# The Lebesgue Measure and Large Cardinals

Jacinto Carlos Ferreira Monteiro de Freitas França
Instituto Superior Técnico - Universidade de Lisboa

23, June, 2017

**Abstract**

One of the significant driving factors behind major developments in set theory has been the Lebesgue measure. The Lebesgue measure is the cornerstone of Lebesgue's theory of integration, which is a generalization of Riemann's theory of integration. Despite the many advantages of the Lebesgue integral, it cannot measure every set of reals. However, an alternative can be found if we adopt a weaker version of the axiom of choice, namely the principle of dependent choices (DC).

Solovay's Theorem establishes that if there is a model of ZFC with an inaccessible cardinal, then there is a model of ZF+DC in which every set of reals is Lebesgue measurable. On the other hand, Shelah's Theorem clarifies that when there is a model of ZF+DC in which every set of reals is Lebesgue measurable, there is also a model of ZFC with an inaccessible cardinal.

The thesis is a detailed exposition of the Solovay and Shelah theorems, and their respective proofs. The prerequisites for the proofs are mainly the model theory of sets; the method of forcing, particularly the Lévy collapse and the random algebra; and descriptive set theory, focused on the aspect of Lebesgue measurability. These prerequisites were presented after a brief overview of the Lebesgue measure. Then the two main theorems were shown with due detail.

**Keywords:** Lebesgue Measure, Inaccessible Cardinal, Principle of Dependent Choices, Lévy Collapse, Random Algebra, Rapid Filter

## 1 Introduction

The Lebesgue measure is the essential component in Henri Lebesgue's theory of integration. The integration techniques commonly taught in introductory calculus courses are based on Riemann's integral. However, Riemann's integral has several limitations, such as its difficulty in dealing with series of functions, or dealing with functions that differ only on a null subset of their common domain. These limitations are eliminated in Lebesgue's theory of integration. The Lebesgue integral is able to measure all the Riemann-integrable functions as well as other functions not encompassed by the Riemann integral. The properties of the Lebesgue integral follow naturally from the properties of the Lebesgue measure.

Despite the many advantages that the Lebesgue measure has, it is not able to measure all sets of reals. There is an example of a nonmeasurable set given by Giuseppe Vitali. Vitali's example makes explicit use of the axiom of choice (AC). It could be the case that AC is not necessary for Vitali's example, but merely sufficient. However, Solovay's Theorem excludes this possibility. Robert Solovay found a model in which all sets of reals are Lebesgue measurable — a property denoted by LM. In order to do this he used a weaker version of the AC, called the principle of dependent choices (DC).

**Theorem 1.1** (Solovay)**.** *If there is a model of* ZFC *with an inaccessible cardinal* $\kappa$*, then there is a model of* $\mathrm{ZF} + \mathrm{DC} + \mathrm{LM}$.

Solovay proved this theorem in 1964. He was the first to put the method of forcing to use after Paul Cohen. Paul Cohen created the method of forcing and used it in 1963 to prove the independence of the continuum hypothesis (CH) from ZFC, as well as the independence of AC from ZF. The method of forcing turned out to be an incredibly versatile and powerful mathematical technique to explore the relative consistency and equiconsistency of set theoretical properties, as well as the independence of such properties from set theoretical axiom systems.

The notion of forcing that Solovay used to prove his theorem was the Lévy collapse $\mathrm{Col}(\omega, <\kappa)$. The Lévy collapse forces all the uncountable cardinals below an inaccessible cardinal $\kappa$ in the ground model $M$ to have cardinality $\aleph_0$ in the extended model $M[G]$. But Solovay's Theorem begs the question: is it necessary for the model of ZFC to have an inaccessible cardinal $\kappa$? Saharon Shelah published an article in 1984 in which he proved that the assumption is necessary.

**Theorem 1.2** (Shelah)**.** *If there is a model of* ZF+DC+LM*, then there is a model of* ZFC *with an inaccessible cardinal* $\kappa$*.*

The Shelah and Solovay theorems[1] establish the equiconsistency of the axiom system ZF+DC+LM and the axiom system ZFC+I, where I stands for "There exists an inaccessible cardinal". The proof of Shelah's Theorem presented in the thesis is more accessible than usual. The original formulation of Shelah's Theorem only assumes the Lebesgue measurability of the $\mathbf{\Sigma^1_3}$ sets of reals, instead of all sets of reals. But if we suppose that all sets of reals are Lebesgue measurable, then the proof becomes simpler because it is not concerned with the technical details related to the projective hierarchy and to relativized analytical hierarchies.

