figure 5: Application of the control laws in Fig. (4). The start of phase $a)$ corresponds to $\eta = 1$, the start of phase $b)$ corresponds to $\eta = 1 + \tau_b$ the start of phase $c)$ corresponds to $\eta = 1 + \tau_c$ and the end of the step occurs at $\eta = 2$. The red and blue dashed line corresponds respectively to the trajectory of the knee and the trajectory of the foot throughout the step. Three dynamic parameters, β, E and ϕ_0 are associated with the standing leg and for the trailing leg five dynamic parameters are associated, τ_b, γ, τ_c, ϵ, λ.

$$\begin{aligned} &\beta = 1 \text{ rad}, \ l = 1 \text{ m}, \ \phi_0 = 2.071 \text{ rad}, \\ &E = 1019.2 \text{ J}, \ m = 80 \text{ Kg}, \ g = 9.8, \ \text{m/s}^2, \end{aligned} \tag{21}$$

and for the trailing leg,

$$\begin{aligned} &\lambda = 0.5, \quad \gamma = 2\pi/3 \text{ rad}, \quad \tau_b = 0.2, \\ &\tau_c = 0.6, \quad \epsilon = 0.2 \text{ rad}. \end{aligned} \tag{22}$$

The plot of the control laws $\theta(\eta)$ and $\mu(\eta)$ according to the parameters in Eqs. (21) and (22), are illustrated in Fig. (4). Figure (5) contains the different snapshots of the robot in each phase transition and also the trajectories of the knee and the foot for the entire duration of the first step.

2.3. Energy lost and Control Mechanism in Step Transitions

Throughout a step the energy of the system is conserved. Upon reaching the end of the step, at $t_{f_{(n)}}$, an instantaneous transition to the next step occurs, which can be described by swapping the standing and the trailing leg. The standing leg has only the polar component associated to the velocity vector due to the pendular motion. When the standing leg reaches the end of a step, the resulting velocity vector of the mass, $\vec{v}$, is not aligned with the velocity at the beginning of the next step. Let us call $\vec{v}_{f_{(n)}} = \vec{v}\left(t_{f_{(n)}}\right)$ the velocity of the mass at the final instant of step n. Similarly, let us call $\vec{v}_{i_{(n)}} = \vec{v}\left(t_{i_{(n)}}\right)$ the velocity of the mass at the initial instant of step n. Following Figs. (6) and (7), there are two important sub phases in this transition. Assuming an elastic collision between the two legs and an inelastic collision between the trailing leg and the floor, still in the current step n, a projection of the final velocity, $v_{f_{(n)}} \sin(\beta)$, is lost instantly when proceeding to the new step, step $n + 1$. After losing the radial velocity component associated to the old step, the energy of the system is lowered in the first sub phase of the step transition. To compensate this effect, in the same instant that the energy is lost, a control mechanism is applied to raise the velocity of the mass in the polar direction to reset the energy of the system according to Eq. (3) keeping the parameter E as the fixed energy associated to the standing leg. Figure (7) illustrates the second sub phase associated to step transition in order for the energy of the system to remain constant.

We can measure the effects of this control system in two equivalent ways. One of the ways involves adding instantly the amount of polar velocity needed, $\vec{v}'_{(n+1)}$, defined as

$$\begin{aligned} \vec{v}'_{(n+1)} &= v'_{(n+1)} \cdot (-\hat{e}_{p_{i_{(n+1)}}}) \\ &= -l\dot{\phi}'_{(n+1)} \cdot (-\hat{e}_{p_{i_{(n+1)}}}) \end{aligned} \tag{23}$$

along with the polar versor $-\hat{e}_{p_{i_{(n+1)}}}$, in such a way that

$$v_{i_{(n+1)}} = v_{f_{(n)}} \cos\beta + v'_{(n+1)}, \tag{24}$$

This velocity can be determined by considering the energy difference at the final instant of a step and at the initial instant of the next step. In a flat horizontal surface, the energy at the final instant of step n can be expressed as

