Investment decisions upon innovative technological products

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Abstract
The scope of this thesis is Financial Mathematics, in particular Investment under Uncertainty. Its goal is to define the optimal investment policy regarding an innovative technological product, by maximizing its expected long run profit, related to the following scenarios:

1. A firm wants to invest and enter the market with a new product;
2. An active firm wants to invest and launch a new product, that totally replaces the old one;
3. An active firm wants to invest and launch a new product, while keeping temporarily the old product.

Moreover, it is assumed that an active firm produces an established product. The investment decision is irreversible, instantaneous, has an associate (sunk) cost and can be made at any time after a desired innovation level is reached.
The methodology of this thesis assumes the demand to evolve as a Geometric Brownian Motion and the innovation level accordingly to a Compound Poisson Process and derives the demand level that justifies the investment decision in each situation along with the respective comparative statics analysis. The sensitivity of optimal investment times and the impact of R&D investment in the innovation process are also analysed both analytically and numerically.
Overall, this thesis expects to support decision teams with technological products' investments, stating when is the best time to invest, the optimal production capacity and the value of the project.
Keywords: Optimal Stopping Problems; Real Options Approach; Investment Under Uncertainty; Technology Innovation.

1. Introduction

Nowadays, society is highly attached to technology, from mobile phones to fancy gadgets - and sometimes scratching the absurd. Nevertheless, all this huge demand brought changes on how IT companies should manage their investments. These should, not only pay attention to the product demand on the market, but also to technology evolution. Therefore investors began to require more complex models to support their decisions. Models must not only consider the current value of the firm, such as the ones based on the Net Present Value (NPV), but also must take into account the potential associated to future events.

In order to a firm survive on the competitive market, it needs either to develop their own technology, by investing on R&D, or to import a desired one, with an associated cost.

This work is focused on the case where the firm chooses to develop their own technology and, hence, define *a priori* the desired innovation level and investment to be made. Since it is impossible to access that a certain level of technology will be reached in a precise amount of time, the innovation process is considered to evolve randomly with time, with a rate that might be influenced by the amount of money invested. More money implies more resources and, hence, an higher evolution rate.

After reaching the desired innovation level, the firm must evaluate under which market conditions it should invest and in which type of products. The final goal of this paper is to define a new decision policy that maximizes the expected long run profit, using real options analysis, and assess the impact of R&D investment on the investment decision.

2. Background concepts

2.1. Optimal Stopping Problems

The main goal of optimal stopping problems consists on finding a stopping time such that a reward or cost function is maximized or minimized, respectively. We treat along this section the case on which we are dealing with a reward function - for further details we recommend the references (K. Ross, 2018) and (∅ksendal, 2014).

We consider a probability space probability space $(\Omega, \mathcal{F}, P)$ associated to the underlying Brownian Motion W, on which $\mathcal{F} = \{\mathcal{F}_t,\ t \geq 0\}$ corresponds to its natural filtration and the unidimensional Ito process $\mathbf{X} = \{X_t,\ t \geq 0\}$ with state space defined on R. $\mathbf{X}$ evolves accordingly to the stochastic differential equation (SDE):

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x \in R, \tag{1}$$

where b and σ are functions that satisfy ItÃť conditions given by:

$$\exists K \in (0,\infty)\ \forall t \in [0,\infty)\ \forall x, y \in R^n:$$
$$|b(t,x,\alpha) - b(t,y,\alpha)| + \|\sigma(t,x,\alpha) - \sigma(t,y,\alpha)\| \leq K|x-y|$$
$$|b(t,x,\alpha)|^2 + \|\sigma(t,x,\alpha)\| \leq K^2(1+|x|^2).$$

Moreover, we define one of the most important concepts regarding optimal stopping problems.

Definition 1. *A function $\tau : \Omega \to [0,\infty]$ is called a stopping time with respect to the filtration $\mathcal{F}$ is $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t,\ \forall t \geq 0$.*

Intuitively we have that our reward function is strongly influenced by a running cost function g, which accounts for the instantaneous earnings before the decision is taken; a terminal function h, corresponding to the long-run earnings or termination payoff associated to the observed value of the Ito process when the decision is incurred; a stopping time τ, upon which we switch from one stage to another, and a initial given state for the underlying Ito process. Here we denote the reward function by J and it is such that

$$J(x,\tau) = E^{X_0=x}\left[\left(\int_0^\tau e^{-rs}g(X_s)\,ds + e^{-r\tau}h(X_\tau)\right)1_{\{\tau<\infty\}}\right] \tag{2}$$

Denoting V as the value function associated to the reward problem, it is such that

$$V(x) = \sup_\tau J(x,\tau) \tag{3}$$

with τ taken to be a stopping time in the set of all $\{\mathcal{F}_t\}$-stopping times. Therefore the optimal stopping time τ^* is such that

$$J(x,\tau^*) = V(x),\ \forall x \in R \tag{4}$$

In order to accomplish that, we suppose a continuation and a stopping region to be respectively given by $C = \{x \in R : x < x^*\}$ and $\mathcal{S} = \{x \in R : x \geq x^*\}$. These are intuitive guesses: since we want to maximize our reward function, we expect that small values of x lead to a smaller value of $h(x)$, for which we verify $h(x) < V(x)$ meaning that is more attractive to *continue* with an instantaneous earning g; where, on the other hand, large values of x conduce to $h(x) = V(x)$, being preferable to change the strategy. Alternatively, continuation and stopping regions might be also defined as

$$C = \{x \in R : h(x) < V(x)\} \tag{5}$$
$$\mathcal{S} = \{x \in R : h(x) = V(x)\}, \tag{6}$$

respectively. We note that there is a vast literature about the definition of continuation and stopping problems, that we omit in the work. We simply motivate C and $\mathcal{S}$, as we present (in a rather informal way) in this section.