The thesis is a detailed exposition of these two theorems by Solovay and Shelah, and of their respective proofs. We began by reviewing the Lebesgue measure in the context of the real line $\mathbb{R}$. Then we proceeded to set theory itself, providing the preliminary notions and results to be able to prove the Solovay and Shelah theorems. We started with model theoretic prerequisites related to set theory. Then we presented the theory of forcing, along with some notable examples, of which the most relevant to the thesis is the Lévy collapse. We finished the preliminary material with descriptive set theory focused on the Lebesgue measure. After the preliminary chapters, we proved Solovay's Theorem and Shelah's Theorem, and explained the ideas directing the course of the proofs. We finished the thesis with a brief overview of important results in set theory related to the Lebesgue measure.

[1]Solovay and Shelah proved many important theorems besides the two mentioned here. However, we adopt the convention of calling these two theorems the Solovay Theorem and the Shelah Theorem.

# 2 Preliminaries of Set Theory

## 2.1 Models of Set Theory

Our starting point is the above mentioned example of a nonmeasurable set of reals given by Vitali. This example involves the AC. As such, if we want to find a model in which all sets of reals are Lebesgue measurable, then we should adopt a weaker version of the AC.

**Definition 2.1** (Principle of Dependent Choices)**.** Let $A$ be a nonempty set, and $R$ be a binary relation on $A$, such that, for every $x \in A$, there is a $y \in A$, satisfying $x \; R \; y$. Then there is a sequence $\langle x_n : n \in \omega \rangle$ of elements of $A$, such that $x_n \; R \; x_{n+1}$.

The purpose of DC is to be able to make a countable number of consecutive choices. This method of creating a sequence is often employed in mathematical analysis with the help of AC. So, even though AC implies DC, most applications of AC in mathematical analysis require only DC. As such, DC is a good substitute for AC in the context of mathematical analysis and, thus, in the context of the Lebesgue integral. The model of ZF+DC+LM that we want to find is built with the help of the notion of ordinal-definability from elements of a set $A$.

**Definition 2.2.** Let $A$ and $X$ be sets.

(1) We say that $X$ is *ordinal-definable from elements of* $A$, or that $X \in \mathrm{OD}_A$, if there is a formula $\varphi$, ordinal numbers $\alpha_1, \ldots, \alpha_n$, and elements $a_1, \ldots, a_k$ of $A$, such that the set $X$ is given by $X = \{y : \varphi(y, \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_k)\}$.

(2) We say that $X$ is *hereditarily ordinal-definable from elements of* $A$ if $\mathrm{TC}(\{X\}) \subset \mathrm{OD}_A$, where $\mathrm{TC}(\{X\})$ is the transitive closure of $\{X\}$. The class of the hereditarily ordinal-definable sets from elements of $A$ is denoted by $\mathrm{HOD}_A$.

**Theorem 2.3.** *Let $A$ be a set.*

(1) $\mathrm{HOD}_A$ *is a transitive model of* ZF*.*

(2) *If there is a well-ordering of $A$ in* $\mathrm{OD}_A$*, then* $\mathrm{HOD}_A$ *is a transitive model of* ZFC*.*

A case of particular importance for the thesis is when $A$ is the set of real numbers. In the context of set theory the real numbers are either binary sequences, ${}^{\omega}2$, or sequences of natural numbers, ${}^{\omega}\omega$. Let us work with ${}^{\omega}\omega$. Since there is no well-ordering of ${}^{\omega}\omega$ in $\mathrm{OD}_{{}^{\omega}\omega}$, the class $\mathrm{HOD}_{{}^{\omega}\omega}$ does not model AC. Therefore, $\mathrm{HOD}_{{}^{\omega}\omega}$ is a good candidate for the model of ZF+DC+LM that we want to obtain. In order to obtain said model we also need an inaccessible cardinal.

**Definition 2.4.** A cardinal $\kappa$ is called an *inaccessible* cardinal if it is a regular, uncountable, and strong limit cardinal.

**Theorem 2.5.** *If $\kappa$ is an inaccessible cardinal, then* $\mathrm{V}_\kappa \models$ ZFC*.*

ZFC cannot prove its own consistency because of Gödel's second incompleteness theorem. Therefore, the previous theorem shows that the existence of an inaccessible cardinal is independent from ZFC. As such, assuming the existence of an inaccessible cardinal in Solovay's Theorem is not redundant.

## 2.2 Forcing and the Lévy Collapse

The idea behind the method of forcing is to start with a transitive model $M$ and to extend it to a new transitive model $M[G]$ by adding a carefully crafted object $G \notin M$. With this object $G$ we force the extended model $M[G]$ to have some desired properties. If $M$ is a transitive model of ZFC, then $M[G]$ is also a transitive model of ZFC. We can assume that $M$ is a countable transitive model of ZFC (abbreviated to CTM), when proving relative consistency results. Let us see an example related to the Lévy collapse.