$$H = mgh + \frac{1}{2} m \vec{v}_{f_{(n)}} \cdot \vec{v}_{f_{(n)}}, \tag{25}$$

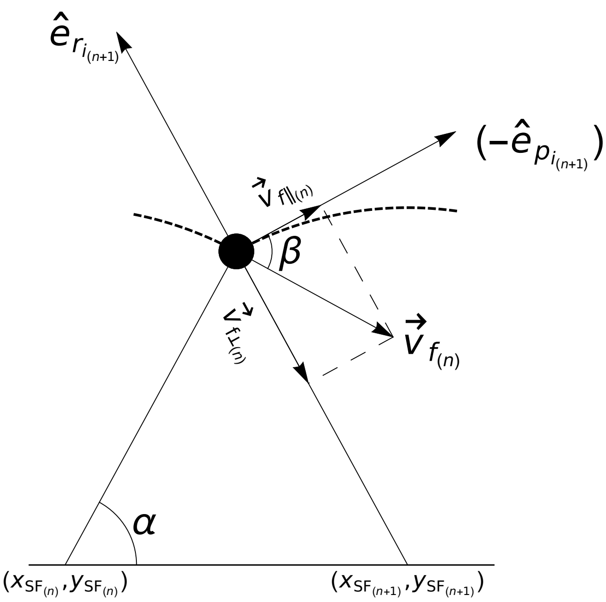

Figure 6: Step transition sub phase 1. The standing leg in the next step loses the radial component associated to the velocity $\vec{v}_{f_{(n)}} = \vec{v}(t_{f_{(n)}})$, the velocity of the mass at the final instant of step n.

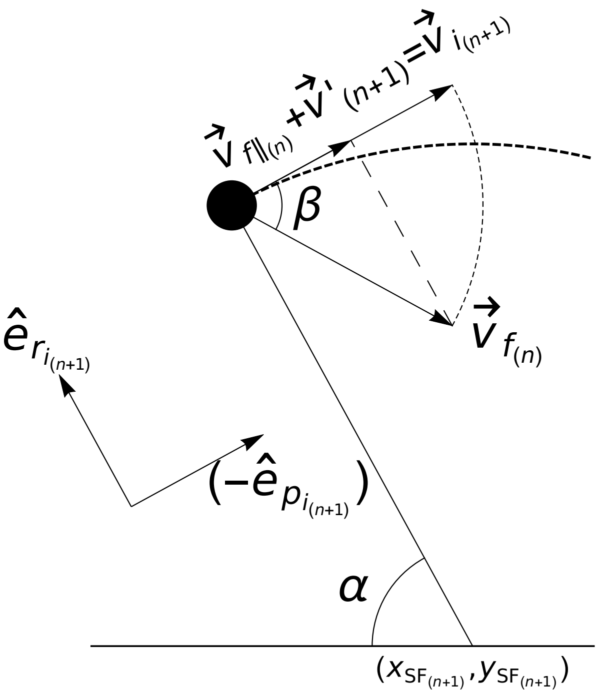

Figure 7: Step transition sub phase 2. A control mechanism is applied to reset the energy of the standing leg. This is done by increasing the polar velocity of the mass by $v'_{(n+1)}$.

with h being the height of the mass. In the beginning of the step we want to conserve the energy of the last step

$$H = mgh + \frac{1}{2} m\vec{v}_{i_{(n+1)}} \cdot \vec{v}_{i_{(n+1)}}. \tag{26}$$

In a flat surface, we can from Eqs. (25) and (26) determine $v'_{(n+1)}$

$$\begin{aligned} v'_{(n+1)} &= v_{f_{(n)}}(1 - \cos\beta) \\ &= \sqrt{\frac{2}{m}} \sqrt{E - mgl\cos\frac{\beta}{2}} \Big(1 - \cos\beta\Big). \end{aligned} \tag{27}$$