As consequence, since the optimal stopping time sets the first instant upon which is advantageous to turnaround, that is when $h(x) = V(x)$ for some $x \in R$, it might be formally defined as

$$\tau^* = \inf\{t \geq 0 : X_t \notin C\} = \inf\{t \geq 0 : X_t \in \mathcal{S}\}. \tag{7}$$

V is such that it verifies the Hamilton-Jacobi-Bellman (HJB) variational equation which, in case of a reward problem, is given by

$$\min\{rV(x) - \mathcal{L}V(x) - g(x), V(x) - h(x)\} = 0,\ x \in R. \tag{8}$$

From (5) and (6), our guess regarding the continuation region is right and that it verifies the leftmost side of (8), that is, $C = \{x \in R : rV(x) - \mathcal{L}V(x) - g(x) = 0\}$.

Notwithstanding, the HJB variational inequality provides us the main tool to characterise the continuation (and stopping) region, we need to make use of our intuition to construct it - in this case it manifests by the assumption that larger values of x lead to the choice of termination payoff.

In order to verify the validation of our guesses concerning C and $\mathcal{S}$, one can use Ito lemma. However V must be such that $V \in C^1(R)$. This is a strong assumption that is not always verified, in particularly, in the boundary between C and $\mathcal{S}$. Fortunately this hypothesis can be relaxed, by using the theorem hereunder.

Theorem 1 (Verification Theorem)**.** *Suppose $\exists\ \phi : R \to R$ such that:*

1. $\phi \in C^1(R)$.
2. $\exists \psi : R \to R$ *measurable function:*
   a) $\forall a > 0 :\ \psi$ *is Lebesgue integrable in* $[-a, a]$*;*
   b) $\forall y \in R :\ \phi'(y) - \phi'(0) = \int_0^y \psi(z)dz$.
3. $\forall x \in R :\ \min\{rV(x) - \mathcal{L}V(x) - g(x), V(x) - h(x)\} = 0$.
4. $\forall x \in R :\ \lim_{t\to\infty} e^{-rt}E[\phi(X_t)] = 0$.

Then,

1. $\forall x \in R :\ \phi(x) \geq J(x,\tau)\ \forall \tau \in \mathcal{S} \Rightarrow \phi(x) \geq V(x)$.
2. *Let C be defined as in* (5) *and τ^* as in* (7)*.*
   Then, $\phi(x) = J(x,\tau^*) = V(x)$ iff τ^* is an optimal stopping time.

Combining the HJB variational inequality with the Verification theorem we obtain a powerful tool to characterise continuation and stopping regions, by the free-boundary problem:

$$\begin{cases} V(x) - h(x) = 0 & , \ x \in \mathcal{S} \\ rV(x) - \mathcal{L}V(x) - g(x) = 0 & , \ x \in C \\ V(x) = h(x) & , \ x \in \partial C \\ V'(x) = h'(x) & , \ x \in \partial C \end{cases} \tag{9}$$

This way we assure that our guess concerning the continuation region is the right one so as the value that triggers our decision, that is, x^*. Finally, the optimal stopping time is deduced from (7) to be such that $\tau^* = \inf\{t \geq 0 : X_t \geq x\}$, which is characterised by (5).

2.2. Real Options approach

To incorporate the irreversibility and the possibility to delay an investment the real option approach was extended to support investment decisions. This chapter was based on the pioneer work of Dixit and Pindyck (Dixit & Pindyck, 1994).

In this approach, investment opportunities are seen as real options: *the firm has the right, but not the obligation, to undertake certain initiatives such as deferring, abandoning, expanding, staging or contracting a capital investment project* (S. A. Ross, Westerfield, Jaffe, & Jordan, 1988). There are three factors assumed to hold during the investment decision:

1. Future rewards are random and thus uncertain;
2. The decision is irreversible, in the sense that it is a sunk cost: the investment expenditure cannot be fully recovered;
3. The decision can be made at any time.

Alternatively to the traditional NPV analysis - on which, since the investment is irreversible, it is seen as a now or never opportunity (without the possibility to postpone the investment) -, the decision can be postponed resulting in an extra value associated to its potential.

Therefore we have that the investment problem can be stated as in (2). Taking into account the formulation in (9), the value function in the continuation region is defined a Cauchy-Euler equation, whose solution, V_C, is given by the sum of the homogeneous solution V_h with a particular solution V_p, that is, $V_C(x) = V_h(x) + V_p(x), \ \forall x \in C$.

The homogeneous solution is found by solving the (homogenous) Cauchy-Euler equation of second order associated, that is, $\frac{\sigma^2}{2}x^2V_h''(x) + \mu x V_h'(x) - rV_h(x) = 0$, whose solution has the form of

$$V_h(x) = ax^{d_1} + bx^{d_2}$$

where d_1 and d_2 are the positive and negative solutions of the quadratic equation $d^2 + \left(\frac{2\mu}{\sigma^2} - 1\right)d - \frac{2r^2}{\sigma^2} = 0$, given by

$$d_{1,2} = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{10}$$

Hence, we obtain that the value function V is of the form:

$$V(x) = \begin{cases} ax^{d_1} + bx^{d_2} + V_p(x) & , \ x < x^* \\ h(x) & , \ x \geq x^* \end{cases} \tag{11}$$

where coefficients a and b and the value that triggers the investment x^* are found by value matching ($V(x) = h(x), \ x \in \partial C$) and smooth pasting ($V'(x) = h'(x), \ x \in \partial C$) conditions along with the ones derived based on the situation treated. Finally, by knowing V we straightforward know how to characterise both continuation and stopping regions.

3. Investing and entering the market with a new product

In this chapter we consider a firm that is not active in the market and aims to invest in a new product, with an innovation level θ. Its instantaneous profit function is given by

$$\pi(X_t) = (\theta - \alpha K)KX_t, \tag{12}$$

where $\alpha > 0$ is a sensibility parameter, X_t corresponds to the demand level observed at the instant $t \geq 0$ and K the quantity wanted to be produced.

