Lisbon's Public Transportation Network and its Fractional Dynamics

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Abstract

Research works demonstrate that transportation networks can be described by fractals, as of writing, or at most over some short period. In this paper, the public transportation network of Lisbon, Portugal, is analysed from 1901 to 2015.

This study can be divided in two stages. In the first stage, fractal dimension and entropy are used to quantify the evolution of the network over time. The results show that both quantities are appropriate to quantify the growth of the network, as the description is compatible with known historical events. These results are also in line with previously published material on other public transportation networks. The second stage focuses on relating the previous results with additional information. This phase includes: establishing correlations between the city development and population numbers with the evolution of the public transportation network; considering different levels of the network based on transportation schedule and passenger capacity; observing the significance of the distance between consecutive stops. The results confirm the correlation between shifts in population in parishes and the development of the network. Using the routes’ schedule allows to stratify the network, providing perspective on the fractal behaviour at different levels and also on how the network evolved to homogenise the capacity of different routes. The distance between consecutive stops follows a non-trivial power-law behaviour.

Keywords: Public Transportation Network, Fractal Dimension, Entropy, Lisbon

1. Introduction

The growth of Lisbon’s public transportation network is closely related to the expansion of the city. The increasing need of mobility drives the expansion of the network, not only in size but also in complexity. The public transportation network considered here comprises four means of transportation: bus, subway, tram and train, each with different limitations and attributes. This leads to interesting dynamics within the network, as a single trip can involve more than one mode of transport. Additionally, not all transports began operating at the same time, and were not at the same stage of development when introduced as they are now. Therefore, there is an interest in studying the evolution of the network through time.

There are several studies on the fractal nature of public transportation networks [3–5, 11, 19, 20], showing that these tools are capable of describing the complexity of these network, as well as its evolution over time. In this work, the same concepts will be applied to Lisbon’s public transportation network, in order to assess its fractional dynamics.

The topics covered in this work consist of: the variation of the fractal dimension and fractional order entropy of the network through time; the relation between these variations and the shift in population within the municipality; the variation of the fractal dimension and fractional order entropy at different levels of the network; the probabilistic distribution of network stops.

The geographical scope of this study is the municipality of Lisbon and the time frame considered is between the years 1901 and 2015. Figure 1 portrays the network at selected years.

The information on the available routes in each year was compiled from [7,10,17,19], while the coordinates were obtained with [1]. Population numbers were extracted from national censuses from 1911 to 2011 available in [13, 14].

2. Box-counting Fractal Dimension

While a fractal object can display a pattern repeated at different scales, ranging from infinitely large to infinitely small, a public transportation system can only show this behaviour at certain scales. For this kind of network, the largest conceivable
scale is the size of a continent; the smallest is the distance between two consecutive bus or tram stops. So, within a reasonable limit, a fractal behaviour can be observed and therefore quantified. One such measure is the fractal dimension, which measures the space a fractal fills as the scale observed goes from the larger to the smaller one. While there are several definitions for fractal dimension, [2, 6, 15], due to its easier numerical implementation, the one considered in this paper is the box-counting fractal dimension.

The algorithm for the box-counting fractal dimension, as explained in [19], is as follows:

1. Repeat
   - cover the fractal object $F$ with a grid composed of equal squares, with size $\epsilon > 0$;
   - count the number of boxes (squares) that cover the fractal $F$, $N_\epsilon(F) \in \mathbb{N}$;
   - decrease $\epsilon$;
2. The fractal dimension $b \in \mathbb{R}$ is the slope of the log-log plot of $N_\epsilon(F)$ vs. $\epsilon$

$$b(F) = - \lim_{\epsilon \to 0^+} \frac{\log N_\epsilon(F)}{\log \epsilon}$$  

Equation (1) shows the relationship between the size of the grid and the fractal dimension, such that $N_\epsilon(F)$ corresponds to the number of boxes that cover the fractal object $F$ for a given $\epsilon$, with $\epsilon$ being the size of the side of each square on the grid. As the network only shows a fractal behaviour within a certain scale, a stopping criteria must be introduced to determine the boundaries of $\epsilon$. By plotting the fractal dimension $b$ versus $\epsilon$ for a particular year, it is possible to find the interval where $\epsilon$ decreases almost linearly with the grid size, and therefore the points obtained with grids outside this interval can be disregarded from the calculation of $b$.