The alternative way to interpret the effects of the control mechanism is to lower or raise the necessary amount of energy to the system in order for it to remain constant. In order to determine the energy increase by the control mechanism, $E'_{(n+1)}$, we need to categorize into two situations, namely, one where the collision of the trailing leg helps the continuation of the step and the one where it hinders. For the case $\beta > \pi/2$, a control mechanism must be implemented to stop the motion of the mass, propagate it in such a way that $\dot{\phi}_{i_{(n+1)}} < 0$ and the energy is not lost on the next step. For the case $\beta < \pi/2$ the robot can use the energy from the step before so that the remaining amount is added. We can write the energy E as

$$\begin{aligned} E &= \frac{1}{2} m v^2_{f_{(n)}} + mgh = \\ &= \begin{cases} +1/2m(v_{f_{(n)}}\cos\beta)^2 + mgh + \\ +E'_{(n+1)} \quad \text{if } 0 < \beta \leq \pi/2, \\ -1/2m(v_{f(n)}\cos\beta)^2 + mgh + \\ +E'_{(n+1)} \quad \text{if } \pi/2 < \beta \leq \pi, \end{cases} \end{aligned} \tag{28}$$

which results the following expression for, $E'_{(n+1)}$, the energy added by the control system

$$E'_{(n+1)} = \begin{cases} \left(E - mgl\cos\frac{\beta}{2}\right)\sin^2\beta \\ \text{if } 0 < \beta \leq \pi/2, \\ \left(E - mgl\cos\frac{\beta}{2}\right)(1 + \cos^2\beta) \\ \text{if } \pi/2 < \beta \leq \pi. \end{cases} \tag{29}$$

3. Movement Along an Irregular Surface

We will introduce a dynamic control technique in which the time elapsed in a step is appropriated for the type of floor we are given. Consider that we have a floor f that can be given as a function of x, $f = f(x)$, where the robot is walking upon. If the robot can walk on an irregular surface, there exists a specific geometric display involving the two legs because of the fixed aperture β in step transitions. Figure (8) displays the geometry of the

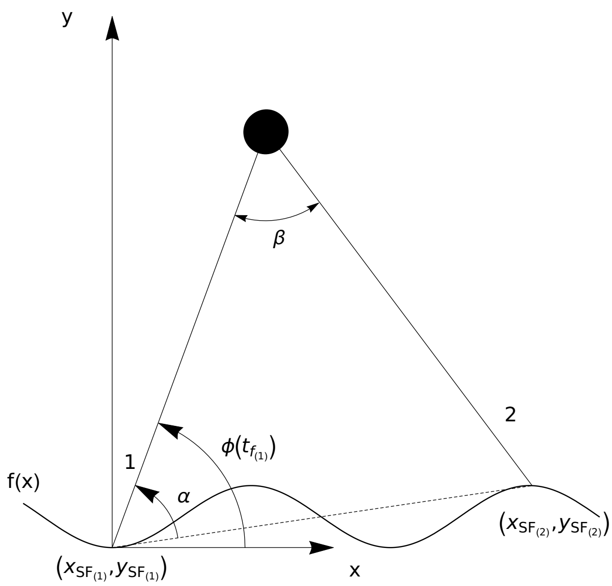

Figure 8: Geometry associated to the final instant of the first step in an irregular surface given as a function of x, $f(x)$. The segment that connects both feet is projected to the respective axis given that the corresponding angle is $\phi(t_{f_{(1)}}) - \alpha$.

robot in step transitions, on an irregular surface. If $(x_{SF_{(1)}}, y_{SF_{(1)}})$ is the position of the foot of the standing leg on the first step, and $(x_{SF_{(2)}}, y_{SF_{(2)}})$ is the position of the foot of the standing leg on the second step, given the geometry present in Fig. (8), we can say that if there exists an angle $\phi(t^*)$ such that the following relations are satisfied

$$x_{SF_{(2)}} - x_{SF_{(1)}} = 2l \cos\alpha \cos\left(\phi(t^*) - \alpha\right), \quad (30)$$

$$y_{SF_{(2)}} - y_{SF_{(1)}} = 2l \cos\alpha \sin\left(\phi(t^*) - \alpha\right), \quad (31)$$

then, $\phi(t^*)$ is the exit angle associated to the first step and t^* is the final time associated to step 1, with step 2 beginning immediately after. However, the system of Eqs. (30) and (31) has 3 unknowns, therefore, it is required to add an equation in order to determine an unique solution to the system. This additional equation, comes from the fact that both the feet have to be in contact with the floor when a step transitions occurs,

$$y_{SF_{(2)}} = f\left(x_{SF_{(2)}}\right), \quad (32)$$

which results on the following condition that must be satisfied for the robot to walk on irregular surfaces given the evolution in time of Eq. (2)

$$y_{SF_{(1)}} + 2l \cos\alpha \sin\left(\phi(t^*) - \alpha\right) = \\ = f\left(x_{SF_{(1)}} + 2l \cos\alpha \cos\left(\phi(t^*) - \alpha\right)\right). \quad (33)$$