3.1. Benchmark model

We want to find when is the optimal investment time such that it leads to the maximization of the expected discounted long-term profit, assuming that this decision is taken in finite time. This can be expressed by the problem with value function F given by

$$F(x) = \sup_{\tau} E^{X_0=x}\left[\left(\int_{\tau}^{\infty} e^{-rs}\pi(X_s)\, ds - e^{-r\tau}\delta K\right)1_{\{\tau<\infty\}}\right] \tag{13}$$

Manipulating its expression and using Fubini's theorem along with the fact that $r > \mu$ we obtain the optimal standard problem

$$F(x) = \sup_{\tau} E^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K)KX_\tau}{r-\mu} - \delta K\right)1_{\{\tau<\infty\}}\right] \tag{14}$$

Its solution is given by

$$F(x) = \begin{cases} ax^{d_1} & , \ x < x_B^* \\ \frac{(\theta-\alpha K)Kx}{r-\mu} - \delta K & , \ x \geq x_B^* \end{cases}, \tag{15}$$

with $a = \left(\frac{K(\theta-\alpha K)x_B^*}{r-\mu} - \delta K\right)(x_B^*)^{-d_1} = \frac{\delta K (x_B^*)^{-d_1}}{d_1 - 1}$ and $x_B^* = \frac{d_1}{d_1-1}\frac{\delta(r-\mu)}{\theta-\alpha K}$.

Assessing how the demand value that triggers the investment, x_B^*, is influenced by market condition, we obtain the following Proposition.

Proposition 1. *The decision threshold x_B^* increases with $r, \sigma, \ K, \ \alpha$ and δ, and decreases with θ and μ.*

3.2. Capacity optimization model

Now we consider a more realistic case, in which the firm wants to take the best of its investment by choosing an appropriate capacity of production. This can be achieved by requiring that the capacity of production leads to the maximisation of the discounted long-term cash-flow. Therefore our goal is now to find when is the optimal (finite) time to invest and which is the optimal capacity associated to it. This can be stated as

$$F^*(x) = \sup_{\tau} E^{X_0=x}\Big[\max_K \Big\{e^{-r\tau}\Big(\int_{\tau}^{\infty} e^{-r(\tau-s)}\pi(X_s)\, ds - \\ - \delta K\Big)\Big\} 1_{\{\tau<\infty\}}\Big]. \quad (16)$$

This expression is simplified as

$$F^*(x) = \sup_{\tau} E^{X_0=x}\left[e^{-r\tau}\max_K \{h(X_\tau, K)\}\, 1_{\{\tau<\infty\}}\right], \quad (17)$$

where $h(x, K) = \frac{(\theta-\alpha K)Kx}{r-\mu} - \delta K$ corresponds to the terminal function. Denoting the capacity that maximizes the terminal function K^*, we have that it is given by

$$K^* := \arg\max_K h(x, K) = \frac{\theta}{2\alpha} - \frac{\delta(r-\mu)}{2\alpha x},\ \forall x \quad (18)$$

Simplifying (17), the optimization problem can be stated as

$$F^*(x) = \sup_{\tau} E^{X_0=x}\left[e^{-r\tau}\frac{(\theta X_\tau - \delta(r-\mu))^2}{4\alpha(r-\mu)X_\tau} 1_{\{\tau<\infty\}}\right], \quad (19)$$

Its solution is given by

$$F^*(x) = \begin{cases} bx^{d_1} & , \ x < x_C^* \\ e^{-r\tau}\frac{(\theta x-\delta(r-\mu))^2}{4\alpha(r-\mu)x} & , \ x \geq x_C^* \end{cases} \quad (20)$$

with $b = \left(\frac{(\theta x-\delta(r-\mu))^2}{4\alpha(r-\mu)x_C^*}\right)(x_C^*)^{-d_1} = \frac{\delta\theta}{\alpha(d_1^2-1)}\left(\frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta}\right)^{-d_1}$ and $x_C^* = \frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta}$.

Proposition 2. *The decision threshold x_C^* increases with r, σ and δ, decreases with θ and has a non-monotonic behaviour with μ. None of any other parameters have effect on x_C^*.*

Note that x_C^* is the demand value that triggers the investment on the capacity optimization model. Replacing its value in (18) we obtain the optimal capacity that the firm must invest on, when the demand reaches x_C^*. Its expression is given by

$$K_C^* = \frac{2\sigma^2\theta}{\alpha\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 3\right) - 2\mu\right)}. \quad (21)$$

Proposition 3. *The optimal capacity level K_C^* increases with μ, σ and θ, decreases with r and α, and it is independent on δ.*

4. Investing in a new product when the firm is already active

We consider now the case in which a firm that is already active, with an established product in the market, and has the opportunity to invest in a new and more profitable product, replacing the old one. The investment can only be done after the innovation level θ is reached.

By an established product we mean that it is so well recognized in the market that its unitary price is not influenced by the demand level. Therefore its instantaneous profit function is deterministic and given by

$$\pi_0 = (1 - \alpha K_0)K_0. \quad (22)$$

When the innovation breakthrough takes place, the firm has the option to invest and immediately start to produce the new product. Since this one is a new product, susceptible to the consumers' demand, its unitary price function is not deterministic and is given by

$$\pi_1(X_t) = (\theta - \alpha K_1)X_t. \quad (23)$$

4.1. Benchmark model

Assuming that the investment decision must be made in finite time, our optimal stopping problem may be written as

$$F(x) = \sup_{\tau} E^{X_0=x}\Big[\int_0^{\tau} \pi_0 e^{-rs} ds + \Big(\int_{\tau}^{\infty} \pi_1(X_s)e^{-rs} ds - \\ - e^{-r\tau}\delta K_1\Big) 1_{\{\tau<\infty\}}\Big], \quad (24)$$

where δK_1 corresponds to investment sunk costs that the firm needs to inccur. Manipulating the expression, (24) takes the form

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau} E^{X_0=x}\Big[e^{-r\tau}\Big(\frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \\ - \Big(\delta K_1 + \frac{\pi_0}{r}\Big)\Big) 1_{\{\tau<\infty\}}\Big], \quad (25)$$

which leads to the simplified value function

$$F(x) = \frac{\pi_0}{r} + \begin{cases} a_2 x^{d_1} & , \ x < x_B^* \\ \frac{(\theta-\alpha K_1)K_1 x}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) & , \ x \geq x_B^* \end{cases} \quad (26)$$

where $a_2 = \left(\delta K_1 + \frac{\pi_0}{r}\right)\frac{(x_B^*)^{-d_1}}{d_1-1}$ and the investment threshold is $x_B^* = \frac{d_1}{d_1-1}\frac{\delta K_1 + \frac{\pi_0}{r}}{\theta-\alpha K_1}\frac{r-\mu}{K_1}$.

