TÉCNICO LISBOA

Superradiance of Bosonic Fermion Condensates

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Resumo

A superradiância é um fenómeno de amplificação de radiação que surge em muitos contextos em física. Um dos contextos em que este fenómeno se manifesta é na chamada difusão superradiante. Em geral, acredita-se que campos fermiónicos não podem ser superradiantemente amplificados, enquanto campos bosónicos podem. No entanto, existem vários casos de sistemas de fermiões com comportamento bosónico, sendo os pares de Cooper da teoria de Bardeen-Cooper-Schrieffer (BCS) da superconductividade e os mesões de física de partículas exemplos disso mesmo. Isto leva-nos a perguntar: Será possível ter um condensado de fermiões capaz de exibir amplificação superradiante? Ou, existirá alguma interacção não-linear entre fermiões que os torne capazes de exibir amplificação superradiante? A resposta a estas questões é de grande importância para testar a ideia de que o processo de Penrose é o análogo corpuscular do fenómeno de difusão superradiante.

Nesta tese, revemos os casos da difusão de um campo escalar carregado numa barreira de potencial electrostático forte (paradoxo de Klein) e num espaço-tempo de Reissner-Nordström, obtendo modos superradiantes em ambos os casos. Para além disso, provamos o facto bem conhecido de que campos de Dirac não exibem superradiância, tanto no paradoxo de Klein como na geometria de Reissner-Nordström.

Por fim, damos, pela primeira vez, um exemplo de uma teoria não-linear para campos de Dirac que admite soluções superradiantes, tanto no paradoxo de Klein como no espaço-tempo de Reissner-Nordström.

Palavras-chave: superradiância, paradoxo de Klein, Reissner-Nordström, condensado bosónico, fermiões, buracos negros.

Abstract

Superradiance is a radiation enhancement process that happens in many contexts in physics. One of its manifestations is what we call superradiant scattering. It is, generally, believed that fermionic fields cannot be superradiantly amplified, whereas bosonic fields can. Nevertheless, there are several examples in nature of fermion systems with bosonic behaviour, for instance, Cooper pairs in Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity and mesons in particle physics. This raises the questions: Is it possible to have a fermion condensate which can exhibit superradiant amplification? Or, is there a non-linear interaction between fermions which enables them to exhibit superradiant amplification? The answer to these questions is of great importance to test the idea that the Penrose process is the particle analogous of superradiant scattering phenomenon.

In this thesis, we review the cases of a charged scalar field scattering in a strong electrostatic potential barrier (Klein paradox) and in a Reissner-Nordström background, obtaining superradiant modes in both cases. Moreover, we prove the well known fact that Dirac fields cannot exhibit superradiant amplification both in the case of Klein paradox as well as in Reissner-Nordström black holes.

Finally, for the first time, we give an example of a non-linear Dirac theory that admits superradiant solutions both in the Klein paradox and in the Reissner-Nordström background.

Keywords: superradiance, Klein paradox, Reissner-Nordström, bosonic condensate, fermions, black holes.

Notation and Conventions

In this thesis, we use the metric signature $(+\,-\,-\,-)$. Moreover, we use units with $c = \hbar = G = k_e = 1$ where c is the speed of light in vacuum, $\hbar$ is the normalized Planck constant, G is the gravitational constant and k_e is the Coulomb constant.

- $\mathbb{1}$ denotes the unit matrix/operator.
- ϵ_{abc} is the so-called Levi-Civita symbol.
- A^* denotes the complex conjugate of A.
- $A^\dagger$ denotes the hermitian conjugate of A.
- q denotes the electric charge of a scalar field.
- e denotes the electric charge of a Dirac (spin-1/2) field.
- ∇_μ is the covariant derivative.
- $Y_l^{m_l}$ is the spherical harmonic of degree l and order m_l where l and m_l are integers satisfying $l > |m_l|$. These functions satisfy the orthonormality relations:

$$\int d\Omega\, (Y_l^{m_l})^* Y_{l'}^{m_{l'}} = \delta_{l,l'}\, \delta_{m_l,m_{l'}} \quad .$$

- σ^i are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

  These matrices are hermitian and unitary and satisfy the relation

$$\sigma^a \sigma^b = i\epsilon_{abc}\, \sigma^c + \delta_{ab}\, \mathbb{1}_2 \quad .$$

- γ^μ are the Dirac 4×4 matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad i = 1,\, 2,\, 3\, .$$

  The Dirac matrices satisfy the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. They also satisfy the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

- $\bar{\Psi} = \Psi^\dagger\gamma^0$ is the Dirac conjugate of the spinor Ψ.
- We use the abbreviations RN for Reisner-Nordström and BH for black hole.

Contents

Introduction

1

1.1 Motivation and Overview

Superradiance is a phenomenon where radiation is enhanced by some system with the capability to dissipate energy. This phenomenon occurs in several contexts in physics, for instance, it can happen in quantum optics [1, 2], in quantum mechanics [3, 4] and in relativity [5]. An interesting manifestation of this phenomenon occurs in the scattering of fields by certain systems, where the scattered field obtains a larger energy than the one the incident field had. So, the diffuser system must have some dissipative mechanism that allows the transference of energy to the field. In this work we are always interested in this kind of superradiance.

It is a fact of nature that all the known particles fall into one of two big families: fermions (particles with half-integer spin) and bosons (particles with integer spin). Quarks and leptons are fermions, while the force carrier particles are bosons. The main difference between these two families is that fermions obey the Pauli exclusion principle which states that two identical fermions cannot be in the same state at the same point of spacetime.

It happens that the phenomenon of superradiance in the scattering of fields depends on which family the field belongs to [6, 7]. In fact, it is generally believed that the scattering of fermionic fields cannot be superradiant. However, since each specific field has its own field equations and can be scattered by different diffuser systems, one cannot give a mathematical general proof of this idea. But one can prove it for some particular cases. For instance, it is known that the scattering of Dirac spin-1/2 fields on a electrostatic potential barrier and on a Kerr-Newman (charged, rotating) black hole (BH) cannot exhibit superradiance [3, 4, 8]. Here we refer to BHs as diffuser systems, but it only makes sense to study superradiance in this context if BHs have some dissipative mechanism which allows energy transfer to the surrounding fields. It turns out that this mechanism exists.

The study of the scattering of charged fields on strong electromagnetic backgrounds is generally called Klein paradox. In 1929, using the Dirac equation, Klein showed that an electron beam propagating in a region with a sufficiently large potential barrier can be transmitted without the exponential damping expected from non-relativistic quantum mechanics [9]. This phenomenon was dubbed Klein paradox by Sauter [10] and it can be explained by the pair production at the potential barrier using quantum field theory [3, 11]. Moreover, as we show in this thesis the Klein paradox is a simple example where a field can be superradiantly amplified. In fact, using quantum field theory it is possible to understand completely the phenomenon of superradiant scattering and its connection with pair creation. It is possible to show that for sufficiently large potential barriers there is a production of scalar and Dirac pairs, explaining the existence of transmitted modes instead of the exponential damping [3, 12]. Also, it is known that superradiance occurs due to pair production and the fact that Dirac fields do not exhibit superradiant amplification relies on the Pauli exclusion principle [6, 3, 12]. The behaviour is fundamentally the same when the fields are scattering on a (possibly charged) Kerr background [13, 14].

The idea of BHs as massive objects with such a large mass that even light could not escape them was first proposed by John Michell, in 1783, and, after, by Pierre-Simon Laplace, in 1796 [15, 16]. However, only in 1915, with the theory of general relativity of Albert Einstein, it was possible to understand how (massless) light interacts with gravity. In fact, it turns out that BHs arise in a very natural way in general relativity. There are solutions of the Einstein equations that contain closed regions with their interior causally disconnected from their exterior, in the sense that what happens in their interior cannot influence their exterior and, so, everything that enters these regions cannot escape them (even light) [17]. One hundred years ago, in 1916, Karl Schwarzschild discovered the first solution of the Einstein field equations with a region of this kind, this is called the Schwarzschild solution [18]. By definition, these regions are BHs and their boundary is the so-called event horizon. Furthermore, it is exactly the event horizon which provides a dissipative mechanism to the BH [6, 19]. This is something very peculiar, because the necessary dissipation for the existence of superradiant amplification is often provided by some kind of friction or viscosity, which always involve some matter or radiation fields. Since BHs appear in several vacuum solutions of the Einstein field equations, the event horizon provides the vacuum with a dissipative mechanism. This is very interesting, because the fields are allowed to extract energy from the vacuum itself by superradiant scattering. Moreover, in principle, this is a real phenomenon since we have strong observational indications that BHs exist. In fact, the first direct confirmation of their existence was the last year's detection, by LIGO, of the gravitational wave signal *GW150914*, originated by the collision and merger of a pair of BHs [20].

The study of BH superradiance started in 1971 with the independent predictions of Zel'dovich and Misner that some waves could be amplified by rotating (Kerr) BHs [5, 21]. Moreover, the work done by Teukolsky was crucial to the study of scattering of fields on Kerr background. Teukolsky showed that linearised perturbations of the Kerr geometry can be compactified in one single separable master equation, which contain the cases of scalar, electromagnetic and gravitational perturbations [22]. Using this master equation, Teukolsky and Press proved that scalar, electromagnetic and gravitational waves scattering on a Kerr BH have superradiant modes [23]. In 1973, Unruh separated the massless Dirac equation on Kerr background and showed that these spin-1/2 (neutrino) fields do not have superradiant modes [24]. This result was extended for massive spin-1/2 (Dirac) fields by Chandrasekhar [25]. In 1976, Page separated the Dirac equation on the more general Kerr-Newman background and, one year later, Lee used his result to show that Dirac fields have no superradiant modes on this background [8, 26]. Another interesting approach to the study of BH superradiance was that of Bekenstein, who saw the connection between this phenomenon and Hawking's area theorem [27]. This argument is so simple and beautiful that we shall outline it here. If the energy-momentum tensor of a (possibly charged) test field propagating on a Kerr-Newman background satisfies the null energy condition [28] at the event horizon, then the energy ΔM, angular momentum ΔJ and electric charge ΔQ absorbed by the BH satisfy [29]:

$$\Delta M \geq \Omega \Delta J + V \Delta Q \quad , \tag{1.1}$$

where Ω is the angular velocity of the BH horizon and V is the electric potential at the horizon. It is easy to see that the ratios of the angular momentum over the energy and of the electric charge over the energy of a wave with frequency ω, azimuthal number m and electric charge e are, respectively, m/ω and e/ω [27]. Then, the inequality (1.1) reads

$$\frac{\Delta M}{\omega} \left(\omega - m\Omega - eV\right) \geq 0 \quad . \tag{1.2}$$

Superradiant modes must extract energy from the BH and, so, $\Delta M < 0$, which implies that ω must satisfy

$$0 < \omega < m\Omega + eV \quad . \tag{1.3}$$

These are, precisely, the modes which extract energy from the BH. Since the energy-momentum tensor of the Dirac field does not satisfy the null energy condition at the event horizon [30], we see that these fields are not contemplated by this proof and, in fact, as we said above, they do not exhibit superradiant amplification when scattering on this background.

In 1971, Roger Penrose theorized a phenomenon called the Penrose process [31]. This is a phenomenon where rotational energy can be extracted from Kerr (rotating) BHs and it is generally believed to be the particle analogue of superradiant scattering. The Penrose process makes use of the fact that Kerr BHs have a region called ergoregion, where a particle can have negative energy with respect to an observer at infinity [6, 17]. The idea of Penrose consists in considering a particle falling into the ergoregion and decaying there in two another particles. Obeying the energy-momentum conservation law, it is possible that one of the particles falls into the BH with negative energy (with respect to an observer at infinity) and the other escapes to infinity with a larger energy than that of the original particle. In fact, it can be shown that, for Reissner-Nordström (charged static) BHs, there exists a generalized ergoregion and a similar energy extraction process is possible [32, 33].