Equation (1) implies that the fractal dimension can be estimated as the exponent of a power law [19]:

$$N_\epsilon(F) = ae^b$$  

In Equation (2), the parameter $a \in \mathbb{R}^+$ is related to the length between the bifurcations of the network. This parameter provides additional information regarding the complexity and interconnectivity of the network.

To implement this algorithm to the public transportation network, the following steps were applied:

1. Each map is padded with empty spaces around it in order to create a square image. This is done by adding background pixels around the centered map until the image has a resolution of $2^k \times 2^k$, where $k \in \mathbb{N}$.

The value of $k$ is chosen so that $2^k$ is the first power of 2 larger than the largest side of the original map.

2. For every year, the first step of the box-counting algorithm is applied. In each iteration, $\epsilon$ decreases in powers of base 2, starting at $2^7$ and ending at $2^2$. Values of $\epsilon / \in [2^2, 2^7]$ do not make sense, since the size of the image is the upper limit to the size a square can have, and the size of the marker of the lines in each image is the smallest a square can be.

3. The results of the previous step are approximated by means of a power law (2) using the least squares method.

The steps mentioned were followed as described to compute the fractal dimension of the entire public transportation network for the whole city. The results from this process are presented in Figure 2 and are discussed along those of Figure 6, further below.

In addition to the entire municipality as a whole, each parish was also analysed individually. The results obtained are presented in Figure 3.

As it can be seen in Figure 3, the median value for the average parish fractal dimension increases over time, but it always hovers around 1, bounded by $[0.93, 1.14]$. This means that the scale of a parish, the public transportation network is mostly a group of lines, and not a compact network. However, from the 1950s onward, upper adjacent points display $b > 1.24$, with a maximum value of $b = 1.33$ in the
As the network grows in a non-uniform fashion through the years, the calculation of the entropy associated with its expansion can give a better understanding of the distribution of the public transportation routes along the years.

The usefulness of this measure of uncertainty led to a lot of interest in generalizing the Shannon entropy and exploring its applications in other fields, spurriing various formulations [18], such as the entropy of fractional order, $S_\alpha$ [16] in Equation (3).

$$S_\alpha = \sum_i \left\{ -\frac{p_i^{-\alpha}}{\Gamma(\alpha + 1)} [\ln(p_i + \psi(1) - \psi(1 - \alpha))] p_i \right\},$$

where $\alpha$ depicts the fractional order, with $-1 < \alpha \leq 1$. The probability of some event occurring is $p_i$ while $\Gamma(\cdot)$ and $\psi(\cdot)$ represent the gamma and digamma functions, respectively. The case $\alpha \to 0$ yields the Shannon entropy ($S = \sum_i -\ln(p_i)p_i$).

To compute $S_\alpha$ for the public transportation network throughout the years, the following algorithm, adapted from [19], was applied:

1. A map for each year is created. These maps are the same as the ones used for the fractal dimension, explained in section 2, but each line of the network is drawn as a succession of equally spaced points, instead of actual lines.
2. A grid of $4 \times 4$ (pixel) squares is superimposed over the map. This resolution for the squares was chosen because the points on the map are roughly $2 \times 2$ (pixel) squares.
3. The probabilities are computed as $p_i = \frac{n_i}{N}$, where $n_i$ is the number of points in each square of the grid and $N$ is the total number of points across all squares.
4. Entropy is computed using Equation (3).

Steps 3 and 4 are repeated for various values of $\alpha \in [0.1, 0.9]$ with increments of 0.01.

This process results in the plot shown in Figure 5.