Proposition 4. *The decision threshold x_B^* increases with δ, decreases with θ and does not have a monotonic behaviour with K_0, K_1, r. Regarding the sensibility parameter α, x_B^* increases with it when $\theta < \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$, and decreases otherwise. Regarding volatility σ, x_B^* increases with it when $d_1 \in (1, \frac{1}{2}(3+\sqrt{5}))$ and decreases when $d_1 > \frac{1}{2}(3+\sqrt{5})$.*

4.2. Capacity optimization model

As previously done, we extend the previous model. The firm is allowed to choose the optimal capacity of the new product. This optimal stopping problem can be stated as

$$\begin{aligned} F^*(x) &= \sup_{\tau} E^{X_0=x}\Big[\max_{K_1}\Big\{\int_0^{\tau} \pi_0 e^{-rs} ds + \\ &\quad + e^{-r\tau}\Big(\int_{\tau}^{\infty} \pi_1(X_s)e^{-rs}ds - \delta K_1\Big)1_{\{\tau<\infty\}}\Big\}\Big] \\ &= \frac{\pi_0}{r} + \sup_{\tau} E^{X_0=x}\Big[e^{-r\tau}\max_{K_1}\Big\{\frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \\ &\quad - \Big(\delta K_1 + \frac{\pi_0}{r}\Big)\Big\}1_{\{\tau<\infty\}}\Big], \end{aligned} \tag{27}$$

Since the expression to be maximized is the same as in the previous scenario, we obtain that the capacity that maximizes the value of the project is the one given in (18).

Replacing its value in (27), the stopping problem takes the form of

$$\begin{aligned} F^*(x) &= \frac{\pi_0}{r} + \sup_{\tau} E^{X_0=x}\Big[e^{-r\tau}\Big(\frac{\pi_0}{r} + \\ &\quad + \frac{(\theta X_\tau - \delta(r-\mu))^2}{4\alpha(r-\mu)X_\tau}\Big)\Big\}1_{\{\tau<\infty\}}\Big]. \end{aligned} \tag{28}$$

Its solution is given by

$$F^*(x) = \frac{\pi_0}{r} + \begin{cases} bx^{d_1} & , \ x < x_C^* \\ \frac{\pi_0}{r} + \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} & , \ x \geq x_C^* \end{cases}, \tag{29}$$

where $b = \left(\frac{K_0(\alpha K_0-1)}{r} + \frac{(\theta x_C^* - \delta(r-\mu))^2}{4\alpha x_C^*(r-\mu)}\right)(x_C^*)^{-d_1}$ and the demand value that triggers the investment is $x_C^* = \frac{r-\mu}{(d_1-1)\theta^2 r}\left(d_1\,(2\alpha\pi_0 + \delta\theta r) + \sqrt{(\delta\theta r)^2 + 4d_1^2\alpha\pi_0\,(\alpha\pi_0 + \delta\theta r)}\right)$.

Proposition 5. *The decision threshold x_C^* increases with δ, decreases asymptotically with θ and has a non-monotonic behaviour with μ, r, α and K_0.*

Replacing the investment threshold x_C^* in (18), we obtain that the capacity that optimizes the long term profit is given by

$$K_C^* = \frac{\theta}{2\alpha} - \frac{\delta(d_1-1)\theta^2 r}{2\alpha\left(\sqrt{\psi} + d_1(2\alpha\pi_0 + \delta\theta r)\right)} \tag{30}$$

where $\psi := 4d_1^2\alpha\pi_0(\delta\theta r + \alpha\pi_0) + \delta^2\theta^2 r^2$.

Due to the complexity of the expression above, we were only able to infer about the influence of the innovation level and the capacity of production of the old product on the optimal capacity to invest on the new product.

Proposition 6. *Asymptotically, the optimal capacity level K_C^* grows in a linear rate with θ. Also, K_C^* has a non-monotonic behaviour with K_0.*

5. Investing in a new product, allowing a simultaneous production period, when the firm is already active

Now we extend the previous scenario. We consider a firm, which has an established product in the market, and wants to find the best time to:

1. Invest and introduce in the market an innovative product with technology level θ, allowing the possibility of a simultaneous production of both old and new products;
2. Abandon the production of the old product, maintaining the new one in the market.

On the first stage only the established product is produced, whose instantaneous profit π_0 was already stated in (22).

During the second stage the firm produces both old and new products, leading to the follow profit functions

$$\pi_0^A(X_t) = (1 - \alpha K_0 - \eta K_1 X_t)K_0 \tag{31}$$

$$\pi_1^A(X_t) = (\theta - \alpha K_1 - \eta K_0 X_t)K_1. \tag{32}$$

A *cannibalisation* (or *horizontal differentiation*) parameter η is introduced to embody the crossed effect between the old and the new product. As we consider both products to be interacting in the same market, η represents the penalty that the quantity associated to a product will influence the price of the other. We consider here that this influence is the same for both products, so we can have a unique cannibalisation parameter η. By observing that $\eta \geq \alpha$ would imply a larger effect on the product price that on the quantity of production itself, we establish η to be such that $\eta < \alpha$.

By adding both profits π_0^A and π_1^A we attain the profit associated to this second stage, that is,

$$\begin{aligned} \pi_A(X_t) &= \pi_0^A(X_t) + \pi_1^A(X_t) \\ &= (1-\alpha K_0)K_0 + (\theta - \alpha K_1)K_1 X_t - 2\eta K_0 K_1 X_t \end{aligned} \tag{33}$$

Finally, on the third state, we consider that the firm abandons the old product and starts producing solely the innovative product. This one is not recognized in the market. The instantaneous profit function associated to it, π_1, is the same as the already stated in (23).

