Complexified Symplectomorphisms and Geometric Quantization

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November 2017

Abstract

In this text, we study complex, symplectic and Kähler geometry. We present an overview of [4]. Starting with a Kähler manifold, we act on its complex structure by pulling back along the complexified flow of an Hamiltonian vector field. We prove that the resulting Kähler metrics form a curve in an (infinite dimensional) Riemannian manifold whose points are Kähler potentials and that this curve is actually a geodesic with respect to the metric (which is the Mabuchi metric). We study Kähler structures on the cotangent bundle of a Lie group G. We explain the known Kähler structures of Hall and Kirwin and of Kirwin, Mourão, and Nunes and relate the two. In [3], the authors prove that one can define Kähler structures on T^*G using Ad-invariant complexifier functions. We prove that the condition of Ad-invariance can be replaced by a more general set of equations. Finally, we give an example of geometric quantization of T^*G. We introduce a line bundle L on T^*G and a polarization $\mathcal{P}_\tau$ defined by pulling back the vertical polarization along the complexified flow of an Hamiltonian vector field. From these structures, we define a Hilbert space whose elements are integrable $\mathcal{P}_\tau$-polarized sections of L.
Keywords: Kähler geometry, geometric quantization, Mabuchi metric, Lie group, cotangent bundle

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1. Geodesics on the Space of Kähler Metrics of a Manifold: an overview of [4]

1.1. Action of a complexified analytic flow on a complex structure

Let (M, J_0) be a compact, complex manifold. In [4], the authors explain how it is possible to introduce a new complex structure on M, J_τ, given a real analytic vector field X. Informally speaking, if $z_0^1, ..., z_0^n$

are holomorphic coordinates associated to J_0, then $e^{\tau X}\cdot z_0^1, ..., e^{\tau X}\cdot z_0^n$ will be coordinates associated to J_τ, where the exponentials should be interpreted as a series in powers of X.

Theorem 1.1 (Mourão and Nunes, [4]). *Let (M, J_0) be a compact, complex manifold, and X be a real analytic vector field on M. There exists a $T > 0$ such that for all $\tau \in B(0, T)$, there exists an integrable almost complex structure J_τ satisfying:*

(i) (Definition of J_τ) Let $p \in M$, and let $(U_\alpha, z_0^1, ..., z_0^n)$ be a J_0-holomorphic coordinate neighborhood of p. Then, there exists an open neighborhood $V_{\alpha,p}$ of p such that:

- *$p \in V_{\alpha,p} \subset \overline{V_{\alpha,p}} \subset U_\alpha$;*
- *$V_{\alpha,p}$ has compact closure;*
- *the series $z_\tau^j \coloneqq e^{\tau X}\cdot z_0^j$ are uniformly convergent in $V_{\alpha,p}$;*
- *$(V_{\alpha,p}, z_\tau^1, ..., z_\tau^n)$ is a J_τ-holomorphic coordinate neighborhood of p.*

(ii) There exists a unique biholomorphism $\varphi_\tau \colon (M, J_\tau) \longrightarrow (M, J_0)$ that satisfies the following property: for all $p \in M$, there exists $(U_\alpha, z_0^1, ..., z_0^n)$ a J_0-holomorphic coordinate neighborhood of p and there exists $(V_{\alpha,p}, z_\tau^1, ..., z_\tau^n)$ a J_τ-holomorphic coordinate neighborhood satisfying the conditions of (i), such that $\varphi_\tau(V_{\alpha,p}) \subset U_\alpha$ and $z_\tau^j = z_0^j \circ \varphi_\tau$.

Proof. See theorems 2.5 and 2.6. in [4]. □

Remark 1.2. The biholomorphisms φ_τ do not depend only on τ, but also on the initial complex structure J_0. If J is any complex structure on M, denote by $\varphi_{\tau,J}$ the unique biholomorphism $\varphi_{\tau,J} \colon (M, J_\tau) \longrightarrow (M, J)$ such that $z^j \circ \varphi_{\tau,J} = e^{\tau X}\cdot z^j$ (in this notation, the biholomorphisms of the previous theorem are written $\varphi_\tau = \varphi_{\tau,J_0}$). If $\tau, \sigma \in \mathbb{C}$, then the following diagram is a commutative diagram of biholomorphisms:

$$\begin{array}{ccc} (M, J_{\tau+\sigma}) & \xrightarrow{\varphi_{\tau,J_\sigma}} & (M, J_\sigma) \\ {\scriptstyle \varphi_{\sigma,J_\tau}}\downarrow & \searrow^{\varphi_{\tau+\sigma,J_0}} & \downarrow{\scriptstyle \varphi_{\sigma,J_0}} \\ (M, J_\tau) & \xrightarrow[\varphi_{\tau,J_0}]{} & (M, J_0) \end{array}$$

We now note that:

- If $t \in \mathbb{R}$, since $(\phi_X^t)^* z^j = e^{tX}\cdot z^j$, then φ_t is the flow of X: $\varphi_t = \phi_X^t$. Therefore it does not depend on J_0. In this case, $\varphi_{t+s} = \varphi_t \circ \varphi_s$.
- If $\tau, \sigma \in \mathbb{C} \setminus \mathbb{R}$, then in general it is not true that $\varphi_{\tau+\sigma} = \varphi_\tau \circ \varphi_\sigma$, which means that φ is not a flow. What we can say about about $\varphi_{\tau+\sigma}$ is that $\varphi_{\tau+\sigma} = \varphi_{\tau+\sigma,J_0} = \varphi_{\tau,J_0} \circ \varphi_{\sigma,J_\tau} = \varphi_{\sigma+\tau,J_0} = \varphi_{\sigma,J_0} \circ \varphi_{\tau,J_\sigma}$.

