Superradiance of Bosonic Fermion Condensates

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Abstract

Superradiance is a radiation enhancement process that happens in many contexts in physics. One of its manifestations is what we call superradiant scattering. It is, generally, believed that fermionic fields cannot be superradiantly amplified, whereas bosonic fields can. Nevertheless, there are several examples in nature of fermion systems with bosonic behaviour, for instance, Cooper pairs in Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity and mesons in particle physics. This makes us ask the questions: Is it possible to have a fermion condensate which can exhibit superradiant amplification? Or, is there a non-linear interaction between fermions which enables them to exhibit superradiant amplification? The answer to these questions is of great importance to test the idea that the Penrose process is the particle analogous of superradiant scattering phenomenon. In this work, we treat the cases of scalar, Dirac and non-linear Dirac fields scattering on an electrostatic potential barrier and on a black hole background. We give an example of a theory describing a fermion (spin-1/2) condensate that admits superradiant solutions.

Keywords: superradiance, Klein paradox, Reissner-Nordström, bosonic condensate, fermions.

Superradiance is a phenomenon where radiation is enhanced by some system with the capability to dissipate energy. This phenomenon occurs in several contexts in physics, for instance, it can happen in quantum optics [1, 2], in quantum mechanics [3, 4] and in relativity [5]. An interesting manifestation of this phenomenon occurs in the scattering of fields by certain systems, where the scattered field obtains a larger energy than the one the incident field had. So, the diffuser system must have some dissipative mechanism that allows the transference of energy to the field. In this work we are always interested in this kind of superradiance.

It is a fact of nature that all the known particles fall into one of two big families: fermions (particles with half-integer spin) and bosons (particles with integer spin). Quarks and leptons are fermions, while the force carrier particles are bosons. The main difference between these two families is that fermions obey the Pauli exclusion principle which states that two identical fermions cannot be in the same state at the same point of spacetime.

It happens that the phenomenon of superradiance in the scattering of fields depends on which family the field belongs to [6, 7]. In fact, it is generally believed that the scattering of fermionic fields cannot be superradiant. However, since each specific field has its own field equations and can be scattered by different diffuser systems, one cannot give a mathematical general proof of this idea. But one can prove it for some particular cases. For instance, it is known that the scattering of Dirac spin-1/2 fields on a electrostatic potential barrier and on a Kerr-Newman (charged, rotating) black hole (BH) cannot exhibit superradiance [3, 4, 8]. Here we refer to BHs as diffuser systems, but it only makes sense to study superradiance in this context if BHs have some dissipative mechanism which allows energy transfer to the surrounding fields. It turns out that this mechanism exists.

The study of the scattering of charged fields on strong electromagnetic backgrounds is generally called Klein paradox. In 1929, using the Dirac equation, Klein showed that an electron beam propagating in a region with a sufficiently large potential barrier can be transmitted without the exponential damping expected from non-relativistic quantum mechanics [9]. This phenomenon was dubbed Klein paradox by Sauter [10] and it can be explained by the pair production at the potential barrier using quantum field theory [3, 11]. Moreover, as we show in this thesis the Klein paradox is a simple example where a field can be superradiantly amplified. In fact, using quantum field theory it is possible to understand completely the phenomenon of superradiant scattering and its connection with pair creation. It is possible to show that for sufficiently large potential barriers there is a production of scalar and Dirac pairs, explaining the existence of transmitted modes instead of the exponential damping [3, 12]. Also, it is known that superradiance occurs due to pair production and the fact that Dirac fields do not exhibit superradiant amplification relies on the Pauli exclusion principle [6, 3, 12]. The behaviour is fundamentally the same when the fields are scattering on a (possibly charged) Kerr background [13, 14].

The idea of BHs as massive objects with such a large mass that even light could not escape them was first proposed by John Michell, in 1783, and, after, by Pierre-Simon Laplace, in 1796 [15, 16]. However, only in 1915, with the theory of general relativity of Albert Einstein, it was possible to understand how (massless) light interacts with gravity. In fact, it turns out that BHs arise in a very natural way in general relativity. There are solutions of the Einstein equations that contain closed regions with their interior causally disconnected from their exterior, in the sense that what happens in their interior cannot influence their exterior and, so, everything that enters these regions cannot escape them (even light) [17]. One hundred years ago, in 1916, Karl Schwarzschild discovered the first solution of the Einstein field equations with a region of this kind, this is called the Schwarzschild solution [18]. By definition, these regions are BHs and their boundary is the so-called event horizon. Furthermore, it is exactly the event horizon which provides a dissipative mechanism to the BH [6, 19]. This is something very peculiar, because the necessary dissipation for the existence of superradiant amplification is often provided by some kind of friction or viscosity, which always involve some matter or radiation fields. Since BHs appear in several vacuum solutions of the Einstein field equations, the event horizon provides the vacuum with a dissipative mechanism. This is very interesting, because the fields are allowed to extract energy from the vacuum itself by superradiant scattering. Moreover, in principle, this is a real phenomenon since we have strong observational indications that BHs exist. In fact, the first direct confirmation of their existence was the last year's detection, by LIGO, of the gravitational wave signal *GW150914*, originated by the collision and merger of a pair of BHs [20].