3.1. Energy Reset and Control Mechanism in a Non-Horizontal Surface

Upon reaching the transition instant between steps, it is necessary to know which velocity $v'_{(n+1)}$ to add in order for the system not lose any energy. It's possible to determine $v'_{(n+1)}$ as well as the initial velocity of the next step, $v_{i_{(n+1)}} = v_{f_{(n)}} \cos\beta + v'_{(n+1)}$. In the case of an irregular surface, v'_{n+1} and $E'_{(n+1)}$ are expressed respectively in Eqs. (34) and (35). The variable $\sigma_{(n)}$ is the angle that the segment which connects the two feet makes with an horizontal floor on the final instant of step n. It can be defined, taking into consideration Figs. (1) and (8),

$$\sigma_{(n)} = \phi\left(t_{f_{(n)}}\right) - \alpha. \quad (36)$$

For floors with a fixed angle ψ, we have that $\sigma_{(n)} = \psi$ and therefore we can relate the walking pattern of the robot with the inclination of the floor.

Let us consider the robot walking downward an inclined ramp with the slope making an angle ψ. If the robot can pass its vertical position, *i.e.*, if $E > mlg$, Eq. (35) has non-trivial solutions in the case $0 < \beta < \pi/2$ and $E'_{(n+1)} = 0$. The case where $E'_{(n+1)} = 0$ means that there is no active control on step transitions to monitor the energy of the robot. This also means that the robot enters a limit cycle in this situation because the energy of the robot on the next step converges to the value of energy in Eq. (35). Figure (9) illustrates this effect where the robot walks down a slope for 5 steps and the phase space $(\phi, \dot{\phi})$ was recorded. The robot enters a limit cycle because the gravitational energy added due to the height difference balances the lost radial projection of the final velocity from the step before. Figure (10) is a density plot which relates the energy necessary to maintain a limit cycle trajectory E/mlg with a given angle of the slope ψ, and an aperture between the legs at the start of the step β.

3.2. Walking Failure Conditions

We will now also explore strategies and conditions to determine *a priori* if the incrementation of a step is possible given the floor conditions in accordance with the parameters of the robot, and formulate the general principles to stable walking for continuous surfaces. Two failing criteria were elaborated for the standing leg. One of the criteria involves the local minimum of the floor the robot is walking on. It is necessary that $\phi\left(t_{f_{(n)}}\right) < \phi\left(t_{i_{(n)}}\right)$, otherwise, the robot is not walking forward but backwards. One way to measure how far is the standing leg from failing the step due to this limitation is to consider the following failing condition.

$$\frac{v'_{(n+1)}}{\sqrt{lg}} = \sqrt{2}\left(\sqrt{\frac{E}{mlg} - \cos\left(\sigma_{(n)} + \frac{\beta}{2}\right)} - \cos\beta\sqrt{\frac{E}{mlg} - \cos\left(\sigma_{(n)} - \frac{\beta}{2}\right)}\right). \tag{34}$$

$$\frac{E'_{(n+1)}}{mlg} = \begin{cases} \frac{E}{mlg}\sin^2\beta - \left(\cos\left(\sigma_{(n)} + \frac{\beta}{2}\right) - \cos\left(\sigma_{(n)} - \frac{\beta}{2}\right)\cos^2\beta\right) & \text{if } 0 < \beta \leq \pi/2, \\ \frac{E}{mlg}(1+\cos^2\beta) - \left(\cos\left(\sigma_{(n)} + \frac{\beta}{2}\right) + \cos\left(\sigma_{(n)} - \frac{\beta}{2}\right)\cos^2\beta\right) & \text{if } \pi/2 < \beta \leq \pi. \end{cases} \tag{35}$$

