Abstract—Many of the methods adopted for AUV navigation rely on dead reckoning and acoustic navigation for position estimation. With the growing need to implement cost and time saving systems, single beacon navigation has recently come to the foreground. In this setup, instead of deploying an array of transponders to the sea floor, only one beacon is set at a known position, in order to aid the vehicle’s dead reckoning system. Here, without the ability to perform trilateration algorithms given the ranges obtained from several transponders, the trajectory taken by the vehicle needs to exhibit some form of excitation in order to provide range measurements that are relevant for positioning. Furthermore, in real applications, the initial estimate of the vehicle’s position, is not know with certainty, and the range measurements obtained are noisy and contain outliers. Considering this, we address the problems of reducing the initial uncertainty relative to the initial position estimation, and driving the vehicle to a desired final position, by developing a two part receding horizon navigation system that solves these problems separately but sequentially. First, we consider a single beacon, single vehicle setup. Here, for each part of the system, we integrate optimal trajectory planning, using the Fisher information matrix as an observability metric, and the extended Kalman filter as a state estimator, and implement and test the complete system. Then, for a cooperative, single beacon setup, where two vehicles exchange information with a beacon and each other, we repeat the preceding work, and compare both studied scenarios.

I. INTRODUCTION

In recent years, autonomous underwater vehicles (AUVs) have been used in order to perform the exploration of certain areas in oceans, rivers and other bodies of water. Without the need to be controlled by an operator and dotted with the ability to be left in mission for days at the time, AUVs are now widely used in scientific and military applications. Equipped with different sensors and communication systems in order to collect information about the areas they roam, it is extremely important that they are equipped with an accurate navigation system for proper motion control and data georeferencing. The fact that GPS signals are not available underwater, makes it necessary for AUVs to be equipped with sensors that allow them to perform self positioning or localization algorithms. Therefore, navigation of autonomous underwater vehicles (Ref. [1]) relies on two major methods: dead reckoning (DR) and acoustic navigation (AN). Some in depth details about these methods and the instruments used can be seen in Ref. [2]. There are several types of AN systems in what concerns the size of the transponder baseline, one of those, the most commonly used, is the long baseline system (LBL). In LBL systems the vehicle interrogates a number of transponders that are moored on the sea floor, often times at the corners of the operation site, at known positions, and listens to the replies to compute the distance of the vehicle to the baseline. Then, using trilateration algorithms, the position of the vehicle can be estimated. The distance between transponders doesn’t exceed a few kilometers and the accuracy of the position measurements is usually better than one meter but, in certain configurations, can reach only a few centimeters. The main disadvantage of this system is the need to deploy, calibrate, and retrieve the transponders from the sea floor, which consists of a complex and time consuming operation.

A. Problem Statement

Consider a fully submerged underwater vehicle initially deployed at a point that is known with some uncertainty, and whose position we want to measure with accuracy, as it moves through the water towards a pre-defined final position. In order to achieve this, without the use of a supporting surface vehicle, a LBL baseline system needs to be mounted on to the sea floor in order to pair the acoustic system to the sensors integrated in the vehicle for positioning. Given the practical considerations presented in the previous section, the need for a cost effective and time saving solution arises. In classical single beacon navigation, an individual beacon is deployed on the sea bed (in a much simplier and cost saving operation in comparison to that required to deploy LBL) in order to aid the estimate of the vehicle’s pose, given by dead reckoning on board sensors (DVL, and AHRS) which are able to measure the vehicle’s velocity with respect to the sea floor, and its angular velocity with respect to its body frame. The vehicle receives, at a certain rate, acoustic signals emitted by the beacon, measuring the range (or distance) at which the beacon is. Without the ability to perform trilateration algorithms, it becomes extremely important that the trajectory of the vehicle, is diverse enough in terms of maneuvers, so that the range measurements obtained are relevant for improving positioning accuracy.

B. Single Beacon AUV Navigation: Previous Work

The problem of single beacon navigation has been studied previously by authors that proposed the solution of creating a virtually long baseline system Ref. [3]–[5]. According to Ref. [3], this can be achieved by determining the position of a
vehicle by advancing multiple ranges from said transponder and along the vehicle’s dead reckoning track. Given the nonlinear nature of the observations (range measurements) an extended Kalman filter was proposed in Ref. [4] to obtain an estimation of the vehicle’s position as well as the ocean currents, assumed to be constant. The authors of Ref. [5] also proposed an algorithm for the computation of the moving beacon trajectory that minimizes the estimation error. In Ref. [6], a multilateration based algorithm with a Kalman filter was presented, creating a virtually long baseline along the vehicle’s dead reckoning track, while simultaneously considering a simple optimization problem that provides a measurement of the vehicle’s position at each time step, making the observation model a linear one. According to the author, it is necessary to estimate currents in order to achieve convergence of the position estimation in single beacon navigation using KF techniques. Regarding integrated planning, estimation and control, to a final desired position simultaneously beacon navigation, in Ref. [9] an algorithm was proposed that “best worst case scenario” in what concerns information available for cooperative positioning using a nonlinear least squares), for cooperative positioning using a single autonomous surface craft. The author of Ref. [9] also described the experimental implementation of an online algorithm for cooperative localization and compares three estimation methods (extended Kalman filter, particle filter and nonlinear least squares), for cooperative positioning using a single autonomous surface craft. The author of Ref. [9] also studies the information available for a cooperative positioning system using two vehicles and concludes that the inclusion of another vehicle slightly improves the systems observability. Finally in Ref. [13], the author performs a cost savings analysis comparing a single beacon approach to the commonly used long baseline System. This study corroborates the assumption that implementing a single beacon system is, in fact, beneficial.