The study of BH superradiance started in 1971 with the independent predictions of Zel'dovich and Misner that some waves could be amplified by rotating (Kerr) BHs [5, 21]. Moreover, the work done by Teukolsky was crucial to the study of scattering of fields on Kerr background. Teukolsky showed that linearised perturbations of the Kerr geometry can be compactified in one single separable master equation, which contain the cases of scalar, electromagnetic and gravitational perturbations [22]. Using this master equation, Teukolsky and Press proved that scalar, electromagnetic and gravitational waves scattering on a Kerr BH have superradiant modes [23]. In 1973, Unruh separated the massless Dirac equation on Kerr background and showed that these spin-1/2 (neutrino) fields do not have superradiant modes [24]. This result was extended for massive spin-1/2 (Dirac) fields by Chandrasekhar [25]. In 1976, Page separated the Dirac equation on the more general Kerr-Newman background and, one year later, Lee used his result to show that Dirac fields have no superradiant modes on this background [8, 26]. Another interesting approach to the study of BH superradiance was that of Bekenstein, who saw the connection between this phenomenon and Hawking's area theorem [27]. This argument is so simple and beautiful that we shall outline it here. If the energy-momentum tensor of a (possibly charged) test field propagating on a Kerr-Newman background satisfies the null energy condition [28] at the event horizon, then the energy ΔM, angular momentum ΔJ and electric charge ΔQ absorbed by the BH satisfy [29]:

$$\Delta M \geq \Omega \Delta J + V \Delta Q \quad , \tag{1}$$

where Ω is the angular velocity of the BH horizon and V is the electric potential at the horizon. It is easy to see that the ratios of the angular momentum over the energy and of the electric charge over the energy of a wave with frequency ω, azimuthal number m and electric charge e are, respectively, m/ω and e/ω [27]. Then, the inequality (1) reads

$$\frac{\Delta M}{\omega} (\omega - m\Omega - eV) \geq 0 \quad . \tag{2}$$

Superradiant modes must extract energy from the BH and, so, $\Delta M < 0$, which implies that ω must satisfy

$$0 < \omega < m\Omega + eV \quad . \tag{3}$$

These are, precisely, the modes which extract energy from the BH. Since the energy-momentum tensor of the Dirac field does not satisfy the null energy condition at the event horizon [30], we see that these fields are not contemplated by this proof and, in fact, as we said above, they do not exhibit superradiant amplification when scattering on this background.

In 1971, Roger Penrose theorized a phenomenon called the Penrose process [31]. This is a phenomenon where rotational energy can be extracted from Kerr (rotating) BHs and it is generally believed to be the particle analogue of superradiant scattering. The Penrose process makes use of the fact that Kerr BHs have a region called ergoregion, where a particle can have negative energy with respect to an observer at infinity [6, 17]. The idea of Penrose consists in considering a particle falling into the ergoregion and decaying there in two another particles. Obeying the energy-momentum conservation law, it is possible that one of the particles falls into the BH with negative energy (with respect to an observer at infinity) and the other escapes to infinity with a larger energy than that of the original particle. In fact, it can be shown that, for Reissner-Nordström (charged static) BHs, there exists a generalized ergoregion and a similar energy extraction process is possible [32, 33].

As we said, the Penrose process is generally believed to be the particle analogue of superradiant scattering phenomenon. However, while the two processes are classical, superradiant amplification seems to carry some quantum features of the field being scattered. In particular, even though the fields are not quantised, superradiant scattering already seems to predict pair production. Now, if we believe that the Penrose process is a real phenomenon, which happens for ordinary matter in nature and, at the same time, we believe it to be the particle analogue of superradiance, we have something to explain. Because we know that all the ordinary (baryonic) matter is made of fermions at the very fundamental level and it is believed that fermions do not exhibit superradiant amplification. This raises the expectation that it may exist some non-linear interaction between the fermions, which restores their capability to exhibit superradiance. In other words, we expect the existence of bosonic fermion condensates with the capability to exhibit superradiant amplification. In fact, the existence of fermion systems with a bosonic behaviour is not very strange and happens in nature. For instance, Cooper pairs in BCS theory of superconductivity and mesons in particle physics are examples of these bosonic fermion condensates.

The existence of these kind of condensates can have interesting applications in astrophysics. In fact, it is known that we can confine superradiantly amplified fields through various mechanisms, like massive fields and anti-de Sitter boundaries [6, 34, 35]. This confinement can originate strong instabilities called BH bombs [36], which have applications in searches for dark matter and physics beyond the Standard Model [6, 37, 38].

In this thesis, we review the scattering of scalar and Dirac fields on an electrostatic potential barrier (Klein paradox) and on a Reissner-Nordström (charged static) BH. The results obtained for these cases are well known [4, 3, 6, 8]. Nevertheless, as far as we know, the only proof of the absence of superradiance for Dirac fields on RN background is obtained as a limit of the more general Kerr-Newman background [8], which uses the formalism of Newman-Penrose [39] to separate the Dirac equation. Here, we use the spherical symmetry of RN geometry to separate the Dirac equation in an easier way and we proceed to prove the absence of superradiance for Dirac fields on a RN background in a new way. Finally, we provide a non-linear Dirac field theory, which is inspired by the Nambu-Jona-Lasinio model [40] and can exhibit superradiant amplification both on the Klein paradox and RN background. Here, we are not concerned about the generality or validity of this theory. Instead, we want to provide a simple theory, which we believe to describe a fermion condensate and, at the same time, allows superradiant scattering solutions. So, we give a proof of concept that it is possible to construct this kind of condensates. In principle, there are other theories more realistic than this one, which allow solutions with the same kind of behaviour.

In this work, we always consider charged test fields, neglecting the electromagnetic field produced by them and their back-reaction on the spacetime geometry. So, we consider that these fields always propagate in a fixed background geometry. This test field approximation is correct at first order in the fields, because their effect on the geometry and on the electromagnetic field is only of second order [6]. Also, we do not expect any qualitative change of our conclusions from using this approximation. In fact, in any viable astrophysical scenario, the first order on the amplitude of the fields is enough.