**Definition 2.6.** Let $\mu$ be a regular cardinal, and let $\kappa \geq \mu$. Let $\mathrm{Col}(\mu, \kappa) = {}^{<\mu}\kappa$ be the set of functions $f : \gamma \to \kappa$, for all $\gamma < \mu$. If $p, q \in \mathrm{Col}(\mu, \kappa)$, let $p \leq q$ iff $p \supset q$. Then $(\mathrm{Col}(\mu, \kappa), \leq)$ is called the *collapse of $\kappa$ to $\mu$*.

If we start with $M$, and define $\mathrm{Col}(\mu, \kappa)$ in $M$, then it is possible to define a family $G$ of $p_\eta \in \mathrm{Col}(\mu, \kappa)$ such that $\bigcup_{\eta<\mu} p_\eta$ is a surjective function $f : \mu \to \kappa$. Such a function exists by virtue of the properties of $G$. Namely $G$ is a filter that is $\mathrm{Col}(\mu, \kappa)$-generic over $M$. Neither $G$ nor $f$ are elements of $M$ but both are elements of $M[G]$. As such, the cardinality of $\mu^M$ in $M[G]$ is $\kappa$. There is a sense in which the collapse $\mathrm{Col}(\omega, \lambda)$ resembles a universal object: If $\lambda$ is an infinite cardinal, and $\mathbb{P}$ is a separative partial order with $|\mathbb{P}| = \lambda$, such that $1_{\mathbb{P}} \Vdash |\check{\lambda}| = \aleph_0$, then there is a dense homomorphism $\pi : \mathrm{Col}(\omega, \lambda) \to \mathbb{P}$.

**Definition 2.7** (Lévy Collapse)**.** Let $\kappa$ be an inaccessible cardinal. Let us define $\mathrm{Col}(\omega, <\kappa)$ as

$$\Big\{\Big(p : D \to \Big(\bigcup\nolimits_{\eta\in[\omega,\kappa[}\ \mathrm{Col}(\omega, \eta)\Big)\Big) : D = \mathrm{dom}(p) \subset [\omega, \kappa[ \;\wedge\; |D| < \omega \;\wedge\; \forall \eta \in D\ p(\eta) \in \mathrm{Col}(\omega, \eta)\Big\}.$$

For $p, q \in \mathrm{Col}(\omega, <\kappa)$, define $p \leq q$ iff $\mathrm{dom}(p) \supset \mathrm{dom}(q)$ and $p(\eta) \supset q(\eta)$, for all $\eta \in \mathrm{dom}(q)$. Then $(\mathrm{Col}(\omega, <\kappa), \leq)$ is called the *Lévy collapse of $\kappa$ to $\omega$*.

The main properties of $\mathrm{Col}(\omega, <\kappa)$ are summarized in the next theorem.

**Theorem 2.8.** *Let $M$ be a CTM, $G$ be $\mathrm{Col}(\omega, <\kappa)$-generic over $M$, $\kappa$ be an inaccessible cardinal, $\varphi(v_1, \ldots, v_n)$ be a formula, and $\check{x_1}, \ldots, \check{x_n} \in M^{\mathrm{Col}(\omega,<\kappa)}$ be canonical names. Then*

(1) $\mathrm{Col}(\omega, <\kappa)$ *is an atomless and separative partial order.*

(2) $\mathrm{Col}(\omega, <\kappa)$ *is homogenous with respect to canonical names.*

(3) *Either* $1 \Vdash_M^{\mathrm{Col}(\omega,<\kappa)} \varphi(\check{x_1}, \ldots, \check{x_n})$*, or* $1 \Vdash_M^{\mathrm{Col}(\omega,<\kappa)} \neg\varphi(\check{x_1}, \ldots, \check{x_n})$.

(4) $\mathrm{Col}(\omega, <\kappa)$ *is $\aleph_0$-complete and satisfies the $\kappa$-c.c.*

(5) *All $M$-cardinals outside of $[\omega, \kappa[$ will remain cardinals in $M[G]$. All $M$-cardinals in $[\omega, \kappa[$ will be ordinals of size $\aleph_0$ in $M[G]$. And $\kappa = \aleph_1$ in $M[G]$.*

Since $\mathrm{Col}(\omega, <\kappa)$ is atomless and separative, it is homogenous with respect to canonical names and $G \notin M$. In turn, separativeness entails point (3), which means that the properties of the elements of $M$ can always be decided in $M[G]$. Furthermore, point (4) states properties of $\mathrm{Col}(\omega, <\kappa)$ that determine how $M$-cardinals are evaluated in $M[G]$.

## 2.3 The Lebesgue Measure and Descriptive Set Theory

Let $s \in {}^{<\omega}\omega$. Then $s * k$ is the extension of $s$ to a sequence with length $|s|+1$, such that $(s*k)(|s|) = k$. A *basic open set* is a set of the form $O(s) = \{x \in {}^{\omega}\omega : s \subset x\}$. The *Baire space* is ${}^{\omega}\omega$ equipped with the topology generated by the basic open sets. The smallest $\sigma$-algebra on ${}^{\omega}\omega$ that contains all of its open sets is called the *Borel algebra* on ${}^{\omega}\omega$ and is denoted by $\mathcal{B}({}^{\omega}\omega)$. Its elements are called the *Borel* sets. The cardinality of $\mathcal{B}({}^{\omega}\omega)$ is $|\mathcal{B}({}^{\omega}\omega)| = 2^{\aleph_0}$.