As we said, the Penrose process is generally believed to be the particle analogue of superradiant scattering phenomenon. However, while the two processes are classical, superradiant amplification seems to carry some quantum features of the field being scattered. In particular, even though the fields are not quantised, superradiant scattering already seems to predict pair production. Now, if we believe that the Penrose process is a real phenomenon, which happens for ordinary matter in nature and, at the same time, we believe it to be the particle analogue of superradiance, we have something to explain. Because we know that all the ordinary (baryonic) matter is made of fermions at the very fundamental level and it is believed that fermions do not exhibit superradiant amplification. This raises the expectation that it may exist some non-linear interaction between the fermions, which restores their capability to exhibit superradiance. In other words, we expect the existence of bosonic fermion condensates with the capability to exhibit superradiant amplification. In fact, the existence of fermion systems with a bosonic behaviour is not very strange and happens in nature. For instance, Cooper pairs in BCS theory of superconductivity and mesons in particle physics are examples of these bosonic fermion condensates.

The existence of these kind of condensates can have interesting applications in astrophysics. In fact, it is known that we can confine superradiantly amplified fields through various mechanisms, like massive fields and anti-de Sitter boundaries [6, 34, 35]. This confinement can originate strong instabilities called BH bombs [36], which have applications in searches for dark matter and physics beyond the Standard Model [6, 37, 38].

In this thesis, we review the scattering of scalar and Dirac fields on an electrostatic potential barrier (Klein paradox) and on a Reissner-Nordström (charged static) BH. The results obtained for these cases are well known [4, 3, 6, 8]. Nevertheless, as far as we know, the only proof of the absence of superradiance for Dirac fields on RN background is obtained as a limit of the more general Kerr-Newman background [8], which uses the formalism of Newman-Penrose [39] to separate the Dirac equation. Here, we use the spherical symmetry of RN geometry to separate the Dirac equation in an easier way and we proceed to prove the absence of superradiance for Dirac fields on a RN background in a new way. Finally, we provide a non-linear Dirac field theory, which is inspired by the Nambu-Jona-Lasinio model [40] and can exhibit superradiant amplification both on the Klein paradox and RN background. Here, we are not concerned about the generality or validity of this theory. Instead, we want to provide a simple theory, which we believe to describe a fermion condensate and, at the same time, allows superradiant scattering solutions. So, we give a proof of concept that it is possible to construct this kind of condensates. In principle, there are other theories more realistic than this one, which allow solutions with the same kind of behaviour.

In this work, we always consider charged test fields, neglecting the electromagnetic field produced by them and their back-reaction on the spacetime geometry. So, we consider that these fields always propagate in a fixed background geometry. This test field approximation is correct at first order in the fields, because their effect on the geometry and on the electromagnetic field is only of second order [6]. Also, we do not expect any qualitative change of our conclusions from using this approximation. In fact, in any viable astrophysical scenario, the first order on the amplitude of the fields is enough.

The field theories that we use throughout this work are always described by an action of the form:

$$S = S_G + S_{EM} + S_M \quad , \tag{1.4}$$

with

$$S_G = \int d^4x\sqrt{-g}\ \frac{R}{16\pi} \quad , \tag{1.5}$$

$$S_{EM} = -\int d^4x\sqrt{-g}\ \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \tag{1.6}$$

where g is the determinant of the metric $g_{\mu\nu}$ of spacetime, R is its scalar curvature and $F_{\mu\nu}$ is the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{1.7}$$

The action S_M:

$$S_M = \int d^4x \sqrt{-g}\, \mathcal{L}_M \quad , \tag{1.8}$$

where $\mathcal{L}_M$ is the lagrangian density of some matter field. This action describes the matter field under analysis. In this work, we will consider three kind of matter fields: scalar fields, Dirac fields and non-linear Dirac fields.

In this thesis, we always use theories S_M which are U(1) invariant. A theory of this kind is such that if it describes the field ξ, then its equations are invariant under the transformation

$$\xi \longrightarrow e^{i\alpha}\xi \quad , \tag{1.9}$$

with α a real constant. So, by Noether's theorem, there is a conserved current associated with this symmetry [41]. We call this current the particle-number current and we use its flux to study the phenomenon of superradiant scattering [7]. We say that there is superradiant amplification if the absolute value of the flux of the reflected particle-number current is larger than that of the incident one.

2 Klein paradox

In this chapter, we study the phenomenon of superradiance when scattering scalar or Dirac charged fields in an electrostatic potential barrier. In particular, we show the well known fact that Dirac fields cannot exhibit superradiance. On the other hand, we prove that charged scalar fields can exhibit superradiance for modes with frequency ω obeying the relation $m < \omega < qV - m$ with V the electric potential, q the charge and m the mass of the field. We also analyse the case of non-linear Dirac fields and prove that there is a superradiant regime. Thus, although linear Dirac fields cannot exhibit superradiance, non-linear Dirac fields can. We interpret these non-linear Dirac fields as describing roughly condensates of interacting Dirac particles. As a motivation for this interpretation, we have the Nambu-Jona-Lasinio model [40], which is based on an action quite similar to ours. This model is used as an effective theory to describe mesons, which are composed by pairs of interacting quarks and anti-quarks, both spin-1/2 fermionic particles, and have a bosonic behaviour as a whole. Moreover, this model is motivated by BCS theory of superconductivity and, in particular, by the concept of Cooper pairs, which are pairs of interacting electrons that have also a bosonic behaviour as a whole.

We consider a two dimensional problem on a four dimensional flat spacetime. So, we have the fields propagating along t and z, with the metric of the Minkowski background [42]

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad , \tag{2.1}$$

and the arbitrary electromagnetic potential

$$A_\mu = (V(z), 0, 0, 0) \quad , \tag{2.2}$$

with the asymptotic behaviour

$$V(z) = \begin{cases} 0, & \text{for } z \to -\infty \\ \widetilde{V} > 0, & \text{for } z \to +\infty \end{cases} \quad . \tag{2.3}$$

Then, we are interested in what happens to incident quasi-monochromatic charged waves (of a specific field) coming from $z = -\infty$ and scattering in this potential barrier. To interpret the waves as incident, reflected and transmitted, we use their group velocity. As is known in wave theory, a wave-packet

$$\varphi(t,z) = \int d\omega A(\omega) e^{-i[\omega t - k(\omega) z]} \quad , \tag{2.4}$$

has group velocity

$$v_g = \left(\frac{\partial k(\omega)}{\partial \omega}\right)^{-1} \quad , \tag{2.5}$$

with the frequency ω real, $A(\omega)$ a complex-valued function and $k(\omega)$ the dispersion relation of the specific field.

Let us define here what we mean by quasi-monochromatic waves of frequency $\widetilde{\omega}$. These are wave-packets with frequencies in an infinitesimal interval around some frequency $\widetilde{\omega}$. These waves are approximately monochromatic

$$\varphi(t,z) \approx A\, e^{-i[\widetilde{\omega} t - k(\widetilde{\omega}) z]} \quad , \tag{2.6}$$

and they have group velocity

$$v_g = \left(\frac{\partial k(\omega)}{\partial \omega}\right)^{-1}_{\omega=\widetilde{\omega}} \quad . \tag{2.7}$$

This is, of course, an approximation, because we cannot define the concept of group velocity for monochromatic waves. But we can make the approximation as good as we want by considering the frequency interval around $\widetilde{\omega}$ as small as needed.

2.1 Scattering of scalar fields

Let us start by considering the scalar theory

$$S_{scalar} = \int dx^4 \left[D_\mu \phi (D^\mu \phi)^* - m^2 |\phi|^2\right] \quad , \tag{2.8}$$

with m the mass of the scalar field and $D_\mu = \partial_\mu + iqA_\mu$, where $q > 0$ is the electric charge of the field. In this theory, we are imposing the minimal coupling between the charged field and the electromagnetic field.

This theory is sometimes called Klein-Gordon theory. It was proposed, in 1926, by Oskar Klein and Walter Gordon to describe relativistic electrons. In fact, now, we know that this theory describes scalar (spin-0) fields, which, in the framework of quantum field theory, are associated with the Higgs boson particle. Furthermore, this theory describes also spinless relativistic composite particles, as the pion.

From the action (2.8), we can obtain the field equations

$$(D^\mu D_\mu + m^2)\phi = 0 \quad , \tag{2.9}$$

and

$$[(D^\mu)^* (D_\mu)^* + m^2]\phi^* = 0 \quad . \tag{2.10}$$

Since the second field equation is just the complex conjugate of the first equation, we focus only on

Equation (2.9).

In the present problem, this field equation can be written as

$$[(\partial_t^2) + 2iqV\partial_t - q^2V^2 - \partial_z^2 + m^2]\phi = 0 \quad , \tag{2.11}$$

or, in the time-independent form,

$$[\partial_z^2 + (\omega - qV)^2 - m^2]\varphi = 0 \quad , \tag{2.12}$$

which is obtained by separation of variables with $\phi(t, z) = e^{-i\omega t}\varphi(z)$.

Let us, now, solve the time-independent field equation in the asymptotic regions $z \to -\infty$ and $z \to +\infty$.

Region I: $z \to -\infty$

In this region, Equation (2.12) reads

$$(\partial_z^2 + \omega^2 - m^2)\varphi_I = 0 \quad , \tag{2.13}$$

which has the general solution

$$\varphi_I(z) = A\, e^{ikz} + B\, e^{-ikz} \quad , \tag{2.14}$$

with

$$k^2 = \omega^2 - m^2 \quad . \tag{2.15}$$

Notice that, in the problem we are considering, we must have waves in this region and, so, $\omega \in X$ with

$$X = \{\omega \,|\, \omega^2 - m^2 > 0\} \quad .$$

Now, we want to write explicitly the solution corresponding to the incident wave and the one corresponding to the reflected wave. It is possible to write the total solution as a sum of incident and reflected waves because Equation (2.11) is linear. Then, remembering that we are considering quasi-monochromatic waves, we use the group velocity to interpret the general solution of Equation (2.11). By the definition of group velocity, we see that, in this region, the general solution is the sum of two waves with symmetric group velocities. The group velocity of the wave associated with the term proportional to e^{ikz} in (2.14) is

$$v_g = \frac{k}{\omega} \quad , \tag{2.16}$$

where we used Equation (2.15) and the definition of group velocity (2.7).

So, we can write the incident wave solution as

$$\phi_I^i = I\, e^{-i[\omega t - k(\omega) z]} \quad , \tag{2.17}$$

and the reflected one as

$$\phi_I^r = R\, e^{-i[\omega t + k(\omega) z]} \quad , \tag{2.18}$$

where

$$k(\omega) = \epsilon\sqrt{\omega^2 - m^2} \quad , \tag{2.19}$$

with $\epsilon = \text{sign}(\omega + m)$. We choose the sign of k in a way that makes the incident and the reflected waves have positive and negative group velocities, respectively, along z. We are allowed to do this, because the dispersion relation (2.15) does not impose a sign for k. This imposition comes from our physical boundary conditions.