The field theories that we use throughout this work are always described by an action of the form:

$$S = S_G + S_{EM} + S_M \quad , \tag{4}$$

with

$$S_G = \int d^4x\sqrt{-g}\ \frac{R}{16\pi} \quad , \tag{5}$$

$$S_{EM} = -\int d^4x\sqrt{-g}\ \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \tag{6}$$

where g is the determinant of the metric $g_{\mu\nu}$ of spacetime, R is its scalar curvature and $F_{\mu\nu}$ is the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{7}$$

The action S_M:

$$S_M = \int d^4x\sqrt{-g}\,\mathcal{L}_M \quad , \tag{8}$$

where $\mathcal{L}_M$ is the lagrangian density of some matter field. This action describes the matter field under analysis. In this work, we will consider three kind of matter fields: scalar fields, Dirac fields and non-linear Dirac fields.

In this thesis, we always use theories S_M which are U(1) invariant. A theory of this kind is such that if it describes the field ξ, then its equations are invariant under the transformation

$$\xi \longrightarrow e^{i\alpha}\xi \quad , \tag{9}$$

with α a real constant. So, by Noether's theorem there is a conserved current associated with this symmetry [41]. We call this current the particle-number current and we use its flux to study the phenomenon of superradiant scattering [7]. We say that there is superradiant amplification if the absolute value of the flux of the reflected particle-number current is larger than that of the incident one.

1. Klein paradox

In this chapter, we study the phenomenon of superradiance when scattering scalar or Dirac charged fields in an electrostatic potential barrier. In particular, we show the well known fact that Dirac fields cannot exhibit superradiance. On the other hand, we prove that charged scalar fields can exhibit superradiance for modes with frequency ω obeying the relation $m < \omega < qV - m$ with V the electric potential, q the charge and m the mass of the field. We also analyse the case of non-linear Dirac fields and prove that there is a superradiant regime. Thus, although linear Dirac fields cannot exhibit superradiance, non-linear Dirac fields can. We interpret these non-linear Dirac fields as describing roughly condensates of interacting Dirac particles. As a motivation for this interpretation, we have the Nambu-Jona-Lasinio model [40], which is based on an action quite similar to ours. This model is used as an effective theory to describe mesons, which are composed by pairs of interacting quarks and anti-quarks, both spin-1/2 fermionic particles, and have a bosonic behaviour as a whole. Moreover, this model is motivated by BCS theory of superconductivity and, in particular, by the concept of Cooper pairs, which are pairs of interacting electrons that have also a bosonic behaviour as a whole.

We consider a two dimensional problem on a four dimensional flat spacetime. So, we have the fields propagating along t and z, with the metric of the Minkowski background [42]

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad , \tag{10}$$

and the arbitrary electromagnetic potential

$$A_\mu = (V(z),\, 0,\, 0,\, 0) \quad , \tag{11}$$

with the asymptotic behaviour

$$V(z) = \begin{cases} 0, & \text{for } z \to -\infty \\ \widetilde{V} > 0, & \text{for } z \to +\infty \end{cases} \quad . \tag{12}$$

Then, we are interested in what happens to incident quasi-monochromatic charged waves (of a specific field) coming from $z = -\infty$ and scattering in this potential barrier. To interpret the waves as incident, reflected and transmitted, we use their group velocity. As is known in wave theory, a wave-packet

$$\varphi(t,z) = \int d\omega A(\omega) e^{-i[\omega t - k(\omega) z]} \quad , \tag{13}$$

has group velocity

$$v_g = \left(\frac{\partial k(\omega)}{\partial \omega}\right)^{-1} \quad , \tag{14}$$

with the frequency ω real, $A(\omega)$ a complex-valued function and $k(\omega)$ the dispersion relation of the specific field.

Let us define here what we mean by quasi-monochromatic waves of frequency $\widetilde{\omega}$. These are wave-packets with frequencies in an infinitesimal interval around some frequency $\widetilde{\omega}$. These waves are approximately monochromatic

$$\varphi(t,z) \approx A\, e^{-i[\widetilde{\omega} t - k(\widetilde{\omega}) z]} \quad , \tag{15}$$

and they have group velocity

$$v_g = \left(\frac{\partial k(\omega)}{\partial \omega}\right)^{-1}_{\omega = \widetilde{\omega}} \quad . \tag{16}$$

This is, of course, an approximation, because we cannot define the concept of group velocity for monochromatic waves. But we can make the approximation as good as we want by considering the frequency interval around $\widetilde{\omega}$ as small as needed.

1.1. Scattering of scalar fields

Let us start by considering the scalar theory

$$S_{scalar} = \int dx^4 \left[D_\mu \phi (D^\mu \phi)^* - m^2 |\phi|^2 \right] \quad , \tag{17}$$

with m the mass of the scalar field and with:

$$D_\mu = \partial_\mu + iqA_\mu \quad ,$$

where $q > 0$ is the electric charge of the field. In this theory, we are imposing the minimal coupling between the charged field and the electromagnetic field.

This theory is sometimes called Klein-Gordon theory. It was proposed, in 1926, by Oskar Klein and Walter Gordon to describe relativistic electrons. In fact, now, we know that this theory describes scalar (spin-0) fields, which, in the framework of quantum field theory, are associated with the Higgs boson particle. Furthermore, this theory describes also spinless relativistic composite particles, as the pion.

It is possible to show that we can write the incident wave solution as

$$\phi_I^i = I\, e^{-i[\omega t - k(\omega) z]} \quad , \tag{18}$$

and the reflected one as

$$\phi_I^r = R\, e^{-i[\omega t + k(\omega) z]} \quad , \tag{19}$$

where

$$k(\omega) = \epsilon\sqrt{\omega^2 - m^2} \quad , \tag{20}$$

with $\epsilon = \mathrm{sign}(\omega + m)$. The transmitted wave can be written as

$$\phi_{II}^t = T\, e^{-i[\omega t - s(\omega) z]} \quad , \tag{21}$$

with $\omega \in X$ and

$$s(\omega) = \widetilde{\epsilon}\sqrt{(\omega - q\widetilde{V})^2 - m^2} \quad , \tag{22}$$

with $\widetilde{\epsilon} = \mathrm{sign}(\omega - q\widetilde{V} + m)$.