1.2. Action of a complexified analytic flow on a Kähler structure

Starting with a Kähler manifold (M, ω, J_0), and applying theorem 1.1 using an Hamiltonian vector field X_h, we get a manifold M equipped with structures (M, ω, J_τ). The contents of the next theorem are that (at least for small $|\tau|$) (M, ω_0, J_τ) is a Kähler manifold.

Theorem 1.3 (Mourão and Nunes, [4]). *Let (M, ω_0, J_0, g_0) be a compact Kähler manifold, with ω_0, J_0, g_0 real analytic. Let $h \in C^{an}(M)$. Then, there exists a $T > 0$ such that:*

(i) For all $\tau \in B(0, T)$ $(M, \omega_0, J_\tau, g_\tau)$ is a Kähler manifold, where:

- *J_τ is the complex structure defined by applying theorem 1.1 to (M, ω_0, J_0, g_0) with the vector field X_h;*
- *$g_\tau \coloneqq \omega_0(\cdot, J_\tau \cdot)$.*

(ii) (Kähler potential for $(M, \omega_0, J_\tau, g_\tau)$) For all $p \in M$ there exists:

- *$(U_\alpha, z_0^1, ..., z_0^n)$ a J_0-holomorphic coordinate neighborhood of p;*
- *$\kappa_0 \colon U_\alpha \longrightarrow \mathbb{R}$ a local Kähler potential for (M, ω_0, J_0);*
- *$V_{\alpha,p}$ an open set such that:*

- $p \in V_{\alpha,p} \subset \overline{V_{\alpha,p}} \subset U_\alpha$;
- $V_{\alpha,p}$ *has compact closure;*
- *For all* $\tau \in B(0,T)$, $\varphi_\tau(V_{\alpha,p}) \subset U_\alpha$;
- *For all* $\tau \in B(0,T)$, κ_τ *defined by*

$$\theta \coloneqq \frac{i}{2}(\partial_0 - \bar{\partial}_0)\kappa_0 \tag{1a}$$

$$\alpha_t \coloneqq \int_0^t e^{sX_h}(\theta(X_h))ds \tag{1b}$$

$$\alpha_\tau \coloneqq \textit{unique complex analytic continuation of } \alpha_t \tag{1c}$$

$$\psi_\tau \coloneqq -\frac{i}{2}e^{\tau X_h} \cdot \kappa_0 + \tau h - \alpha_\tau \tag{1d}$$

$$\kappa_\tau \coloneqq -2 \operatorname{Im} \psi_\tau \tag{1e}$$

is well defined on $V_{\alpha,p}$ *(because the Lie series in (1b), (1c) and (1d) are uniformly convergent), and is a local Kähler potential for* (M, ω_0, J_τ).

Proof. Take the T from theorem 1.1, and restrict it such that $\omega_0(\cdot, J^\tau \cdot)$ is positive definite for all $\tau \in B(0,T)$ (this can be done because of continuity). $\omega_0(\cdot, J^\tau \cdot)$ being symmetric is equivalent to $\omega_0\left(\frac{\partial}{\partial z_\tau^j}, \frac{\partial}{\partial z_\tau^k}\right) = 0$, which is true because of the properties of the exponential, $e^{\tau X_h}\omega = \omega$ (X_h is Hamiltonian, therefore it preserves the symplectic form), and $e^{\tau X_h}\frac{\partial}{\partial z_0^j} = \frac{\partial}{\partial z_\tau^j}$. The proof that κ_τ is a Kähler potential is by computation. One needs to prove the following formulas, in the given order:

$$d\theta = -\omega_0 \tag{2}$$

$$\alpha_\tau = \sum_{k=1}^{\infty} \frac{\tau^k}{k!} X_h^{k-1}(\theta(X_h)) \tag{3}$$

$$d\alpha_\tau = e^{\tau d\iota_{X_h}}\theta - \theta \tag{4}$$

$$e^{\tau L_{X_h}}\theta = \theta - \tau dh + d\alpha_\tau \tag{5}$$

$$\theta^{(0,1)_\tau} = \bar{\partial}_\tau \psi_\tau \tag{6}$$

$$i\partial_\tau \bar{\partial}_\tau \kappa_\tau = -d\theta \tag{7}$$

□

1.3. The space of Kähler metrics

Definition 1.4. The **space of Kähler metrics on** M **in the cohomology class** $[\omega_0]$ is:

$$\mathcal{H}(\omega_0, J_0) \coloneqq \{\varphi^*\omega_0 \mid \varphi \in \mathrm{Diff}(M), [\varphi^*\omega_0] = [\omega_0], (M, \varphi^*\omega, J_0) \text{ is Kähler}\}. \tag{8}$$

The **space of Kähler potentials on** M **with base point** $[\omega_0]$ is:

$$\mathcal{K}(\omega_0, J_0) \coloneqq \{\phi \in C^\infty(M) \mid \tilde{g} \coloneqq (\omega_0 + i\partial_0\bar{\partial}_0\phi)(\cdot, J_0\cdot) \text{ is positive definite}\}. \tag{9}$$

We now list the facts about these spaces that will be relevant to us.