Region II: $z \to +\infty$

In this other region, Equation (2.12) reads

$$[\partial_z^2 + (\omega - q\widetilde{V})^2 - m^2]\varphi_{II} = 0 \quad , \tag{2.20}$$

which has the general solution

$$\varphi_{II}(z) = Ae^{isz} + Be^{-isz} \quad , \tag{2.21}$$

with

$$s^2 = (\omega - q\widetilde{V})^2 - m^2 \quad . \tag{2.22}$$

While in the region I only wave solutions are allowed by our boundary conditions, here it is not necessary to have a travelling transmitted wave. In fact, it is possible, with $\omega \in X$, to have stationary waves attenuated or amplified along z. But we exclude the solutions corresponding to these amplified stationary waves, because they give rise to instabilities (the amplitude goes to infinity when $z \to +\infty$) and are, therefore, out of the domain of validity of the test field approximation.

Now, we see that, in this region, the general solution of Equation (2.11) corresponds either to the sum of two waves travelling with symmetric group velocities or to the sum of an attenuated and an amplified stationary waves. In the case of travelling waves (ω such that $\omega \in X$ and $(\omega - q\widetilde{V})^2 > m^2$), the group velocity of the wave associated with the term proportional to e^{isz} in (2.21) is

$$v_g = \frac{s}{\omega - q\widetilde{V}} \quad , \tag{2.23}$$

where we used Equation (2.22) and the definition of group velocity (2.7).

Then, the transmitted wave can be written as

$$\phi_{II}^t = T\, e^{-i[\omega t - s(\omega) z]} \quad , \tag{2.24}$$

with $\omega \in X$ and

$$s(\omega) = \widetilde{\epsilon}\sqrt{(\omega - q\widetilde{V})^2 - m^2} \quad , \tag{2.25}$$

with $\widetilde{\epsilon} = \text{sign}(\omega - q\widetilde{V} + m)$. Again, as in region I, we choose the sign of s to be consistent with our physical boundary conditions. In fact, it is very easy to verify that if this solution is a travelling wave (ω such that $\omega \in X$ and $(\omega - q\widetilde{V})^2 > m^2$), it has positive group velocity and if it is a stationary wave (ω such that $\omega \in X$ and $(\omega - q\widetilde{V})^2 \leq m^2$), it is an attenuated one.

Conserved z-current

Let us introduce the particle-number z-current of the scalar field ϕ:

$$j^z = -\frac{i}{2}(\phi^* \partial_z \phi - \phi\, \partial_z \phi^*) \quad . \tag{2.26}$$

This current is the z-component of the Noether's conserved current associated with the U(1) symmetry of the scalar field ϕ.

It is easy to see that this current is conserved along z:

$$\partial_z j^z = -\frac{i}{2}(\phi^* \partial_z^2 \phi - \phi\, \partial_z^2 \phi^*) = 0 \quad , \tag{2.27}$$

where in the last equality we used Equation (2.12) and its complex conjugate.

To analyse the existence of superradiant modes, we can calculate the z-current of the incident, reflected and transmitted waves and, using its conservation along z, search for modes with the absolute value of the reflected current larger than that of the incident one.

The incident z-current is

$$(j^z)^i = -\frac{i}{2}[(\phi_I^i)^* \partial_z \phi_I^i - \phi_I^i\, (\partial_z \phi_I^i)^*] = |I|^2 k \quad , \tag{2.28}$$

the reflected one is

$$(j^z)^r = -\frac{i}{2}[(\phi_I^r)^* \partial_z \phi_I^r - \phi_I^r\, (\partial_z \phi_I^r)^*] = -|R|^2 k \quad , \tag{2.29}$$

and the transmitted is

$$(j^z)^t = -\frac{i}{2}[(\phi_{II}^t)^* \partial_z \phi_{II}^t - \phi_{II}^t\, (\partial_z \phi_{II}^t)^*] = |T|^2 \text{Re}(s) \quad . \tag{2.30}$$

In the asymptotic regions I and II, the scalar field ϕ is given by $\phi_I = \phi_I^i + \phi_I^r$ and $\phi_{II} = \phi_{II}^t$, respectively. Thus, from the fact that the z-current is conserved along z, we have that

$$(j^z)_I = (j^z)_{II} \quad , \tag{2.31}$$

where $(j^z)_I$ and $(j^z)_{II}$ denote the z-current j^z in the asymptotic regions I and II, respectively. Moreover, it is very easy to show that

$$(j^z)_I = (j^z)^i + (j^z)^r \quad , \tag{2.32}$$

and that

$$(j^z)_{II} = (j^z)^t \quad . \tag{2.33}$$

Then, from Equation (2.31):

$$(j^z)^i + (j^z)^r = (j^z)^t \quad , \tag{2.34}$$

and substituting Equations (2.28), (2.29) and (2.30), it follows that

$$|I|^2 - |R|^2 = \frac{\mathrm{Re}(s)}{k}\,|T|^2 \quad . \tag{2.35}$$

Let us compare the reflected z-current with the incident one and analyse in what conditions we can have superradiance. Consider the quantity

$$\left|\frac{(j^z)^r}{(j^z)^i}\right| = \left|\frac{R}{I}\right|^2 = 1 - \frac{\mathrm{Re}(s)}{k}\left|\frac{T}{I}\right|^2 \quad , \tag{2.36}$$

where we used Equation (2.35) in the last equality. By definition, superradiance is present when $|(j^z)^r| > |(j^z)^i|$ or, equivalently,

$$0 > \frac{\mathrm{Re}(s)}{k} = \widetilde{\epsilon}\,\epsilon\,\frac{\mathrm{Re}\left(\sqrt{(\omega - q\widetilde{V})^2 - m^2}\right)}{\sqrt{\omega^2 - m^2}} \quad . \tag{2.37}$$

Thus, we see that the superradiant regime is such that the transmitted wave is a travelling wave ($s \in \mathbb{R}$) and $\widetilde{\epsilon}\,\epsilon < 0$. This is possible only if $q\widetilde{V} > 2m$ and for frequencies ω satisfying

$$m < \omega < q\widetilde{V} - m \quad . \tag{2.38}$$

So, modes of the scalar field satisfying the above condition will exhibit superradiant amplification, provided that $q\widetilde{V} > 2m$. Notice that the superradiant regime coincides with the regime of negative transmitted z-current ($(j^z)^t < 0$). The superradiant modes (2.38) we obtained are in agreement with the ones obtained in Ref. [3].

From (2.36), we see that for transmitted stationary waves (s purely imaginary) there is total reflection ($(j^z)^r = (j^z)^i$). This is something that we will find again, in the next section, when scattering Dirac fields in a similar potential.

2.2 Scattering of Dirac fields

In this case, we are interested in the theory

$$S_{Dirac} = \int dx^4 (i\bar{\Psi}\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi) \quad , \tag{2.39}$$

with m the mass of the Dirac field and, as in the scalar field case, $D_\mu = \partial_\mu + ieA_\mu$, where $e > 0$ is the electric charge of the field.

This theory was proposed, in 1928, by Paul Dirac. It describes spin-1/2 particles, like electrons and quarks. In fact, this theory was a great success, since it predicted the value of the gyromagnetic ratio of the electron in a completely rigorous way and also the existence of a new kind of matter, the so-called antimatter. This antimatter turned out to be observed experimentally, in 1932, by Carl Anderson.

From the action (2.39), we can derive the field equations

$$i\gamma^\mu D_\mu \Psi - m\Psi = 0 \quad , \tag{2.40}$$

and

$$iD^*_\mu \bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0 \quad . \tag{2.41}$$

It is easy to see that the second field equation is obtained from the first by taking its hermitian conjugate and right multiply it by γ^0. So, we focus on the first field equation (2.40).

In our particular problem, this equation can be written as

$$[i\gamma^0(\partial_t + ieV) + i\gamma^3\partial_z - m\mathbb{1}]\Psi = 0 \quad . \tag{2.42}$$

We can separate this equation with $\Psi(t, z) = e^{-i\omega t}\chi(z)$, obtaining the time-independent field equation

$$[i\gamma^3\partial_z + (\omega - eV)\gamma^0 - m\mathbb{1}]\chi = 0 \quad , \tag{2.43}$$

where $\chi(z)$ is a 4-spinor.

Using the ansatz

$$\chi(z) = \begin{pmatrix} u \\ v \end{pmatrix} e^{irz} \quad , \tag{2.44}$$

where u and v are 2-spinors, we obtain the matrix equation

$$\begin{pmatrix} \omega - eV - m & -r\sigma^3 \\ r\sigma^3 & -\omega + eV - m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \quad . \tag{2.45}$$

Since we want a solution other than the trivial one, the matrix determinant must vanish. So, we obtain

$$r^2 = (\omega - eV)^2 - m^2 \quad , \tag{2.46}$$

where we used $\sigma^3\sigma^3 = \mathbb{1}$. Moreover, we have the indeterminate solution

$$v = \frac{r}{\omega - eV + m}\,\sigma^3 u \quad . \tag{2.47}$$

Let us now use these results to find the asymptotic solutions of the time-independent field equation in the regions $z \to -\infty$ and $z \to +\infty$.

Region I: $z \to -\infty$

In this region, Equation (2.43) reads

$$[i\gamma^3\partial_z + \omega\gamma^0 - m\mathbb{1}]\chi_I = 0 \quad , \tag{2.48}$$

whose general solution can be written as

$$\begin{aligned}\chi_I(z) = \left[A\begin{pmatrix} u_+ \\ \frac{k}{\omega+m}u_+\end{pmatrix} + B\begin{pmatrix} u_- \\ -\frac{k}{\omega+m}u_-\end{pmatrix}\right]e^{ikz}+ \\ +\left[C\begin{pmatrix} u_+ \\ -\frac{k}{\omega+m}u_+\end{pmatrix} + D\begin{pmatrix} u_- \\ \frac{k}{\omega+m}u_-\end{pmatrix}\right]e^{-ikz} \quad ,\end{aligned} \tag{2.49}$$

with

$$k^2 = \omega^2 - m^2 \quad , \tag{2.50}$$

and

$$\begin{aligned}\sigma^3 u_+ &= u_+ \quad , \\ \sigma^3 u_- &= -u_- \quad .\end{aligned} \tag{2.51}$$

Above we used the fact that every 2-spinor u can be written as a linear combination of the 2-spinors u_+ and u_-. This is because these are two eigenvectors of σ^3 associated with different eigenvalues and σ^3 is a 2×2 hermitian matrix.

As in the scalar field case, our physical boundary conditions imply that we must have waves in this region and so $\omega \in X$ with

$$X = \{\omega \,|\, \omega^2 - m^2 > 0\} \quad . \tag{2.52}$$

Now, since the field equation (2.42) is linear, we want to write its solution in this region as a sum of incident and reflected waves. It is easy to see that the general solution of Equation (2.42) is a sum of two waves travelling with symmetric group velocities. Furthermore, the group velocity of the wave associated with the term proportional to e^{ikz} in (2.49) is

$$v_g = \frac{k}{\omega} \quad , \tag{2.53}$$

where we used the definition of group velocity (2.7) and Equation (2.50).

So, the incident wave solution is

$$\Psi_I^i = \left[I_+\begin{pmatrix} u_+ \\ \frac{k}{\omega+m}u_+\end{pmatrix} + I_-\begin{pmatrix} u_- \\ -\frac{k}{\omega+m}u_-\end{pmatrix}\right]e^{-i[\omega t - k(\omega)z]} \quad , \tag{2.54}$$

and the reflected one is

$$\Psi_I^r = \left[R_+\begin{pmatrix} u_+ \\ -\frac{k}{\omega+m}u_+\end{pmatrix} + R_-\begin{pmatrix} u_- \\ \frac{k}{\omega+m}u_-\end{pmatrix}\right]e^{-i[\omega t + k(\omega)z]} \quad , \tag{2.55}$$

where

$$k(\omega) = \epsilon\sqrt{\omega^2 - m^2} \quad , \tag{2.56}$$

with $\epsilon = \mathrm{sign}(\omega + m)$. We choose the sign of k in a way that makes the incident and reflected waves have positive and negative group velocities, respectively.