The mathematical formulation of this scenario is given by

$$\begin{aligned} F(x) &= \sup_{\tau_1} E^{X_0=x}\Big[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \sup_{\tau_2} E^{X_{\tau_1}=x_{\tau_1}}\Big[\\ &\int_{\tau_1}^{\tau_2} \pi_A(X_s)e^{-rs}ds + \int_{\tau_2}^{\infty} \pi_1(X_s)e^{-rs}ds\ 1_{\{\tau_2<\infty\}} - e^{-r\tau_1}\delta K_1\Big]1_{\{\tau_1<\infty\}}\Big] \end{aligned} \tag{34}$$

Changing integration variables and solving the deterministic integral concerning π_0, we obtain from (34) that

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} E^{X_0=x}\Big[e^{-r\tau_1}\Big(\sup_{\tau_2} E^{X_{\tau_1}}\Big[\int_0^{\tau_2-\tau_1} (\pi_A(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds + + \int_{\tau_2-\tau_1}^{\infty} (\pi_1(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds\Big] - \delta K_1\Big)\Big]. \tag{35}$$

Observe that in (35) we have two optimal stopping times to derive. First, we solve the optimal problem w.r.t. to the *most recent* optimal stopping time, that is the time to remove the old product, τ_2. Then, using the obtained solution we solve the optimal stopping problem w.r.t. the optimal stopping time to introduce the innovative product, τ_1.

Consider F_2 the value function associated to the optimal stopping time τ_2. From (35), it follows that its expression corresponds to

$$F_2(x) = \sup_{\tau_2} E^{X_{\tau_1}=x_{\tau_1}}\Big[\int_0^{\tau_2-\tau_1} \Big(\pi_A(X_{\tau_1+s}) - \pi_0\Big) e^{-rs} ds + + \int_{\tau_2-\tau_1}^{\infty} \Big(\pi_1(X_{\tau_1+s}) - \pi_0\Big) e^{-rs} ds\Big] \tag{36}$$

Denote τ as the time, after the introduction of the innovative product, the firm should remove the old product, that is $\tau := \tau_2 - \tau_1$. Using the Strong Markov property associated to the demand process, the Tower Rule and Fubini's Theorem, (36) simplifies to

$$F_2(x) = \sup_{\tau} E^{X_0=x}\left[\int_0^{\tau} g(X_s) e^{-rs} ds + e^{-r\tau} h(X_\tau)\right] \tag{37}$$

$$= \sup_{\tau} E^{X_0=x}\Big[\int_0^{\tau} \Big(\pi_0^A(X_s) + \pi_1^A(X_s) - \pi_0\Big) e^{-rs} ds + + e^{-r\tau}\Big(\frac{(\theta - \alpha K_1)K_1}{r-\mu} X_\tau - \frac{\pi_0}{r}\Big)\Big], \tag{38}$$

Since the HJB variational inequality associated to (38) implies that the non-homogeneous PDE

$$\frac{\sigma^2}{2} x^2 F_2''(x) + \mu x F_2'(x) - r F_2(x) + g(x) = 0 \tag{39}$$

holds for any demand value in the continuation region, it follows that its solution F_2 takes the form of

$$F_2(x) = F_{2,h}(x) + F_{2,p},\ \forall x \in C, \tag{40}$$

where $F_{2,h}$ corresponds to the solution to the homogeneous version of the PDE in (39), $F_{2,p}$ to a particular solution of (39) and C to the continuation region of the form $\{x \in R : x < x_2^*\}$.

We first calculate the particular solution $F_{2,p}$. By considering $F_{2,p}''(x) = 0$ and using expression (39), a particular solution is found to be

$$F_{2,p}(x) = \frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r - \mu} x \xrightarrow[x\to 0]{} 0 \tag{41}$$

Since there is no possibility of having a project having a negative value it follows $F_2(x) \geq 0,\ \forall x \in C$. From (41), we obtain that $F_{2,h} = a_2 x^{d_1} + b_2 x^{d_2}$ simplifies to $F_{2,h} = a_2 x^{d_1}$, guaranteeing this way that $\lim_{x\to 0} F_2(x) = \lim_{x\to 0} F_{2,h}(x) + F_{2,p}(x) = 0$.

Therefore its solution is given by

$$F^*(x) = \begin{cases} a_2 x^{d_1} + \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} x & ,\ x < x_2^* \\ \frac{(\theta-\alpha K_1)K_1}{r-\mu} x - \frac{\pi_0}{r} & ,\ x \geq x_2^* \end{cases} \tag{42}$$

with $a_2 = \left(\frac{2\eta K_0 K_1}{r-\mu} x^* - \frac{\pi_0}{r}\right)(x_2^*)^{-d_1}$ and the demand level that triggers the exiting of the old product $x_2^* = \frac{d_1}{d_1 - 1} \frac{1-\alpha K_0}{2\eta K_1} \frac{r-\mu}{r}$.

Replacing (42) in (35), we obtain that the value function associated to this third scenario is given by

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} E^{X_0=x}\Big[e^{-r\tau_1}\Big(\frac{(\theta - \alpha K_1)K_1}{r - \mu} X_{\tau_1} + + \left(a X_{\tau_1}^{d_1} - \frac{2\eta K_0 K_1}{r-\mu} X_{\tau_1}\right) 1_{\{X_{\tau_1} < x_2^*\}} - - \frac{\pi_0}{r} 1_{\{X_{\tau_1} \geq x_2^*\}} - \delta K_1\Big)\Big], \tag{43}$$

which is again an optimal problem with null running cost function. However, it has two terminal cost functions defined on disjoint domains. Thereafter, we obtain two optimal times associated to the adoption of the innovative product, depending on whether we allow a simultaneous production period or not. These are depending on the following demand thresholds:

- $x_{1,A}^*$: the threshold that triggers the investment and the addition of the new product to the market, starting a period of simultaneous production. Therefore it is associated to the region $X_{\tau_1} < x_2^*$ so as to the following stopping problem

$$\sup_{\tau_1} E^{X_0=x}\left[e^{-r\tau_1}\left(\frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} X_{\tau_1} + a_2 X_{\tau_1}^{d_1} - \delta K_1\right)\right] \tag{44}$$

  where the decision is made on finite time τ with probability 1. Note that when the demand observes larger values than x_2^* its respective value is given by the expression on the rightmost side of (43).

- $x_{1,R}^*$: the threshold associated to the region $X_{\tau_1} \geq x_2^*$, for which the firm invest in the new product by immediately replacing the one established in the market.

