Energy in general relativity: a comparison between quasilocal definitions

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Using a 3+1 spacetime decomposition, we derive Brown-York's and Lynden-Bell-Katz's quasilocal energy definitions. Then, we analyse the properties of the two definitions in specific spacetimes and derive what laws of black hole mechanics come from each definition. Comparing the results in the Newtonian limit, we find a suitable interpretation for the localization of gravitational energy for each definition. Using reasonable arguments, we show which definition is more appropriate and consistent with some phenomena, like Mercury's perihelion precession. Finally, we suggest a way to unify the two definitions by modifying the expression for matter energy in special relativity.

I. INTRODUCTION

In general relativity, it is not possible to define a local gravitational energy-momentum tensor. However, some non-tensorial local objects that describe interesting features of gravitational energy have been found, such as the coordinate-dependent Einstein pseudotensor [1], or the Isaacson tensor for gravitational waves [2, 3]. Despite this, it is sometimes possible to define a spacetime's global mass. This was done by Arnowitt, Deser and Misner (ADM) for spatial infinity [4] of an asymptotically flat spacetime and by Bondi and Sachs in the case of null infinity [5]. Moreover, we can define the total energy enclosed by a surface. Such an energy definition is said to be quasilocal. Many quasilocal energy definitions exist in general relativity, an interesting review is given by [6]. For instance, Hawking, Geroch, Misner and Sharp, Penrose, Bartnik, Brown and York, Lynden-Bell and Katz, among others, have different quasilocal energy definitions.

Due to the large number of quasilocal energy definitions, it is important to have a method that allow us to compare them. Such a method is presented here, and is used to compare Brown-York [7] and Lynden-Bell-Katz [8] quasilocal definitions.

This paper is organized as follows. In Section II, we define Brown-York's quasilocal energy and Lynden-Bell and Katz's quasilocal energy. In Section III, we compare the two definitions in specific spacetimes and calculate the law of black hole mechanics that follow from each definition. In Section IV, we discuss the results and give reasonable arguments in order to choose which definition is better.

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II. DEFINITION OF BROWN-YORK'S AND LYNDEN-BELL AND KATZ'S ENERGY

A. The 3+1 decomposition

In general relativity, there is a rigorous formalism that allow us to foliate a region M of spacetime in a family of spacelike hypersurfaces Σ, labeled by t, using adapted coordinates. This formalism is called the 3+1 decomposition.

Although this formalism yields many results , we only write explicitly the ADM decomposition of the metric, which is closely connected to the 3+1 decomposition. The line element ds^2 can be then written as

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b \\ &= -N^2\, dt^2 + h_{ij}\left(dx^i + V^i dt\right)\left(dx^j + V^j dt\right) , \end{aligned} \quad (1)$$

where g_{ab} is the spacetime metric, h_{ij} is the induced metric on Σ, N is called the lapse function and V^i is called the shift vector. For static, spherically symmetric spacetimes, ds^2 can be written as

$$ds^2 = -N^2\, dt^2 + h^2\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) , \quad (2)$$

where N and h are functions of r only.

B. Brown-York quasilocal energy

Brown and York derived their energy definition by applying the Hamilton-Jacobi formalism to the action of general relativity with boundary term. The expression they obtained was

$$E_{\rm BY} = \frac{1}{8\pi}\int_B d^2x\sqrt{\sigma}\,(k - k_0)\ . \quad (3)$$

where B is the two-boundary of Σ, σ is the determinant of the induced metric σ_{ab} in B, k is the trace of the extrinsic curvature of B, and k_0 is the trace of the extrinsic curvature of B, if it was embedded in a flat spacetime. Considering the case of a static, spherically symmetric spacetime, with metric given in Eq. (2), the energy becomes simply