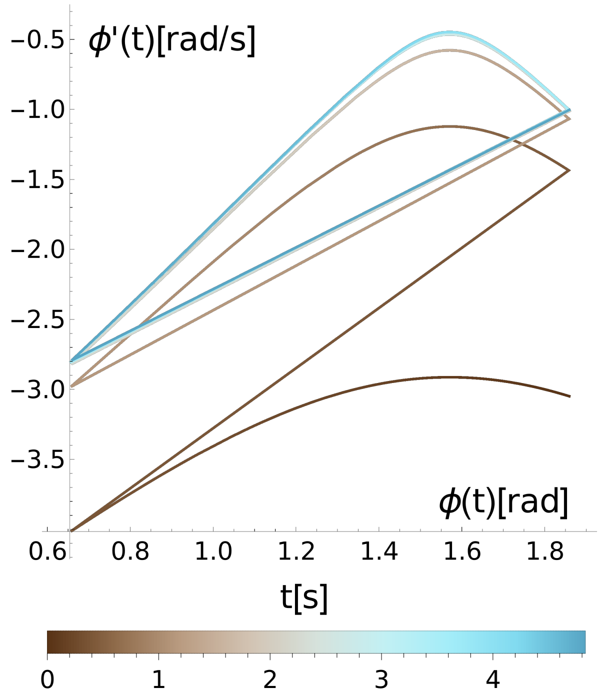

Figure 9: Phase space $(\phi, \dot{\phi})$ of the robot walking down a slope for 5 steps with no control mechanism. We can see that the energy associated to the movement, from step to step, is converging since the differences between each step become smaller. The parameters associated were the following: $\psi = -\pi/10\,\text{rad}$, $\beta = 1.205\,\text{rad}$, $l = 1\,\text{m}$, $m = 80\,\text{Kg}$, $g = 9.8\,\text{m/s}^2$, $E_0 = 1123.5\,\text{J}$.

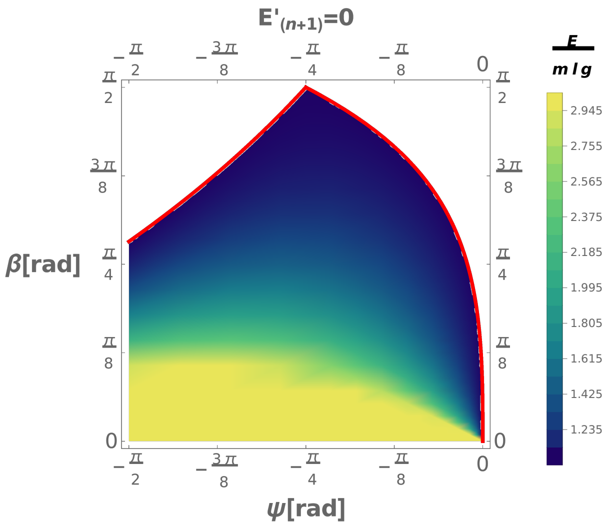

Figure 10: Regions of parameter space associated to the existence of limit cycles. The colours correspond to E/mlg and represent the energy necessary to maintain a limit cycle trajectory with a given ψ and β. Highlighted at red are the solutions in which $E/mlg = 1$.

Failure Condition 1. *Given the standing leg at a start of a step, if the foot of an imaginary trailing leg rotated of an angle β in the counter-clockwise direction with the current standing leg is below the floor, the mass of the robot in this step will collide with the surface in study, failing the step.*

With the parameters $\beta = 0.525\,\text{rad}$ and $l = 1\,\text{m}$, Fig. (11) illustrates an example of a case where the Failing Condition 1 is satisfied and the mass collides with the floor, making it impossible to increment a step.