Using the z-component of the particle-number current associated with the scalar field ϕ:

$$j_z = -\frac{i}{2}(\phi^* \partial_z \phi - \phi\, \partial_z \phi^*) \quad , \tag{23}$$

we can show that

$$\left|\frac{j_z^r}{j_z^i}\right| = 1 - \frac{\mathrm{Re}(s)}{k}\left|\frac{T}{I}\right|^2 \quad , \tag{24}$$

where j_z^i and j_z^r are the incident and reflected z-currents, respectively.

By definition, we have superradiant scattering when $|j_z^r| > |j_z^i|$. So, superradiance is possibly only if $q\widetilde{V} > 2m$ and the superradiant modes are:

$$m < \omega < q\widetilde{V} - m \quad . \tag{25}$$

These superradiant modes are in agreement with the ones obtained in Ref. [3].

1.2. Scattering of Dirac fields

In this case, we are interested in the theory

$$S_{Dirac} = \int dx^4 (i\bar{\Psi}\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi) \quad , \tag{26}$$

with m the mass of the Dirac field, $\bar{\Psi}$ is the Dirac conjugate of the spinor Ψ and, as in the scalar field case,

$$D_\mu = \partial_\mu + ieA_\mu \quad ,$$

where $e > 0$ is the electric charge of the field.

This theory was proposed, in 1928, by Paul Dirac. It describes spin-1/2 particles, like electrons and quarks. In fact, this theory was a great success, since it predicted the value of the gyromagnetic ratio of the electron in a completely rigorous way and also the existence of a new kind of matter, the so-called antimatter. This antimatter turned out to be observed experimentally, in 1932, by Carl Anderson.

We can show that the incident wave solution is

$$\Psi_I^i = \left[I_+ \begin{pmatrix} u_+ \\ \frac{k}{\omega+m} u_+ \end{pmatrix} + \right. \\ \left. + I_- \begin{pmatrix} u_- \\ -\frac{k}{\omega+m} u_- \end{pmatrix} \right] e^{-i[\omega t - k(\omega) z]} \quad , \tag{27}$$

and the reflected one is

$$\Psi_I^r = \left[R_+ \begin{pmatrix} u_+ \\ -\frac{k}{\omega+m} u_+ \end{pmatrix} + \right. \\ \left. + R_- \begin{pmatrix} u_- \\ \frac{k}{\omega+m} u_- \end{pmatrix} \right] e^{-i[\omega t + k(\omega) z]} \quad , \tag{28}$$

where

$$k(\omega) = \epsilon\sqrt{\omega^2 - m^2} \quad , \tag{29}$$

with $\epsilon = \mathrm{sign}(\omega + m)$. The transmitted wave is

$$\Psi_{II}^t = \left[T_+ \begin{pmatrix} u_+ \\ \frac{s}{\omega - e\widetilde{V} + m} u_+ \end{pmatrix} + \right. \\ \left. + T_- \begin{pmatrix} u_- \\ -\frac{s}{\omega - e\widetilde{V} + m} u_- \end{pmatrix} \right] e^{-i[\omega t + s(\omega) z]} \quad , \tag{30}$$

with

$$s(\omega) = \widetilde{\epsilon}\sqrt{(\omega - e\widetilde{V})^2 - m^2} \quad , \tag{31}$$

where $\widetilde{\epsilon} = \mathrm{sign}(\omega - e\widetilde{V} - m)$.

Using the z-component of the particle-number current associated with the Dirac field Ψ:

$$j^z = \frac{1}{2}\bar{\Psi}\gamma^3\Psi \quad , \tag{32}$$

we can show that

$$\left|\frac{(j^r)^z}{(j^i)^z}\right| = \\ = 1 - \frac{\omega + m}{k}\frac{\mathrm{Re}(s)}{\omega - e\widetilde{V} + m}\frac{|T_+|^2 + |T_-|^2}{|I_+|^2 + |I_-|^2} \quad , \tag{33}$$

where $(j^i)^z$ and $(j^r)^z$ are the incident and reflected z-currents, respectively.

Thus, we see that $(j^r)^z \leq (j^i)^z$ and, so, there are no superradiant modes of the Dirac field. This is in agreement with the results of Refs. [3, 4].

1.3. Scattering of non-linear Dirac fields

In this section, we want to consider the scattering of a fermion condensate. We use the usual Dirac free field action with an additional interaction term proportional to $(\bar{\Psi}\Psi)^2$. This term is such that the U(1) symmetry of Ψ is preserved. Then, we have a Noether's conserved current associated with this symmetry. The z-component of this current can be shown to be equal to the one of the last section. So, let us consider the non-linear Dirac field theory

$$S=\int dx^4[i\,\bar{\Psi}\gamma^\mu D_\mu\Psi-\frac{\lambda}{2}(\bar{\Psi}\Psi)^2] \quad , \tag{34}$$

with all the quantities defined as in the last section and with the coupling

$$\lambda(z)=\widetilde{\lambda}\,e^2A_\mu A^\mu=\begin{cases}0, & \text{for } z\to-\infty\\ \widetilde{\lambda}\,e^2\widetilde{V}^2>0, & \text{for } z\to+\infty\end{cases} \quad , \tag{35}$$

where $\widetilde{\lambda}>0$ is a real constant. The reason to consider this kind of coupling is that we want the field equation to be linear at $z\to-\infty$, in a way that makes it possible to write the solution as the sum of incident and reflected waves.