**Definition 2.9.** The *Lebesgue measure* on $\mathcal{B}({}^{\omega}\omega)$ is a function $m_{\mathcal{L}} : \mathcal{B}({}^{\omega}\omega) \to [0,1]$, defined by recursion as follows:

(1) If $s = \emptyset$, then $m_{\mathcal{L}}(O(\emptyset)) = m_{\mathcal{L}}({}^{\omega}\omega) = 1$.

(2) If $m_{\mathcal{L}}(O(s))$ is defined, then $m_{\mathcal{L}}(O(s * k)) = \frac{1}{2^{k+1}} \cdot m_{\mathcal{L}}(O(s))$.

(3) Let $\{A_i : i \in \omega\}$ be a collection of pairwise disjoint sets, such that $m_{\mathcal{L}}(A_i)$ is defined for every $i$. Then $m_{\mathcal{L}}(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} m_{\mathcal{L}}(A_i)$.

(4) If $m_{\mathcal{L}}(A)$ is defined, then $m_{\mathcal{L}}({}^{\omega}\omega - A) = 1 - m_{\mathcal{L}}(A)$.

In order to extend the definition of $m_{\mathcal{L}}$ beyond $\mathcal{B}({}^{\omega}\omega)$ we resort to null sets. We say that $A \subset {}^{\omega}\omega$ is a *null set* when $\inf\{m_{\mathcal{L}}(U) : A \subset U \wedge U \text{ is open }\} = 0$.

**Definition 2.10.** A set $A \subset {}^{\omega}\omega$ is *Lebesgue measurable*, or *$\mathcal{L}$-measurable*, if there is a Borel set $B$, such that the symmetric difference $A \Delta B$ is a null set. In this case, the *$\mathcal{L}$-measure* of $A$ is $m_{\mathcal{L}}(A) = m_{\mathcal{L}}(B)$.

It follows that there is a close relation between Borel sets and $\mathcal{L}$-measurable sets. In order to study $\mathcal{L}$-measurability in the context of set theory it is important to have a notion of forcing that involves the Borel sets. That notion of forcing is the random algebra.

Let $A, B \in \mathcal{B}({}^{\omega}\omega)$. We write $A \approx B$ when $A \Delta B$ is a null set. This is an equivalence relation. The equivalence class of $A$ is $[A] = \{B \in \mathcal{B}({}^{\omega}\omega) : B \approx A\}$. We say that $[A] \leq [B]$ iff $A - B$ is a null set.

**Definition 2.11.** The *random algebra* on ${}^{\omega}\omega$ is the set $\mathcal{B}_0({}^{\omega}\omega) = \{[A] : m_{\mathcal{L}}(A) > 0\}$, equipped with the partial order $\leq$ between equivalence classes.

**Proposition 2.12.** *$(\mathcal{B}_0({}^{\omega}\omega), \leq)$ is an atomless and separative partial order that satisfies the c.c.c.*

**Proposition 2.13.** *Let $M$ be a CTM, and let $G$ be a $\mathcal{B}_0({}^{\omega}\omega)$-generic filter. Then there is an unique real $x_G \in {}^{\omega}\omega \cap M[G]$, such that, for all the Borel sets $B$ that are encoded by sequences in $M$, we have*

$$x_G \in B^* \Leftrightarrow [B] \in G,$$

*where each $B^*$ is the unique Borel set in $M[G]$ that corresponds to $B \in M$. In particular, $M[x_G] = M[G]$. The $x_G$ is called a* random real *over $M$.*

The set of random reals can be characterized in terms of the null Borel sets in $M$.

**Proposition 2.14.** *Let $M$ be a CTM. Then $x \in {}^{\omega}\omega$ is a random real over $M$ iff $x \notin B^*$, for all the Borel sets $B \in (\mathcal{B}({}^{\omega}\omega))^M$ which are null in $M$.*

Conversely, each real that is not random over $M$ must be in a Borel set $B^* \in M[G]$, such that $B^M$ is null in $M$. The next proposition is a key step in the proof of Solovay's Theorem.

**Proposition 2.15.** *If $(2^{\aleph_0})^M$ is countable in $M[G]$, then $A = \{x \in {}^{\omega}\omega : x \text{ is not random over } M\}$ is a null set.*

# 3 Solovay's Theorem

## 3.1 Introduction

**Theorem 3.1** (Solovay)**.** *If there is a model of* ZFC *with an inaccessible cardinal $\kappa$, then there is a model of* $\mathrm{ZF} + \mathrm{DC} + \mathrm{LM}$.