II. THEORETICAL BACKGROUND

A. Optimal Estimation

An estimator takes into consideration the inputs a priori introduced to the system and the measurement of its outputs in order to produce an estimate \( \hat{x} \) of its state. Firstly, the estimator takes into consideration the model of the system, creating a prediction. Then, the difference between the predicted output and the measurement is considered and multiplied by a gain \( L(k) \) in order to correct the initial prediction, making up for the unknown disturbances. The estimator’s dynamics can, in this way, be divided into two steps: prediction and correction. For the discrete state space representation of a system, the estimator’s dynamics (see Ref. [14]) are given by

\[
\begin{align*}
\text{Prediction: } & \quad \bar{x}(k) = A\bar{x}(k-1) + Bu(k-1) \\
\text{Correction: } & \quad \hat{x}(k) = \bar{x}(k) + L(k)(y(k) - C\bar{x}(k))
\end{align*}
\]

The term filter, refers to an estimator with ability to filter out said disturbances, providing an estimate of the system’s state that takes these into account. At each instant, the error associated with the estimation is given by \( e(k) = x(k) - \hat{x}(k) \). The value of the estimator’s gain \( L(k) \) that minimizes the error’s covariance \( P(k) = E[e(k)\, e(k)^T] \) is often referred to as the Kalman gain. The so called Kalman filter algorithm (KF) is the optimal estimator that at each step, computes and applies to the current estimation of the state, the Kalman gain for a linear system such as the one described in Eq. [1]. The extended Kalman filter algorithm (EKF) is the extension of the KF when the system’s dynamics are non linear. For system’s like

\[
\begin{align*}
x(k+1) &= f(x(k), u(k)) + v(k) \\
y(k) &= h(x(k)) + w(k)
\end{align*}
\]

where \( v(k) \) and \( w(k) \) are the process and observation additive noise, respectively and are assumed to be Gaussian, zero mean and have constant covariance matrices \( E[v(k)v(k)^T] = Q \geq 0 \) and \( E[w(k)w(k)^T] = R > 0 \), the extended Kalman filter algorithm that estimates the state is governed by

\[
\begin{align*}
\text{Prediction: } & \quad \bar{x}(k) = f(\bar{x}(k-1), u(k-1)) \\
& \quad \bar{P}(k) = F(k)\bar{P}(k-1)F(k)^T + Q \\
\text{Correction: } & \quad L(k) = \bar{P}(k)H(k)^T \left[H(k)\bar{P}(k)H(k)^T + R\right]^{-1} \\
& \quad \hat{x}(k) = \bar{x}(k) + L(k)[y(k) - h(\bar{x}(k))] \\
& \quad \bar{P}(k) = (I - L(k)H(k))\bar{P}(k)
\end{align*}
\]

where:

\[
\begin{align*}
F(k) &= \frac{\partial f}{\partial x} |_{\bar{x}(k-1),u(k-1)} \\
H(k) &= \frac{\partial h}{\partial x} |_{\bar{x}(k)}
\end{align*}
\]

EKF implementation example: Consider a vehicle moving underwater, at a fixed velocity \( v_{abs} \) and variable heading rate \( r(k) \). The navigation system that estimates the vehicle’s position \( p(k) = [p_x(k) \ p_y(k)]^T \) at each discrete time instant \( k \) relies on measurements obtained from the DVL mounted onto the vehicle, the heading rate input used to control the position, as well as the range measurements obtained from exchanging acoustic information to a single beacon positioned to the ocean.
discretization rate of time of 60 seconds are presented in figures 1 and 2. In order to described trajectories, the results obtained for a simulation estimation error are summarized in table I. For two of the sets of control inputs. The values of the control input in of the vehicle’s movement when it is actuated with different we considered the measurements obtained from the simulation compare how the estimator behaves for different trajectories, position coordinates before the correction step. In order to H and F system’s nonlinear characteristics. To this end, matrices $F(k)$ and $H(k)$ described in Eq. (4) are given by

$$F(k) = \begin{bmatrix} 1 & 0 & T_s v_{abs} \sin(\psi(k)) \\ 0 & 1 & -T_s v_{abs} \cos(\psi(k)) \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$H(k) = \begin{bmatrix} \frac{\bar{p}_x(k)}{\sqrt{p_x^2(k) + p_y^2(k)}} & \frac{\bar{p}_y(k)}{\sqrt{p_x^2(k) + p_y^2(k)}} & 0 \end{bmatrix}$$

where $\bar{p}_x(k)$ and $\bar{p}_y(k)$ are the predictions at time $k$ of the position coordinates before the correction step. In order to compare how the estimator behaves for different trajectories, we considered the measurements obtained from the simulation of the vehicle’s movement when it is actuated with different sets of control inputs. The values of the control input in each simulation, as well as the results obtained regarding the estimation error are summarized in table I. For two of the described trajectories, the results obtained for a simulation time of 60 seconds are presented in figures[1] and [2]. In order to apply the EKF algorithm, the system was simulated at a discretization rate of $T_s = 1 s$ (assuming that all the measurements are obtained at the same rate) with an initial state of: $x(0) = [-9.7 \ m \ 0.2 \ m \ 0 rad]$. The measurements obtained from the exchange of acoustic signals with the beacon were corrupted by a white Gaussian noise of zero mean and covariance $R = \sigma^2 = 0.05 \ m$. The algorithm was applied to the initial estimation $\hat{x}(0) = [-10 \ m \ 0 \ m \ 0 rad]^T$ and the initial position estimate covariance matrix is $\hat{P}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which corresponds to a circular uncertainty zone shown in blue in this section’s figures, with a confidence level of 68.26% (Ref. [15]).

By analyzing the results obtained, it is possible to see how, when applied the same algorithm, different input signals (and therefore different trajectories) yield to different results in terms of mean square error between the estimated and the true trajectories. Intuitively, it is possible to see how this phenomenon occurs: when there is uncertainty in the initial position estimate, the algorithm tries to correct it by heavily relying on the range measurements. In this initial stage it is imperative that the maneuvers performed by the vehicle allow it to obtain measurements that are not ambiguous in the range sense.