Considering that the investment decision is made in finite time with probability 1, this situation leads to the following stopping problem

$$\sup_{\tau_1} E^{X_0=x}\left[e^{-r\tau_1}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_{\tau_1}-\frac{\pi_0}{r}-\delta K_1\right)\right] \quad (45)$$

Solving both optimal stopping problems, we obtain that the value function F in (35) may be written as

$$\begin{aligned}
F(x) &= \frac{\pi_0}{r}+\\
&+ a_{1,A}x^{d_1}1_{\{x<x_{1,A}\ \wedge\ \eta<\eta^*\}}+\\
&+ a_2x^{d_1}+\frac{(\theta-\alpha K_1)K_1-2\eta K_0K_1}{r-\mu}x-\delta K_1 1_{\{x_{1,A}\le x<x_2^*\ \wedge\ \eta<\eta^*\}}+\\
&+ a_{1,R}x^{d_1}1_{\{x<x_{1,R}^*\ \wedge\ \eta\geq\eta^*\}}+\\
&+ \frac{(\theta-\alpha K_1)K_1x}{r-\mu}-\frac{\pi_0}{r}-\delta K_1 1_{\{(x>x_2^*\ \wedge\ \eta<\eta^*)\ \dot{\vee}\ (x>x_{1,R}^*\ \wedge\ \eta\geq\eta^*)\}}
\end{aligned}$$

where the demand thresholds corresponds to $x_{1,A}^*=\frac{d_1}{d_1-1}\frac{\delta K_1(r-\mu)}{(\theta-\alpha K_1)K_1-2\eta K_0K_1}$, $x_{1,R}^*=\frac{d_1}{d_1-1}\frac{\delta K_1+\frac{\pi_0}{r}}{\theta-\alpha K_1}\frac{r-\mu}{K_1}$, $x_2^*=\frac{d_1}{d_1-1}\frac{1-\alpha K_0}{2\eta K_1}\frac{r-\mu}{r}$, the coefficients to $a_{1,A}=a_2x_2^*+\frac{((\theta-\alpha K_1)K_1-2\eta K_1K_0)}{r-\mu}(x_{1,A}^*)^{1-d_1}-\delta K_1(x_{1,A}^*)^{-d_1}$, $a_{1,R}=\left(\delta K_1+\frac{\pi_0}{r}\right)\frac{(x_B^*)^{-d_1}}{d_1-1}$ and $a_2=\left(\frac{2\eta K_0K_1}{r-\mu}x^*-\frac{\pi_0}{r}\right)(x_2^*)^{-d_1}$ and the cannibalization threshold to $\eta^*=\frac{(1-\alpha K_0)(\theta-\alpha K_1)}{2(\delta K_1r+\pi_0)}$.

The value η^* corresponds to the cannibalization value such that $x_{1,A}^*=x_{1,R}^*$. For cannibalization values η smaller than η^* the firm should adopt a simultaneous production period followed by the total replacement of the old product in the market. For cannibalization values η greater than η^* the firm should invest and immediately replace the old product for the innovative one. This scenario corresponds to the one studied previously on the benchmark model of Section 4.

Concerning both investment thresholds $x_{1,A}^*$ and x_2^*, the following propositions hold.

Proposition 7. *The decision threshold $x_{1,A}^*$ increases with η, δ, σ, α, K_0 and K_1 and decreases with θ.*

Proposition 8. *The decision threshold x_2^* increases with σ and decreases with η, α, K_0 and K_1.*

6. Simulation study of the optimal investment times

We now evaluate the optimal investment times by simulating sample paths and assessing the time until the investment should be done. First consider we study the influence of initial demand values on optimal investing times, by analysing statistical measures of the simulated series (in particular: mean, median and standard deviation) Secondly, we analyse how the estimated mean of optimal investment times behaves with volatility and assess its relation with changes either on the associated threshold value or optimal capacity level.

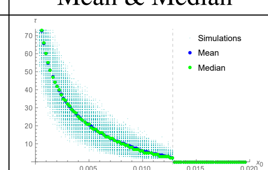
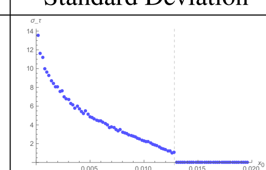
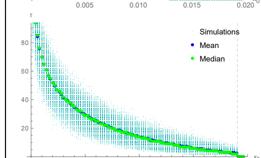
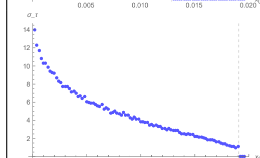
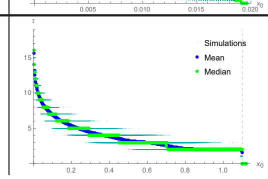
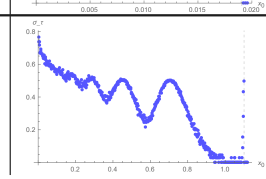
6.1. Sensitivity of the optimal investment times w.r.t. initial demand value, x_0

The following methodology was implemented in order to assess the influence of the initial value chosen on the waiting time until its optimal to invest on the new product:

1. Fix parameters of interest regarding the scenario considered;
2. Simulate 1000 sample paths of demand processes using `generalStopTime` function.
   Output: collection of values $(x_{0j},\{\tau_{ji}\}_{i\in\{1,\ldots,1000\}})$;
3. For each x_{0j},
   calculate mean, median and standard deviation of observed optimal investment times τ_j using obtained values $\{\tau_{ji}\}_{i\in\{1,\ldots,1000\}}$;
4. Plot the results obtained:
   • Simulated investment times along with the estimated mean and median;
   • Simulated standard variation, calculated from the sample variance.

Since the results obtained were similar between the first and the second scenarios, treated on sections 3 and 4, respectively, we only mention the results w.r.t. the first and the third scenarios.

Table 1
Sensitivity analysis of estimated mean, median and standard deviation of the optimal investment time for each of the three situations studied, regarding different initial values x_0 and the benchmark (1-B) and capacity optimization models (1-CO) regarding the scenario on which a firm wants to enter the market with an innovative product and the scenario on which an active firm wants to enter the market with a new product, allowing a simultaneous production period (3).