- As stated in [1], by the $\partial\bar{\partial}$-lemma any other Kähler metric that is cohomologous to ω_0 can be written using a global Kähler potential. This implies that $\mathcal{H}(\omega_0, J_0)$ can be identified with the space of Kähler potentials, modulo constants:

$$\mathcal{H}(\omega_0, J_0) \cong \mathcal{K}(\omega_0, J_0)/\mathbb{R} \tag{10a}$$

$$\cong \left\{\phi \in C^\infty(M) \mid \tilde{g} \coloneqq (\omega_0 + i\partial_0\bar{\partial}_0\phi)(\cdot, J_0\cdot) \succ 0, \int_M \phi\omega_0^n = 0\right\}. \tag{10b}$$

- $\mathcal{H}(\omega_0, J_0)$ can be regarded as an infinite dimensional manifold, where a tangent vector $\delta\phi_0$ to $\mathcal{H}(\omega_0, J_0)$ at ϕ_0 is a function on M. To see this, consider a curve in $\mathcal{H}(\omega_0, J_0)$, $t \mapsto \phi_t \in C^\infty(M)$ (where we have chosen a family of representatives for the equivalence classes such that $\int_M \phi_t\omega_0^n = 0$, which makes the map $t \mapsto \phi_t$ smooth). The derivative with respect to t is another function:

$$\delta\phi_0 \coloneqq \frac{d}{dt}\Big|_{t=0} \phi_t \in C^\infty(M). \tag{11}$$

- $\mathcal{H}(\omega_0, J_0)$ can be equipped with a Riemannian metric called the **Mabuchi metric**, which is given by

$$\langle \delta_1\phi, \delta_2\phi\rangle = \int_M \frac{1}{n!}(\delta_1\phi \cdot \delta_2\phi)\ \omega_\phi \wedge ... \wedge \omega_\phi, \tag{12}$$

where $\omega_\phi = \omega_0 + i\partial_0\bar{\partial}_0\phi$. As proven in [1], the Riemannian manifold $\mathcal{H}(\omega_0, J_0)$ admits a unique Levi-Civita connection. A curve $\{\phi_t\}_{t\in I}$ is a geodesic for this connection if and only if

$$\ddot{\phi}_t = \frac{1}{2}\|\nabla_{\tilde{g}_t}\dot{\phi}_t\|^2_{\tilde{g}_t}, \tag{13}$$

where $\nabla_{\tilde{g}_t}$ is the gradient, and $\|\cdot\|_{\tilde{g}_t}$ is the norm, both with respect to the metric $\tilde{g}_t = (\omega_0 + i\partial_0\bar{\partial}_0\phi_t)(\cdot, J_0\cdot)$.

1.4. The path of Kähler metrics (generated by a complex flow of an Hamiltonian vector field) is a geodesic

Let (M, ω_0, J_0) be Kähler. Let $h \in C^{an}(M)$, and consider its Hamiltonian vector field X_h. Let T be as in theorem 1.3. For all $\tau = it \in B(0,T)$, where $t \in \mathbb{R}$, by acting with the complexified flow of X_h we obtain new Kähler structures (ω_0, J_{it}) on M. We consider the path $\{(\omega_0, J_{it})\}_{t\in(-T,T)}$. Consider the path of isomorphic Kähler structures $\{((\varphi_{it}^{-1})^*\omega_0, (\varphi_{it}^{-1})^*J_{it})\}_{t\in(-T,T)} = \{(\omega_{it}, J_0)\}_{t\in(-T,T)}$. Define $\Phi_t \coloneqq \varphi_{it}$.

Definition 1.5. For each $t \in]-T, T[$, define a function ϕ_t on M in the following way. Let $p \in M$. Let U, V be neighborhoods of $\varphi_\tau^{-1}(p)$ as in theorem 1.3, with local Kähler potentials $\kappa_0 \colon U \longrightarrow \mathbb{R}$ and $\kappa_\tau \colon V \longrightarrow \mathbb{R}$. Then, in the neighborhood $\varphi_\tau(V)$ of p

$$\phi_t|_{\varphi_\tau^{-1}(V)} = \kappa_\tau \circ \varphi_\tau - \kappa_0. \tag{14}$$

Proposition 1.6.

(i) ϕ_t *is well defined.*

(ii) $\omega_{it} = \omega_0 + i\partial_0\bar{\partial}_0\phi_t$.

Proof. Let κ_0 and κ_0' be local Kähler potentials for ω_0 with respect to J_0. Using the fact that closed forms are locally exact and that $\partial_0\bar{\partial}_0(\kappa_0 - \kappa_0') = 0$, one can show that locally $\kappa_0 - \kappa_0'$ is a holomorphic function plus an anti-holomorphic function. Using this fact, and expressions (1a) to (1e) that define κ_τ in terms of κ_0, one can show that $\kappa_\tau' - \kappa_\tau = (\kappa_0' - \kappa_0)\circ\varphi_\tau$. Then, from the definition of Kähler potential it is easy to show that $\omega_0 + i\partial_0\bar{\partial}_0(\kappa_{it}\circ\Phi_t^{-1} - \kappa_0) = \omega_{it}$. □

Theorem 1.7 (Mourão and Nunes, [4])**.** ϕ_t *satisfies:*

$$\ddot{\phi}_t = \frac{1}{2}\|\nabla_{\tilde{g}_{it}}\dot{\phi}_t\|^2_{\tilde{g}_{it}}. \tag{15}$$

Therefore, the path of Kähler metrics $\{(\omega_0, J_{it})\}_{t\in(-T,T)}$ *is a geodesic.*

Proof. One needs to prove the following formulas, in the given order:

$$\frac{d}{dt}\alpha_{it} = iX_h(\alpha_{it}) + i\theta(X_h) \tag{16}$$

$$\frac{d}{dt}\kappa_{it} = -X_h(\psi_{it} + \bar{\psi}_{it}) - 2h + 2\theta(X_h) \tag{17}$$

$$\dot{\phi}_t = \left(\frac{d}{dt}\kappa_{it} + \partial_{it}\kappa_{it}(-iX_h) + \bar{\partial}_{it}\kappa_{it}(iX_h)\right)\circ\Phi_t^{-1} \tag{18}$$

$$\dot{\phi}_t = -2h\circ\Phi_t^{-1} \tag{19}$$

$$\ddot{\phi}_t\circ\Phi_t = 2\|\nabla_{g_{it}}h\|^2_{g_{it}} \tag{20}$$

$$\|\nabla_{g_{it}}h\|^2_{g_{it}} = \frac{1}{4}\|\nabla_{\tilde{g}_{it}}\dot{\phi}_t\|^2_{\tilde{g}_{it}}\circ\Phi_t \tag{21}$$