The key ingredients in the proof of Solovay's Theorem are the Lévy collapse $\mathrm{Col}(\omega, < \kappa)$, with $\kappa$ inaccessible, the homogeneity of $\mathrm{Col}(\omega, < \kappa)$, the random algebra and the Solovay sets.

## 3.2 Solovay Sets and Random Reals

A real number $x$ is *generic over $M$* if $M[x]$ is the smallest generic extension of $M$ containing $x$.

**Definition 3.2.** Let $M$ be a CTM, and $A \subset {}^{\omega}\omega$. We say that $A$ is *Solovay over $M$* if there is a formula $\varphi$, and parameters $a_1, \ldots, a_k \in M$, such that if $x \in {}^{\omega}\omega$ is generic over $M$, then

$$x \in A \ \text{ iff } \ M[x] \vDash \varphi(x, a_1, \ldots, a_k).$$

The first goal is to find a sufficient condition for all Solovay sets $A$ over $M$ to be $\mathcal{L}$-measurable. In order to do that we start by partitioning $A$ into random reals and nonrandom reals.

**Lemma 3.3.** *Let $M$ be a CTM, and $A \subset {}^{\omega}\omega$ be Solovay over $M$. Then there is a Borel set $B \subset {}^{\omega}\omega$, such that $x \in A$ iff $x \in B$, for every $x \in {}^{\omega}\omega$ that is random over $M$.*

It follows that the elements of $A \Delta B$ are all nonrandom reals. By Proposition 2.16 we get the following:

**Corollary 3.4.** *Let $M$ be a CTM, $\mathbb{P} \in M$ be a partial order, $G \subset \mathbb{P}$ be a $\mathbb{P}$-generic filter, and $(2^{\aleph_0})^M$ be countable in $M[G]$. If $A \subset {}^{\omega}\omega$ is Solovay over $M$, then $A$ is $\mathcal{L}$-measurable.*

## 3.3 The Strategy For the Proof of Solovay's Theorem

It is possible to prove that all the sets of reals $A \in M[G] \cap \mathrm{OD}_{{}^{\omega}\omega \cap M[G]}$ are Solovay sets. If we forced $(2^{\aleph_0})^M$ to be countable in $M[G]$, then the set of nonrandom reals in $A$ would be null, making $A$ an $\mathcal{L}$-measurable set. Then we would extract the substructure $\mathrm{HOD}^{M[G]}_{{}^{\omega}\omega \cap M[G]}$ from $M[G]$ and be done with the proof.

However, it is not so simple because there are technical difficulties that arise in relation to the Solovay sets. It is not possible to prove that each set of reals $A \in M[G] \cap \mathrm{OD}_{{}^{\omega}\omega \cap M[G]}$ is Solovay over the same submodel of $M[G]$. Each $A$ is Solovay over a submodel $M_A$ of $M[G]$ that depends on $A$. Additionally, in each of the $M_A$ we need $(2^{\aleph_0})^{M_A} = \aleph_0^{M[G]}$ to prove that $A$ is $\mathcal{L}$-measurable. This is where the inaccessible cardinal has a crucial role.

If $\kappa$ is an inaccessible cardinal, and $\aleph_\alpha < \kappa$, then $2^{\aleph_\alpha} < \kappa$. In order to obtain $(2^{\aleph_0})^{M_A} = \aleph_0^{M[G]}$ for every $M_A$, we should collapse all the possible cardinalities $(2^{\aleph_0})^{M_A}$. This can be accomplished with the Lévy collapse $\mathrm{Col}(\omega, < \kappa)$, where $\kappa$ is an inaccessible cardinal. The Lévy collapse will force the nonrandom reals of a Solovay set $A$ over $M_A$ to be a null set, making $A$ an $\mathcal{L}$-measurable set. This is how the forcing notions $\mathrm{Col}(\omega, < \kappa)$ and $\mathcal{B}_0({}^{\omega}\omega)$ "interact" in the proof of Solovay's Theorem.

We need a careful application of the Lévy collapse. For each $A$, the submodel $M_A$ of $M[G]$ is a generic extension of $M$, such that $M \subsetneq M_A \subsetneq M[G]$. We will find a general method to factorize $M[G]$ so that, in each particular factorization related to $A$, it includes $M_A$ as a factor. This factorization method allows us to prove that each set of reals $A \in M[G] \cap \mathrm{OD}_{^\omega\omega\cap M[G]}$ is Solovay over $M_A$, where $(2^{\aleph_0})^{M_A} = \aleph_0^{M[G]}$, and thus prove the $\mathcal{L}$-measurability of $A$.

Once the technical difficulties are removed, the proof is straightforward. After the Lévy collapse we extract the substructure $N = \mathrm{HOD}^{M[G]}_{^\omega\omega\cap M[G]}$ from $M[G]$, and prove it is a model of ZF+DC in which all sets of reals are $\mathcal{L}$-measurable.