The fractional entropy increases with $\alpha$ until a maximum value at $\alpha = 0.74$, at which point it starts to decrease as $\alpha$ continues to increase, decreasing faster as $\alpha$ grows larger.
In Figure 6, the periods established in Figure 2 are superimposed over the fractional order entropy, for $\alpha = 0.74$.

Comparing these results with those of Figure 2 provides some insight on the evolution of the network:

- **1st Period (1900 – 1920):** The evolution of the fractal dimension almost mirrors that of the entropy, growing faster with the inception of the network and eventually settling with a near constant value as it approaches 1920.

- **2nd Period (1920 – 1944):** After some closings, the network remains almost the same till the end of World War II draws near.

- **3rd Period (1944 – 1959):** Year 1944 sees the introduction of the bus service in Lisbon. This results in rapid growth of the network, as shown by the jump in both the fractal dimension and the entropy. Finally, in 1959 the subway system is opened, adding even more lines in the centre of the city.

- **4th Period (1959 – 1998):** This period begins with a decrease in entropy, once again due to the closing of several tram routes. However, the entropy quickly grows again, as bus routes start to cover the perimeter of Lisbon and reach new locations on the western and north-eastern sides of the municipality. The network stagnates by the end of this period, as shown by the slowing of the growth rate for the fractal dimension and the entropy.

- **5th Period (1998 – 2015):** The fractal dimension and the entropy increase faster in 1998 due to the growth of the subway network. A big reorganisation of the buses from 2007 can be seen. Parameter $\alpha$ is mostly constant, as new intersections are not being created.

The computations for the entropy values are also extended to each parish individually, with the results obtained shown in Figures [7, 8]. These figures displays how the fractional order entropy changed in 50 years, from 1965 to 2015. This interval is part of the 4th and 5th periods of Figure 6.

These figures show the shift of the focus of the network over the last 50 years. Figures [7, 8] show, respectively, the values of entropy for each parish,
with Figure 8 displaying each parish colored according to the change in entropy experienced between these years. Red colored parishes show a decrease in entropy, while green colored parishes reveal an increase in entropy. With the exception of Belém, the red colored parishes are located at the city’s downtown, mostly adjacent to each other. Of these parishes, only São Vicente and Misericórdia are green, with the increase in entropy amounting to 10% in both situations. This overall decrease in entropy at these locations is compatible with the known streamlining of the bus route, as well as the expansion to outer parishes.

The computation of the entropy values was done by fixing $\alpha$ to the value that maximizes the entropy for each parish, as done before for the entire city. This means that $\alpha \in [0.41, 0.65]$, according to the size of the parish.

4. Relation between fractal dimension and fractional order entropy with population

During the time span [1901, 2015] there were 11 national censuses, with the first taking place on 1911 and the last on 2011. Figure 9 shows how the population of Lisbon evolved during those years.

![Figure 9: Evolution of the population of Lisbon during [1901, 2015].](image)

Table 1: Correlation coefficients between population and entropy/fractal dimension for Lisbon.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\rho$</th>
<th>$r_s$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractal Dimension $a$</td>
<td>0.09</td>
<td>-0.22</td>
<td>-0.11</td>
</tr>
<tr>
<td>Fractal Dimension $b$</td>
<td>0.16</td>
<td>-0.10</td>
<td>-0.02</td>
</tr>
<tr>
<td>$S_\alpha$</td>
<td>-0.02</td>
<td>-0.30</td>
<td>-0.24</td>
</tr>
</tbody>
</table>

The low values of $\rho$ across all variables on Table 1 are coherent with the non-linear nature of the variables considered, as displayed in Figures 2, 6 and 9. Only $b$, the parameter responsible of quantifying the extension of the network, appears to be weakly correlated with the population. Other correlation coefficients for this variable also show negative values near 0, suggesting almost no linear relationship between $b$ and population numbers. In regards to parameter $a$, both $r_s$ and $\tau$ suggest a weak downhill relationship between this variable and population numbers.

When it comes to entropy, $r_s$ and $\tau$ are similar and negative, indicating a possible weak relation between the entropy and the population.