$$E_{\mathrm{BY}}(r) = r\left(1 - \frac{1}{h}\right) . \qquad (4)$$

C. Lynden-Bell-Katz quasilocal energy

Lynden-Bell and Katz used a different approach to define gravitational energy. They defined an expression for the total "matter" energy in general relativity, by generalizing an expression valid in special relativity, and then subtracted that energy to the total ADM mass of the spacetime. The result should give the gravitational energy of the system. This approach can be made quasilocal through Israel's surface layer formalism [9, 10]. We then get for the energy inside a two-boundary B

$$E_{\mathrm{LBK}} = \frac{1}{8\pi}|\xi| \int_B d^2x\sqrt{\sigma}\,(k - k_0) \; . \qquad (5)$$

where $|\xi| = \sqrt{-\xi_a\xi^a}$, ξ^a being the timelike vector field, and the other quantities have been defined in Eq. (3). Considering the case of a static, spherically symmetric spacetime, with metric given in Eq. (2), the energy becomes simply

$$E_{\mathrm{LBK}}(r) = N\,r\left(1 - \frac{1}{h}\right) . \qquad (6)$$

III. COMPARISON OF BROWN-YORK ENERGY AND LYNDEN-BELL-KATZ ENERGY IN SPECIFIC CASES

Using Eqs. (4) and (6), we can compare the two definitions in specific spacetimes and derive the law of black hole mechanics that follow from each one.

A. Definition of energy density for static, spherically symmetric spacetimes

To compare the two quasilocal definitions, we shall compare the energy densities generated by each energy in the Newtonian limit. For that, we need a definition for energy density from a quasilocal energy expression. We consider static, spherically symmetric spacetimes because all the spacetimes that are going to be analyzed share this property.

The expression for energy density can be found by evaluating the energy dE enclosed in the volume between two very similar concentric spheres with radial coordinate r and $r + dr$. By spherical symmetry, the density ρ becomes

$$\rho = \frac{1}{4\pi h\, r^2}\frac{dE}{dr} \, , \qquad (7)$$

where h is defined in Eq. (2).

B. Schwarzschild spacetime

We can now start comparing the two definitions. We begin by considering a Schwarzschild spacetime. The metric of a Schwarzschild can be written, in Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}}\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) , \qquad (8)$$

where m is the standard mass parameter.

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (4),

$$E_{\mathrm{BY}}(r) = r\left(1 - \sqrt{1 - \frac{2m}{r}}\right) . \qquad (9)$$

Calculating the density ρ_{BY} and approximating to second order in $\frac{m}{r}$, we get

$$\rho_{\mathrm{BY}} \approx -\frac{m^2}{8\pi r^4} = -\frac{|\boldsymbol{g}|^2}{8\pi} \, , \qquad (10)$$

where $\boldsymbol{g} = -\frac{m}{r^2}\,\boldsymbol{e_r}$ is the Newtonian gravitational field, $\boldsymbol{e_r}$ being the unit radial vector. It is interesting to note that the total Newtonian gravitational potential energy U can be calculated with

$$U = \int_{\mathrm{space}} -\frac{|\boldsymbol{g}|^2}{8\pi}\, dV \, . \qquad (11)$$

This means that Brown-York's energy in Schwarzschild spacetime corresponds to the interpretation that the gravitational energy is all stored in the field.

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (6),

$$E_{\mathrm{LBK}}(r) = \sqrt{1 - \frac{2m}{r}}\left(1 - \sqrt{1 - \frac{2m}{r}}\right) r \, . \qquad (12)$$

Calculating the density ρ_{LBK} associated with this energy, and approximating to second order in $\frac{m}{r}$, we get

$$\rho_{\mathrm{LBK}} \approx \frac{m^2}{8\pi r^4} = \frac{|\boldsymbol{g}|^2}{8\pi} \, . \qquad (13)$$

This density is the opposite of the result of Eq. (10). However, this is less justifiable according to Newtonian theory. To solve this problem, Lynden-Bell and Katz interpret that gravitational energy is contained in the matter and in the field, and summing the two gives the total Newtonian gravitational potential energy.