Since the non-linear coupling vanishes at $z\to-\infty$, the incident solution is

$$\Psi^i_I=\left[I_+\begin{pmatrix}u_+\\ \frac{k}{\omega+m}u_+\end{pmatrix}+ +I_-\begin{pmatrix}u_-\\ -\frac{k}{\omega+m}u_-\end{pmatrix}\right]e^{-i[\omega t-k(\omega)z]} \quad , \tag{36}$$

and the reflected one is

$$\Psi^r_I=\left[R_+\begin{pmatrix}u_+\\ -\frac{k}{\omega+m}u_+\end{pmatrix}+ +R_-\begin{pmatrix}u_-\\ \frac{k}{\omega+m}u_-\end{pmatrix}\right]e^{-i[\omega t+k(\omega)z]} \quad , \tag{37}$$

where

$$k(\omega)=\omega \quad . \tag{38}$$

Moreover, we are considering the transmitted solution

$$\Psi^t_{II}=T\begin{pmatrix}u_+\\ \eta\,u_+\end{pmatrix}e^{-i\omega t} \quad , \tag{39}$$

where

$$\eta=-\sqrt{1-\frac{\omega-e\,\widetilde{V}}{\lambda|T|^2}}<0 \quad , \tag{40}$$

with ω satisfying

$$\omega<e\,\widetilde{V}+\lambda|T|^2 \quad . \tag{41}$$

Using the z-component of the particle-number current associated with this non-linear Dirac field Ψ:

$$j^z=\frac{1}{2}\bar{\Psi}\gamma^3\Psi \quad , \tag{42}$$

we can show that

$$\left|\frac{(j^r)^z}{(j^i)^z}\right|=1-\eta\left(\frac{|T|^2}{|I_+|^2+|I_-|^2}\right) \quad . \tag{43}$$

But, since $\eta<0$, we have $|(j^r)^z|>|(j^i)^z|$. Thus, for modes

$$\omega<e\,\widetilde{V}+\lambda|T|^2 \quad , \tag{44}$$

we have superradiant solutions.

2. Superradiance on black hole backgrounds

It is well known that static, charged BHs are described by the so-called Reissner-Nordström (RN) geometry. In spherical coordinates, for $r>r_+$, the RN geometry is represented by the squared line element

$$ds^2=f\,dt^2-f^{-1}\,dr^2-r^2(\,d\theta^2+\sin^2\theta\,d\varphi^2) \quad . \tag{45}$$

Here,

$$f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2} \quad , \tag{46}$$

where M and Q are the mass and electric charge of the BH, respectively. In these coordinates, there is an event horizon radius at

$$r=r_+=M+\sqrt{M^2-Q^2} \quad . \tag{47}$$

Furthermore, the charge Q sources a spherically symmetric electromagnetic field

$$A_\mu=(\,V(r),\vec{0}\,) \qquad \text{with} \qquad V(r)=\frac{Q}{r} \quad . \tag{48}$$

As in the case of the Klein paradox, we use the test field approximation. Then, we ignore the back-reaction of the fields on the geometry of the spacetime. Although we use charged fields, we also ignore the electromagnetic field produced by them. These approximations are justified by the fact that these effects are of second order on the charged fields and, so, for sufficiently weak (small amplitude) fields, these effects can be neglected. Moreover, in astrophysical relevant setups, the electromagnetic field produced by this kind of charged fields have negligible effect on the geometry [43].

2.1. Scattering of scalar fields

Let us consider the scalar field theory

$$S_{scalar}=\int dx^4\sqrt{-g}\,\left[\,g_{\mu\nu}D^\nu\phi(D^\mu\phi)^*-m^2|\phi|^2\right] \quad , \tag{49}$$

with

$$D_\mu=\nabla_\mu+iqA_\mu \quad ,$$

and all the other quantities defined as in the Klein paradox case.

It is possible to show that the incident wave solution is

$$\phi^i=\sum_{l,m_l}\frac{I}{r}\,Y_l^{m_l}\,e^{-i[\omega t+k(\omega)r]} \quad , \tag{50}$$

and the reflected one is

$$\phi^r=\sum_{l,m_l}\frac{R}{r}\,Y_l^{m_l}\,e^{-i[\omega t-k(\omega)r]} \quad , \tag{51}$$

where

$$k(\omega)=\epsilon\sqrt{\omega^2-m^2} \quad , \tag{52}$$

with $\epsilon = \text{sign}(\omega + m)$. The transmitted wave can be written as

$$\phi^t_{II} = \sum_{l,m_l} \frac{T}{r_+} Y_l^{m_l} \exp\Big[- i\omega t - - is(\omega)\frac{r_+^2}{r_+ - r_-}\log(r - r_+)\Big] \quad , \tag{53}$$

where

$$s(\omega) = \widetilde{\epsilon}\, |\omega - qV_+| = \omega - qV_+ \quad , \tag{54}$$

with $\widetilde{\epsilon} = \text{sign}(\omega - qV_+)$ and $V_+ = V(r_+)$. The I, R and T appearing in the above solutions are complex functions of l and m_l, but, for the sake of simplicity, we do not represent this, explicitly, in our notation.

Using the particle-number current:

$$j^\mu = -\frac{i}{2}\left[\phi^* D^\mu \phi - \phi\,(D^\mu\phi)^*\right] \quad , \tag{55}$$

and its flux $\mathcal{F}$ over a spherical surface of radius r (denoted by S_r), with $r \to +\infty$:

$$\mathcal{F} = \lim_{r\to+\infty} \int_{S_r} d\Omega\, r^2 j^r \quad , \tag{56}$$

one can show that

$$\left|\frac{\mathcal{F}^r}{\mathcal{F}^i}\right| = 1 - \frac{s}{k}\frac{\sum_{l,m_l}|T|^2}{\sum_{l,m_l}|I|^2} \quad , \tag{57}$$

where $\mathcal{F}^i$ and $\mathcal{F}^r$ are the fluxes of the incident and reflected currents, respectively.

By definition, there is superradiant scattering when $|\mathcal{F}^r| > |\mathcal{F}^i|$. Thus, the scalar field have the superradiant modes

$$m < \omega < qV_+ \quad . \tag{58}$$

These superradiant modes are equal to the ones obtained in Refs. [6, 27].