Considering each parish individually yields the results presented in Figures 10 and 11.

![Figure 10: Map of $\rho$ between $b$ and population numbers (parish).](image)

![Figure 11: Map of $\rho$ between $S_\alpha$ and population numbers (parish).](image)

Both figures show that around 80% of the parishes display a moderate linear relationship ($|\rho| \geq 0.50$) between the evolution of the population numbers and of the fractal dimension/entropy. It is evident in both cases that parishes along the river show a negative correlation coefficient, while the more positive values are associated with outskirt parishes to the north of the municipality.

These results suggest a correlation between the expansion of the public transportation network and
the shift of the population from the downtown of the city to the outskirts parishes. Also, the fractal dimension and entropy provide compatible results. To explore this further, a single year was considered (2011) and a relation between fractal dimension and entropy was explored, as portrayed in Figure 12.

Figure 12: Fractal dimension vs Fractional order entropy (2011).

This figure shows that parishes with large values for the fractal dimension usually have high values for $S_\alpha$. By doing a linear regression and plotting the result over the data points (the orange line in Figure 12) there are 4 outliers that stand out: Alvalade, Avenidas Novas, Benfica and Marvila (labelled accordingly in Figure 12).

To understand why these parishes have an outlier behaviour, an additional variable was considered, the area of the parish (in km$^2$), obtained from [14]. Plotting this variable along with the population numbers and $S_\alpha$ yields the results shown in Figure 13.

Figure 13: Population vs Area vs Fractional order entropy (2011).

Figure 13 shows the relation between area, population and $S_\alpha$ for each parish. It is visible that the fractional order entropy is not related solely to either area or population numbers.

The dotted line in Figure 13 denotes a separation between the 6 largest and more populated parishes from the rest. To the left of this line most parishes are near parishes with similar values of $S_\alpha$. The small downtown parishes are concentrated on the southwest corner of the plot and have the lowest entropy values. Another group of parishes that share close values of $S_\alpha$ are those with an area over 2.5 km$^2$ and under 4.2 km$^2$ and population between $[15, 22] \times 10^3$ citizens.

The results depicted in Figure 12 in addition to those represented on Figure 13 suggest that fractal dimension and $S_\alpha$ are related with the size and level of population of each parish. In addition to this, Figure 13 shows that it is possible to separate the 24 parishes in two groups base on area and population and one of this groups displays low to medium entropy values while the second group is composed of parishes with high values for the entropy. However, population numbers and area are not enough to explain the results obtained for the fractal dimension and entropy of each parish, since they are insufficient to describe each parish itself.

Parishes were then partitioned using the $k$-means clustering algorithm. The features used were the area, population and location of the parish and the number of clusters utilized was $k = 2$, as this was the value found (by trial and error) to better suit this small data set. Since the fractal dimension and the fractional order entropy were not considered they have no bearing on the results of the clustering. Figure 14 shows the results of the clustering plotted in a fractal dimension vs. fractional order entropy plot.

Figure 14: Fractal dimension vs Fractional order entropy (clusters).

Figure 14 contains the same plot as Figure 12, however each point is now colored according to the cluster it was assigned to, based on its features. It is shown that outside of the three labelled outliers parishes partitioned according to the features described previously also share similar fractal dimension and $S_\alpha$. This means that both the fractal dimension and the fractional order entropy of the public transportation network may be related not only with the network complexity but also with the area, population and location of each parish.
5. Layered Network

Until now all routes served by at least one public mode of transportation were equally considered. But this is not realistic. Some are served by frequent transportation means, with high passenger capacity. Others are poorly served and cannot transport but few people each day.

For this application two scenarios were analysed: the fractal dimension of the bus network over the years, layered according to known schedule information (from [8]) and the entropy for the entire network in 2015, estimated values for the passenger capacity, computed based on the schedule of each mode of transport and its standard vehicle passenger capacity (with data from [7,17]).