C. Reissner-Nordström spacetime

The metric of a Reissner-Nordström spacetime can be written, using Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \frac{1}{1 - \frac{2m}{r} + \frac{q^2}{r^2}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{14}$$

where m is the standard mass parameter and q is the electric charge of the singularity in geometric units.

The Brown-York quasilocal energy for a surface of constant r , using Eq. (4),

$$E_{\mathrm{BY}}(r) = r \left(1 - \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}\right) . \tag{15}$$

Calculating the density ρ_{BY} and approximating to second order in $\frac{m}{r}$ and in $\frac{q}{r}$, we get

$$\rho_{\mathrm{BY}} \approx -\frac{m^2}{8\pi r^4} + \frac{q^2}{8\pi r^4} = -\frac{|\boldsymbol{g}|^2}{8\pi} + \frac{|\boldsymbol{E}|^2}{8\pi} , \tag{16}$$

where $\boldsymbol{E} = \frac{q}{r^2} \boldsymbol{e_r}$ is the electric field produced by the charge q. Eq. (16) contains the result found in Eq. (10) and the electromagnetic energy density. This result is remarkable and very elegant since the gravitational and electromagnetic energy densities have exactly the same structure, differing only in the sign.

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (6),

$$E_{\mathrm{LBK}}(r) = \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}} \left(1 - \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}\right) r . \tag{17}$$

Calculating the density ρ_{LBK} and approximating to second order in $\frac{m}{r}$ and in $\frac{q}{r}$, we get

$$\rho_{\mathrm{LBK}} \approx \frac{m^2}{8\pi r^4} + \frac{q^2}{8\pi r^4} = \frac{|\boldsymbol{g}|^2}{8\pi} + \frac{|\boldsymbol{E}|^2}{8\pi} . \tag{18}$$

where $\boldsymbol{E}$ is the electric field produced by the charge q, as in Eq. (16). Eq. (18) contains the result found in Eq. (13) and the electromagnetic energy density. The interpretation of the gravitational energy term is the same as the interpretation in Eq. (13). However, even though the gravitational energy term differs by a sign between Brown-York and Lynden-Bell-Katz energy densities, they both yield the expected result for the electromagnetic energy density.

D. De Sitter spacetime

The metric of a de Sitter spacetime can be written, using Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right) dt^2 + \frac{1}{1 - \frac{r^2}{l^2}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{19}$$

where $l = \sqrt{\frac{3}{\Lambda}}$ is the cosmological radius, Λ being the cosmological constant that enters Einstein's equations.

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (4),

$$E_{\mathrm{BY}}(r) = r \left(1 - \sqrt{1 - \frac{r^2}{l^2}}\right) . \tag{20}$$

Calculating the density ρ_{BY} and approximating to second order in $\left(\frac{r}{l}\right)^2$, we get

$$\rho_{\mathrm{BY}} \approx \frac{3}{8\pi l^2} - \frac{r^2}{32\pi l^4} = \frac{\Lambda}{8\pi} - \frac{1}{8\pi}\left(\frac{\Lambda r}{6}\right)^2 . \tag{21}$$

We can interpret the right hand side of Eq. (21) in the following way. The first term, $\frac{\Lambda}{8\pi}$, is just the energy density associated with the cosmological constant, or dark energy, and the second term is associated with the gravitational field produced by the dark energy (since it has the dependence in Λ^2). This result is the expected Newtonian limit, since the gravitational field created by a continuous distribution of mass with density $\frac{\Lambda}{8\pi}$, as is the case in the De Sitter spacetime, is $\boldsymbol{g} = -\frac{\Lambda r}{6} \boldsymbol{e_r}$.