Where the following fact is useful to prove equations (18) and (20): if $f \in C^\infty(M)$, then it is possible to prove that $\frac{d}{dt}(f\circ\varphi_{it}^{-1}) = \left(\partial_{it}f(-iX_h) + \bar{\partial}_{it}f(iX_h)\right)\circ\varphi_{it}^{-1}$. □

2. Kähler structures on T^*G

2.1. Kähler structures on Cotangent Bundles

In [2], Hall and Kirwin describe a Kähler structure on a tubular open set of T^*M, where M is a real analytic compact Riemannian manifold.

Theorem 2.1 (Hall and Kirwin, [2])**.** *Let M be a compact, real analytic manifold equipped with a real analytic Riemannian metric g. Let $T^{*,R}M$ denote the tubular neighborhood of radius R of the zero section, $T^{*,R}M \coloneqq \{\alpha \in T^*M \mid g(\alpha^\#, \alpha^\#) < R^2\}$. Let $h\colon T^*M \longrightarrow \mathbb{R}$ be defined by $h(\alpha) = \frac{1}{2}g(\alpha^\#, \alpha^\#)$. Let $\beta \in \Omega^2(M)$ be closed, and define $\omega_\beta = \omega_{ST} - \pi^*\beta$, where ω_{ST} denotes the standard symplectic form on T^*M. Then, ω_β is a symplectic form, and there exists an R such that:*

*(i) There exists an unique complex structure on $T^{*,R}M$, J_{MCS}, (called the **magnetic complex structure**), such that*

$$T_{1,0}(T^{*,R}M) = e^{-iL_{X_h}}(\ker d\pi \otimes \mathbb{C}), \tag{22}$$

where X_h is the Hamiltonian vector field of h with respect to ω_β.

(ii) $(T^{*,R}M, \omega_\beta, J_{MCS})$ is Kähler.

Let G be a compact Lie Group, equipped with a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$. In [3], Kirwin, Mourão and Nunes describe a Kähler structure on T^*G, that can be obtained by changing the complex structure by means of a Thiemann complexifier function. We give a short description of the complexification of a compact Lie group and present some necessary definitions.

Remark 2.2 (Complexification of a compact Lie group)**.** Let G be a compact Lie group. Then G can be regarded as a matrix Lie group. The matrix Lie group $G_\mathbb{C} = \{ge^{iX} : g \in G, X \in \mathfrak{g}\}$ with product, differential structure and complex structure given by those of the matrices with complex entries is the universal complexification of G. If we define

$$\begin{aligned} \psi_\tau &\colon G \times \mathfrak{g} \longrightarrow G_\mathbb{C} \\ &(g, Y) \longmapsto ge^{\tau Y} \end{aligned} \tag{23}$$

then for $\operatorname{Im}\tau \neq 0$, ψ_τ are diffeomorphisms.

Remark 2.3. The map

$$\begin{aligned} \phi &\colon T^*G \longrightarrow G \times \mathfrak{g} \\ &\alpha \longmapsto (\pi(\alpha), ((dL_{\pi(\alpha)})^*\alpha)^\#) \end{aligned} \tag{24}$$

is a diffeomorphism.

Definition 2.4. Let $h\colon T^*G \longrightarrow \mathbb{R}$ be left invariant: $h((dL_g)^*\alpha) = h(\alpha)\ \forall \alpha \in T^*G, g \in G$. We refer to h as the **Thiemann complexifier function**. We define:

- u, the **gradient** of h:

$$\begin{aligned} u &\colon \mathfrak{g} \longrightarrow \mathfrak{g} \\ &Y \longmapsto (dh)_{Y^\flat}, \end{aligned} \tag{25}$$

  where since $\mathfrak{g}^* = T_e^*G \subset T_{Y^\flat}(T^*G)$ we have that $(dh)_{Y^\flat} \in (T_{Y^\flat}(T^*G))^* \subset (\mathfrak{g}^*)^* = \mathfrak{g}$.

- H, the **jacobian** of h:

$$\begin{aligned} H &\colon \mathfrak{g} \longrightarrow \operatorname{End}(\mathfrak{g}) \\ Y &\longmapsto \begin{pmatrix} H(Y)\colon \mathfrak{g} \longrightarrow \mathfrak{g} \\ \qquad W \longmapsto (du)_Y(W) \end{pmatrix}. \end{aligned} \tag{26}$$

- φ_h, the **map associated** to h:

$$\begin{aligned} \varphi_h \colon T^*G &\longrightarrow T^*G \\ \alpha &\longmapsto \begin{pmatrix} \varphi_h(\alpha) \colon T_{\pi(\alpha)}G \longrightarrow \mathbb{R} \\ Y \longmapsto (dh)_\alpha(Y^\flat) \end{pmatrix}, \end{aligned} \tag{27}$$

  where $Y^\flat \in T^*_{\pi(\alpha)}G \subset T_\alpha(T^*G)$.

Theorem 2.5 (Kirwin, Mourão and Nunes, [3])**.** *Let G be a compact Lie Group, equipped with a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Let $h \colon T^*G \longrightarrow \mathbb{R}$ be left invariant, right invariant (Ad-invariant when seen as a function $\mathfrak{g} \longrightarrow \mathbb{R}$), such that H is positive definite everywhere and has a positive lower bound for the norm. Then,*

(i) φ_h is a diffeomorphism.