### 3.4 Solovay's Technical Lemma

If $G$ is a $\mathrm{Col}(\omega, <\kappa)$-generic filter, then we define the restrictions $G \restriction [\lambda, k[$ and $G \restriction \lambda$ by restricting the support of $p \in G$ to $[\lambda, k[$ and $\lambda$, respectively. So $(G \restriction \lambda) \times (G \restriction [\lambda, k[)$ is isomorphic to $G$.

**Lemma 3.5.** *Let $M$ be a CTM, $G$ be a $\mathrm{Col}(\omega, <\kappa)$-generic filter, and $N$ be a transitive set with $\omega \in N$.*

(1) *For each $f \in {}^\omega N \cap M[G]$, there is a cardinal $\lambda < \kappa$, such that $f \in M[G \restriction \lambda]$.*

(2) *In particular, if $x \in {}^\omega\omega \cap M[G]$, then there is a cardinal $\lambda < \kappa$, such that $x \in M[G \restriction \lambda]$.*

This lemma can be adapted to a finite number of reals. That is, if $x_1, \dots, x_n \in {}^\omega\omega \cap M[G]$, then there is some $\lambda < \kappa$, such that $x_1, \dots, x_n \in M[G \restriction \lambda]$. The next lemma is the specific factorization lemma that we need to surpass the technical difficulties mentioned above. To prove this lemma we require the above mentioned property of $\mathrm{Col}(\omega, \lambda)$ that resembles universality.

**Lemma 3.6.** *Let $M$ be a CTM, $\mathbb{P} \in H_\kappa^{M[G\restriction\lambda]}$ be a partial order where $\lambda < \kappa$, and $x \in {}^\omega\omega \cap M[G]$ be $\mathbb{P}$-generic over $M[G \restriction \lambda]$. Then there is a filter $H \in M[G]$ which is $\mathrm{Col}(\omega, <\kappa)$-generic over $M[G \restriction \lambda][x]$, such that $M[G] = M[G \restriction \lambda][x][H]$.*

### 3.5 Solovay's Theorem

**Theorem 3.7.** *Let $M$ be a CTM, $\kappa$ be an inaccessible cardinal, and $G$ be a $\mathrm{Col}(\omega, <\kappa)$-generic filter. Then every set of reals $A \in M[G] \cap \mathrm{OD}_{^\omega\omega\cap M[G]}$ is $\mathcal{L}$-measurable.*

We merely outline the steps in proof of this theorem. If $A \in M[G] \cap \mathrm{OD}_{^\omega\omega\cap M[G]}$, then there is a corresponding formula $\varphi_A$ such that $x \in A$ iff $M[G] \vDash \varphi_A(x, \alpha_1, \dots, \alpha_k, x_1, \dots, x_l)$. By the technical lemma, we obtain a factorization $M[G] = M[G \restriction \lambda][x][H]$. Using the Forcing Theorem, we obtain $x \in A$ iff $\exists p \in H\ p \Vdash \varphi_A(\check{x}, \check{\alpha}_1, \dots, \check{\alpha}_k, \check{x}_1, \dots, \check{x}_l)$. Since $\mathrm{Col}(\omega, <\kappa)$ is homogeneous the element 1 of $\mathrm{Col}(\omega, <\kappa)$ decides $\varphi_A$. Given that the forcing relation is definable, there is a formula $\psi_A$, such that $x \in A$ iff $M[G \restriction \lambda][x] \vDash \psi_A(x)$. This means that $A$ is Solovay over $M[G \restriction \lambda]$. Thus $A$ is $\mathcal{L}$-measurable. Note that the homogeneity of $\mathrm{Col}(\omega, <\kappa)$ allows us to make a decisive step.

**Theorem 3.8** (Solovay)**.** *Let $N = \mathrm{HOD}^{M[G]}_{^\omega\omega\cap M[G]}$. If $M$ is a model of* ZFC + *"$\kappa$ is an inaccessible cardinal", then $N \vDash \mathrm{ZF} + \mathrm{DC} + \mathrm{LM}$.*

Once more, we merely outline the steps in proof of this theorem. Since $N \subset \mathrm{OD}^{M[G]}_{^\omega\omega\cap M[G]}$, every set of reals in $N$ is $\mathcal{L}$-measurable. Additionally, we know that $N \vDash \mathrm{ZF}$. Thus, all that is left to prove is that

$N \models$ DC. Let $A \in N$, and let $R$ be a binary relation on $A$, such that, for each $x \in A$, there is a $y \in A$, satisfying $x \; R \; y$. We want to prove that there is a sequence $\langle x_n : n \in \omega \rangle$ of elements of $A$ that belongs to $N$. This involves the definition of the first term $x_0$ from four objects: a formula, a real number, and two ordinal numbers. If $x_n$ is defined, then $x_{n+1}$ can be defined by recursion and minimization of the elements $y \in A$ such that $x_n \; R \; y$. The fact that $x_{n+1}$ can be defined by recursion and minimization implies that the sequence is definable in $N$.