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (6),

$$E_{\mathrm{LBK}}(r) = \sqrt{1 - \frac{r^2}{l^2}} \left(1 - \sqrt{1 - \frac{r^2}{l^2}}\right) r . \tag{22}$$

Calculating the density ρ_{LBK} and approximating to second order in $\left(\frac{r}{l}\right)^2$, we get

$$\rho_{\mathrm{LBK}} \approx \frac{3}{8\pi l^2} - \frac{11 r^2}{32\pi l^4} = \frac{\Lambda}{8\pi} + \frac{1}{8\pi}\left(\frac{\Lambda r}{6}\right)^2 - \frac{19 \Lambda^2 r^2}{288\pi} . \tag{23}$$

We can interpret the right hand side of Eq. (23) in the following way. The first term, $\frac{\Lambda}{8\pi}$, is just the energy density associated with the cosmological constant, or dark energy, as in Eq. (21). The second term is associated with the positive energy density stored in the gravitational field produced by the dark energy. The third term is the negative gravitational energy density stored in the dark energy. In order to be consistent with the Lynden-Bell and Katz interpretation for the localization of the gravitational energy, the sum of the last two terms should be equal to twice the gravitational potential energy in the matter, given by $\frac{1}{2}\frac{\Lambda}{8\pi}\Phi$, where Φ is the gravitational potential, plus the positive gravitational field energy density, given by $\frac{|\boldsymbol{g}|^2}{8\pi}$, where $\boldsymbol{g}$ is the gravitational field. However, from a Newtonian perspective, it is not possible to find an adequate method to get this expression.

E. Schwarzschild-de Sitter spacetime

The metric of a Schwarzschild-de Sitter spacetime can be written, using Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right) dt^2 + \frac{1}{1 - \frac{2m}{r} - \frac{r^2}{l^2}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{24}$$

where $l = \sqrt{\frac{3}{\Lambda}}$ is the cosmological radius, and m is the standard mass parameter of the black hole.

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (4),

$$E_{\rm BY}(r) = r\left(1 - \sqrt{1 - \frac{2m}{r} - \frac{r^2}{l^2}}\right) . \tag{25}$$

Calculating the density $\rho_{\rm BY}$ and approximating to second order in $\left(\frac{r}{l}\right)^2$ and in $\frac{m}{r}$, we get

$$\rho_{\rm BY} \approx \frac{3}{8\pi l^2} - \frac{m^2}{8\pi r^4} - \frac{m}{8\pi r l^2} - \frac{r^2}{32\pi l^4} \tag{26}$$

$$= \frac{\Lambda}{8\pi} - \frac{1}{8\pi}\left(\frac{m}{r^2} + \frac{\Lambda r}{6}\right)^2 . \tag{27}$$

With our previous results, we can interpret the terms of the right hand side of Eq. (26). The first term, $\frac{\Lambda}{8\pi}$, is the energy associated with dark energy, just as in Eq. (21). The second term is the energy associated with the gravitational field. Note that the gravitational field energy contribution is consistent with the value of gravitational field created by the point mass in the origin and by the continuous distribution of matter. Indeed, $\boldsymbol{g} = -\left(\frac{m}{r^2} + \frac{\Lambda r}{6}\right) \boldsymbol{e_r}$.

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (6),

$$E_{\rm LBK}(r) = \sqrt{1 - \frac{2m}{r} - \frac{r^2}{l^2}}\left(1 - \sqrt{1 - \frac{2m}{r} - \frac{r^2}{l^2}}\right) r . \tag{28}$$

Calculating the density $\rho_{\rm LBK}$ and approximating to second order in $\left(\frac{r}{l}\right)^2$ and in $\frac{m}{r}$, we get

$$\rho_{\rm LBK} \approx \frac{3}{8\pi l^2} + \frac{m^2}{8\pi r^4} + \frac{m}{8\pi r l^2} - \frac{11 r^2}{32\pi l^4}$$

$$= \frac{\Lambda}{8\pi} + \frac{1}{8\pi}\left(\frac{m}{r^2} + \frac{\Lambda r}{6}\right)^2 - \frac{19\Lambda^2 r^2}{288} . \tag{29}$$

The interpretation of the terms on the right hand side of Eq. (29) is the same as in Eq. (26), as long as we follow the Lynden-Bell-Katz interpretation for the gravitational energy. The first term, $\frac{\Lambda}{8\pi}$, is the energy associated with dark energy, just as in Eq. (23). The second term is the (positive) energy associated with the gravitational field produced by the mass m and by the continuous dark energy distribution. The third term represents the negative gravitational energy density stored in the dark energy, that also appeared in Eq. (23).