We are interested in developing robust relations between the parameters of the robot and the properties of the floor. For this reason, we will use the modulus function to study the limit case in which a step can be incremented, since it allows us to study the valleys or peaks in which the derivative of the floor does not pass a certain value. The general types of floor we are interested can be defined as,

$$f(x) = \pm\tan(\psi)|x|, \tag{37}$$

where ψ is the angle of the of the floor with an horizontal line in the interval $\psi \in [0, \pi/2]$. To give an example of a valley in a critical situation regard-

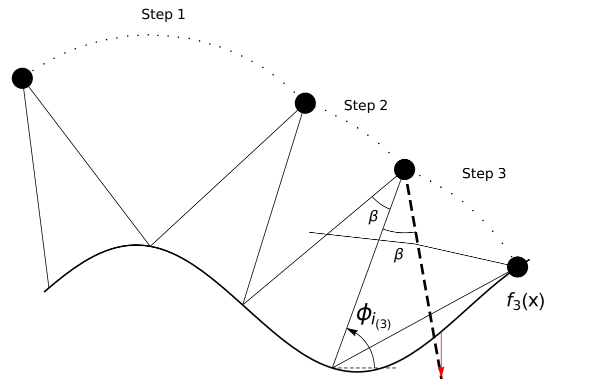

Figure 11: Step 3 unsuccessfully incremented, with $\beta = 0.525\,\text{rad}$ and $l = 1\,\text{m}$ and $f_3(x) = 0.3\cos(0.2 + 3x)$. The foot of the imaginary trailing leg is below the floor, as indicated by the red arrow, and therefore, the mass collides with the floor making it impossible to increment a step.

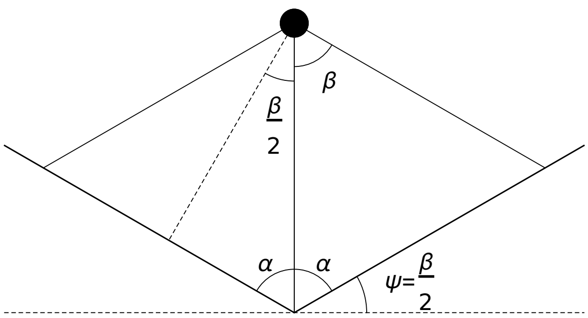

Figure 12: Initial and final states of the robot in the limit which Failure Condition 1 provides a successful iteration of the standing leg, with a floor f of type valley, by taking the plus sign of Eq. (37), for $\beta = \pi/3$ and $\psi = \pi/6$. In this case, the time elapsed in the step is null since we are in the critical situation in which the step can be incremented.

ing our Failure Condition 1, Fig. (12) is an illustration of the case when the standing leg is oriented perpendicularly to the horizontal axis of reference. A sufficient condition can be formulated in which floors of the type valley should obey in order for a step to successfully be incremented.

Sufficient Condition 1. *Given a generic continuous floor f, where in a limited region $x \in [a, b]$ it has only one local minimum, if,*

$$\max |f'(x)| < \tan \beta/2, \tag{38}$$

then a solution for the next step can be successfully found regarding the incrementation of the standing leg in the interval considered.

The other failing criteria involves the local maximum of a surface function. Given a floor f we know that the standing leg has a minimum angle which it cannot exceed. This angle corresponds to the amplitude the tangent line of the floor passing through the foot. It can be given as

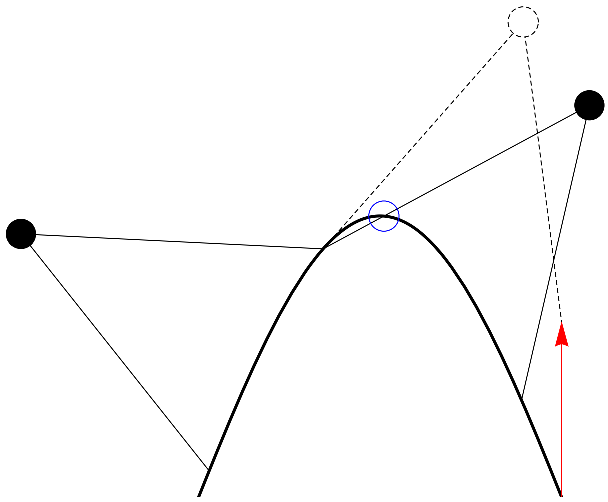

Figure 13: Unsuccessful iteration of the step considering Failure Condition 2 with $\beta = 0.85\,\text{rad}$ and $f(x) = \sin(2.5(0.44 + x))$. The blue circle in the standing leg at the final instant of the step implies that an invalid collision with the floor occurred.

$$\phi_{Min_{(n)}} = \arctan f'(x_{SF_{(n)}}). \tag{39}$$

In case the standing leg goes below $\phi_{Min_{(n)}}$, it will collide internally with the floor, invaliding the step of the robot. Figure (13) illustrates how the standing leg can hit the floor considering this limitation of the robot. We can formulate the respective failing criteria to measure how far is the robot from failing the step similarly to the case of local minimums.