Region II: $z \to +\infty$

In this region, Equation (2.43) reads

$$[i\gamma^3\partial_z + (\omega - e\widetilde{V})\gamma^0 - m\mathbb{1}]\chi_{II} = 0 \quad , \tag{2.57}$$

which has the general solution

$$\begin{aligned} \chi_{II}(z) = & \left[A \begin{pmatrix} u_+ \\ \frac{s}{\omega - e\widetilde{V} + m} u_+ \end{pmatrix} + B \begin{pmatrix} u_- \\ -\frac{s}{\omega - e\widetilde{V} + m} u_- \end{pmatrix} \right] e^{isz} + \\ & + \left[C \begin{pmatrix} u_+ \\ -\frac{s}{\omega - e\widetilde{V} + m} u_+ \end{pmatrix} + D \begin{pmatrix} u_- \\ \frac{s}{\omega - e\widetilde{V} + m} u_- \end{pmatrix} \right] e^{-isz} \quad , \end{aligned} \tag{2.58}$$

with

$$s^2 = (\omega - e\widetilde{V})^2 - m^2 \quad . \tag{2.59}$$

Now, the discussion proceeds exactly in the same way as that of the scalar field case. The general solution of Equation (2.42) corresponds either to the sum of two waves travelling with symmetric group velocities or to the sum of an attenuated and an amplified stationary waves. Again, in the case of travelling waves ($\omega \in X$ and $(\omega - e\widetilde{V})^2 > m^2$), the group velocity of the wave associated with the term proportional to e^{isz} in Equation (2.58) is

$$v_g = \frac{s}{\omega - e\widetilde{V}} \quad , \tag{2.60}$$

where we used the group velocity definition (2.7) and Equation (2.59).

Then, the transmitted wave is

$$\Psi_{II}^t = \left[T_+ \begin{pmatrix} u_+ \\ \frac{s}{\omega - e\widetilde{V} + m} u_+ \end{pmatrix} + T_- \begin{pmatrix} u_- \\ -\frac{s}{\omega - e\widetilde{V} + m} u_- \end{pmatrix} \right] e^{-i[\omega t + s(\omega) z]} \quad , \tag{2.61}$$

with

$$s(\omega) = \widetilde{\epsilon}\sqrt{(\omega - e\widetilde{V})^2 - m^2} \quad , \tag{2.62}$$

where $\widetilde{\epsilon} = \mathrm{sign}(\omega - e\widetilde{V} + m)$. It is easy to see that, with the sign chosen for s, the transmitted wave is either a travelling wave (when $\omega \in X$ and $(\omega - e\widetilde{V})^2 > m^2$) with positive group velocity or an attenuated stationary wave (when $\omega \in X$ and $(\omega - e\widetilde{V})^2 \leq m^2$).

Conserved z-current

Now, we introduce the particle-number z-current of the Dirac field Ψ:

$$j^z = \frac{1}{2}\bar{\Psi}\gamma^3\Psi \quad . \tag{2.63}$$

This current is the z-component of the Noether's conserved current associated with the U(1) symmetry of the Dirac field Ψ.

We see that this current is conserved along z, because

$$\begin{aligned}\partial_z j^z &= \frac{1}{2}\left[(\partial_z\bar{\Psi})\gamma^3\Psi + \bar{\Psi}\gamma^3(\partial_z\Psi)\right]\\ &= \frac{i}{2}\left[-\Big((\omega - eV)\bar{\Psi}\gamma^0\Psi - m\bar{\Psi}\Psi\Big) + \Big((\omega - eV)\bar{\Psi}\gamma^0\Psi - m\bar{\Psi}\Psi\Big)\right] = 0 \quad ,\end{aligned} \tag{2.64}$$

where we used Equation (2.43) and its hermitian conjugate right multiplied by γ^0.

As we have done in the case of the scalar field, to check if there exists a superradiant regime, we need to calculate the z-current of the incident, reflected and transmitted waves. Then, we make use of the conservation of this current along z to compare the magnitudes of the incident and reflected z-currents.

The incident z-current is

$$(j^i)^z = \frac{1}{2}\bar{\Psi}^i_I\gamma^3\Psi^i_I = \frac{k}{\omega + m}(|I_+|^2 + |I_-|^2) \quad , \tag{2.65}$$

the reflected one is

$$(j^r)^z = \frac{1}{2}\bar{\Psi}^r_I\gamma^3\Psi^r_I = -\frac{k}{\omega + m}(|R_+|^2 + |R_-|^2) \quad , \tag{2.66}$$

and the transmitted is

$$(j^t)^z = \frac{1}{2}\bar{\Psi}^t_{II}\gamma^3\Psi^t_{II} = \frac{\mathrm{Re}(s)}{\omega - e\widetilde{V} + m}(|T_+|^2 + |T_-|^2) \quad , \tag{2.67}$$

where we used the properties

$$\begin{aligned} u_+^\dagger u_- &= u_-^\dagger u_+ = 0 \quad ,\\ u_+^\dagger u_+ &= u_-^\dagger u_- = 1 \quad ,\end{aligned} \tag{2.68}$$

which follow from the definitions of u_+ and u_- in Equation (2.51) and the fact that $\sigma^3\sigma^3 = \mathbb{1}$ and σ^3 is hermitian.

In the asymptotic regions I and II, the Dirac field Ψ is given by $\Psi_I = \Psi^i_I + \Psi^r_I$ and Ψ^t_{II}, respectively. From the conservation of the z-current, we have that

$$(j^z)_I = (j^z)_{II} \quad , \tag{2.69}$$

where $(j^z)_I$ and $(j^z)_{II}$ denote the z-current j^z in the asymptotic regions I and II, respectively. Furthermore, it is straightforward to show that

$$(j^z)_I = (j^i)^z + (j^r)^z \quad , \tag{2.70}$$

and that

$$(j^z)_{II} = (j^t)^z \quad . \tag{2.71}$$

From Equation (2.69):

$$(j^i)^z + (j^r)^z = (j^t)^z \quad , \tag{2.72}$$

and substituting Equations (2.65), (2.66) and (2.67), we obtain

$$|I_+|^2 + |I_-|^2 - |R_+|^2 - |R_-|^2 = \frac{\omega + m}{k} \frac{\mathrm{Re}(s)}{\omega - e\widetilde{V} + m} \left(|T_+|^2 + |T_-|^2\right) \quad . \tag{2.73}$$

Finally, we compare the magnitudes of the incident and reflected z-currents:

$$\left|\frac{(j^r)^z}{(j^i)^z}\right| = \frac{|R_+|^2 + |R_-|^2}{|I_+|^2 + |I_-|^2} = 1 - \frac{\omega + m}{k} \frac{\mathrm{Re}(s)}{\omega - e\widetilde{V} + m} \left(\frac{|T_+|^2 + |T_-|^2}{|I_+|^2 + |I_-|^2}\right) \quad , \tag{2.74}$$

where in the last equality we used Equation (2.73).

As we said in the scalar field section, for modes associated with transmitted stationary waves ($\omega \in X$ and $(\omega - e\widetilde{V})^2 \leq m^2$), there is, again, total reflection $|(j^i)^z| = |(j^r)^z|$, because s is purely imaginary. Furthermore, since $\mathrm{sign}(k) = (\omega + m)$ and $\mathrm{sign}(s) = (\omega - e\widetilde{V} + m)$ for $s \in \mathbb{R}$, then $|(j^r)^z| \leq |(j^i)^z|$. So, we conclude that for the scattering of Dirac fields there are no superradiant modes. This is in agreement with the results of Refs. [3, 4].

2.3 Scattering of non-linear Dirac fields

In this section, we want to consider the scattering of a fermion condensate. We use the usual Dirac free field action with an additional interaction term proportional to $(\bar{\Psi}\Psi)^2$. This term is such that the U(1) symmetry of Ψ is preserved. Then, we have a Noether's conserved current associated with this symmetry. The z-component of this current can be shown to be equal to the one of the last section. So, let us consider the non-linear Dirac field theory

$$S = \int dx^4 [i\,\bar{\Psi}\gamma^\mu D_\mu \Psi - \frac{\lambda}{2}(\bar{\Psi}\Psi)^2] \quad , \tag{2.75}$$

with all the quantities defined as in section (2.2) and with the coupling

$$\lambda(z) = \widetilde{\lambda}\, e^2 A_\mu A^\mu = \begin{cases} 0, & \text{for } z \to -\infty \\ \widetilde{\lambda}\, e^2 \widetilde{V}^2 > 0, & \text{for } z \to +\infty \end{cases} \quad , \tag{2.76}$$

where $\widetilde{\lambda} > 0$ is a real constant. The reason to consider this kind of coupling is that we want the field equation to be linear at $z \to -\infty$, in a way that makes it possible to write the solution as the sum of incident and reflected waves.

From this action, we obtain the field equation

$$i\gamma^\mu D_\mu \Psi - \lambda(\bar{\Psi}\Psi)\Psi = 0 \quad , \tag{2.77}$$

and its hermitian conjugate right multiplied by γ^0.

In this case, the above equation reads

$$[i\gamma^0(\partial_t + ieV) + i\gamma^3\partial_z - \lambda(\bar{\Psi}\Psi)\mathbb{1}]\Psi = 0 \quad . \tag{2.78}$$

Using $\Psi(t,z) = Ne^{-i\omega t}\chi(z)$, we obtain the time-independent field equation

$$[i\gamma^3\partial_z + (\omega - eV)\gamma^0 - \lambda(\bar{\chi}\chi)\mathbb{1}]\chi = 0 \quad , \tag{2.79}$$

where $\chi(z)$ is a 4-spinor.

Using the ansatz

$$\chi(z) = \begin{pmatrix} u \\ v \end{pmatrix} e^{irz} \quad , \tag{2.80}$$

where u and v are 2-spinors, we have

$$\bar{\chi}\chi = u^\dagger u - v^\dagger v = |u|^2 - |v|^2 \quad , \tag{2.81}$$

and we obtain the matrix equation

$$\begin{pmatrix} \omega - eV - \lambda|N|^2(|u|^2 - |v|^2) & -r\sigma^3 \\ r\sigma^3 & -\omega + eV - \lambda|N|^2(|u|^2 - |v|^2) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \quad . \tag{2.82}$$

Since we want a solution other than the trivial one, the matrix determinant must vanish. So, we obtain

$$\lambda^2|N|^4(|u|^2 - |v|^2)^2 - (\omega - eV)^2 + r^2 = 0 \quad . \tag{2.83}$$

Furthermore, using $u = u_+$ and assuming $v = \eta\, u_+$, with $\eta \in \mathbb{R}$, this expression reads

$$\lambda^2|N|^4(1 - \eta^2)^2 - (\omega - eV)^2 + r^2 = 0 \quad , \tag{2.84}$$

and the solution of the matrix equation is the η which satisfies

$$\omega - eV - \lambda|N|^2(1 - \eta^2) - \eta\, r = 0 \quad , \tag{2.85}$$

with u_+ as defined in section (2.2). Now, we focus on the asymptotic regions I and II.