F. Interior Schwarzschild spacetime

The metric of an interior Schwarzschild spacetime can be written, using Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -N^2 \, dt^2 + \frac{1}{1 - \frac{8\pi\rho_0}{3} r^2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{30}$$

where N^2 is given by

$$N^2 = \left[\frac{3}{2}\sqrt{1 - \frac{8\pi\rho_0}{3} r_0^2} - \frac{1}{2}\sqrt{1 - \frac{8\pi\rho_0}{3} r^2}\right]^2 . \tag{31}$$

This metric describes the interior of a star of constant gravitational mass density ρ_0 and with radius r_0. Consistency with the outside Schwarzschild solution requires that

$$m = \frac{4}{3}\pi r_0^3 \, \rho_0 . \tag{32}$$

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (4),

$$E_{\rm BY}(r) = r\left(1 - \sqrt{1 - \frac{8\pi\rho_0}{3} r^2}\right) = r\left(1 - \sqrt{1 - \frac{2m}{r_0^3} r^2}\right) . \tag{33}$$

By analyzing the weak field limit of the quasilocal energy at r_0, and requiring consistency with the interpretation for the localization of gravitational energy in Newtonian gravity, we arrive to following relation between the ADM mass m and the rest mass m^* of the star

$$m = m^* - \frac{3}{5}\frac{(m^*)^2}{r_0} . \tag{34}$$

The second term of the right hand side of Eq. (34) is just the total gravitational potential energy of the star. Therefore, in this weak field limit, the gravitational mass measured at infinity is just the sum of the rest mass and the gravitational potential energy of the star. Thus, gravitational potential energy is itself a source of gravitational energy. Calculating the density $\rho_{\rm BY}$ and approximating to second order in $\frac{mr^2}{r_0^3}$, we get

$$\rho_{\rm BY} \approx \frac{m}{\frac{4}{3}\pi r_0^3} - \frac{1}{8\pi}\frac{m^2 r^2}{r_0^6} . \tag{35}$$

The first term is just ρ_0, the density of the star. The second term is just the gravitational energy density stored in the gravitational field. Note that, in the Newtonian theory, the gravitational field inside a homogeneous sphere of mass m and radius r_0 is $\boldsymbol{g} = -\frac{mr}{r_0^3}\boldsymbol{e_r}$. Therefore, Eq. (35) is the expected expression for the energy density according to the Brown-York interpretation.

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (6),

$$E_{\rm LBK}(r) = \left(\frac{3}{2}\sqrt{1-\frac{8\pi\rho_0}{3}r_0^2} - \frac{1}{2}\sqrt{1-\frac{8\pi\rho_0}{3}r^2}\right)\left(1-\sqrt{1-\frac{8\pi\rho_0}{3}r^2}\right) r \tag{36}$$

$$= \left(\frac{3}{2}\sqrt{1-\frac{2m}{r_0}} - \frac{1}{2}\sqrt{1-\frac{2m}{r_0^3}r^2}\right)\left(1-\sqrt{1-\frac{2m}{r_0^3}r^2}\right) r\,. \tag{37}$$