### B. The Posterior Cramér-Rao Lower Bound

So far, we have described filters that can estimate a system’s state in the presence of uncertainty. However, how does one evaluate the efficacy of the estimation in what concerns the obtained measurements? By resorting to the tools of estimation theory (Ref. [16]), we start by defining the lower bound of the covariance of the estimates obtained from the implementation of an unbiased estimator (see Ref. [17]). This value is the Cramér-Rao lower bound (CRLB) (see Ref. [18])

$$E[(\hat{x} - x)(\hat{x} - x)^T] \geq J^{-1}$$

---

**TABLE I: Control input applied and MSE of the trajectory estimations**

<table>
<thead>
<tr>
<th>Trajectory</th>
<th>$v_{abs}[m/s]$</th>
<th>$r[k][rad/s]$</th>
<th>MSE[m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trajectory 1</td>
<td>0.5</td>
<td>0.05 $\cos\left(\frac{2\pi k}{10}\right)$</td>
<td>0.0691</td>
</tr>
<tr>
<td>Trajectory 2</td>
<td>0.5</td>
<td>0.001</td>
<td>0.0785</td>
</tr>
</tbody>
</table>
where \( \hat{x} \) is an unbiased estimate of a non random parameter \( x \) and \( J \) is the Fisher Information Matrix (FIM). The matrix inequality in Eq. (8) can be also described by saying that \( D(k) \) given by

\[
D(k) = E[(\hat{x} - x)(\hat{x} - x)^T] - J^{-1} \geq 0
\]

(9)
is a positive semi-definite matrix. The FIM can be described as the quantification of the existing information about a certain parameter for estimation. If the estimated parameter’s covariance matrix equals the PCRLB, the estimator is considered efficient (all the existing information about the system has been extracted, according to Ref. [16]). The FIM is defined as

\[
J = E[(\nabla_x \ln p(y|x))(\nabla_x \ln p(y|x))^T]
\]

(10)
where \( p(y|x) \) is the conditional likelihood function of the observations (measurements) \( y \), given the state \( x \) (see Ref. [19]).

The CRLB for random parameters is referred to as the Van Trees (Ref. [20]) Crâmer-Rao lower bound, or the posterior Crâmer-Rao lower bound (Ref. [21]), given by

\[
PCRLB^{-1} = CRLB^{-1} + E[(\nabla_x \ln p(x))(\nabla_x \ln p(x))^T]
\]

(11)
where \( AP^{-1} \) takes into account the a priori information, regarding parameter \( x \). Consider the following system with nonlinear dynamics:

\[
x(k + 1) = f(x(k), u(k)) + w(k) \\
y(k) = h(x(k)) + v(k)
\]

(13)
for \( k = 0, ..., K \), where \( x(k) \in \mathbb{R}^n \) is denoted the state of the system at time \( k \), corrupted by state noise \( w(k) \in \mathbb{R}^n \) and \( y(k) \in \mathbb{R}^p \) is the observation made at time \( k \), corrupted by measurement noise \( v(k) \in \mathbb{R}^n \). Following the method introduced by Tichavsky et al. in Ref. [21], in order to obtain the posterior Crâmer-Rao lower bound (PCRLB), for an arbitrary \( k \) the Fisher information matrix is computed with the joint probability density function of the vector composed of all the state vector of said system \( X(k) = [x(0)^T, ..., x(k)^T]^T \), and the vector composed of all the observations (outputs) of the system \( Y(k) = [y(0)^T, ..., y(k)^T]^T \), given by

\[
p(X(k), Y(k)) = p(x(0)) \prod_{j=1}^{K} p(y(j)|x(j)) \prod_{i=1}^{K} p(x(i)|x(i-1))
\]

(14)
By decomposing \( X(k) \) as \( X(k) = [X(k-1)^T \ x(k)^T]^T \), the information matrices of \( X(k) \) and \( x(k) \) are given by:

\[
FIM(X(k)) = \begin{bmatrix}
F_k^1 & F_k^2 \\
F_k^3 & F_k^4
\end{bmatrix}
\]

(15)
and

\[
FIM(x(k)) = F_k^3 - F_k^{4T} F_k^{-1} F_k^2
\]

(16)
where

\[
F_k^1 = E[(\nabla_{X(k-1)} \ln p(X(k), Y(k))) (\nabla_{X(k-1)} \ln p(X(k), Y(k)))^T],
\]

(17)
\[
F_k^2 = E[(\nabla_{x(k)} \ln p(X(k), Y(k))) (\nabla_{X(k-1)} \ln p(X(k), Y(k)))^T],
\]

(18)
\[
F_k^3 = E[(\nabla_{x(k)} \ln p(X(k), Y(k))) (\nabla_{x(k)} \ln p(X(k), Y(k))^T]
\]

(19)
and the information matrix of \( x(k + 1) \), given by

\[
FIM(X(k + 1)) = \begin{bmatrix}
F_k^1 & F_k^2 \\
F_k^3 & F_k^4
\end{bmatrix}
\]

(20)
here,

\[
D_k^1 = E[(\nabla_{x(k+1)} \ln p(x(k+1)|x(k))) (\nabla_{x(k+1)} \ln p(x(k+1)|x(k))^T],
\]

(21)
\[
D_k^2 = E[(\nabla_{x(k+1)} \ln p(x(k+1)|x(k))) (\nabla_{x(k)} \ln p(x(k+1)|x(k))^T],
\]

(22)
\[
D_k^3 = E[(\nabla_{x(k+1)} \ln p(x(k+1)|x(k))) (\nabla_{x(k+1)} \ln p(y(k+1)|x(k)^T)^T + E[(\nabla_{x(k+1)} \ln p(y(k+1)|x(k))) (\nabla_{x(k+1)} \ln p(y(k+1)|x(k+1))^T]
\]

(23)
Consequently, the information matrix associated with \( x(k + 1) \) is given by the recursive expression:

\[
FIM(x(k + 1)) = D_k^2 - D_k^{2T} ( FIM(x(k)) + D_k^1)^{-1} D_k^2
\]