Region I: $z \to -\infty$

In this region, we proceed in exactly the same way as in section (2.2). Then, the incident solution is

$$\Psi_I^i = \left[I_+ \begin{pmatrix} u_+ \\ \frac{k}{\omega} u_+ \end{pmatrix} + I_- \begin{pmatrix} u_- \\ -\frac{k}{\omega} u_- \end{pmatrix} \right] e^{-i[\omega t - k(\omega) z]} \quad , \tag{2.86}$$

and the reflected one is

$$\Psi_I^r = \left[R_+ \begin{pmatrix} u_+ \\ -\frac{k}{\omega} u_+ \end{pmatrix} + R_- \begin{pmatrix} u_- \\ \frac{k}{\omega} u_- \end{pmatrix} \right] e^{-i[\omega t + k(\omega) z]} \quad , \tag{2.87}$$

where

$$k(\omega) = \omega \quad . \tag{2.88}$$

Region II: $z \to +\infty$

In this region, we consider the particular transmitted solution

$$\Psi_{II}^t = T \begin{pmatrix} u_+ \\ \eta\, u_+ \end{pmatrix} e^{-i[\omega t - s(\omega) z]} \quad , \tag{2.89}$$

with Equations (2.84) and (2.85) reading, respectively,

$$\lambda^2 |T|^4 (1 - \eta^2)^2 - (\omega - e\,\widetilde{V})^2 + s^2 = 0 \quad , \tag{2.90}$$

and

$$\omega - e\,\widetilde{V} - \lambda |T|^2 (1 - \eta^2) - \eta\, s = 0 \quad . \tag{2.91}$$

Using these two equations we can show that

$$(\eta^2 + 1) s^2 - 2\eta s (\omega - e\,\widetilde{V}) = 0 \quad . \tag{2.92}$$

It is very easy to see that Equations (2.91) and (2.92) admit the particular solution:

$$s = 0 \quad , \tag{2.93}$$

$$\eta = -\sqrt{1 - \frac{\omega - e\,\widetilde{V}}{\lambda |T|^2}} < 0 \quad , \tag{2.94}$$

with ω satisfying

$$\omega < e\,\widetilde{V} + \lambda |T|^2 \quad . \tag{2.95}$$

Conserved z-current

In this section, we use the same particle-number z-current of the last section. Again, in this non-linear case, this current is the z-component of the Noether's conserved current associated with the U(1) symmetry of Ψ. The current is conserved along z, because

$$\begin{aligned} \partial_z j^z &= \frac{1}{2} \left[(\partial_z \bar{\Psi}) \gamma^3 \Psi + \bar{\Psi} \gamma^3 (\partial_z \Psi) \right] \\ &= \frac{i}{2} \left[-\Big((\omega - eV) \bar{\Psi} \gamma^3 \Psi + \lambda (\bar{\Psi}\Psi)^2 \Big) + \Big((\omega - eV) \bar{\Psi} \gamma^3 \Psi + \lambda (\bar{\Psi}\Psi)^2 \Big) \right] = 0 \quad , \end{aligned} \tag{2.96}$$

where we used Equation (2.79) and its hermitian conjugate right multiplied by γ^0.

Finally, we want to calculate the z-current of the incident, reflected and transmitted solutions and analyse if they admit a superradiant regime.

The incident z-current is

$$(j^i)^z = \frac{1}{2}\bar{\Psi}_I^i \gamma^3 \Psi_I^i = |I_+|^2 + |I_-|^2 \quad , \tag{2.97}$$

the reflected one is

$$(j^r)^z = \frac{1}{2}\bar{\Psi}_I^r \gamma^3 \Psi_I^r = -|R_+|^2 - |R_-|^2 \quad , \tag{2.98}$$

and the transmitted is

$$(j^t)^z = \frac{1}{2}\bar{\Psi}_{II}^t \gamma^3 \Psi_{II}^t = \eta |T|^2 \quad , \tag{2.99}$$

where we used the properties

$$\begin{aligned} u_+^\dagger u_- &= u_-^\dagger u_+ = 0 \quad , \\ u_+^\dagger u_+ &= u_-^\dagger u_- = 1 \quad . \end{aligned} \tag{2.100}$$

In exactly the same way as in the last section, one can show that the conservation of the z-current along z implies

$$(j^i)^z + (j^r)^z = (j^t)^z \quad . \tag{2.101}$$

Then, substituting Equations (2.97), (2.98) and (2.99), we obtain

$$|I_+|^2 + |I_-|^2 - |R_+|^2 - |R_-|^2 = \eta |T|^2 \quad . \tag{2.102}$$

Now, we have all we need to compare the magnitudes of the incident and reflected z-currents:

$$\left| \frac{(j^r)^z}{(j^i)^z} \right| = \frac{|R_+|^2 + |R_-|^2}{|I_+|^2 + |I_-|^2} = 1 - \eta \left(\frac{|T|^2}{|I_+|^2 + |I_-|^2} \right) \quad , \tag{2.103}$$

where we used Equation (2.102) in the last equality. But, since $\eta < 0$, we have $|(j^r)^z| > |(j^i)^z|$. Then, for modes

$$\omega < e\,\widetilde{V} + \lambda |T|^2 \quad , \tag{2.104}$$

we have solutions that exhibit superradiant amplification.

At this moment, we accomplished the main objective of this thesis. We gave an example of a fermionic condensate that exhibits superradiance. In principle, there are many other theories (maybe, more realistic than this one) that admit a similar behaviour. For example, the same theory that we used but with a negative coupling constant $\widetilde{\lambda} < 0$ also has a superradiant regime.

In the following chapter, we generalize what we have done in this chapter for a charged BH background.

3 Superradiance on black hole backgrounds

In this chapter, we show that all the qualitative conclusions about the existence of superradiance that we obtained in the last chapter are preserved to the case of a charged BH. In fact, it is known that, in the more general case of rotating charged BHs, scalar fields do exhibit superradiance and Dirac fields do not [27, 8]. Moreover, we believe that the non-linear Dirac theory which we introduced in the last chapter also has a superradiant regime in that more general background.

It is well known that static, charged BHs are described by the so-called Reissner-Nordström (RN) geometry. In spherical coordinates, for $r > r_+$, the RN geometry is represented by the squared line element

$$ds^2 = f\, dt^2 - f^{-1}\, dr^2 - r^2(\, d\theta^2 + \sin^2\theta\, d\varphi^2) \quad . \tag{3.1}$$

Here,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad , \tag{3.2}$$

where M and Q are the mass and electric charge of the BH, respectively. In these coordinates, there is an event horizon radius at

$$r = r_+ = M + \sqrt{M^2 - Q^2} \quad . \tag{3.3}$$

Furthermore, the charge Q sources a spherically symmetric electromagnetic field

$$A_\mu = (\, V(r), \vec{0}\,) \qquad \text{with} \qquad V(r) = \frac{Q}{r} \quad . \tag{3.4}$$

As in the last chapter, we use the test field approximation. Then, we ignore the back-reaction of the fields on the geometry of the space-time. Although we use charged fields, we also ignore the electromagnetic field produced by them. These approximations are justified by the fact that these effects are of second order on the charged fields and, so, for sufficiently weak (small amplitude) fields, these effects can be neglected. Moreover, in astrophysical relevant setups, the electromagnetic field produced by this kind of charged fields have negligible effect on the geometry [43].

3.1 Scattering of scalar fields

Let us start with the scalar field theory

$$S_{scalar} = \int dx^4 \sqrt{-g} \left[g_{\mu\nu} D^\nu \phi (D^\mu \phi)^* - m^2 |\phi|^2 \right] \quad , \tag{3.5}$$

with $D_\mu = \nabla_\mu + iqA_\mu$ and all the other quantities defined as in section (2.1). From this action, we obtain the field equation

$$D_\mu D^\mu \phi + m^2 \phi = 0 \quad , \tag{3.6}$$

and its complex conjugate.

With a little algebra, it is straightforward to show that Equation (3.6) reads

$$\begin{aligned} \partial_t^2 \phi + 2iq\frac{Q}{r}\partial_t \phi - 2\frac{f^2}{r}\partial_r \phi - \cot\theta \frac{f}{r^2}\partial_\theta \phi - f f' \partial_r \phi - \\ - f^2 \partial_r^2 \phi - \frac{f}{r^2}\left(\partial_\theta^2 \phi + \frac{1}{\sin^2\theta}\partial_\varphi^2 \phi \right) - q^2 \frac{Q^2}{r^2}\phi + m^2 f \phi = 0 \quad . \end{aligned} \tag{3.7}$$

Let us define the operator

$$L^2 = -\Delta_{S^2} = -\partial_\theta^2 - \cot\theta\, \partial_\theta - \frac{1}{\sin^2\theta}\, \partial_\varphi^2 \quad , \tag{3.8}$$

where Δ_{S^2} is the laplacian in spherical coordinates.

Now, we consider the ansatz

$$\phi(t, r, \theta, \varphi) = \sum_{l, m_l} e^{-i\omega t}\, Y_l^{m_l}(\theta, \varphi) \frac{\psi(r)}{r} \quad , \tag{3.9}$$

where the quasi-monochromatic approximation introduced in the beginning of the previous chapter is being used. Although ψ is a function of l and m_l, we omit this dependence in our notation for the sake of simplicity.

Using the property of spherical harmonics:

$$L^2 Y_l^{m_l} = l(l+1) Y_l^{m_l} \quad , \tag{3.10}$$

we substitute the ansatz (3.9) in Equation (3.7) to give

$$f^2 \frac{d^2}{dr^2}\psi + f f' \frac{d}{dr}\psi - \left[\frac{f f'}{r} + \frac{f}{r^2}\, l(l+1) + m^2 f - (\omega - qV)^2 \right] \psi = 0 \quad . \tag{3.11}$$

Introducing the so-called tortoise coordinate r_* defined by

$$\frac{d\,r}{dr_*} = f \quad , \tag{3.12}$$

Equation (3.11) transforms into

$$\frac{d^2}{dr_*^2}\psi - \left(\frac{f}{r^2}\,l(l+1) + \frac{ff'}{r} + m^2 f - (\omega - qV)^2\right)\psi = 0 \quad . \tag{3.13}$$

One can ask why we need this tortoise coordinate. In fact, the main reason to use this coordinate is to obtain a second order differential field equation without the first order derivative of ψ. In this way, as we will show later, there is a simple conserved current along r_*, which allows us to relate the asymptotic solutions of Equation (3.7) at $r_* \to -\infty$ and $r_* \to +\infty$.

We remark that we can write f as

$$f = \frac{1}{r^2}(r - r_+)(r - r_-) \quad , \tag{3.14}$$

with

$$r_- = M - \sqrt{M^2 - Q^2} \leq r_+ \quad . \tag{3.15}$$

Then, we can integrate the inverse of Equation (3.12) to obtain

$$r_* = C + r - \frac{r_-^2}{r_+ - r_-}\log(r - r_-) + \frac{r_+^2}{r_+ - r_-}\log(r - r_+) \quad , \tag{3.16}$$

where C is a real constant. So, we see that the region $r_+ < r < +\infty$ under analysis is mapped to $-\infty < r_* < +\infty$ in the new coordinate.

Now, we want to study the asymptotic behaviour of ϕ in the region I ($r_* \to +\infty \Leftrightarrow r \to +\infty$) and region II ($r_* \to -\infty \Leftrightarrow r \to r_+$).

Region I: $\boldsymbol{r_* \to +\infty \Leftrightarrow r \to +\infty}$

In this region, Equation (3.13) reads

$$\frac{d^2}{dr_*^2}\psi_I + \left(\omega^2 - m^2\right)\psi_I = 0 \quad , \tag{3.17}$$

where we made use of the fact that the limits of f and f' when $r \to +\infty$ are finite. This differential equation has the general solution

$$\psi_I(r) = A\,e^{ikr_*} + B\,e^{-ikr_*} = A\,e^{ikr} + B\,e^{-ikr} \quad , \tag{3.18}$$

with

$$k^2 = \omega^2 - m^2 \quad . \tag{3.19}$$

The analysis of this region is analogue to the one of the region I of section (2.1), with the substitution $z \to -r_*$. The reason for the minus sign is that while in the Klein paradox we considered incident waves coming from $z = -\infty$ to $z = +\infty$, here, the incident waves are coming from $r_* = +\infty$ to $r_* = -\infty$. With this sign we transform waves with positive (negative) group velocities along z to waves with negative (positive) group velocities along r_*.