Model	Mean & Median	Standard Deviation
1B		
1CO		
3		

From the obtained results presented on Table 1, we conclude:

- The smaller the initial demand x_0, the longer the firm needs to wait to invest;
- The larger the demand value that triggers the investment, the longer the firm needs to wait to be advantageous to invest;
- The mean and the median of investment times are approximately equal, suggesting that the distribution of investment times is approximately symmetric;
- Both mean and median are of the same order of magnitude as the standard deviation. This suggests a high variability of the results obtained. Increasing the amount of sample paths simulated we observe no significant decrease of the variability, only a considerable longer computational running time.

The noticeable logarithmic behaviour of the simulated optimal times' mean comes from the fact that

$$E^{X_0}[X_\tau] = X_0 e^{\mu\tau} \;\Leftrightarrow\; \tau = \frac{1}{\mu}\log\left|\frac{E^{X_0}[X_\tau]}{X_0}\right| \tag{46}$$

Considering the estimate of the optimal investment time given by the sample mean of the simulated exercising times (with respect to each initial value tested; here denoted by $\hat{\tau}$) and noticing that X_τ, written above, corresponds to a threshold (with respect to a particular investment situation; here denoted by x^*), it follows that

$$\hat{\tau} = \frac{1}{\mu}\log\left|\frac{x^*}{X_0}\right|, \quad \text{considering } x^* > X_0. \tag{47}$$

The case $x^* \leq X_0$ is not here addressed since it immediately implies that the firm invests right away, resulting in a null expected waiting time.

Taking now into account the optimal time to stop the production of the established product, studied on Chapter **??**, we obtain the results presented on Table 2.

Table 2
Sensitivity analysis of estimated mean, median and standard deviation of optimal time to decide stop the production of the established product when considering an observed demand $x^*_{1,A}$.

Mean	Median	Standard Deviation
5.49	5.00	0.77

These values are read as time units after the entrance of the innovative product in the market, which is attained once the demand reaches $x^*_{1,A}$. Therefore a single initial value is considered is this case.

6.2. Sensitivity of the optimal investment times with respect to volatility, σ

The following methodology was implemented in order to assess the influence of the uncertainty of the market on the waiting time until its optimal to invest on the new product:

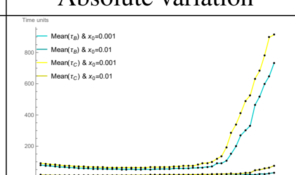
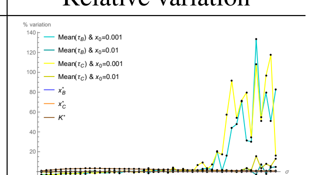
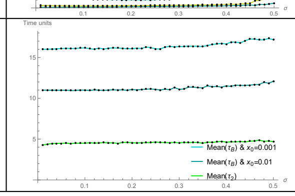
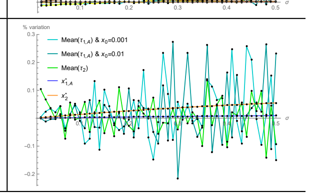
1. Fix parameters of interest regarding the scenario considered;
2. Simulate 500 sample paths of demand processes using `estTau` function.
   Output: collection of values $(\sigma_j, \bar{\tau}_j)$ with $\bar{\tau}_j$ representing the mean of the investment times concerning the 500 sample paths simulated w.r.t. σ_j;
3. For each x_{0j},
   calculate mean, median and standard deviation of observed optimal investment times τ_j using obtained values $\{\tau_{ji}\}_{i\in\{1,\ldots,1000\}}$;
4. Plot the results obtained:
   • Estimated expected optimal investment times w.r.t. σ;
   • variation rate of estimated expected optimal investment times w.r.t. σ calculated accordingly to

$$\triangle_n\% = \frac{f(\sigma_j) - f(\sigma_{j-1})}{\sigma_j - \sigma_{j-1}} \times 100\% \tag{48}$$

   with $\sigma_j = 0.01 + j \times 0.01, \quad j = \left\{0, \ldots, \frac{\sigma Max}{\sigma Step} = 50\right\}$, where f corresponds to the sample mean of optimal times simulated w.r.t. a certain volatility.

Again, only the first and third scenarios are here represented.

Table 3
Sensitivity analysis of the (estimated) mean of optimal investment time w.r.t. the demand's volatility, regarding different initial values $x_0 \in \{0.001,\ 0.01\}$ *and the benchmark model of the first investment scenario and the third investment scenario.*

Ch.	Absolute variation	Relative variation
1		
3		

Regarding the case of the first investment scenario, treated on Section 3, the following conclusions are pointed:

1. The mean time to undertake the decision is stable for a significant range of values σ (i.e., the mean seems not to change significantly for values $\sigma \leq 0.35$). But then it shows a considerable increasing trend, more pronounced on the optimal capacity model;

2. The observed mean of the investment times has a non-monotonic behaviour:

   - For small values of σ (approximately smaller than 0.2), as slight decreasing tendency, which leads to anticipate the investment decision, is observed as the volatility increases. This is not an expected result from real option approach - for which an increasing volatility leads to a late decision (Dixit & Pindyck, 1994) -, however it might be justifiable by the crossed-effect of a small rate growing of the thresholds for small values of σ and the impact of a small volatility level on the demand process;
   - For large values of σ (approximately greater than 0.2), we observe that a larger uncertainty on the market seems to likely postpone the investment decision;

3. There seems to exist no relation between the behaviour of the mean time to undertake the decision and the demand threshold level or the optimal capacity associated.

The values obtained regarding the third scenario do not vary as the ones in the first scenario, due to the values of parameters fixed.

6.3. Sensitivity of the optimal investment times with respect to the crossed-effect between initial demand value, x_0, and volatility, σ

Additionally we study the crossed-effect of both initial demand value and volatility. This is achieved by analysing the frequency of the optimal investment times obtained through their histogram representation, simulating 500 demand sample paths until they reach either the respective demand threshold or the time horizon, here set as 1500 time units.

The results obtained among scenarios were similar. Therefore only the first scenario is here presented.