(ii) Define $J^\tau = (\psi_\tau \circ \phi \circ \varphi_h)^* J_{G_\mathbb{C}}$. If $\operatorname{Im}\tau > 0$ then (T^*G, ω, J^τ) is Kähler.

2.2. Relation between theorems 2.1 and 2.5

Let G be a compact real analytic Lie group with a real analytic bi-invariant Riemannian metric. Let $\beta \in \Omega^2(M)$ be closed and $h : T^*G \longrightarrow \mathbb{R}$ satisfying the conditions of theorem 2.5. In this case, both theorems give a Kähler structure on an open subset of T^*G. An interesting question is whether the theorems "intersect", this is, if the Kähler structure of Hall and Kirwin for some β is isomorphic to the Kähler structure of Kirwin, Mourão and Nunes for some h. Assume that the answer is yes. Then, there exists a Kähler isomorphism $\Phi \colon T^{*,R}G \longrightarrow U \subset T^*G$. Since Φ is a Kähler isomorphism, we have that:

$$\omega_\beta = \Phi^* \omega_{ST} = \omega_{ST} - \pi^*\beta, \tag{28}$$

$$J_{MCS} = \Phi^* J^\tau. \tag{29}$$

Writing $J_{MCS} = \Phi^* J^\tau$ in terms of the (1,0) tangent spaces it is possible to show that $e^{\bar{\tau} L_{X_h^{ST}}}(\ker d\pi \otimes \mathbb{C}) = e^{-iL_{X^{ST}_{E\circ\Phi}}} \circ d\Phi(\ker d\pi \otimes \mathbb{C})$ (where $E(\alpha) = \frac{1}{2} g(\alpha^\#, \alpha^\#)$), which is then a necessary condition for the two theorems "intersecting".

2.3. The theorem of Hall and Kirwin in the case of a Lie group

Consider the following conjecture, which is stating that theorem 2.1 works for $R = \infty$ in the case of a Lie group with bi-invariant metric.

Conjecture 2.6. Let G be a compact, real analytic Lie group equipped with a real analytic, left invariant Riemannian metric g. Let $h \colon T^*M \longrightarrow \mathbb{R}$ be given by $h(\alpha) = \frac{1}{2} g(\alpha^\#, \alpha^\#)$. Let $\beta \in \Omega^2(G)$ be closed, and define $\omega_\beta = \omega_{ST} - \pi^*\beta$. Then,

(i) ω_β is a symplectic form.

(ii) There exists an unique complex structure on T^*G, J_{MCS}, (called the **magnetic complex structure**), such that

$$T_{1,0}(T^*G) = e^{-iL_{X_h}}(\ker d\pi) \otimes \mathbb{C}, \tag{30}$$

where X_h is the Hamiltonian vector field of h with respect to ω_β.

(iii) $(T^*G, \omega_\beta, J_{MCS})$ is Kähler.

We identify T^*G to $G \times \mathfrak{g}$ using left translation and the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the metric. This allows us to define the following useful basis of vector fields and of forms: let $\{T_1, ..., T_n\}$ be a $\langle \cdot, \cdot \rangle$-orthonormal basis of $\mathfrak{g}$ with associated coordinates $\{y^1, ..., y^n\}$, $\{X_1, ..., X_n\}$ be the left invariant

vector fields on G such that $(X_j)_e = T_j$, and $\{\omega^1, ..., \omega^n\}$ be the basis of differential forms in G that is dual to $\{X_1, ..., X_n\}$. Then,

$$T_{(g,Y)}(G \times \mathfrak{g}) = \text{span}\{X_1, ..., X_n\} \oplus \text{span}\left\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\right\}, \tag{31}$$

$$T^*_{(g,Y)}(G \times \mathfrak{g}) = \text{span}\{\omega^1, ..., \omega^n\} \oplus \text{span}\left\{dy^1, ..., dy^n\right\}. \tag{32}$$

Then, to try and prove the conjecture, in principle we could follow the following method. Compute explicitly the vectors $\{e^{-iL_{X_h}}\frac{\partial}{\partial y^j}\}_{j=1,...,n}$. Define J_{MCS} by declaring that these vectors are a basis of $T_{1,0}(T^*G)$. Check that this J_{MCS} is indeed a complex structure, and that it is compatible with ω_β, by proving that

$$\begin{cases} \omega_\beta(Z_j, Z_k) = 0 \quad \forall j,k \\ [i\omega_\beta(\bar{Z}_j, Z_k)] \succ 0 \end{cases}, \tag{33}$$

where $Z_j = e^{-iL_{X_h}}\frac{\partial}{\partial y^j}$. Unfortunately, $e^{-iL_{X_h}}\frac{\partial}{\partial y^j}$ is very hard to compute even in simple cases of β, so we cannot take any conclusions about conjecture 2.6.

2.4. The theorem of Kirwin, Mourão and Nunes in the case of non Ad-invariant h

Assume the hypothesis of theorem 2.5 except Ad-invariance of h. As seen in [3], φ_h being a diffeomorphism depends only on the conditions about H. Therefore, φ_h is still a diffeomorphism in this case, and $J^\tau = (\psi_\tau \circ \phi \circ \varphi_h)^* J_{G_\mathbb{C}}$ is a complex structure. Identifying $T^*G \cong G \times \mathfrak{g}$ as in the previous section, and noting that $T_{(x,Y)}(G \times \mathfrak{g}) \cong T_xG \oplus T_Y\mathfrak{g} \overset{dL_{x^{-1}} \times \text{id}}{\cong} \mathfrak{g} \oplus \mathfrak{g}$, then $(d\varphi_h)_\alpha$, $(d\psi_\tau)_{(g,u(Y))}$, and $J_{G_\mathbb{C}}$ can be written as endomorphisms of $\mathfrak{g} \oplus \mathfrak{g}$, which can be written as 2×2 matrices of endomorphisms of $\mathfrak{g}$. From [3],