# 4 Shelah's Theorem

## 4.1 Introduction

**Theorem 4.1** (Shelah)**.** *If there is a model of* ZF+DC+LM*, then there is a model of* ZFC *with an inaccessible cardinal.*

The proof of Shelah's Theorem is by contradiction: we suppose that ZF+DC+LM is consistent and that $\omega_1^V$ is not inaccessible in $L$, and then show that there is a set of reals that is not $\mathcal{L}$-measurable. This set of reals is a rapid filter on $\omega$. Filters on $\omega$ can be thought of as subsets of ${}^{\omega}2$. As such, a real number $a \in {}^{\omega}2$ is considered, when appropriate, as the characteristic *function* of the corresponding subset $a^{-1}(\{1\}) \subset \omega$; or as that very *subset* $a^{-1}(\{1\}) \subset \omega$.

## 4.2 Rapid Filters Are Not Lebesgue Measurable

Let $A \subset {}^{\omega}2$. Suppose that, for every $a \in A$, if $a \restriction (\omega - n) = b \restriction (\omega - n)$ for some $n$, then $b \in A$. Then $A$ is called a *tail set*. In a tail set, the membership is not affected when two elements have different finite "tails". The $\mathcal{L}$-measurable tail sets satisfy an important property.

**Proposition 4.2.** *If $A$ is an $\mathcal{L}$-measurable tail set, and $B$ is any $\mathcal{L}$-measurable set, then $A$ and $B$ are independent, i.e., $m_{\mathcal{L}}(A \cap B) = m_{\mathcal{L}}(A).\, m_{\mathcal{L}}(B)$.*

**Corollary 4.3** (Kolmogorov's $0-1$ Law)**.** *If $A$ is an $\mathcal{L}$-measurable tail set, then either $m_{\mathcal{L}}(A) = 0$ or $m_{\mathcal{L}}(A) = 1$.*

**Definition 4.4.** A filter $F \subset {}^{\omega}2$ is called *rapid* if for every increasing sequence $\langle n_i : i \in \omega \rangle$ of natural numbers, there is an $a \in F$ such that $|a \cap n_i| \leq i$, for all $i \in \omega$.

Note that $a$ and $n_i$ are being regarded in the intersections $a \cap n_i$ as subsets of $\omega$. The next proposition holds because of Kolmogorov's $0-1$ Law.

**Proposition 4.5.** *Let $F$ be a filter on $\omega$ such that $\omega - n \in F$, for every $n$. If $F$, regarded as a subset of ${}^{\omega}2$, is $\mathcal{L}$-measurable, then $m_{\mathcal{L}}(F) = 0$.*

One of the properties of rapid filters is that $\omega - n \in F$, for every $n$.

**Lemma 4.6.**

(1) *Suppose that $F \subset {}^{\omega}2$ intersects every compact subset of ${}^{\omega}2$ of positive $\mathcal{L}$-measure. Then the outer measure of $F$ is positive, $m^*_{\mathcal{L}}(F) > 0$.*

(2) *Let* $F \subset {}^{\omega}2$ *be a rapid filter on* $\omega$. *Then* $F$ *intersects every compact subset of* ${}^{\omega}2$ *of positive* $\mathcal{L}$*-measure.*

**Corollary 4.7.** *If* $F$ *is a rapid filter on* $\omega$*, then* $F$ *is not* $\mathcal{L}$*-measurable.*

Since rapid filters are not $\mathcal{L}$-measurable, we shall define a rapid filter to be used as the nonmeasurable set required in the proof by contradiction of Shelah's Theorem.

## 4.3 Defining the Rapid Filter

Let us assume in this subsection that $A \subset {}^{\omega}2$ is an uncountable, well-orderable set.

**Definition 4.8.** Let $W \subset {}^{<\omega}2$, and let $a \in {}^{\omega}2$.

(1) We say that $W$ *captures* $a$ if $\exists n \forall m \geq n \ (a \upharpoonright m \in W)$.

(2) We say that $W$ *captures* $A$ if $W$ captures every $a$ in $A$.

(3) We say that $W$ *splits on* $n$ if there is an $s \in W$, with length $|s| = n$, such that the extensions $s * 0$ and $s * 1$ are both in $W$.

**Definition 4.9.** Let $F \subset {}^{\omega}2$ be the set defined as follows:

$$a \in F \quad \text{iff} \quad \exists W \subset {}^{<\omega}2 \ (W \text{ captures } A \text{ and } W \text{ splits on } n \text{ iff } a(n) = 1)$$

**Lemma 4.10.** $F$ *is a filter on* $\omega$.

Under the right circumstances, which we cannot yet guarantee, we can show that $F$ is a rapid filter.