(24)
and the initial information matrix \( FIM(x(0)) \) is related to the initial probability density function of \( x(0) \). When the noise processes \( v(k) \) and \( w(k) \) of Eq. [13] are white Gaussian processes with zero mean and covariance matrices \( Q(k) \) and \( R(k) \) respectively, the logarithmic probability density functions in \( p(x(k+1)|x(k)) \) and \( p(y(k+1)|x(k+1)) \) are given by

\[
\ln p(y(k+1)|x(k+1)) = c3 - 1/2 \left( y(k+1) - h(x(k+1)) \right)^T Q^{-1}(k) \left( y(k+1) - h(x(k+1)) \right)
\]

(25)
\[
\ln p(x(k+1)|x(k)) = c4 - 1/2 \left( x(k+1) - f(x(k)) \right)^T R^{-1}(k+1) \left( x(k+1) - f(x(k)) \right)
\]

(26)
where \( c3 \) and \( c4 \) are constants, and after some computations the resulting FIM recursive expression is given by

\[
FIM(x(k + 1)) = H^T(k+1) R^{-1}(k+1) H(k+1) + F(k) FIM^{-1}(x(k)) F(k)^T + Q(k)^{-1}
\]

(27)
Consider the model presented previously in Eq. (5), where we consider that there is uncertainty in the first estimation of the system’s state, white Gaussian noise \( w(k) \sim N(0, \sigma^2) \) in the range measurements but no noise in the model, and the expression computed for the PCRLB Eq. (27). Given this, we have
\[
Q(k) = 0_{n \times n} \\
R(k) = \sigma^2
\]
and the FIM for the estimation of parameter \( x(k) \) at the end of the trajectory (for \( k = K \)), is given by
\[
FIM(x(K)) = FIM(x(0)) + \\
\frac{1}{\sigma^2} \sum_{i=1}^{K} \left[ \begin{array}{cc}
p^2_x(i) & p_x(i)p_y(i) \\
p_x(i)p_y(i) & p^2_y(i)
\end{array} \right]
\]
where \( d^2(i) = p^2_x(i) + p^2_y(i) \). As referred before, the initial Fisher information matrix is related to the probability density function associated with the initial estimation, which is, in the case of a Gaussian distribution around an estimate, the inverse of the initial covariance matrix Ref. [21]. The expression presented in Eq. (29) demonstrates that the efficiency of the estimator, when there are single beacon range measurements, is related to the distance \( d(k) \) at which the vehicle is regarding the beacon, as well as its coordinates at a certain time \( k \). For unbiased estimates, the values of the eigenvalues of the PCRLB, and the inverse of the FIM, correspond to the radius of its position uncertainty. Due to these considerations, the determinant of the Fisher information matrix is a good metric for the information available for positioning.

### III. Single Beacon Single Vehicle Navigation

Suppose it is required to drive a vehicle to a final desired position \( p_f \) of interest. The vehicle is able to obtain range measurements from a single beacon positioned at the origin of the referential corrupted by white Gaussian noise \( w(k) \). Given the nature of the obtained measurements, we will consider the system to be discrete, sampled at a rate of \( T_s \), assuming that, for simplicity, all the measurements can be obtained at the same rate. The system’s model is given by
\[
p(k + 1) = p(k) + T_s u(k) \\
y(k) = d(k) + w(k) \\
w(k) \sim N(0, \sigma^2)
\]
where \( u(k) \) is the linear velocity control applied to the vehicle, the system state corresponds to the vehicle’s position \( p(k) \) at time \( k \), and is decomposed into the position along each axis \( x(k) = p(k) = \left[ p_x(k) \ p_y(k) \right]^T \), the output of the system is the range measurements obtained from a single beacon positioned at the origin of the referential corrupted by white Gaussian noise \( w(k) \). In reality, we do not have access to the vehicle’s position, but only to the expected value of it. At each time instant, we can have the vehicle’s estimated position \( \hat{p}_0 \), together with the covariance of that estimate, denoted \( P_0 \). It is therefore important that, before starting to control the vehicle to a desired position, we may have a good estimate of it’s location. Furthermore, in order to control the vehicle to its final position, one must derive a control law that, at each step, takes into consideration the error between the current and the final positions. This becomes harder when there is uncertainty regarding the position of the vehicle. In order words, the control law computed specifically for the estimated position is applied to the vehicle at its true position, which can result in the failure of said control law when the uncertainty associated with that position is considerable. In order to solve the combined navigation and control problem formulated above, without discretizing the uncertainty area, we divided the overall problem into two distinct sub-problems: imparting motions to the vehicle as means to reduce the uncertainty its location and 2) once the position is known with sufficient accuracy, engage a control system to steer it to the desired final target position.

#### A. Part 1: Trajectory Planning to Reduce Initial Uncertainty

In this part of the navigation system, we intend to reduce the initial position uncertainty associated with the vehicle, by applying a control law that allows the vehicle to obtain range measurements that enable it to improve the estimate of its position. As referred in section [II-B] by maximizing the determinant of the FIM, we are guaranteeing that the best possible maneuvers are being performed in terms of information provided to the estimator. Here, we will maximize the logarithm of the determinant instead, given that it preserves maximum and minimum values, at the same time as it cuts out exponents in the objective function. The optimal trajectory when the vehicle is in the initial estimated position \( \hat{p}_0 = FIM^{-1}(p_0) \) with a associated uncertainty of \( \tilde{P}_0 = FIM^{-1}(p_0) \), lies in the solution of the following optimization problem
\[
\min_{u(0),\ldots,u(K)} -\log \left[ \det \left( FIM \left[ p(K) \right] \right) \right] \\
\text{s.t.} \\
p(k + 1) = p_0 + T_s \sum_{i=0}^{k} u(i) \sigma(||u(i)||, u_{\text{max}})
\]
where, as computed in section [II-B] the Fisher information matrix at the final simulation point \( K \) for the single range measurement model is given by the expression in equation [29], \( \sigma(||s||, c) \) is a saturation function such that \( s \sigma(||s||, c) \) is the normalized value of \( s \) in the range \([-c, c]\)
\[
\sigma(s, c) = \begin{cases} 
0 & \text{for } s = 0 \\
1 & \text{for } 0 < |s| < c \\
\frac{c}{|s|} & \text{otherwise}
\end{cases}
\]
and \( u_{\text{max}} \) is the maximum norm of the input signal such that \(||u(k)||| \leq u_{\text{max}}\).