$$(d\varphi_h)_\alpha = \begin{pmatrix} I & 0 \\ 0 & H(Y) \end{pmatrix}, \tag{34}$$

$$(d\psi_\tau)_{(g,u(Y))} = \begin{pmatrix} e^{-\tau_1 \text{ad}_{u(Y)}} \cos(\tau_2 \text{ad}_{u(Y)}) & \frac{I - e^{-\tau_1 \text{ad}_{u(Y)}} \cos(\tau_2 ad_{u(Y)})}{\text{ad}_{u(Y)}} \\ -e^{-\tau_1 \text{ad}_{u(Y)}} \sin(\tau_2 \text{ad}_{u(Y)}) & \frac{e^{-\tau_1 \text{ad}_{u(Y)}} \sin(\tau_2 \text{ad}_{u(Y)})}{\text{ad}_{u(Y)}} \end{pmatrix}, \tag{35}$$

$$J_{G_\mathbb{C}} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{36}$$

Using these formulas we can compute the $(+i)$-eigenvectors of J^τ, which are a basis of $T_{1,0}(T^*G)$. The following vectors form such a basis:

$$Z_j^\tau := \sum_{k=1}^n \left[\frac{\left(1 - e^{\bar{\tau}\text{ad}_{u(Y)}}\right)}{\text{ad}_{u(Y)}}\right]_j^k X_k + \sum_{k=1}^n \left[H(Y)^{-1}\right]_j^k \frac{\partial}{\partial y^k}. \tag{37}$$

Then, ω and J^τ are compatible if and only if $\omega(Z_j^\tau, Z_k^\tau) = 0 \ \forall j,k$ and $[i\omega(\bar{Z}_j^\tau, Z_k^\tau)] \succ 0$. These equations take the form:

$$-\frac{I - e^{-\bar{\tau}\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} H(Y)^{-1} - H(Y)^{-1}\frac{I - e^{\bar{\tau}\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} + \frac{I - e^{-\bar{\tau}\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} \text{ad}_Y \frac{I - e^{\bar{\tau}\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} = 0, \tag{38}$$

$$-i\frac{I - e^{-\tau\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} H(Y)^{-1} - iH(Y)^{-1}\frac{I - e^{\bar{\tau}\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} + i\frac{I - e^{-\tau\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} \text{ad}_Y \frac{I - e^{\bar{\tau}\text{ad}_{u(Y)}}}{\text{ad}_{u(Y)}} \succ 0. \tag{39}$$

Equations (38) and (39) are true if h is Ad-invariant. For another example, if $h(Y) = \frac{1}{2}\langle Y, Y\rangle + \langle Y, Z\rangle$, it is possible to see that this h satisfies equations (38) and (39) if and only if $\text{ad}_Z = 0$.

3. An Example of Quantization of T^*G

3.1. General setup

In this section we perform geometric quantization on the cotangent bundle of a Lie group. Let:

- G be a compact Lie group with Lie algebra $\mathfrak{g}$, equipped with a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$.
- $\mathfrak{h} \subset \mathfrak{g}$ be a maximal abelian subalgebra.
- $\{T_1, ..., T_n\}$ be an orthonormal basis of $\mathfrak{g}$, such that $\{T_1, ..., T_r\}$ is a basis of $\mathfrak{h}$, and y^j, X_j, ω^j be defined as in subsection 2.3.
- $h \colon \mathfrak{g} \longrightarrow \mathbb{R}$, the complexifier function, be defined as $h(y^1, ..., y^n) = \frac{1}{2} \sum_{j,k=1}^{n} H_{jk} y^j y^k$, where H is symmetric, $H_{jk} = 0$ if j or $k \geq r+1$, and the $r \times r$ matrix with entries $H_{jk}, j, k = 1, ..., r$ is positive definite.

3.2. Prequantum line bundle and polarization

We consider the line bundle whose total space is $L = T^*G \times \mathbb{C}$, and whose projection map is $\pi \colon T^*G \times \mathbb{C} \longrightarrow T^*G$. For this line bundle, $\Gamma(T^*G \times \mathbb{C}) = C^\infty(T^*G; \mathbb{C})$. We define h to be the hermitian structure that on the fiber of L above $\alpha \in T^*G$ is $h(z, w) = z\bar{w}$. Let ∇ be the connection given by $\nabla_X s = X(s) + i\theta(X)s$. It is possible to prove that $R(X, Y)s = -i\omega(X, Y)s$ (the curvature of ∇ is $-i\omega$), and that ∇ and h are compatible.

Consider the **vertical polarization** of T^*G, $\mathcal{P}_0 \coloneqq \ker d\pi \otimes \mathbb{C} = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n} \right\} \subset T(T^*G) \otimes \mathbb{C}$. Consider the following distribution:

$$\mathcal{P}_\tau = \operatorname{span}_{\mathbb{C}} \left\{ e^{\bar{\tau} L_{X_h}} \frac{\partial}{\partial y^1}, ..., e^{\bar{\tau} L_{X_h}} \frac{\partial}{\partial y^n} \right\}. \tag{40}$$

The conditions imposed on h make it so that it is possible to compute

$$e^{\bar{\tau} L_{X_h}} \frac{\partial}{\partial y^i} = \begin{cases} -\sum_{j=1}^{r} \bar{\tau} H_{ji} X_j + \frac{\partial}{\partial y^i} - \sum_{j=r+1}^{n} \bar{\tau} \left[\operatorname{ad}_Y H \right]_i^j \frac{\partial}{\partial y^j} & \text{if } i = 1, ..., r \\ \sum_{j=r+1}^{n} \left[e^{\bar{\tau} \operatorname{ad}_{H(Y)}} \right]_i^j \frac{\partial}{\partial y^j} & \text{if } i = r+1, ..., n. \end{cases} \tag{41}$$