**Lemma 4.11.** *Suppose that the union of any sequence of* $\omega_1$ *null sets is null. Then, for any increasing sequence* $\langle n_i : i \in \omega \rangle$*, there is a* $W \subset {}^{<\omega}2$*, such that* $W$ *captures* $A$ *and* $|\{s \in W : |s| = n_i\}| \leq i$*, for all* $i \in \omega$.

**Lemma 4.12.** *Suppose that for any increasing sequence* $\langle n_i : i \in \omega \rangle$*, there is a* $W \subset {}^{<\omega}2$*, such that* $W$ *captures* $A$ *and* $|\{s \in W : |s| = n_i\}| \leq i$*, for all* $i \in \omega$*. Then* $F$ *is rapid.*

We can summarize the results so far as follows.

**Proposition 4.13.** *If* $A \subset {}^{\omega}2$ *is an uncountable, well-orderable set, and the union of* $\omega_1$ *null sets is null, then there exists a rapid filter* $F$.

## 4.4 Shelah's Theorem

In light of the previous proposition we need to make sure that two facts follow from the hypotheses of Shelah's Theorem: (1) The union of $\omega_1$ null sets is null; (2) There is an uncountable well-orderable set $A \subset {}^{\omega}2$.

**Proposition 4.14.** *Let every set of reals be* $\mathcal{L}$*-measurable. Then the union of* $\omega_1$ *null sets is null.*

**Corollary 4.15.** *Let every set of reals be* $\mathcal{L}$*-measurable. If* $A \subset {}^{\omega}2$ *is an uncountable, well-orderable set, then there is a rapid filter, i.e., there is a set that is not* $\mathcal{L}$*-measurable.*

The only missing piece to prove a contradiction is the existence of an uncountable, well-orderable set $A \subset {}^{\omega}2$. If we assume that $\omega_1^V$ is not inaccessible in L, then we can find a well-ordered set whose cardinality is $\omega_1$ in L$[x]$, for a specific $x$. Once this is established, we obtain a contradiction, which means that $\omega_1^V$ must be inaccessible in L. Then Shelah's Theorem follows.

**Theorem 4.16** (Shelah)**.** *If there is a model of* ZF+DC+LM*, then we have* L $\vDash$ *"$\omega_1^V$ is inaccessible".*

We just outline the main steps in the proof. If we suppose that L $\vDash$ "$\omega_1^V$ is not inaccessible", then we end up concluding that $\omega_1^V$ is a successor in L. Let L $\vDash \omega_1^V = \alpha^+$, where $\alpha < \omega_1^V$ is countable in V, and let $x \in {}^{\omega}2$ be a code for a well-ordering of $\omega$ of order-type $\alpha$. Then L$[x] \vDash$ "$\alpha$ is countable". Since we have L$[x] \vDash \omega_1^V = \alpha^+$, we conclude that $\omega_1^V$ is the first uncountable cardinal of $L[x]$ (which is less or equal to $|{}^{\omega}2|$ in $L[x]$). Therefore, the set ${}^{\omega}2 \cap$ L$[x]$ is uncountable. And given that $L[x]$ is well-orderable, ${}^{\omega}2 \cap$ L$[x]$ is a well-ordered uncountable set of reals. Therefore there exists a rapid filter, i.e., a nonmeasurable set of reals. Since this is a contradiction, we must have L $\vDash$ "$\omega_1^V$ is inaccessible".

# 5 Conclusions

The Lebesgue measure inspired many advances in set theory, of which Solovay's Theorem and Shelah's Theorem are prominent examples. These two theorems establish the equiconsistency of the axiom systems ZFC+I and ZF+DC+LM. This equiconsistency result also means that ZF+DC+LM has the consistency strength of an inaccessible cardinal. The inaccessible cardinals are large cardinals and large cardinals form a (partial) hierarchy. As such, the axiom system ZF+DC+LM ranks at the level of the inaccessible cardinals in the partial hierarchy of large cardinals.

# 6 Main References


1 M. Bekkali. Topics in Set Theory, volume 1476 of Lecture Notes in Mathematics. Springer, Berlin, 1991.

2 T. Jech. Set Theory. Springer, Berlin, $3^{rd}$ edition, 2002.

3 A. Kanamori. The Higher Infinite. Springer, Berlin, $2^{nd}$ edition, 2009.

4 K. Kunen. Set Theory, An Introduction to Independence Proofs. Elsevier, Amsterdam, 1980.

5 M. Ricou. Measure and Integration. Unpublished Lecture Notes in Portuguese, Instituto Superior Técnico, 2013.

6 R. Schindler. Set Theory, Exploring Independence and Truth. Springer, Berlin, 2014.

7 B. Semmes. The Raisonnier-Shelah construction of a nonmeasurable set. Master's thesis, University of Amsterdam, 1997.