#### B. Part 2: Trajectory Planning to Reach a Final Position

After assuring that the position of the vehicle is known with some degree of certainty, it is necessary to drive the vehicle to it’s desired final position. To do so, we use a null-space behavioral approach introduced in Ref. [11] and Ref. [10]. This
approach is based on the method of geometrically prioritizing tasks described in Antonelli et al Ref. [22].

**Geometrically Prioritizing Tasks:** Consider the continuous time system where the goal is to drive the vehicle’s position \( p(t) \) to a certain final point \( p_f \) while maximizing information about positioning, given noisy range measurements to a beacon at the origin

\[
\dot{p}(t) = u(t) \\
y(t) = d(t) + w(t) \quad w(t) \sim N(0, \sigma^2).
\]

The work in Ref. [22], describes how the vehicle can be steered to a destination point \( p_f \), while adding an extra component to the control that allows to perform an extra task (in this case, to increase the observability of the vehicle’s trajectory by maximizing the range information). In this way, the control law applied to the vehicle is given by

\[
u(t) = -K e(t) + N_e v(t)
\]

where \( e(t) = p(t) - p_f \), \( K \) is the proportional controller gain such that \( K > 0 \) and \( N_e v(t) \) is the orthogonal projection of vector \( v(t) \) in \( e(t) \), represented in Fig. 3 given by

\[
N_e = I - \frac{e(t)e^T(t)}{e^T(t)e(t)} \in \mathbb{R}^{n \times n}
\]

such that,

\[
e^T(t)N_e v(t) = \left(e^T(t) - \frac{e^T(t)e(t)e^T(t)}{e^T(t)e(t)} \right) = 0_{n \times n}.
\]

For this continuous time system, it is easy to prove that for any value of vector \( v(t) \), the origin \( e = 0_{n \times 1} \) is asymptotically stable. To this end, and choosing the Lyapunov candidate function (Ref. [23])

\[
V(t) = \frac{1}{2} e^T(t)e(t)
\]

with derivative

\[
\dot{V}(t) = e^T(t)\dot{e}(t) = e^T(t)u(t)
\]

we can conclude that

\[
\dot{V}(t) = -e^T(t)Ke(t) + e^T(t)N_e v(t) = -e^T(t)Ke(t) < 0
\]

which solves the problem of controlling the position of the vehicle to the final position \( p_f \). Interestingly enough, the above control law does not yield convergence of the vehicle to its final destination when a discretization of the system is used. This fact had hitherto been unknown in the literature. To better understand the issue at hand, we first consider the discrete time equations in Eq. (30), with a discretization time \( T_s \). As the system is discrete, an appropriate candidate Lyapunov function must be found. Let

\[
V(k) = e^T(k)e(k)
\]

and consider the control law

\[
u(k) = -K e(k) + N_e v(k)
\]

where the same conditions of the continuous case apply. Given

\[
e(k+1) = p(k+1) - p_f = p(k) - p_f + T_s (-K e(k) + N_e v(k)) = e(k)(1 - T_s K) + T_s N_e v(k)
\]

the variation of the Lyapunov candidate function is given by

\[
V(k+1) - V(k) = \left(e(k)(1 - T_s K) + T_s N_e v(k)\right)^T \left(e(k)(1 - T_s K) + T_s N_e v(k)\right)
\]

\[
= (1 - T_s K)^2 e^T(k)e(k) - e^T(k)e(k) + 2 T_s^2 e^T(k)N_e v(k)
\]

This means that in order to show asymptotic stability of the origin or, equivalently, that \( V(k+1)-V(k) \) is negative definite, some constraints must be imposed. To verify this, notice that \( v(k) \) must be chosen so that

\[
V(k+1) - V(k) = (1 - T_s K)^2 \|e(k)\|^2 + T_s^2 \|N_e v(k)\|^2 < 0
\]

(44)

at all times. Given the Fisher information matrix Eq. (29) and the conditions needed to control the vehicle stated above, the optimization problem that returns the velocity vector \( v(k) \), \( k = 0,..., K \) that maximizes the information available for positioning, while driving the vehicle to \( p_f \), is given by the following formulation:

\[
\min_{v(0),...,v(K)} \ -\log \det \left( FIM \left[ p(K) \right] \right) \quad \text{s.t.}
\]

\[
p(k+1) = p_0 + T_s \sum_{i=0}^k u(i)
\]

\[
u(k) = -K e(k) \sigma(||K e(k)||, \alpha u_{\text{max}}) + N_e v(k) \sigma(||N_e v(k)||, (1 - \alpha) u_{\text{max}})
\]

\[
(1 - T_s K)^2 \|e(k)\|^2 + T_s^2 \|N_e v(k)\|^2 < 0
\]

(45)

where \( \alpha \) is the percentage of the input control norm that can be dedicated to steering the vehicle to the destination point \( p_f \) and the saturation function \( \sigma(s, c) \), is described in Eq. (32).

Furthermore, some considerations need to be made regarding the null-space approach adopted. Given the discretization made to the system, when the vehicle approaches the final desired position, the orthogonal component of \( u(k) \) starts to take precedence over the overall control law, in the sense that its norm is much larger than that of \( K e(k) \). This means that, as the vehicle approaches \( p_f \), the algorithm starts to focus on maximizing information rather than smoothly driving the vehicle to its goal. The discrete nature of the algorithm is such that, during the final stages, the implementation of the optimal trajectory results in the vehicle “roaming” around \( p_f \).
In order to assure that the vehicle is smoothly driven to \( p_f \), we introduce a final safety zone \( Z_f \) around the final position, governed by Eq. (46) and (47).

\[
\text{if } p(k) \notin Z_f : \quad u(k) = -K e(k)\sigma(||K e(k)||, \alpha_{u_{\text{max}}}) + N_{e}v(k)\sigma(||N_{e}v(k)||, (1 - \alpha)u_{\text{max}}) \\
\text{if } p(k) \in Z_f : \quad u(k) = -K e(k)\sigma(||K e(k)||, u_{\text{max}}) \\
\begin{align*}
& \text{if } e(k)^T e(k) < z_f \quad \text{then } p(k) \in Z_f \\
& \text{if } e(k)^T e(k) \geq z_f \quad \text{then } p(k) \notin Z_f
\end{align*}
\]

This implementation is such that from the moment when the vehicle enters this safety zone, the algorithm neglects the \( N_{e}v(k) \) component, driving the vehicle’s trajectory straight to the objective position.