Failure Condition 2. *Given the foot of an imaginary trailing leg rotated of an angle β in the counterclockwise direction from the standing leg on its minimum angle $\phi = \phi_{Min_{(n)}}$, according to Eq. (39). If the foot of this trailing leg is above the floor, the standing leg will collide internally with the surface in study, making the robot fail the step.*

The result of the least restrictive case regarding Failure Condition 2, that enables a relationship between β and ψ is presented in Fig. (14). In this figure, there is freedom for the foot of the standing leg to not be on the vertex of the peak, but it can be shown that if that occurs, we are not maximizing the configurations in which Failure Condition 2 renders a valid step, and therefore, we are unable to state a sufficient condition. The sufficient condition regarding Failure Condition 2 can be directly extracted from Fig. (14).

Sufficient Condition 2. *Given a generic continuous floor f, where in a limited region $x \in [a, b]$ it has only*

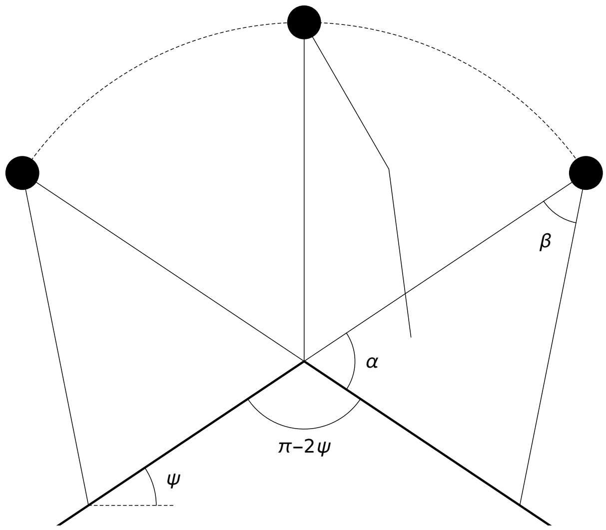

Figure 14: Incrementation of the standing leg regarding Failure Condition 2 with the critical slope ψ only depending of β. The associated parameters are, $\psi = 3\pi/16$ rad, $\beta = \pi/4$ rad.

one local maximum, if,

$$\max |f'(x)| < \tan \frac{\pi - \beta}{4}, \tag{40}$$

then a solution for the next step can be successfully found regarding the incrementation of the standing leg in the interval considered.

We can state a more general condition for any type of floor, whether it is composed by a maximum, minimum, or a mixture of both, with the two sufficient conditions derived. The intersection of both sufficient conditions maximizes the number of floors the robot can walk on at $\psi = \pi/6$ and $\beta = \pi/3$.

Sufficient Condition 3. *Given a generic continuous floor $f(x)$, if,*

$$\max |f'(x)| < \min \left(\tan \frac{\pi - \beta}{4}, \tan \frac{\beta}{2} \right) \tag{41}$$

then, the standing leg can be incremented successfully and also for any surface function of the type $f(x - x_0)$ with $x_0 \in \mathbb{R}$.

4. Conclusions

Even though this walking model is far from complete, for in fact, it corresponds to a very minimalistic view on how we can parametrize bipedal walking for robots, it still grasps the essential qualities and characteristics of the motion associated to walking, the main problems regarding irregular surfaces and how we can tackle them. To increase the level of adaptability of the robot, we can also replace the point feet by a more realistic foot capable of imitating an human feet. Another subtle change that can help the conservation of the energy lost is to add springs to the legs which with the actuation of the feet can help bring more stability to the robot. By exploring the inverted pendulum model we noticed some of the most basic aspects of the human locomotion which can help the understanding of more complex walking models and give us an insight on how to approach the problems presented here and in other contexts.

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