Since we must have waves coming from $r = +\infty$, the condition $\omega \in X$ must hold with

$$X = \{\omega \,|\, \omega^2 - m^2 > 0\} \quad . \tag{3.20}$$

Then, following the procedure that we have done in the region I of section (2.1), we obtain the incident wave solution of Equation (3.7):

$$\phi_I^i = \sum_{l,m_l} \frac{I}{r} Y_l^{m_l} \, e^{-i[\omega t + k(\omega) r_*]} = \sum_{l,m_l} \frac{I}{r} Y_l^{m_l} \, e^{-i[\omega t + k(\omega) r]} \quad , \tag{3.21}$$

and the reflected wave solution

$$\phi_I^r = \sum_{l,m_l} \frac{R}{r} Y_l^{m_l} \, e^{-i[\omega t - k(\omega) r_*]} = \sum_{l,m_l} \frac{R}{r} Y_l^{m_l} \, e^{-i[\omega t - k(\omega) r]} \quad , \tag{3.22}$$

where

$$k(\omega) = \epsilon \sqrt{\omega^2 - m^2} \quad , \tag{3.23}$$

with $\epsilon = \mathrm{sign}(\omega + m)$. The I and R appearing in the above solutions are complex functions of l and m_l, but, for the sake of simplicity, we do not represent this, explicitly, in our notation.

With the sign chosen for k, the incident and reflected waves have negative and positive group velocities, respectively, satisfying the boundary conditions of the problem.

Region II: $\boldsymbol{r_* \to -\infty \Leftrightarrow r \to r_+}$

In this another region, Equation (3.13) reads

$$\frac{d^2}{dr_*^2}\psi_{II} + (\omega - qV_+)^2 \psi_{II} = 0 \quad , \tag{3.24}$$

where

$$V_+ = \frac{Q}{r_+} \quad . \tag{3.25}$$

and we used that $f(r_+) = 0$ and $f'(r_+)$ is finite. This differential equation has the general solution

$$\begin{aligned} \psi_{II}(r) &= A\, e^{isr_*} + B\, e^{-isr_*} \\ &= A \exp\left(i\, s \frac{r_+^2}{r_+ - r_-} \log(r - r_+)\right) + B \exp\left(-i\, s \frac{r_+^2}{r_+ - r_-} \log(r - r_+)\right) \quad , \end{aligned} \tag{3.26}$$

with

$$s^2 = (\omega - qV_+)^2 \quad , \tag{3.27}$$

where we used that

$$r_* = \frac{r_+^2}{r_+ - r_-} \log(r - r_+) \quad . \tag{3.28}$$

Since s is real, we see that we only have wave solutions (with the radial coordinate r_*) for the above differential equation. Then, in this region, the general solution of Equation (3.13) is the sum of two travelling waves with symmetric group velocities. The group velocity of the wave associated with the term proportional to e^{isr_*} in (3.26) is

$$v_g = \frac{s}{\omega - qV_+} \quad , \tag{3.29}$$

where we used Equation (3.27) and the definition of group velocity (2.7).

Then, the transmitted wave can be written as

$$\phi^t_{II} = \sum_{l,m_l} \frac{T}{r_+} Y_l^{m_l}\, e^{-i[\omega t + s(\omega) r_*]} = \sum_{l,m_l} \frac{T}{r_+} Y_l^{m_l} \exp\left(-i\left[\omega t + s(\omega)\frac{r_+^2}{r_+ - r_-}\log(r - r_+)\right]\right) \quad , \tag{3.30}$$

where

$$s(\omega) = \widetilde{\epsilon}\,|\omega - qV_+| = \omega - qV_+ \quad , \tag{3.31}$$

with $\widetilde{\epsilon} = \mathrm{sign}(\omega - qV_+)$. The T in the above solution is a complex function of l and m_l, but, for the sake of simplicity, we do not represent this explicitly in our notation.

It is easy to see that, with this sign for s, the transmitted waves have negative group velocity along r_* and, then, are entering the event horizon. So, our physical boundary conditions are satisfied.

Conserved currents

As we said previously, we introduced the tortoise coordinate in such a way that a simple conserved current exists. This current is given by

$$j_* = -\frac{i}{2}\left(\psi^* \frac{d}{dr_*}\psi - \psi \frac{d}{dr_*}\psi^*\right) \quad , \tag{3.32}$$

for each l and m_l.

It is easily seen that this current is conserved along r_*:

$$\begin{aligned} \frac{d}{dr_*} j_* = -\frac{i}{2}\left(\psi^* \frac{d^2}{dr_*^2}\psi - \psi \frac{d^2}{dr_*^2}\psi^*\right) = -\frac{i}{2}\left[\left(\frac{f}{r^2}\,l(l+1) + \frac{ff'}{r} + m^2 f - (\omega - qV)^2\right) - \right. \\ \left. - \left(\frac{f}{r^2}\,l(l+1) + \frac{ff'}{r} + m^2 f - (\omega - qV)^2\right)\right] |\psi|^2 = 0 \quad , \end{aligned} \tag{3.33}$$

where we used Equation (3.13) and its complex conjugate.

With $\psi_I = \psi_I^i + \psi_I^r$ where $\psi_I^i = I\, e^{-i[\omega t + k(\omega) r_*]}$ and $\psi_I^r = R\, e^{-i[\omega t - k(\omega) r_*]}$, it is very easy to show that

$$(j_*)_I = (j_*)_I^i + (j_*)_I^r \quad . \tag{3.34}$$

Moreover, using $\psi_{II} = \psi_{II}^t$ with $\psi_{II}^t = T\, e^{-i[\omega t + s(\omega) r_*]}$, it is obvious that

$$(j_*)_{II} = (j_*)_{II}^t \quad . \tag{3.35}$$

Since the current is conserved along r_*, the relation

$$(j_*)_I^i + (j_*)_I^r = (j_*)_{II}^t \quad , \tag{3.36}$$

holds.

So, using the incident current

$$(j_*)^i_I = -\frac{i}{2}\left[(\psi^i_I)^*\frac{d}{dr_*}\psi^i_I - \psi^i_I\left(\frac{d}{dr_*}\psi^i_I\right)^*\right] = -|I|^2 k \quad , \tag{3.37}$$

the reflected current

$$(j_*)^r_I = -\frac{i}{2}\left[(\psi^r_I)^*\frac{d}{dr_*}\psi^r_I - \psi^r_I\left(\frac{d}{dr_*}\psi^r_I\right)^*\right] = |R|^2 k \quad , \tag{3.38}$$

and the transmitted current

$$(j_*)^t_{II} = -\frac{i}{2}\left[(\psi^t_{II})^*\frac{d}{dr_*}\psi^t_{II} - \psi^t_{II}\left(\frac{d}{dr_*}\psi^t_{II}\right)^*\right] = -|T|^2 s \quad , \tag{3.39}$$

from Equation (3.36), we obtain

$$|I|^2 - |R|^2 = \frac{s}{k}|T|^2 \quad . \tag{3.40}$$

We remark that this current that we introduced above is not a physical current and should, instead, be interpreted as a mathematical conserved quantity that allows us to relate the asymptotic solutions of Equation (3.7) in regions I and II.

To analyse the phenomenon of superradiance we need a physical current. Here, we use the particle-number current which is given by

$$j^\mu = \frac{i}{2}\left[\phi^* D^\mu\phi - \phi\,(D^\mu\phi)^*\right] \quad . \tag{3.41}$$

This current is the Noether's conserved current associated with the U(1) symmetry of the scalar field ϕ.

It is easily seen that this current is covariantly conserved:

$$\begin{aligned}\nabla_\mu\, j^\mu &= \frac{i}{2}[\phi^*\nabla_\mu D^\mu\phi - \phi\,\nabla_\mu(D^\mu\phi)^* + iq\phi A^\mu\nabla_\mu\phi^* + iq\phi^* A^\mu\nabla_\mu\phi \\ &= \frac{i}{2}[\phi^* D_\mu D^\mu\phi - \phi\,(D_\mu D^\mu\phi)^*] = \frac{i}{2}[-m^2|\phi|^2 + m^2|\phi|^2] = 0 \quad ,\end{aligned} \tag{3.42}$$

where we used Equation (3.6) and its complex conjugate.

The flux $\mathcal{F}$ of the particle-number current flowing out of a spherical surface of radius r (denoted by S_r) with $r \to +\infty$ is given by

$$\mathcal{F} = \lim_{r\to+\infty}\int_{S_r} d\Omega\, r^2 j^r \quad , \tag{3.43}$$

because the RN metric is asymptotically flat.

Moreover, the incident particle-number current is given by

$$\begin{aligned}(j^r)^i_I &= \frac{i}{2}\left[(\phi^i_I)^* D^r\phi^i_I - \phi^i_I\,(D^r\phi^i_I)^*\right] = -\frac{i}{2}\left[(\phi^i_I)^*\partial_r\phi^i_I - \phi^i_I\,\partial_r(\phi^i_I)^*\right] \\ &= -\frac{k}{r^2}\sum_{l,m_l}\sum_{l',m_{l'}} I^* I'(Y^{m_l}_l)^* Y^{m_{l'}}_{l'} \quad ,\end{aligned} \tag{3.44}$$

and the reflected current is

$$
\begin{aligned}
(j^r)_I^i &= \frac{i}{2}\left[(\phi_I^r)^* D^r \phi_I^r - \phi_I^r\,(D^r\phi_I^r)^*\right] = -\frac{i}{2}\left[(\phi_I^r)^*\partial_r\phi_I^r - \phi_I^r\,\partial_r(\phi_I^r)^*\right] = \\
&= -\frac{k}{r^2}\sum_{l,m_l}\sum_{l',m_{l'}} R^* R' (Y_l^{m_l})^* Y_{l'}^{m_{l'}} \quad ,
\end{aligned}
\tag{3.45}
$$

where we used again that the RN metric is asymptotically flat.

Now, by the orthonormality relations of the spherical harmonics, the flux of the incident current ($\mathcal{F}^i$) is given by

$$
\mathcal{F}^i = -k\sum_{l,m_l} |I|^2 \quad , \tag{3.46}
$$

and the flux of the reflected current ($\mathcal{F}^r$) reads

$$
\mathcal{F}^r = k\sum_{l,m_l} |R|^2 \quad . \tag{3.47}
$$

We consider the quantity

$$
\left|\frac{\mathcal{F}^r}{\mathcal{F}^i}\right| = \frac{\sum_{l,m_l}|R|^2}{\sum_{l,m_l}|I|^2} = 1 - \frac{s}{k}\frac{\sum_{l,m_l}|T|^2}{\sum_{l,m_l}|I|^2} \quad , \tag{3.48}
$$

where we used Equation (3.40) in the last equality.

By definition, superradiance is present when the absolute value of the flux of the reflected current at $r = +\infty$ is larger than the incident one or, equivalently, $|\mathcal{F}^r| > |\mathcal{F}^i|$. Then, the superradiant regime is such that

$$
0 > \frac{s}{k} = \epsilon\,\widetilde{\epsilon}\,\frac{|\omega - qV_+|}{\sqrt{\omega^2 - m^2}} \quad . \tag{3.49}
$$

This implies that superradiance is present when $\epsilon\,\widetilde{\epsilon} < 0$ and, so, for ω such that:

$$
m < \omega < qV_+ \quad . \tag{3.50}
$$

These superradiant modes are equal to the ones obtained in Refs. [6, 27].