**C. Receding Horizon Navigation System Implementation**

To take into account the initial position and measurement uncertainty we implemented the navigation system using a receding horizon strategy. The optimization algorithm is run every time an estimate of the EKF algorithm is available hence, a receding horizon strategy. The optimization algorithm is run on the navigation system. If the threshold is never achieved, then the algorithm will transition to its second part when the iteration reaches the size of the previously set horizon. The architecture of the system is presented in Fig. 4. For the model presented in Eq. (30), the EKF parameters described in Eq. (4) are

\[
F(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
H(k) = \begin{bmatrix} \frac{\dot{p}_x(k)}{\sqrt{p_x^2(k) + p_y^2(k)}} & \frac{\dot{p}_y(k)}{\sqrt{p_x^2(k) + p_y^2(k)}} \end{bmatrix}
\]

where \( \dot{p}_x(k) \) and \( \dot{p}_y(k) \), are the \( x \) and \( y \) coordinates of the position estimation at time \( k \), before the correction step. In order to test both parts of the system, they were simulated with \( u_{\text{max}} = 1 \, \text{m/s} \), assuming that the range measurements were, in both cases, corrupted by white Gaussian noise \( w(k) \sim N(0, \sigma^2) \), \( \sigma^2 = 0.05 \, \text{m} \) and the values in Table II.

The described optimization problem presented (as well as all the optimization problems in this paper) was solved using *Matlab’s Optimization Toolbox* (see Ref. [24]). Observing the results in Fig. 4 we can see that the movement of the vehicle presents several sudden changes in direction. This is not an accurate representation of the movements that can, in fact, be performed by the AUV. We must find a representative model that is able to, once implemented the navigation system, provide a smooth and feasible trajectory.

### TABLE II: Navigation system implementation initial conditions: part 1

<table>
<thead>
<tr>
<th>( p_0[m] )</th>
<th>( \dot{p}_0[m] )</th>
<th>( \dot{p}_0[m] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.7</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

![Fig. 4: Navigation system implementation for part 1](image)

**IV. SINGLE BEACON SINGLE VEHICLE NAVIGATION: MODEL EXTENSION**

Consider the same problem presented in III. Now, let’s assume that we also have access to the measurements obtained by the heading rate sensor, that measures the angular velocity with respect to the \( \psi \) axis. The vehicle and measurement model becomes

\[
p(k + 1) = p(k) + T_s ||u(k)|| \begin{bmatrix} \sin \psi(k + 1) \\ \cos \psi(k + 1) \end{bmatrix} \\
\psi(k + 1) = \psi(k) + T_s \dot{r}(k) \\
y(k) = d(k) + w(k) \\
w(k) \sim N(0, \sigma^2)
\]

Here, we are able to repeat the procedure performed in section III-C in order to develop a two part navigation system that can correctly estimate a vehicle’s state, while driving it to the final desired position \( p_f \).

**A. Part 1: Reducing Initial Uncertainty**

Now, the optimization algorithm that returns the sequence of inputs \( r(k), \, k = 0, \ldots, K \) that will drive the vehicle to the optimal trajectory which maximizes the information available for positioning is given by

\[
\min_{r(0) \ldots, r(K)} -\log |\text{det} (FIM[p(K)])| \\
\text{s.t.} \quad p(k + 1) = p_0 + T_s \sum_{i=0}^{k} u_{\text{max}} \begin{bmatrix} \sin \psi(i + 1) \\ \cos \psi(i + 1) \end{bmatrix} \\
\psi(k + 1) = \psi_0 + T_s \sum_{i=0}^{k} r(i) \\
||r(k)|| \leq r_{\text{max}}
\]

where \( r_{\text{max}} \) is the maximum yaw rate that the vehicle can take, which controls the “smoothness” of the vehicle’s movements and \( u_{\text{max}} \) is the constant linear velocity applied. For the same conditions presented in section III-C and considering that there is no uncertainty on the initial heading \( \psi_0 = \dot{\psi}_0 = 0 \)
rad and \( r_{\text{max}} = 0.2 \text{ rad/s} \), the resulting trajectory is shown in Fig. 5.

Fig. 5: Navigation system implementation for part 1: extended model

B. Part 2: Reaching a Final Position

Following the null-space approach described in [3], the optimization algorithm that returns the optimal control inputs \( r(k), k = 0, \ldots, K \) and \( v(k), k = 0, \ldots, K \) that maximize the available information for positioning and, at the same time, implemented with parameters \( r \) and \( T \) part 1, followed by part 2, once a certain uncertainty threshold obtained are presented in figure 6.

\[
\min_{r(0), \ldots, r(K)} \min_{v(0), \ldots, v(K)} -\log \left[ \text{det} \left( FIM \left[ p(K) \right] \right) \right] \tag{52}
\]

\[
p(k + 1) = p_0 + T_s \sum_{i=0}^{k} u(i)
\]

\[
u(k) = -K_c(k) \sigma(||K_c(k)||, \alpha u_{\text{max}}) + N_c(v(k)) \sigma(||N_c(v(k))||, (1 - \alpha) u_{\text{max}})
\]

\[
\psi(k + 1) = \psi_0 + T_s \sum_{i=0}^{k} r(i)
\]

\[
\psi(k + 1) = \frac{u_1(k)}{u_2(k)}
\]

\[
||r(k)|| \leq r_{\text{max}}
\]

TABLE III: Navigation system implementation initial conditions: part 2

<table>
<thead>
<tr>
<th>( p_0 \approx p_0[m] )</th>
<th>( \dot{p}_0[m] )</th>
<th>( p_f[m] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10 [ \text{m} ]</td>
<td>0.0025 [ \text{m} ]</td>
<td>0 [ \text{m} ]</td>
</tr>
<tr>
<td>0 [ \text{m} ]</td>
<td>0.0025 [ \text{m} ]</td>
<td>10 [ \text{m} ]</td>
</tr>
</tbody>
</table>

For the conditions presented in table III, the system was implemented with parameters \( r_{\text{max}} = 0.2 \text{ rad/s}, \psi_0 = \frac{\pi}{2} \), \( \text{rad} \), \( K = 1 \), \( \sigma^2 = 0.05 \text{ m}^2 \) and \( z_f = 5 \text{ m} \), and the results obtained are presented in figure 6.