For the fractal dimension, different layers of the network are computed using percentiles computed from the routes’ capacity. This is done in increments of 10% for the percentiles, ranging from 0% to 90%. The result is a set of 10 different configurations of the network, each depicting a different range of route capacity observed in the network. Figure 15 illustrates some of the maps obtained for the year 2015.

![Figure 15: Public transportation network weighted by passenger capacity.](image)

The fractal dimension for each map can be computed following the algorithm from section 2. The maps can be constructed for the two cases mentioned previously, the bus network over time or the complete network in 2015.

Computing the fractal dimension from maps such as the ones illustrated in Figure 15 using the bus network weighted by the bus frequency yields the results shown in Figure 16.

Figure 16 shows how the fractal dimension of the bus network decreases as the routes with less passenger capacity are discarded, for three years, 1955, 1975 and 2015. While the complexity of the network and the capacity increase over the years, the relation between these variables remains similar through time. Using the slopes, it is possible to observe instances of similar behaviour over three capacity intervals:

- **100% to 80%**: With the passing of time the slopes of the segments within this interval increase, leading the segments to become more horizontal. This points to the network being more fully utilized, as removing the less used routes in 1955 lead to a dip in the fractal dimension larger than the one that occurs in 2015. This is caused not only by the smaller number of routes active in the earlier years of the bus network but also by the meager frequency of buses on routes that connected the center of the municipality to the outskirts.

- **80% to 70%**: In 1955 there is a steep increase in the slope of this segment, indicating that the core of the network is within this interval. As the bus frequency among routes becomes more homogeneous, this increase becomes less significant, as it can be seen by the decreasing value of the slope for this segment over the years. Additionally, this interval extends to the following segment (70% to 60%) in 2015, as a result of the more streamlined and balance configuration of that year.

- **70% onwards**: For this interval the slope decreases again. The value of each segment is almost constant for 2015, while in 1975 there is another decrease around the 40% mark. Due to the plot being cutoff at the value of 1 for the fractal dimension, it is not possible to assess how it behaves for lower capacity values.

For the computation of the fractional order entropy a different method was employed. The computation is done in the same fashion as described in section 3, but instead of drawing every route as a sequence of equally spaced points based on distance, in this case, the points are equally spaced based on the route bus frequency/passenger capacity. The
total number of points allotted to each route can be approximated by

\[ N_{p_i} \approx f_p \times \log(BF_i) \tag{4} \]

where \( N_{p_i} \) is the total number of points for route \( i \), \( f_p \) is a scale factor, that is adjusted to ensure that points are neither too close or too far apart (\( f_p \) has the same value for all the routes of the network), and \( BF_i \) is the bus frequency on route \( i \) (when dealing with the entire network it is the passenger capacity of route \( i \)).

Applying this process to the entire network of year 2015 yields the results presented in Figure 17.

![Figure 17: Fractional order entropy for the network weighted by passenger capacity.](image)

- **Region A**: The decrease in \( S_\alpha \) in this region is not as evident as it was in the case of fractal dimension. This happens because many of the routes that disappear in this segment are small routes located near the center of the city, where there are also the main routes. While excluding the smaller routes does affect \( p_i \) the main routes that remain keep \( p_i \) from decreasing drastically.

- **Region B**: The adjacent segments in this region of the plot have very distinct slope values, unlike what was observed for the fractal dimension.

- **Region C**: Entropy continues to decrease but at a slower rate than previously, as a result of the network being reduced from the subway network to some segments of subway lines aligned at the center of the city, in the south – north direction.

### 6. Inter-Station Distance

The Lévy distribution is a subclass of the family of \( \alpha \)-stable distribution, characterized by the following parameters: \( \alpha \in [0, 2] \), the characteristic exponent that describes the tail of the distribution; \( \beta \in [-1, 1] \), the skewness parameter, specifies if the distribution is either right (\( \beta > 0 \)) or left (\( \beta < 0 \)) skewed; \( \gamma_\alpha > 0 \), the scale parameter and \( \delta \in \mathbb{R} \), the location parameter [9].