2.2. Scattering of Dirac fields

Here, we consider the curved spacetime Dirac theory:

$$S_{\text{Dirac}} = \int dx^4 \sqrt{-g}\left[\, i\,\bar\Psi G^\mu \widetilde{\mathcal{D}}_\mu \Psi - m\bar\Psi\Psi\,\right] \quad , \tag{59}$$

with m the mass of the Dirac field, G^μ the Dirac matrices in curved spacetime and $\widetilde{\mathcal{D}}_\mu = \partial_\mu + ieA_\mu - \Gamma_\mu$, where $e > 0$ is the electric charge of the field and Γ_μ is the so-called spin connection.

The incident wave solution is given by

$$(\Psi)^i_I = \sum_{j,\,k}\left[I^+\,(\Psi^+_{j\,k})^i + I^-\,(\Psi^-_{j\,k})^i\right] \quad , \tag{60}$$

with

$$(\Psi^+_{j\,k})^i = \frac{1}{r}\exp\left[-i\left(\omega\, t + \epsilon\sqrt{\omega^2 - m^2}\, r\right)\right] \times \times \begin{pmatrix} \chi^k_{j-1/2} \\ -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2}\end{pmatrix} \quad , \tag{61}$$

$$(\Psi^-_{j\,k})^i = \frac{1}{r}\exp\left[-i\left(\omega\, t + \epsilon\sqrt{\omega^2 - m^2}\, r\right)\right] \times \times \begin{pmatrix} \chi^k_{j+1/2} \\ -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j-1/2}\end{pmatrix} \quad , \tag{62}$$

where $\epsilon = \text{sign}(\omega + m)$. The $\chi^k_{j-1/2}$ and $\chi^k_{j+1/2}$ are the so-called spinor spherical harmonics. The reflected wave solution is given by

$$(\Psi)^r_I = \sum_{j,\,k}\left[R^+\,(\Psi^+_{j\,k})^r + R^-\,(\Psi^-_{j\,k})^r\right] \quad , \tag{63}$$

with

$$(\Psi^+_{j\,k})^r = \frac{1}{r}\exp\left[-i\left(\omega\, t - \epsilon\sqrt{\omega^2 - m^2}\, r\right)\right] \times \times \begin{pmatrix} \chi^k_{j-1/2} \\ \sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2}\end{pmatrix} \quad , \tag{64}$$

$$(\Psi^-_{j\,k})^r = \frac{1}{r}\exp\left[-i\left(\omega\, t - \epsilon\sqrt{\omega^2 - m^2}\, r\right)\right] \times \times \begin{pmatrix} \chi^k_{j+1/2} \\ \sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j-1/2}\end{pmatrix} \quad . \tag{65}$$

The transmitted wave solution is

$$(\Psi)^t_{II} = \sum_{j,\,k}\left[T^+\,(\Psi^+_{j\,k})^t + T^-\,(\Psi^-_{j\,k})^t\right] \quad , \tag{66}$$

with

$$(\Psi^+_{j\,k})^t = \frac{1}{\sqrt{r_+}\,(r_+ - r_-)^{\frac14}\,(r - r_+)^{\frac14}} \times \times \exp\left[-i\Big(\omega\, t + (\omega - eV_+) r_*\Big)\right] \begin{pmatrix} \chi^k_{j-1/2} \\ -\chi^k_{j+1/2}\end{pmatrix} \quad , \tag{67}$$

$$(\Psi^-_{j\,k})^t = \frac{1}{\sqrt{r_+}\,(r_+ - r_-)^{\frac14}\,(r - r_+)^{\frac14}} \times \times \exp\left[-i\Big(\omega\, t + (\omega - eV_+) r_*\Big)\right] \begin{pmatrix} \chi^k_{j+1/2} \\ -\chi^k_{j-1/2}\end{pmatrix} \quad , \tag{68}$$

where

$$r_* = \frac{r_+^2}{r_+ - r_-}\log(r - r_+) \quad . \tag{69}$$

The I^+, I^-, R^+, R^-, T^+ and T^- in the above solutions are complex-valued function of j and k, but to simplify the notation we do not represent this dependence explicitly.

Using the particle-number current:

$$J^{\mu} = \frac{1}{2}\bar{\Psi}G^{\mu}\Psi \quad , \tag{70}$$

we obtain that

$$\left|\frac{\mathcal{F}^r}{\mathcal{F}^i}\right| = 1 - \sqrt{\frac{\omega+m}{\omega-m}}\,\frac{|T^+|^2+|T^-|^2}{|I^+|^2+|I^-|^2} \quad . \tag{71}$$

Since $|\mathcal{F}^r| \leq |\mathcal{F}^i|$, we see that Dirac fields do not have superradiant modes in this background. This result was know as a limit of the more general Kerr-Newman background [8]. Nevertheless, in this thesis, we show it in a simpler way, making use of the spherically symmetry of RN to separate the Dirac equation and decoupling the field equations with an appropriate change of variables.

2.3. Scattering of non-linear Dirac fields

Let us consider the non-linear Dirac theory:

$$\begin{aligned} S = \int dx^4\sqrt{-g}\,[\,i\,\bar{\Psi}G^{\mu}\widetilde{\mathcal{D}}_{\mu}\Psi - \\ - m\bar{\Psi}\Psi - \frac{\lambda}{2}(\bar{\Psi}\Psi)^2] \quad , \end{aligned} \tag{72}$$

with all the quantities defined in the same way as in the last section and with the coupling

$$\lambda(r) = \widetilde{\lambda}\,e^2 A_{\mu}A^{\mu} = \widetilde{\lambda}\,e^2\,\frac{Q^2}{r^2} > 0 \quad , \tag{73}$$

where $\widetilde{\lambda} > 0$ is a real constant.