It is interesting to note that, by analyzing the weak field limit of the quasilocal energy at r_0, as was done in the case of the Brown-York energy, and requiring consistency with the interpretation for the localization of gravitational energy in Newtonian gravity, we arrive to the same relation between the ADM mass m and the rest mass m^* of the star, given in Eq. (34). Calculating the density $\rho_{\rm LBK}$ and approximating to second order in $\frac{mr^2}{r_0^3}$, we get

$$\rho_{\rm LBK} \approx \frac{m}{\frac{4}{3}\pi r_0^3} + \frac{1}{8\pi}\left(\frac{mr}{r_0^3}\right)^2 + \left(\frac{3m^2r^2}{8\pi r_0^6} - \frac{9m^2}{8\pi r_0^4}\right)\,. \tag{38}$$

As in Eq. (35), the first term is just ρ_0, the density of the star, the second term contains the positive gravitational field energy density, and the third term is the gravitational energy stored in the matter, which can be calculated with the Newtonian potential associated with a sphere of constant density. As expected, the sign of the gravitational field term is reversed in relation to Eq. (35), because of the difference in the interpretation of the localization of gravitational energy.

G. Black hole laws of mechanics and thermodynamics

We consider a Reissner-Nordström spacetime, with metric given in Eq. (14). We derive the laws of mechanics and thermodynamics that follow directly from the different energy definitions.

Using Brown-York quasi-local definition, we arrive to

$$dE_{\rm BY} = -s\,d(4\pi r^2) + \frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\left(\frac{\kappa}{8\pi}dA_+ + (\Phi_+ - \Phi)\,dq\right)\,, \tag{39}$$

where

$$s = \frac{1}{8\pi r}\left[\left(\frac{1-\frac{m}{r}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right) - 1\right] \tag{40}$$

is the gravitational surface pressure,

$$\kappa = \frac{\sqrt{m^2-q^2}}{r_+^2} \tag{41}$$

is the surface gravity of the black hole, r_+ is the outer event horizon, $A_+ = 4\pi r_+^2$ is the surface area of the outer horizon, and $\Phi_+ = \frac{q}{r_+}$ and $\Phi = \frac{q}{r}$ are the electric potentials at r_+ and r. The factor $\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}$ can be interpreted as the blue-shift of the energy from infinity to the sphere of radial parameter r. When $r \to \infty$, we get

$$dE_{\rm BY} = \frac{\kappa}{8\pi}dA_+ + \Phi_+\,dq\,. \tag{42}$$

This equation is straightforwardly turned into the first law of thermodynamics for a black hole. We just have to replace in Eq. (42) the Bekenstein-Hawking entropy $S_{\rm BH} = \frac{A_+}{4}$ and the Hawking temperature $T_{\rm BH} = \frac{\kappa}{2\pi}$ to get

$$dE_{\rm BY} = T_{\rm BH}\,dS_{\rm BH} + \Phi_+\,dq\,. \tag{43}$$

Therefore, the Brown-York quasilocal energy allows a natural derivation of the first law of mechanics and thermodynamics for a Reissner-Nordström black hole.

Using Lynden-Bell-Katz energy definition, we arrive to

$$dE_{\rm LBK} = -s\,d(4\pi r^2) + \alpha\left(\frac{\kappa}{8\pi}dA_+ + (\Phi_+ - \Phi)\,dq\right)\,, \tag{44}$$

where s is defined as the surface pressure for this definition,

$$\alpha = \left(1 - \frac{1-\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right)\,, \tag{45}$$

and the other quantities were defined in Eq. (39). Note that here α does not have the natural blue-shift interpretation that the factor $\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}$ has in Eq. (39). Letting $r \to \infty$, we get the first law of black hole mechanics that we had also find in (42)

$$dE_{\rm LBK} = \frac{\kappa}{8\pi}dA_+ + \Phi_+\,dq\,. \tag{46}$$

Therefore, the Lynden-Bell-Katz quasilocal energy also allows a natural derivation of the first law of mechanics and thermodynamics for a Reissner-Nordström black hole.