C. Complete Navigation Algorithm Implementation

The complete navigation system is completed by performing part 1, followed by part 2, once a certain uncertainty threshold \( T_f \) (or the maximum number of iterations) is achieved. A scheme of how both parts are connected is found in Fig. 7.

It was implemented with the parameters in table IV and considering a known initial heading \( \psi = \pi \text{ rad}, r_{\text{max}} = 0.2 \text{ rad/s}, u_{\text{max}} = 1 \text{ m/s}, K = 1, \sigma^2 = 0.05 \text{ m}^2, T_s = 1 \text{ s} \) and \( z_f = 5 \text{ m} \). The resulting simulation can be seen in figure 9.

\[
p_1(k + 1) = p_1(k) + T_s ||u_1(k)|| \begin{bmatrix} \sin \psi_1(k + 1) \\ \cos \psi_1(k + 1) \end{bmatrix}
\]

\[
\psi_1(k + 1) = \psi_1(k) + T_s r_1(k)
\]

\[
p_2(k + 1) = p_2(k) + T_s ||u_2(k)|| \begin{bmatrix} \sin \psi_2(k + 1) \\ \cos \psi_2(k + 1) \end{bmatrix}
\]

\[
\psi_2(k + 1) = \psi_2(k) + T_s r_2(k)
\]

\[
y(k) = \begin{bmatrix} d_1(k) \\ d_2(k) \\ d_3(k) \end{bmatrix} + \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix}
\]

where \( d_1(k) = \sqrt{p_x^2(k) + p_y^2(k)} \), is the range measurement of vehicle 1 with respect to the beacon at the origin.

TABLE IV: Complete navigation system implementation initial conditions

<table>
<thead>
<tr>
<th>( p_0[m] )</th>
<th>( \dot{p}_0[m] )</th>
<th>( p_f[m] )</th>
<th>( p_1[m] )</th>
<th>( T_f[m] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-19 [ \text{m} ]</td>
<td>-20 [ \text{m} ]</td>
<td>0 [ \text{m} ]</td>
<td>0 [ \text{m} ]</td>
<td>20 [ \text{m} ]</td>
</tr>
</tbody>
</table>

V. COOPERATIVE SINGLE BEACON NAVIGATION

Some applications require the concerted operation of more than one vehicle in order to perform all the necessary tasks in a given mission. In view of this, it is interesting to further expand the developed navigation system for several vehicles that, besides exchanging range information to the beacon, exchange information on the ranges among them. Consider two vehicles moving underwater, and assume that their initial positions are known with some uncertainty, where the uncertainty is described as before, the model that describes the dynamics of both vehicles is now given by

\[
p_1(k + 1) = p_1(k) + T_s ||u_1(k)|| \begin{bmatrix} \sin \psi_1(k + 1) \\ \cos \psi_1(k + 1) \end{bmatrix}
\]

\[
\psi_1(k + 1) = \psi_1(k) + T_s r_1(k)
\]

\[
p_2(k + 1) = p_2(k) + T_s ||u_2(k)|| \begin{bmatrix} \sin \psi_2(k + 1) \\ \cos \psi_2(k + 1) \end{bmatrix}
\]

\[
\psi_2(k + 1) = \psi_2(k) + T_s r_2(k)
\]

\[
y(k) = \begin{bmatrix} d_1(k) \\ d_2(k) \\ d_3(k) \end{bmatrix} + \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix}
\]
$d_2(k) = \sqrt{p_{2x}(k) + p_{2y}(k)}$, is the range measurement of vehicle 2 with respect to the beacon at the origin, and $d_3(k) = \sqrt{(p_{1x}(k) - p_{2x}(k))^2 + (p_{1y}(k) - p_{2y}(k))^2}$, is the range measurement of vehicle 1 with respect to vehicle 2, corrupted by white Gaussian noise processes characterized by $w_1(k) \sim N(0, \sigma_1^2)$, $w_2(k) \sim N(0, \sigma_2^2)$ and $w_3(k) \sim N(0, \sigma_3^2)$.

Considering these models, the system's state is now given by $x(k) = [p_1(k) \ \psi_1(k) \ \psi_2(k)]^T$, the abbreviated state given by $x'(k) = [p_{1x}(k) \ p_{1y}(k) \ p_{2x}(k) \ p_{2y}(k)]^T$, and the parameters of the extended Kalman filter described in Eq. 27, we can compute the Fisher information matrix for the new measurement model. Following the computations presented in section II-B after $K$ iterations the FIM, that considers the two vehicle model and the three range measurements is given by

$$FIM(x'(K)) = FIM(x'(0)) + \sum_{i=1}^{K} \begin{bmatrix} c_1(i) & c_2(i) & c_3(i) & c_4(i) \\ c_2(i) & c_5(i) & c_6(i) & c_7(i) \\ c_3(i) & c_6(i) & c_8(i) & c_9(i) \\ c_4(i) & c_7(i) & c_9(i) & c_{10}(i) \end{bmatrix}$$