3.3. T^*G is foliated by Kähler manifolds

For $j = 1, ..., r$, define:

$$Z_j^\tau = e^{\bar{\tau} X_h} \frac{\partial}{\partial y^j}, \tag{42}$$

$$A_j^\tau = \operatorname{Re} Z_j^\tau, \tag{43}$$

$$B_j^\tau = \operatorname{Im} Z_j^\tau, \tag{44}$$

and consider the following distribution on T^*G:

$$\Sigma = \operatorname{span}_{\mathbb{R}} \left\{ A_1^\tau, ..., A_r^\tau, B_1^\tau, ..., B_r^\tau \right\}. \tag{45}$$

Then it is easy to prove using equation (41) that Σ is involutive, hence integrable by a foliation $\mathcal{F}$. Denote by L_α the leaf of the foliation that contains $\alpha \in T^*G$.

Theorem 3.1. *Let $\alpha \in T^*G$, and consider the leaf of $\mathcal{F}$ that contains α: $\iota \colon L_\alpha \longrightarrow T^*G$. Then:*

(i) $\iota^*\omega$ is a symplectic form on L_α.

(ii) $J : TL_\alpha \longrightarrow TL_\alpha$ defined by

$$J(Z_j^\tau) = iZ_j^\tau \tag{46a}$$

$$J(\bar{Z}_j^\tau) = -i\bar{Z}_j^\tau \tag{46b}$$

is a complex structure on L_α.

(iii) If $\tau_2 > 0$, then $(L_\alpha, \iota^*\omega, J)$ is a Kähler manifold.

Proof. $d\iota^*\omega = \iota^* d\omega = 0$ proves that $\iota^*\omega$ is closed, and if we define Ω^j_τ to be the forms that satisfy $\Omega^j_\tau(Z^\tau_k) = \delta^j_k$ and $\Omega^j_\tau(\bar{Z}^\tau_k) = 0$, then it is possible to show that

$$\iota^*\omega = \sum_{j,k=1}^{r} 2i\tau_2 H_{jk}\Omega^j_\tau \wedge \bar{\Omega}^k_\tau \tag{47}$$

which proves that $\iota^*\omega$ is nondegenerate since H_{jk} is positive definite. Using the definition of J, it is easy to see that $J^2 = -\mathrm{id}$ and using equation (41) it is possible to show that the Nijenhuis tensor of J is zero. Hence J is integrable. Again, by proving that

$$\iota^*\omega(Z^\tau_j, Z^\tau_k) = 0, \tag{48}$$
$$i(\iota^*\omega)(\bar{Z}^\tau_j, Z^\tau_k) \succ 0, \tag{49}$$

one shows that $\iota^*\omega$ and J are compatible. □

3.4. The Hilbert spaces $\mathcal{H}_\tau$ and the natural isomorphism between them

The Hilbert space associated to the vertical polarization $\mathcal{P}_0$ can be defined simply as $\mathcal{H}_0 = L^2(G, dx)$, where dx is the Haar measure of G.

Remark 3.2. We state some relevant facts about $L^2(G, dx)$. Consider the set of all irreducible unitary representations of G: $\{\lambda_\alpha \colon G \longrightarrow GL(V)\}_{\alpha\in I}$. Also, consider its partition into equivalence classes: $\{[\lambda_\alpha]\}_{\alpha\in J}$. For each class $[\lambda_\alpha]$, choose a representative λ_α and a basis $\{v^\alpha_1, ..., v^\alpha_{d_\alpha}\}$ of V_α. Define the **representative functions** $\lambda^{jk}_\alpha \colon G \longrightarrow \mathbb{C}$ the following way: $\lambda^{jk}_\alpha(g)$ is the (j,k) entry of the matrix that represents $\lambda_\alpha(g)$ in the basis $\{v^\alpha_1, ..., v^\alpha_{d_\alpha}\}$. Then, it is a fact that the space spanned by the representative functions of a given class of representations does not depend on the chosen basis. Also, it is a consequence of representation theory that $\{\lambda^{jk}_\alpha\}_{\alpha\in J, j,k\in 1,...,d_\alpha}$ is an orthogonal basis of $L^2(G, dx)$:

$$L^2(G, dx) = \overline{\mathrm{span}_{\mathbb{C}}\{\lambda^{jk}_\alpha\}_{\alpha\in J, j,k\in 1,...,d_\alpha}}. \tag{50}$$

Remark 3.3. Each representation λ_α induces a representation of the Lie algebra, which is the derivative at the identity $d_e\lambda_\alpha$. $\{d_e\lambda_\alpha(X)\}_{X\in\mathfrak{h}}$ is a set of simultaneously diagonalizable matrices, because they commute ($\mathfrak{h}$ is abelian) and they are anti-hermitian. So, for each α, we can choose a basis $\{v^\alpha_1, ..., v^\alpha_{d_\alpha}\}$ of V_α such that for all $X \in \mathfrak{h}$ $d_e\lambda_\alpha(X)$ is diagonal:

$$d_e\lambda_\alpha(X) = \begin{pmatrix} i\lambda^\alpha_1(X) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & i\lambda^\alpha_{d_\alpha}(X) \end{pmatrix}. \tag{51}$$

$i\lambda^\alpha_j \colon \mathfrak{g} \longrightarrow \mathbb{C}$ are linear functionals, which give the eigenvalue of the matrix in the basis $\{v^\alpha_1, ..., v^\alpha_{d_\alpha}\}$.