IV. DISCUSSION: PICKING THE BEST DEFINITION

After analyzing many features of the Brown-York and Lynden-Bell-Katz energy definitions, we can now analyze the results and choose the most appropriate energy definition. We will first analyze the crucial differences of the two energies in the Newtonian limit and then check if there are ways to experimentally distinguish the two definitions. Finally, we propose a solution for a problem that appears when comparing the energy densities.

A. Differences between the energy densities in the Newtonian limit

We can define a Newtonian gravitational energy density $\rho_{\rm grav}$ by

$$\rho_{\rm grav} = a\left(\frac{1}{2}\rho\,\Phi\right) + b\left(-\frac{|\boldsymbol{g}|^2}{8\pi}\right), \qquad (47)$$

for any real numbers a and b such that $a+b=1$. In the pure Newtonian theory, there is no way to choose the correct pair (a,b).

The Brown-York energy density for a Schwarzschild spacetime in the weak field limit, calculated in Eq. (10), corresponds to $b=1$, and therefore $a=0$. On the other hand, the Lynden-Bell-Katz energy density for the same spacetime in the weak field limit corresponds to $b=-1$, and therefore $a=2$. This difference is consistent through all spacetimes.

These results have a simple interpretation. They mean that, in the case of the Brown-York energy definition, all the gravitational energy is stored in the field, whereas in the Lynden-Bell-Katz definition, the energy is stored in the matter (twice the total energy) and in the field (the negative of the total energy). Therefore, on pure simplicity grounds, there is no doubt that the Brown-York energy in simpler than Lynden-Bell-Katz's. However, this criterion is very weak, and we need an experimental difference in order to choose undoubtedly the best energy.

B. The most suitable definition for black hole thermodynamics

In Section III G, we compared the laws of mechanics for black holes generated by the two energy definitions. We can now analyze which one is the most suitable for this situation.

In the far away limit, i.e. when $r\to\infty$, we obtain the same result, given in Eqs. (42) and (46). Therefore, we have to analyze and compare the results for any radial parameter r, that are given in Eqs. (39) and (44). The difference between the results is the factor in front of the term $\frac{\kappa}{8\pi}dA_+ + (\Phi_+ - \Phi)\,dq$. In the case of the Brown-York definition, the factor is $\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}$, which is the gravitational blue-shift from infinity to the surface of radius r. It is not so surprising to get this result, and it has this natural *a posteriori* explanation. In the case of the Lynden-Bell-Katz definition, the factor α is given in Eq. (45). Unlike the Brown-York case, it does not have such an intuitive interpretation. Therefore, even though the two definitions have the same far away limit, Brown-York's is more naturally interpreted and therefore more adapted to black-hole thermodynamics than Lynden-Bell-Katz's.

C. Mercury's perihelion precession: influence of a gravitational energy density

This argument is semi-classical and is related to the physical consequences of having a positive or negative definite gravitational energy density stored in the field. Since Brown-York's energy definition is related to having a negative gravitational energy density stored in the field and Lynden-Bell-Katz's is related to a positive gravitational energy density stored in the field, if we find such an observable distinction we can choose the most appropriate definition.

We consider the influence of a positive or negative gravitational energy density in the perihelion advance of the planets, assuming that gravitational energy acts gravitationally just like any other type of energy. We can show that a negative gravitational energy density, which appears in the Brown-York interpretation, causes an advance of the perihelion equal to a twelfth of the general relativistic advance, whereas a positive gravitational energy density, which appears in the Lynden-Bell-Katz interpretation, causes a regression of the perihelion equal to a twelfth of the general relativistic advance in absolute value.

This argument would work very well if the only effect brought by general relativity were this weight of gravitational energy itself. There are in fact other effects that explain the full general relativity prediction, namely the influence of gravitational field pressure, special relativistic dynamics and gravitational time dilation. Nonetheless, it is only possible to explain the full general relativistic prediction if we consider that gravitational energy density is negative. This strongly suggests that the correct quasilocal energy definition should predict such a negative definite gravitational field energy density. Therefore, considering this argument, the most appropriate energy definition is Brown-York's.