(54)

where

$$c_1(i) = (1/\sigma_1^2)a_x^2(i) + (1/\sigma_2^2)f_x^2(i),$$
$$c_2(i) = (1/\sigma_1^2)a_x(i)a_y(i) + (1/\sigma_2^2)f_x(i)f_y(i),$$
$$c_3(i) = (1/\sigma_1^2)f_x(i)g_x(i),$$
$$c_4(i) = (1/\sigma_2^2)f_y(i)g_y(i),$$
$$c_5(i) = (1/\sigma_1^2)a_y^2(i) + (1/\sigma_2^2)f_y^2(i),$$
$$c_6(i) = (1/\sigma_1^2)a_y(i)a_y(i) + (1/\sigma_2^2)f_y(i)f_y(i),$$
$$c_7(i) = (1/\sigma_2^2)f_y(i)g_y(i),$$
$$c_8(i) = (1/\sigma_2^2)b_x^2(i) + (1/\sigma_3^2)g_x^2(i),$$
$$c_9(i) = (1/\sigma_2^2)b_x(i)g_y(i) + (1/\sigma_3^2)g_x(i)g_y(i),$$
$$c_{10}(i) = (1/\sigma_3^2)b_x^2(i) + (1/\sigma_3^2)g_y^2(i).$$

(55)

and

$$a_j(i) = \frac{p_{1y}(i)}{d_1(i)}, \quad f_j(i) = \frac{p_{1x}(i)-p_{2x}(i)}{d_1(i)}, \quad b_j(i) = \frac{p_{2y}(i)}{d_2(i)}, \quad g_j(i) = \frac{-(p_{1x}(i)-p_{2x}(i))}{d_2(i)}$$

(56)

In order to assess the efficacy of the cooperative algorithm, comparatively to the one of the single beacon setup, we implemented the first part of the cooperative navigation system. This was done by expanding the optimization algorithm given in Eq. 51 in order to solve for the sequence of inputs $r(k)$, $k = 0, ..., K$ and $v(k)$, $k = 0, ..., K$ that, once applied, allow the vehicle to maximize the determinant of the FIM derived in Eq. 54. For the same conditions as section III-A and the initial conditions in table V the system was simulated, and the eigenvalues of matrix $D(k)$ (described in Eq. 9) were compared to the ones of the matrix $D(k)$ obtained in section III. At best, the eigenvalues of $D(k)$ are equal to zero (when the covariance of the estimation equals the PCRLB). The rate at which said eigenvalues tend to zero, allows us to compare the rate at which a system is able to decrease the uncertainty regarding the position estimation.

TABLE V: Initialization variables applied in first part of the cooperative navigation system

<table>
<thead>
<tr>
<th></th>
<th>$p_1(0)$ [m]</th>
<th>$p_1(0)$ [m]</th>
<th>$\dot{P}_1(0)$ [m]</th>
<th>$\psi_1(0)$ [rad]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[9.7]</td>
<td>[10]</td>
<td>[1 0]</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>[0.2]</td>
<td>[0]</td>
<td>[0 1]</td>
<td>0</td>
</tr>
<tr>
<td>$p_2(0)$ [m]</td>
<td>$p_2(0)$ [m]</td>
<td>$\dot{P}_2(0)$ [m]</td>
<td>$\psi_2(0)$ [rad]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-9]</td>
<td>[-10]</td>
<td>[1 0]</td>
<td>\pi</td>
</tr>
</tbody>
</table>

Comparing the evolution of the eigenvalues for both cases, we can see in Fig. 9 how the two vehicle model is able to reduce the position uncertainty at a faster rate. Furthermore, we verified that the value of the determinant of the FIM is always larger for the two vehicle case.
VI. Conclusion and Future Work

This paper addressed the problem of single beacon range-based AUV navigation, in a setup where the range measurements obtained from a single beacon are noisy and contain outliers, and there is uncertainty regarding the initial estimate of the vehicle’s position. In addition, we also addressed the problem of steering the vehicle to a final target position, in the face of initial vehicle position uncertainty. We first tackled the problem using a simple model for the vehicle, and divided the combined navigation and control system in two separate complementary parts: part 1, aimed at reducing initial uncertainty in the vehicle’s position, and part 2, that has the objective of driving the vehicle to a desired position. In part 1, we showed that by solving an optimization problem for trajectory planning, with the determinant of the Fisher information matrix as a cost function, and the model characteristics as constraints at each time step, and by applying an EKF algorithm to estimate the vehicle’s position in a closed loop, receding horizon setup, the system was able to reduce the initial uncertainty. In part 2, inspired by previous results available in the literature, we used a null-space based approach to drive the vehicle to its final position, while guaranteeing sufficiently exciting vehicle motions for observability purposes. Contrary to what is common belief, we found that a simple modification of the algorithms used in continuous time to adapt them to a discrete time setting is not valid. We overcame this hurdle by changing the existing algorithms to include an added stability constraint and a final safety zone approach. However, the trajectory planning algorithm proposed can, in certain circumstances, yield non-smooth paths for the vehicle. In order to create a more realistic vehicle model, and deal with the issues regarding abrupt changes in direction, we included the vehicle’s heading in the system state. We repeated the procedure of integrating both parts of the algorithm by including in all its components, the necessary changes for the model extension. Moreover, we tested the complete system. Finally, we implemented a cooperative navigation system, using the extended vehicle model in order to assess the possible advantage of adding another vehicle and measurements of the range between the two vehicles. For a two vehicle, single beacon setup, we verified that the system is able to reduce the initial uncertainty regarding the vehicles’ positions, at a faster rate than in the single vehicle setup.

In view of the above, we can conclude that the receding horizon navigation system developed holds considerable promise for real-life implementations. Given its architecture, one can pre-define data regarding the initial uncertainty and the duration of both parts of the algorithm, as needed, making it a flexible navigation system that does not require the exact knowledge of the uncertainty area parameters. Furthermore, we concluded that the inclusion of more than one vehicle is, in fact, beneficial. Future work consists of the validation of the described system in a practical scenario, in order to fully access the efficacy of the solution for single beacon navigation proposed. The incorporation of other constraints such as inter-vehicle collision avoidance and maximum time to maneuver also warrant further research work.

References