To define the Hilbert space associated to the polarization $\mathcal{P}_\tau$, we consider the set of $\bar{\mathcal{P}}_\tau$-polarized sections. It is possible to prove that:

$$\left\{s \in \Gamma(L) \colon \nabla_X s = 0 \quad \forall X \in \mathfrak{X}(\bar{\mathcal{P}}_\tau)\right\} = \{e^{-i\tau\hat{h}} f \colon f \in C^\infty(G; \mathbb{C})\}, \tag{52}$$

where $\hat{h}(f) = (i\nabla_{X_h} + h)f$, and $C^\infty(G;\mathbb{C}) \subset C^\infty(T^*G;\mathbb{C})$. The half form correction suggests that we define the following inner product on this set:

$$\langle e^{-i\tau\hat{h}} f, e^{-i\tau\hat{h}} g\rangle_\tau = \sqrt{\left(\frac{\tau}{\pi}\right)^r \det H} \int_{G\times\mathfrak{h}} \left(e^{-i\tau\hat{h}} f\right) \overline{\left(e^{-i\tau\hat{h}} g\right)} dxdy. \tag{53}$$

Choose a set of λ_α as in remark 3.2 and for each α an associated basis $\{v^\alpha_1, ..., v^\alpha_{d_\alpha}\}$ as in remark 3.3.

Proposition 3.4.

$$\langle e^{-i\tau\hat{h}}\lambda^{jk}_\alpha, e^{-i\tau\hat{h}}\lambda^{lm}_\beta\rangle_\tau = \frac{\delta_{jl}\delta_{km}\delta_{\alpha\beta}}{\dim V_\alpha} e^{2\tau_2 h((\lambda^\alpha_k)^\flat)} \tag{54}$$

Proof. The proof consists in using the definition of the inner product, and using the fact that the λ's are representations to write $e^{-i\tau\hat{h}}\lambda_\alpha^{jk}(x,Y) = \lambda_\alpha^{jk}(x)e^{i\tau\lambda_j^\alpha(u(Y))}$. This splits the integral in an integral over G (which can be computed using the fact that $\{\lambda_\alpha^{jk}\}_{\alpha\in J,j,k\in 1,\dots,d_\alpha}$ is an orthogonal set and that λ_α^{jk} has norm squared $1/\dim V_\alpha$) and an integral over $\mathfrak{g}$ (which can be computed using the Gaussian integral formula). □

$\mathcal{H}_\tau$ should be the Hilbert space that contains $\mathcal{P}_\tau$-polarized sections of L that are L^2 with respect to (53). From proposition 3.4, we can conclude that

$$\begin{aligned}\mathcal{H}_\tau &= \overline{\left\{s\in\Gamma(L)\colon \nabla_X s = 0 \quad \forall X\in\mathfrak{X}(\bar{\mathcal{P}}_\tau), \|s\|_\tau<\infty\right\}} && (55)\\ &= \left\{e^{-i\tau\hat{h}}f \mid f\in L^2(G,dx)\right\} && (56)\\ &= \overline{\operatorname{span}_{\mathbb{C}}\left\{e^{-i\tau\hat{h}}\lambda_{d_\alpha}^{jk}\right\}_{\alpha\in J,j,k=1,\dots,d_\alpha}}. && (57)\end{aligned}$$

We want to define a unitary isomorphism $\mathcal{H}_0 \longrightarrow \mathcal{H}_\tau$. Proposition 3.4 shows that $e^{-i\tau\hat{h}}$ preserves orthogonality but not the norm. Because of this, we introduce the linear map Q_h, which is the unique linear map that acts on the basis $\{\lambda_\alpha^{jk}\}_{\alpha\in J,j,k\in 1,\dots,d_\alpha}$ of $\mathcal{H}_0$ as:

$$\begin{aligned}Q_h\colon \mathcal{H}_0 &\longrightarrow \mathcal{H}_0 && (58)\\ \lambda_\alpha^{jk} &\longmapsto h((\lambda_k^\alpha)^\flat)\lambda_\alpha^{jk}\end{aligned}$$

Theorem 3.5. *The map*

$$U_\tau \coloneqq e^{-i\tau\hat{h}}\circ e^{i\tau Q_h}\colon \mathcal{H}_0 \longrightarrow \mathcal{H}_\tau \quad (59)$$

is a unitary isomorphism of Hilbert spaces.

Proof. U_τ is bijective, because it has inverse $U_\tau^{-1} = e^{-i\tau Q_h}\circ e^{-i\tau\hat{h}}$. U_τ is linear because $\hat{h}$ and Q_h are linear. We have to show that U_τ preserves the inner product. It suffices to show that it preserves the norm of the elements of a basis. Since

$$U_\tau(\lambda_\alpha^{jk}) = e^{-i\tau\hat{h}}\left(e^{i\tau Q_h}(\lambda_\alpha^{jk})\right) = e^{-i\tau\hat{h}}\left(e^{i\tau h((\lambda_k^\alpha)^\flat)}\lambda_\alpha^{jk}\right) = e^{i\tau h((\lambda_k^\alpha)^\flat)}e^{-i\tau\hat{h}}(\lambda_\alpha^{jk})$$

we can easily prove that $\|U_\tau(\lambda_\alpha^{jk})\|_\tau^2 = \|\lambda_\alpha^{jk}\|_0^2$. □

Acknowledgements

The author would like to thank João Nunes for all the help as advisor for the thesis associated to this extended abstract, as well as the members of the Examination Committee, Miguel Abreu, José Mourão and Sílvia Anjos. The work presented originated from a scholarship from Centre for Mathematical Analysis, Geometry, and Dynamical Systems (CAMGSD).

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