The characteristic function (CF) is shown in Equations (5) and (6) for this family of distributions:

\[
\phi(t) = \exp \left( -\gamma_\alpha |t|^\alpha \left[ 1 - i\beta \text{sgn}(t) \frac{\pi \alpha}{2} \right] \right) + i\delta t \tag{5}
\]

\[
\phi(t) = \exp \left( -\gamma_\alpha |t|^\alpha \left[ 1 + i\beta \text{sgn}(t) \frac{\pi}{\delta} \log t \right] + i\delta t \right), \quad \text{for } \alpha \neq 1 \tag{6}
\]

Using the CF, it is possible to estimate the parameters of the distribution.

Figure 18 (where \( x \) is the distance between stations in km and \( CDF \) stands for cumulative distribution function) shows that every considered groups of inter-station distances display a power-like decay within a certain interval of values. These intervals correspond to the almost linear segment in the log-log plot.

![Figure 18: Log-log plot of the right tail of the empirical CDF.](image)
stand out from the others, as the parameters appear to be more in line with a Cauchy distribution. A possible reason for this result is that the subway operates underground and thus is not subject to the same restrictions in regards to the location of its stops.

Figure 19 shows the probability density function for each set of data fitted to the corresponding data.

Figure 19: Fitted PDF for each means of transportation.

As indicated by the parameter values on Table 2 the probability density function (PDF) for the distribution of inter-station distances, for the bus+tram (a) takes a similar shape to that of the entire network (d). It is also observable the influence of this pair of means of transportation on the entire network; however the complete network has more points with larger values for the distance, due to the inclusion of the train set (b). This Figure also allows to graphically observe the difference between the subway PDF and the others. Figure 19 (c) shows how the PDF is more akin to a Cauchy PDF, as expected based on the parameter values from Table 2.

7. Conclusions
Both the fractal dimension and the fractional order entropy were found to vary in accordance with the history of the public transportation network.

The values computed for Lisbon’s public transportation network were in line with what is available in the literature for other cities. The different levels of development of the network in each parish was also studied. Downtown parishes were found to present smaller values for fractal dimension and fractional order entropy, while outskirt parishes presented larger values for both variables. The shift in the network to provide better access from the farthest parishes to the city downtown was also observed through the fractional order entropy.

Overall, it was concluded that fractal dimension and entropy were capable of describing the increase in complexity and extension of the network as a whole and as each parish’s segments.

When considering the entire city, a weak relation was found between the growth of the public transportation network and the municipality’s population numbers over time. However, when considering each individual parish, stronger linear correlations were found. In the case of outskirt parishes, the population increased over time, and so did the network in those segments. On the other hand, the downtown parishes presented weaker correlations between the decreasing population numbers and the small alterations in the network in those locations. From these results it was concluded that as the network evolves in complexity and extent, the population is able to be located in parishes farther from the city’s center and downtown and still have high mobility within the municipality.

It was shown that grouping parishes by their population size, relative distance to the city downtown and area yielded groups with similar values for the fractal dimension and fractional order entropy. This indicates that the development of the segments of the network in each parish is not only a function of its population but also of its location and area.

By discarding the less affluent routes, the network is reduced to its main arteries, allowing the influence of each mode of transport to become evident. In particular, it was found that the decay in fractal dimension could be divided in three intervals.

Extending this idea to various instances of the network though its history shows how more robust the network becomes, in terms of its fractional characteristics. It was shown that not only the fractal dimension of the network increases over time, but also the difference between its layers also decreases, meaning that the network becomes more homogeneous, regarding the estimated capacity of its routes.

An $\alpha$-stable probability distribution was found
to fit the consecutive inter-station distance dataset. This result suggests that the distribution of the inter-station distance follows a non-trivial power-law, a result compatible with what is found in the literature.

To further improve on this work, the following recommendations are provided: enhance the precision of the network approximation utilized; include more means of transportation from surrounding municipalities that travel to and from Lisbon; extend the scope from the municipality of Lisbon to the surrounding municipalities; use development indexes to relate the state of the network to each parish.

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