Since the non-linear coupling vanishes at $r \to \infty$, we have the incident wave solution

$$\begin{aligned} \Psi^i = \frac{I}{r}\exp\left[-i\left(\omega\,t + \epsilon\sqrt{\omega^2-m^2}\,r\right)\right] \times \\ \times \begin{pmatrix} \chi^k_{j-1/2} \\ -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2} \end{pmatrix} \quad , \end{aligned} \tag{74}$$

and the reflected solution given by

$$\begin{aligned} \Psi^r = \frac{R}{r}\exp\left[-i\left(\omega\,t - \epsilon\sqrt{\omega^2-m^2}\,r\right)\right] \times \\ \times \begin{pmatrix} \chi^k_{j-1/2} \\ \sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2} \end{pmatrix} \quad , \end{aligned} \tag{75}$$

with $\epsilon = \mathrm{sign}(\omega+m)$. Furthermore, we are considering the transmitted solution

$$\begin{aligned} \Psi^t = Te^{-i\omega\,t}\frac{1}{\sqrt{r_+}\,(r-r_+)^{\frac{1}{4}}\,(r_+-r_-)^{\frac{1}{4}}} \times \\ \times \begin{pmatrix} \chi^k_{j-1/2}(\theta,\varphi) \\ -\eta\,\chi^k_{j+1/2}(\theta,\varphi) \end{pmatrix} \quad , \end{aligned} \tag{76}$$

where

$$\eta = -\sqrt{1 - \frac{\omega - e\,V_+}{\bar{\lambda}|T|^2}} < 0 \quad , \tag{77}$$

with ω satisfying

$$\omega < e\,V_+ + \bar{\lambda}|T|^2 \quad . \tag{78}$$

Above we are considering the particular solutions with $j = k = +\frac{1}{2}$. Here, the I, R and T are complex constants.

Using the particle-number current associated with this non-linear Dirac field:

$$J^{\mu} = \frac{1}{2}\bar{\Psi}G^{\mu}\Psi \quad , \tag{79}$$

we can show that

$$\left|\frac{\mathcal{F}^r}{\mathcal{F}^i}\right| = 1 - \eta\sqrt{\frac{\omega+m}{\omega-m}}\,\frac{|T|^2}{|I^+|^2+|I^-|^2} \quad . \tag{80}$$

Thus, we see that, in this non-linear case, there exists a solution with $|\mathcal{F}^r| > |\mathcal{F}^i|$ for

$$\omega < e\,V_+ + \bar{\lambda}|T|^2 \quad . \tag{81}$$

So, we showed, now on a RN background, that there exist fermion condensates which can exhibit superradiant amplification.

3. Conclusions

In this thesis, we showed that a charged scalar field has superradiant modes both when scattering on a strong electrostatic potential barrier (Klein paradox) and on a RN background. This conclusion was already known and the superradiant modes that we obtained are in agreement with the ones of Refs. [3, 6]. Furthermore, we also proved that Dirac fields do not exhibit superradiant amplification both in the case of Klein paradox as well as in RN background. This fact is well known, but, from what we know, there was no direct proof of the absence of superradiant amplification for Dirac fields scattering on RN geometry [3, 4]. In fact, Lee proved the absence of superradiance for Dirac fields scattering on the more general Kerr-Newman BH [8]. However, since RN is spherically symmetric, we can prove this absence directly in a easier way. In particular, we do not need to use the Newman-Penrose formalism to separate the Dirac equation. Instead, we follow a procedure that explores the spherically symmetry of the problem as in Ref. [44].

We accomplished the main objective of this work, which was to answer the questions: Is it possible to have a fermion condensate which can exhibit superradiant amplification? Or, is there a non-linear interaction between fermions which enables them to exhibit superradiant amplification? It turns out that the answer to these two questions is yes. In fact, in this thesis, we provided a non-linear Dirac field theory which we believe to describe a fermion condensate and has solutions with superradiant amplification both in the Klein paradox and in the RN background. This serves only as a proof of concept that such a fermion condensate can exist, but this conclusion is very important. Because it gives consistency to the usual interpretation of the Penrose process as the particle analogue of superradiant amplification phenomenon. In fact, since the Penrose process is a classical phenomenon, we expect it to happens with ordinary (classical) matter. But, if the Penrose process is the particle analogous of superradiant amplification, this ordinary matter must exhibit also superradiance. Now, since ordinary (baryonic) matter is made of fermions at the fundamental level, we expect superradiance to be restored by some kind of non-linear interaction between the fermions. In this work, we saw that, in fact, this can happen.

References

[1] R. H. Dicke. Coherence in Spontaneous Radiation Processes. *Phys. Rev.*, 93:99–110, Jan 1954.

[2] M. Gross and S. Haroche. Superradiance: An essay on the theory of collective spontaneous emission. *Physics Reports*, 93(5):301 – 396, 1982.

[3] Corinne A Manogue. The Klein paradox and superradiance. *Annals of Physics*, 181(2):261 – 283, 1988.

[4] W. Greiner. *Quantum electrodynamics of strong fields.* Springer Berlin Heidelberg, Berlin, Heidelberg, 1985.

[5] Y. B. Zeldovich. Generation of Waves by a Rotational Body. *Prisma Zh. Eksp. Teor. Fiz*, **14**:270, 1971.

[6] Richard Brito, Vitor Cardoso, and Paolo Pani. Superradiance. *Lect. Notes Phys.*, 906:pp.1–237, 2015.

[7] Hongsu Kim. The absence of fermionic superradiance (a simple demonstration). *Journal of Cosmology and Astroparticle Physics*, 2008(11):007, 2008.

[8] C.H. Lee. Massive spin-12 wave around a Kerr-Newman black hole. *Physics Letters B*, 68(2):152 – 156, 1977.