We conclude the following aspects:

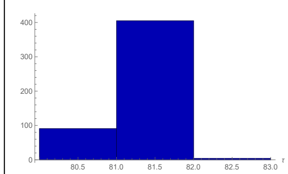
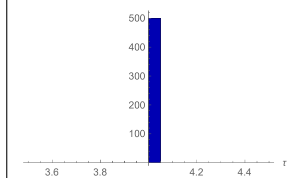
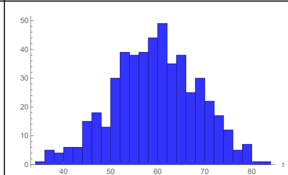
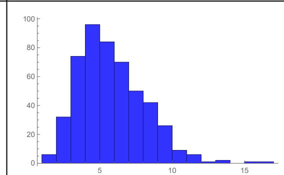
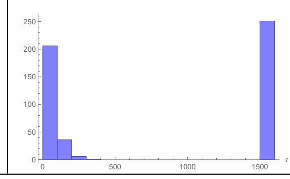
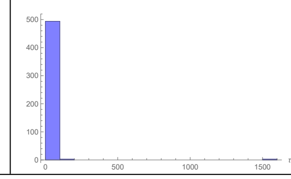
1. The smaller the uncertainty in the market, the lower is the variability of the waiting time to invest. However, as previously concluded from Table 3, this is not associated to the lowest waiting times. When considering $\sigma = 0.001$ and $x_0 = 0.1$ we observe that for all the demand processes simulated, the firm enters the market after 4 time units. This is not a surprising, as we are choosing a really low value of volatility, leading to an almost deterministic process;

2. The larger the volatility, the larger is the number of demand sample paths that do not reach the demand level that triggers the investment within the horizon defined (of 1500 time units). This result is in accordance to the significant growth of the waiting time when on markets with a large uncertainty, as observed on Table 3;

3. A significant reduction on the waiting time until investment is observed when the initial demand value increases. This result is also noticeable in Table 1, on which the waiting time seems to have a logarithmic dependence on the initial value chosen;

4. Assesing the normality of the optimal investment times,, recurring to a Shapiro-Wilk test, we obtain that solely for $x = 0.001$ and $\sigma = 0.1$, the normality is verified for the usual significance levels (1%, 5%, 10%, 15%) - since we obtain a p-value $\simeq 0.264$.

Table 4
Histograms of simulated optimal investment times of the benchmark model w.r.t. the situation for which a firm is not active in the market and wants to enter it with a new product (scenario 1), considering initial demand values $x_0 = \{0.001, 0.01\}$ and demand's volatilities $\sigma = \{0.001, 0.1, 0.5\}$.

σ	$x = 0.001$	$x = 0.01$
0.001		
0.1		
0.5		

7. Value of the project: the influence of the number of innovation jumps

Here we calculate the influence the R&D investment on the value function, taking into account the waiting time until the innovation breakthrough takes place. This allows us to state a relation between the R&D investment and the time a firm needs to wait until it is in a favourable situation to invest.

Innovation levels are assumed to increase by jumps. Therefore the innovation process $\Theta = \{\theta_t,\ t \geq 0\}$ is defined as an homogenous Compound Poisson Process, with constant rate given by $\lambda(R) = R^\gamma$, with R corresponding to the investment in the R&D department, and γ a sensitivity parameter taking values in $(0, 1]$. It can be then expressed as

$$\theta_t = \theta_0 + uN_t,\ t \geq 0. \tag{49}$$

with θ_0 denoting the state of technology at the initial point in time, $u > 0$ a fixed jump size and $\{N_t,\ t \geq 0\}$ the jump process which follows a Poisson Process with rate $\lambda(R)$.

Considering now S_n to be the random variable that represents the waiting time until the n-th jump in the process is observed, accordingly to (K. Ross, 2018), S_n follows an Erlang distribution with shape parameter n and rate parameter $\lambda(R)$, the same as in the jump process, that is,

$$S_n = \min\{t \geq 0 : N(t) = n\} \sim Erlang(n, \lambda(R)). \tag{50}$$

Denoting F as the value function associated to a certain investment situation and V as the maximized expected discounted value function minus the R&D investment needed to be made, is now given by

$$\begin{aligned} V_n(x) &= \max_R E[e^{-rS_n} F(x) - R] \\ &= \max_R \left\{ \int_0^\infty f_{S_n}(t) e^{-rt} F(x) dt - R \right\} \\ &= \max_R \left\{ \int_0^\infty \frac{\lambda(R)^n t^{n-1}}{(n-1)!} e^{-\lambda(R)t} e^{-rt} F(x) dt - R \right\} \\ &= \max_R \left\{ \left(\frac{R^\gamma}{R^\gamma + r} \right)^n F(x) - R \right\} \end{aligned} \tag{51}$$

where f_{S_n} corresponds to the probability density function of an Erlang with shape parameter n and rate parameter $\lambda(R)$.

Since V_n is expected to be greater or equal to 0 $\forall n \in N$, the following restriction must hold

$$R^{\gamma n} F(x) - R(R^\gamma + r)^n \geq 0 \ \Leftrightarrow\ F(x) \geq \frac{(R^\gamma + r)^n}{R^{\gamma n - 1}}. \tag{52}$$

Then the optimal investment in R&D centres, R^*, should be such that the following holds:

$$\frac{\partial}{\partial R} \left(\left(\frac{R^\gamma}{R^\gamma + r} \right)^n F(x) - R \right) = \frac{\gamma F(x) nr \left(\frac{R^\gamma}{r + R^\gamma} \right)^n - rR + R^{\gamma+1}}{rR + R^{\gamma+1}} = 0$$

$$\frac{\partial^2}{\partial R^2} \left(\left(\frac{R^\gamma}{R^\gamma + r} \right)^n F(x) - R \right) = \frac{\gamma F(x) nr \left(\frac{R^\gamma}{r + R^\gamma} \right)^n (r(\gamma n - 1) - (\gamma + 1) R^\gamma)}{R^2 (r + R^\gamma)^2} < 0$$

Due to the complexity of both expressions above represented, we are not able to deduce an expression for stationary points neither to assess if the second partial derivative is negative. And, consequently, we need to resort to numerical calculations. Main results can be consulted in the thesis.

8. Conclusion

The main conclusions are presented along this paper. We highlight the need and the relevance of decision support research, taking into account the needs and realities of nowadays. IT companies are an example, but we truly hope this work can also be useful for any company whose products depend on a desired characteristic.

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