3.2 Scattering of Dirac fields

We start this section by giving a very brief introduction to the Dirac equation in curved spacetime following Ref. [44]. Along this section, we use Latin indexes for locally inertial coordinates and Greek indexes for general coordinate systems. Furthermore, we use a tilde when we want to reinforce that some object is being taken relatively to the locally inertial coordinates.

Dirac equation in curved spacetime

We begin by choosing a set of locally inertial coordinates $\{\widetilde{x}^a_X\}$ at every spacetime point X. Then, the metric in any general coordinate system is

$$g_{\mu\nu} = e^a_{\ \mu}(x)e^b_{\ \nu}(x)\,\eta_{ab} \quad , \tag{3.51}$$

where

$$e^a_{\ \mu}(X) = \left(\frac{\partial\,\widetilde{x}^a_X(x)}{\partial x^\mu}\right)_{x=X} \quad , \tag{3.52}$$

is the so-called tetrad at the point X.

Moreover, if we change the general coordinate system from x^μ to x'^μ, then $e^a_{\ \mu}$ changes to

$$e'^a_{\ \mu} = \frac{\partial x^\nu}{\partial x'^\mu}\,e^a_{\ \nu} \quad , \tag{3.53}$$

meaning that the tetrad $e^a_{\ \mu}$ forms four covariant vector fields.

Given any contravariant vector field $B^\mu(x)$, it is easily seen that we can use the tetrad to express the components of $B^\mu(x)$ relatively to the local inertial coordinates at x as

$$\widetilde{B}^a = e^a_{\ \mu}B^\mu \quad . \tag{3.54}$$

In fact, it is straightforward to generalize this way of using tetrads to express the components of an arbitrary tensor field relatively to locally inertial coordinates (see Ref. [44] for details). These components behave like scalars under general coordinate transformations, since the set of locally inertial coordinates remains fixed under this transformation.

Now, the Principle of Equivalence requires special relativity to hold locally on locally inertial frames. Then, $\widetilde{B}^a$ behaves like a contravariant vector under local Lorentz transformations $\Lambda^a_{\ b}(x)$ at x:

$$\widetilde{B}'^a = \Lambda^a_{\ b}\,\widetilde{B}^b \quad , \tag{3.55}$$

with the obvious generalisation for the locally inertial components of a general tensor field. Moreover, it is easy to show that

$$e'^a_{\ \mu} = \Lambda^a_{\ b}\,e^b_{\ \mu} \quad , \tag{3.56}$$

under the local Lorentz transformation $\Lambda(x)$.

Inspired by the Principle of Equivalence, one would try to generalize the Dirac equation from flat spacetime to curved spacetime by imposing the equation in local inertial coordinates to be locally equal to the one in flat spacetime:

$$\left(i\gamma^a e_a^{\ \mu}\frac{\partial}{\partial x^\mu} - m\right)\Psi = 0 \quad , \tag{3.57}$$

where γ^a are the usual Dirac matrices.

Nevertheless, it happens that this generalisation is wrong. The reason is that the Dirac equation must satisfy the Principle of Lorentz Covariance in the sense that this equation must have the same form in all local inertial coordinates. As we show now, this generalization of the Dirac equation does not satisfy the Principle of Lorentz Covariance.

The spinor Ψ transforms as

$$\Psi' = S(\Lambda(x))\Psi \quad , \tag{3.58}$$

under the local Lorentz transformation $\Lambda(x)$, where $S(\Lambda)$ is a matrix satisfying

$$S^{-1}(\Lambda)\gamma^a S(\Lambda) = \Lambda^a{}_b \gamma^b \quad . \tag{3.59}$$

Then, under the Lorentz transformation $\Lambda(x)$, Equation (3.57) transforms to

$$\left(i\gamma^a e_a{}^\mu \frac{\partial}{\partial x^\mu} - m\right)\Psi + i\gamma^a e_a{}^\mu S^{-1}(\Lambda(x))\left(\frac{\partial}{\partial x^\mu} S(\Lambda(x))\right)\Psi = 0 \quad , \tag{3.60}$$

where we used Equation (3.59). We see that, because of the last term, this Dirac equation depends on the local inertial coordinates which we are using. But, now, we know what to do to generalize the Dirac equation in the right way.

Instead of the derivative

$$e_a{}^\mu \frac{\partial}{\partial x^\mu}\Psi \quad , \tag{3.61}$$

we can use a generalised derivative

$$e_a{}^\mu \mathcal{D}_\mu \Psi = e_a{}^\mu \left(\frac{\partial}{\partial x^\mu} - \Gamma_\mu\right)\Psi \quad , \tag{3.62}$$

where Γ_μ is a matrix called spin connection which, under the Lorentz transformation $\Lambda(x)$, transforms to

$$\Gamma'_\mu = S(\Lambda(x))\Gamma_\mu S^{-1}(\Lambda(x)) + S^{-1}(\Lambda(x))\left(\frac{\partial}{\partial x^\mu} S(\Lambda(x))\right) \quad . \tag{3.63}$$

With this modification it is very easy to see that the Dirac equation is Lorentz covariant.

Thus, the Dirac equation that is believed to describe spin-1/2 fields in curved spacetime is

$$\left(iG^\mu \mathcal{D}_\mu - m\right)\Psi = 0 \quad , \tag{3.64}$$

where we defined the curved spacetime Dirac matrices $G^\mu = \gamma^a e_a{}^\mu$. Notice that these matrices satisfy the anti commutation relations

$$\{G^\mu, G^\nu\} = \{\gamma^a, \gamma^b\}\, e_a{}^\mu e_b{}^\nu = 2\,\eta^{ab} e_a{}^\mu e_b{}^\nu = 2\,g^{\mu\nu} \quad . \tag{3.65}$$

It is possible to show (see Ref. [44] for details) that, for a spinor Ψ, the spin connection is given by

$$\Gamma_\alpha = -\frac{1}{8}[\gamma^a, \gamma^b]\, g_{\mu\nu}\, e_a{}^\mu \nabla_\alpha e_b{}^\nu + a_\alpha \mathbb{1}_4 \quad , \tag{3.66}$$

where a_α is an arbitrary constant covariant vector and $\mathbb{1}_4$ is the identity matrix.

Dirac equation in RN spacetime

Let us define here the locally orthogonal vector fields:

$$\begin{aligned}
\vec{X}_r &= \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} \quad ,\\
\vec{X}_\theta &= \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z} \quad ,\\
\vec{X}_\varphi &= -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} \quad ,
\end{aligned} \tag{3.67}$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors in Cartesian coordinates.

We use the following tetrad of RN:

$$\begin{aligned}
e_a^{\ t} &= \left(\frac{1}{\sqrt{f}},\, 0,\, 0,\, 0\right) \quad ,\\
e_a^{\ r} &= \left(0,\, \sqrt{f}\sin\theta\cos\varphi,\, \sqrt{f}\sin\theta\sin\varphi,\, \sqrt{f}\cos\theta\right) \quad ,\\
e_a^{\ \theta} &= \left(0,\, \frac{1}{r}\cos\theta\cos\varphi,\, \frac{1}{r}\cos\theta\sin\varphi,\, -\frac{\sin\theta}{r}\right) \quad ,\\
e_a^{\ \varphi} &= \left(0,\, -\frac{1}{r}\frac{\sin\varphi}{\sin\theta},\, \frac{1}{r}\frac{\cos\varphi}{\sin\theta},\, 0\right) \quad ,
\end{aligned} \tag{3.68}$$

and we choose the constant covariant vector:

$$a_\alpha = \left(0,\, -\frac{3\sqrt{f}+6}{2r\sqrt{f}},\, -\frac{9}{4}\cot\theta,\, 0\right) \quad . \tag{3.69}$$

Then, one can show that, in this case, the Dirac equation (3.64) reads

$$\left[\gamma^0\left(\frac{i}{S(r)}\frac{\partial}{\partial t} - \frac{e}{S(r)}V(r)\right) + \gamma^r\left(iS(r)\frac{\partial}{\partial r} + \frac{i}{r}(S(r)-1) + \frac{i}{2}S'(r)\right) + i\gamma^\theta\frac{\partial}{\partial\theta} + i\gamma^\varphi\frac{\partial}{\partial\varphi} - m\right]\Psi = 0 \quad , \tag{3.70}$$

where $S(r) = \sqrt{f(r)}$ and $S'(r)$ is the radial derivative of S. The γ^r, γ^θ and γ^φ appearing in the above equation are defined by

$$\begin{aligned}
\gamma^r &= \vec{\gamma}.\,\vec{X}_r = \gamma^1\sin\theta\,\cos\varphi + \gamma^2\sin\theta\,\sin\varphi + \gamma^3\cos\theta \quad ,\\
\gamma^\theta &= \vec{\gamma}.\,\vec{X}_\theta = \frac{1}{r}\left(\gamma^1\cos\theta\,\cos\varphi + \gamma^2\cos\theta\,\sin\varphi - \gamma^3\sin\theta\right) \quad ,\\
\gamma^\varphi &= \vec{\gamma}.\,\vec{X}_\varphi = \frac{1}{r\sin\theta}\left(-\gamma^1\sin\varphi + \gamma^2\cos\varphi\right) \quad ,
\end{aligned} \tag{3.71}$$

with

$$\vec{\gamma} = \gamma^1\hat{x} + \gamma^2\hat{y} + \gamma^3\hat{z} \quad . \tag{3.72}$$

Moreover, we used Dirac fields minimally coupled with the electromagnetic field A_μ of (3.4). So, we made the substitution

$$\partial_\mu \to \partial_\mu + ieA_\mu \quad , \tag{3.73}$$

in the original equation.

We point out that in Ref. [45] they obtain the same Dirac equation (3.70) by starting from a similar tetrad of RN.

Now we use a procedure that explores the spherically symmetry of the problem to separate Equation (3.70). This procedure is the same that is used in Ref. [45].

Separation of the Dirac equation in RN spacetime

To separate the equation we start by defining the angular momentum operator $\vec{L}$ and some other related operators:

$$\begin{aligned} \vec{L} &= -i(\vec{r} \times \vec{\nabla}) \quad , \\ L_{\pm} &= L_x \pm L_y \quad , \\ L^2 &= -\Delta_{S^2} = L_+ L_- + L_z^2 - L_z = L_- L_+ + L_z^2 + L_z \quad , \end{aligned} \tag{3.74}$$

with Δ_{S^2} the laplacian in spherical coordinates. These operators act on the spherical harmonics $Y_l^k(\theta, \varphi)$ as

$$\begin{aligned} L^2 Y_l^k &= l(l+1) Y_l^k \quad , \\ L_z Y_l^k &= k\, Y_l^k \quad , \\ L_{\pm} &= \sqrt{l(l+1) - k(k \pm 1)}\, Y_l^{k \pm 1} \quad , \end{aligned} \tag{3.75}$$

so, the Y_l^k are simultaneous eigenfunctions of L_z and L^2. Furthermore, the spherical harmonics form a basis for square integrable functions over S^2, in the sense that every square integrable function over S^2 can be written as a linear combination of spherical harmonics. Here, S^2 denotes a 2-dimensional spherical surface of unit radius.