[9] O. Klein. Die reflexion von elektronen an einem potentialsprung nach der relativistischen dynamik von dirac. *Zeitschrift fr Physik*, **53**(3-4):157–165, 1929.

[10] Fritz Sauter. Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs. *Zeitschrift für Physik*, 69(11):742–764, 1931.

[11] A. Calogeracos and N. Dombey. History and physics of the Klein paradox. *Contemporary Physics*, 40(5):313–321, 1999.

[12] Alex Hansen and Finn Ravndal. Klein's Paradox and Its Resolution. *Physica Scripta*, 23(6):1036, 1981.

[13] T. Damour. Klein Paradox and Vacuum Polarization. In *Marcel Grossmann Meeting on the Recent Progress of the Fundamentals of General Relativity Trieste, Italy, July 7-12, 1975*, pages 459–482, 1975.

[14] Bruno Arderucio. Superradiance: Classical, Relativistic and Quantum Aspects. Master's thesis, University of São Paulo, 2014.

[15] John Michell. *Philosophical Transactions of the Royal Society of London*, 74:35–57, 1784.

[16] W. Israel. *Dark stars: the evolution of an idea.* Cambridge University Press, 1987.

[17] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation.* W. H. Freeman San Francisco, 1973.

[18] Karl Schwarzschild. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 7:189–196, 1916.

[19] Mauricio Richartz, Silke Weinfurtner, A. J. Penner, and W. G. Unruh. Generalized superradiant scattering. *Phys. Rev. D*, 80:124016, Dec 2009.

[20] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.*, 116:061102, Feb 2016.

[21] C. W. Misner. Interpretation of Gravitational-Wave Observations. *Phys. Rev. Lett.*, 28:994–997, Apr 1972.

[22] Saul A. Teukolsky. Rotating Black Holes: Separable Wave Equations for Gravitational and Electromagnetic Perturbations. *Phys. Rev. Lett.*, 29:1114–1118, October 1972.

[23] S. A. Teukolsky and W. H. Press. Perturbations of a rotating black hole. III - Interaction of the hole with gravitational and electromagnetic radiation. *APJ*, 193:443–461, October 1974.

[24] W. Unruh. Separability of the Neutrino Equations in a Kerr Background. *Phys. Rev. Lett.*, 31:1265–1267, Nov 1973.

[25] S. Chandrasekhar. The Solution of Dirac's Equation in Kerr Geometry. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 349(1659):571–575, 1976.

[26] Don N. Page. Dirac equation around a charged, rotating black hole. *Phys. Rev. D*, 14:1509–1510, Sep 1976.

[27] Jacob D. Bekenstein. Extraction of Energy and Charge from a Black Hole. *Phys. Rev. D*, 7:949–953, Feb 1973.

[28] Stephen Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, 1973.

[29] José Natário, Leonel Queimada, and Rodrigo Vicente. Test fields cannot destroy extremal black holes. *Classical and Quantum Gravity*, 33(17):175002, 2016.

[30] Gábor Zsolt Tóth. Weak cosmic censorship, dyonic Kerr–Newman black holes and Dirac fields. *Classical and Quantum Gravity*, 33(11):115012, 2016.

[31] R. Penrose and R. M. Floyd. Extraction of Rotational Energy from a Black Hole. *Nature Physical Science **229**, 177*, 1971.

[32] G. Denardo and R. Ruffini. On the energetics of Reissner Nordstrøm geometries. *Physics Letters B*, 45(3):259 – 262, 1973.

[33] Manjiri Bhat, Sanjeev Dhurandhar, and Naresh Dadhich. Energetics of the Kerr-Newman black hole by the penrose process. *Journal of Astrophysics and Astronomy*, 6(2):85–100, 1985.

[34] Steven Detweiler. Klein-Gordon equation and rotating black holes. *Phys. Rev. D*, 22:2323–2326, Nov 1980.

[35] Vitor Cardoso and Óscar J. C. Dias. Small Kerr anti-de Sitter black holes are unstable. *Phys. Rev. D*, 70:084011, Oct 2004.

[36] William H. Press and Saul A. Teukolsky. Floating Orbits, Superradiant Scattering and the Black-hole Bomb. *Nature*, 238:211–212, 1972.

[37] Paolo Pani, Vitor Cardoso, Leonardo Gualtieri, Emanuele Berti, and Akihiro Ishibashi. Black-Hole Bombs and Photon-Mass Bounds. *Phys. Rev. Lett.*, 109:131102, Sep 2012.

[38] Richard Brito, Vitor Cardoso, and Paolo Pani. Massive spin-2 fields on black hole spacetimes: Instability of the Schwarzschild and Kerr solutions and bounds on the graviton mass. *Phys. Rev. D*, 88:023514, Jul 2013.

[39] Ezra Newman and Roger Penrose. An Approach to Gravitational Radiation by a Method of Spin Coefficients. *Journal of Mathematical Physics*, 3(3), 1962.

[40] Y. Nambu and G. Jona-Lasinio. Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I. *Phys. Rev.*, 122:345–358, Apr 1961.

[41] Max Bañados and Ignacio Reyes. A short review on Noether's theorems, gauge symmetries and boundary terms. *International Journal of Modern Physics D*, 25(10):1630021, 2016.

[42] Ray D'Inverno. *Introducing Einstein's relativity*. Oxford University Press, 1992.

[43] Philipp Mösta, Carlos Palenzuela, Luciano Rezzolla, Luis Lehner, Shin'ichirou Yoshida, and Denis Pollney. Vacuum electromagnetic counterparts of binary black-hole mergers. *Phys. Rev. D*, 81:064017, Mar 2010.

[44] Felix Finster, Joel Smoller, and Shing-Tung Yau. Particlelike solutions of the Einstein-Dirac equations. *Phys. Rev. D*, 59:104020, Apr 1999.