D. A possible interpretation for the difference in the energies

We now recall the two quasilocal energy definitions in their most fundamental form, which are given in Eqs. (3) and (5). Note that

$$E_{\rm LBK} = |\xi| E_{\rm BY}\,, \qquad (48)$$

which means that $E_{\rm LBK}$ can be interpreted as a red-shifted $E_{\rm BY}$ from B to infinity.

It is interesting to note that, if we redefine the "matter" energy in special relativity by multiplying by $|\xi|^n$, the definition remains valid, since in special relativity $|\xi|=1$. Taking $n=-1$ leads us to the Brown-York energy. In a more recent article, Lynden-Bell, Katz and Bičák argue that the correct total matter energy definition should take $n=-1$, and this naturally leads to Brown-York results [11]. For example, they find that the quasilocal energy at the horizon of a Schwarzschild black hole is $2m$.

This interesting result suggests that in reality Brown-York and Lynden-Bell-Katz quasilocal energy definitions are equivalent and yield the same results, and therefore the same interpretation on the localization of the gravitational energy in the classical limit.

V. CONCLUSION

We compared the Brown-York and Lynden-Bell-Katz quasilocal energy definitions in specific spacetimes, and calculated in each case the total energy density. Analyzing the results in the Newtonian limit, we provided a different interpretation for the localization of gravitational energy for each energy definition. Namely, Brown-York's definition was consistent with having all the gravitational energy stored in the field, whereas Lynde-Bell-Katz's energy was consistent with having gravitational energy stored in the matter and in the field. Then, taking a semi-classical approach where we considered that gravitational energy should also act gravitationally as any other form of matter energy in Newton's theory, we found that Brown-York's interpretation lead to an advance of the perihelion of a twelfth of the full general relativistic advance, whereas Lynden-Bell-Katz's interpretation lead to a retrograde precession of a twelfth of the full general relativistic precession. Finally, analyzing carefully the expressions of the two energies, we gave an interpretation for the difference between them. We also suggested a way to unify the two energies, by reducing Lynden-Bell-Katz's expression to Brown-York's.

ACKNOWLEDGEMENTS

We thank Fundação para a Ciência e Tecnologia (FCT), Portugal, for financial support through Grant No. UID/FIS/00099/2013.

[1] A. Einstein, "Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), 831 (1915).
[2] R. A. Isaacson, "Gravitational Radiation in the Limit of High Frequency. I. The Linear Approximation and Geometrical Optics", Phys. Rev. **166**, 1263 (1968).
[3] R. A. Isaacson, "Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Effective Stress Tensor", Phys. Rev. **166**, 1272 (1968).
[4] R. Arnowitt, S. Deser and C. W. Misner, "Republication of: The dynamics of general relativity", Gen. Rel. Grav. **40**, 1997 (2008).
[5] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems", Proceedings of the Royal Society of London Series A **269**, 21 (1962).
[6] L. B. Szabados, "Quasi-local energy-momentum and angular momentum in general relativity", Living Rev. Relativity **12** (2009) 4.
[7] J. D. Brown and J. W. York, "Quasilocal energy and conserved charges derived from the gravitational action", Phys. Rev. D **47**, 1407 (1993).
[8] D. Lynden-Bell and J. Katz, "Gravitational field energy density for spheres and black holes", Monthly Notices of the Royal Astronomical Society **213**, 21 (1985).
[9] W. Israel, "Singular hypersurfaces and thin shells in general relativity", Il Nuovo Cimento B (1965-1970) **44**, 1 (1966).
[10] Ø. Grøn, "On Lynden-Bell and Katz's definition of gravitational field energy", Gen. Rel. Grav. **18**, 889 (1986).
[11] D. Lynden-Bell, J. Katz and J. Bičák, "Gravitational energy in stationary spacetimes", Class. Quant. Grav. **23**, 7111 (2006)