Let us introduce the matrices

$$\begin{aligned} \sigma^r &= \vec{\sigma}.\vec{X}_r = \sigma^1 \sin\theta \, \cos\varphi + \sigma^2 \sin\theta \, \sin\varphi + \sigma^3 \cos\theta \quad , \\ \sigma^\theta &= \vec{\sigma}.\vec{X}_\theta = \frac{1}{r} \left(\sigma^1 \cos\theta \, \cos\varphi + \sigma^2 \cos\theta \, \sin\varphi - \sigma^3 \sin\theta \right) \quad , \\ \sigma^\varphi &= \vec{\sigma}.\vec{X}_\varphi = \frac{1}{r \sin\theta} \left(-\sigma^1 \sin\varphi + \sigma^2 \cos\varphi \right) \quad , \end{aligned} \tag{3.76}$$

where the σ^1, σ^2 and σ^3 are the Pauli matrices. With a little algebra, it is easy to show the properties

$$\begin{aligned} \sigma^r \sigma^\theta &= i \, \sin\theta \, \sigma^\varphi \quad , \\ \sigma^r \sigma^\varphi &= -\frac{i}{\sin\theta} \, \sigma^\theta \quad , \end{aligned} \tag{3.77}$$

and

$$\begin{aligned} (\sigma^r)^2 &= \mathbb{1} \quad , \\ (\sigma^r)^\dagger &= \sigma^r \quad . \end{aligned} \tag{3.78}$$

It is also useful to notice that

$$\vec{r} \times \vec{\nabla} = -r \left(\frac{\vec{X}_\theta}{\sin\theta} \frac{\partial}{\partial\varphi} - \sin\theta\, \vec{X}_\varphi \frac{\partial}{\partial\theta} \right) \quad . \tag{3.79}$$

Then, using properties (3.77) and (3.79), we obtain the relation

$$\begin{aligned} \sigma^r \vec{\sigma}.\, (\vec{r} \times \vec{\nabla}) &= -r\, \sigma^r \left(\frac{\sigma^\theta}{\sin\theta} \frac{\partial}{\partial\varphi} - \sin\theta\, \sigma^\varphi \frac{\partial}{\partial\theta} \right) \\ &= -i\, r \left(\sigma^\theta \frac{\partial}{\partial\theta} + \sigma^\varphi \frac{\partial}{\partial\varphi} \right) \quad , \end{aligned} \tag{3.80}$$

and, finally,

$$\begin{aligned} \sigma^\theta \frac{\partial}{\partial\theta} + \sigma^\varphi \frac{\partial}{\partial\varphi} &= -\frac{\sigma^r}{r} (\vec{\sigma}.\, \vec{L}) \\ &= -\frac{\sigma^r}{r} (K - 1) \quad , \end{aligned} \tag{3.81}$$

with the operator K defined by

$$K = (\vec{\sigma}.\, \vec{L}) + 1 \quad . \tag{3.82}$$

Let us now define the so-called spinor spherical harmonics [46]:

$$\chi^k_{j-1/2}(\theta,\varphi) = \sqrt{\frac{j+k}{2j}}\, Y^{k-1/2}_{j-1/2}(\theta,\varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{j-k}{2j}}\, Y^{k+1/2}_{j-1/2}(\theta,\varphi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \tag{3.83}$$

$$\chi^k_{j+1/2}(\theta,\varphi) = \sqrt{\frac{j+1-k}{2j+2}}\, Y^{k-1/2}_{j+1/2}(\theta,\varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sqrt{\frac{j+1+k}{2j+2}}\, Y^{k+1/2}_{j+1/2}(\theta,\varphi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \tag{3.84}$$

$$j = 1/2,\ 3/2,\ 5/2,\ (...)\,.$$

$$k = -j,\ -j+1,\ (...),\ j-1,\ j\,.$$

From the fact that the spherical harmonics form a basis for squared integrable functions over S^2, it is easily seen that these spinors form a basis for squared integrable functions over S^2 for each of the two components of the spinor. Furthermore, these spinors are orthonormal in the sense that

$$\begin{aligned} &\int_{S^2} (\chi^k_{j\mp1/2})^\dagger \chi^{k'}_{j'\mp1/2}\, d\Omega = \delta_{jj'} \delta^{kk'} \quad , \\ &\int_{S^2} (\chi^k_{j\mp1/2})^\dagger \chi^{k'}_{j'\pm1/2}\, d\Omega = 0 \quad . \end{aligned} \tag{3.85}$$

With a little algebra we can show that

$$J^2\, \chi^k_{j\mp1/2} = \left(\vec{L} + \frac{1}{2}\vec{\sigma} \right)^2 \chi^k_{j\mp1/2} = j\,(j+1)\, \chi^k_{j\mp1/2} \quad , \tag{3.86}$$

$$J_z\, \chi^k_{j\mp1/2} = \left(L_z + \frac{1}{2}\sigma^3 \right) \chi^k_{j\mp1/2} = k\, \chi^k_{j\mp1/2} \quad , \tag{3.87}$$

$$K\, \chi^k_{j\mp1/2} = \Big((\vec{\sigma}.\vec{L}) + 1 \Big)\, \chi^k_{j\mp1/2} = \pm \left(j + \frac{1}{2} \right) \chi^k_{j\mp1/2} \quad , \tag{3.88}$$

$$K\,\sigma^r\,\chi^k_{j-1/2} = -\left(j+\frac{1}{2}\right)\sigma^r\,\chi^k_{j-1/2} \quad , \tag{3.89}$$

so, the $\chi^k_{j\mp1/2}$ are eigenspinors of the square of the total angular momentum, J^2, and of the projection of the total angular momentum along the z-axis, J_z. Furthermore, by Equations (3.88) and (3.89), the $\sigma^r\,\chi^k_{j-1/2}$ must be proportional to $\chi^k_{j+1/2}$. But, by the properties (3.78) and (3.85), the $\sigma^r\,\chi^k_{j-1/2}$ has norm equal to one. This implies

$$\sigma^r\,\chi^k_{j-1/2} = \chi^k_{j+1/2} \quad , \tag{3.90}$$

$$\sigma^r\,\chi^k_{j+1/2} = \chi^k_{j-1/2} \quad , \tag{3.91}$$

where the second equality follows immediately by applying σ^r to the first equality and using the second property of (3.78).

Now we return to the Dirac equation. Let us consider the two ansatzes for the Dirac spinors:

$$\Psi^+_{j\,k\,\omega}(t,r,\theta,\varphi) = e^{-i\omega t}\frac{S^{-1/2}}{r}\begin{pmatrix} \chi^k_{j-1/2}(\theta,\varphi)\;\Phi^+_{j\,k\,\omega\,1}(r) \\ i\,\chi^k_{j+1/2}(\theta,\varphi)\;\Phi^+_{j\,k\,\omega\,2}(r)\end{pmatrix} \quad , \tag{3.92}$$

$$\Psi^-_{j\,k\,\omega}(t,r,\theta,\varphi) = e^{-i\omega t}\frac{S^{-1/2}}{r}\begin{pmatrix} \chi^k_{j+1/2}(\theta,\varphi)\;\Phi^-_{j\,k\,\omega\,1}(r) \\ i\,\chi^k_{j-1/2}(\theta,\varphi)\;\Phi^-_{j\,k\,\omega\,2}(r)\end{pmatrix} \quad , \tag{3.93}$$

with the two-spinors $\Phi^+_{j\,k\,\omega}$ and $\Phi^-_{j\,k\,\omega}$.

Using the definition of the Dirac matrices and the relation (3.81), Equation (3.70) reads

$$\begin{aligned}\Bigg[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}\left(\frac{i}{S}\frac{\partial}{\partial t} - \frac{e}{S}V\right) + \begin{pmatrix} 0 & \sigma^r \\ -\sigma^r & 0\end{pmatrix}\left(iS\frac{\partial}{\partial r} + \frac{i}{r}(S-1) + \frac{i}{2}S'\right) - \\ -\frac{i}{r}\begin{pmatrix} 0 & \sigma^r(K-1) \\ -\sigma^r(K-1) & 0\end{pmatrix} - m\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1}\end{pmatrix}\Bigg]\Psi^\pm_{j\,k\,\omega} = 0 \quad .\end{aligned} \tag{3.94}$$

Now, using the explicit formulas for the two ansatzes (3.92) and (3.93), with the relations (3.90) and (3.91), after a little algebra we obtain the matrix equations

$$S\frac{d}{dr}\Phi^\pm_{j\;k\;\omega} = \left[\frac{\omega - eV}{S}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \pm \frac{2j+1}{2r}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} - m\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right]\Phi^\pm_{j\;k\;\omega} \quad . \tag{3.95}$$

These equations are in accordance with those obtained in Ref. [47]. Furthermore, we can compactify these two equations in one master equation by defining

$$\Phi_{j\,k\,\omega} = \begin{cases} \Phi^-_{-j\,k\,\omega} & j = -1/2,\ -3/2,\ -5/2,\ (...) \\ \Phi^+_{j\,k\,\omega} & j = 1/2,\ 3/2,\ 5/2,\ (...)\end{cases} \quad , \tag{3.96}$$

$$\widetilde{j}(j) = \mathrm{sign}(j)\left(\frac{2|j|+1}{2}\right) \in \mathbb{Z}\setminus\{0\} \quad , \tag{3.97}$$

obtaining

$$S\frac{d}{dr}\Phi_{j\,k\,\omega} = \left[\frac{\omega - eV}{S}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\tilde{j}}{r}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\Phi_{j\,k\,\omega} \quad . \tag{3.98}$$

This is a coupled system of two first order linear ODEs in the components of

$$\Phi_{j\,k\,\omega}(r) = \begin{pmatrix} \Phi_{j\,k\,\omega\,1}(r) \\ \Phi_{j\,k\,\omega\,2}(r) \end{pmatrix} = \begin{pmatrix} F_j(r) \\ G_j(r) \end{pmatrix} \quad , \tag{3.99}$$

where in the last step we relaxed the notation. Explicitly, the coupled system reads

$$S\frac{d}{dr}F_j - \frac{\tilde{j}}{r}F_j = -\left(\frac{\omega - eV}{S} + m\right)G_j \quad , \tag{3.100}$$

$$S\frac{d}{dr}G_j + \frac{\tilde{j}}{r}G_j = \left(\frac{\omega - eV}{S} - m\right)F_j \quad . \tag{3.101}$$

By changing the variable to $u_*(r)$, with

$$\frac{du_*}{dr} = \frac{1 - \frac{eV}{\omega} + \frac{mS}{\omega}}{S^2} \quad , \tag{3.102}$$

after some algebra, we obtain

$$\frac{d^2}{du_*^2}F_j - \left[\frac{d}{du_*}\left(\frac{\tilde{j}}{r}\frac{S}{1 - \frac{eV}{\omega} + \frac{mS}{\omega}}\right) - \omega^2\left(\frac{1 - \frac{eV}{\omega} - \frac{mS}{\omega}}{1 - \frac{eV}{\omega} + \frac{mS}{\omega}}\right) + \frac{\tilde{j}^2}{r^2}\left(\frac{S}{1 - \frac{eV}{\omega} + \frac{mS}{\omega}}\right)^2\right]F_j = 0 \quad . \tag{3.103}$$

Similarly, changing the variable to $v_*(r)$, with

$$\frac{dv_*}{dr} = \frac{1 - \frac{eV}{\omega} - \frac{mS}{\omega}}{S^2} \quad , \tag{3.104}$$

we obtain

$$\frac{d^2}{dv_*^2}G_j + \left[\frac{d}{dv_*}\left(\frac{\tilde{j}}{r}\frac{S}{1 - \frac{eV}{\omega} - \frac{mS}{\omega}}\right) + \omega^2\left(\frac{1 - \frac{eV}{\omega} + \frac{mS}{\omega}}{1 - \frac{eV}{\omega} - \frac{mS}{\omega}}\right) - \frac{\tilde{j}^2}{r^2}\left(\frac{S}{1 - \frac{eV}{\omega} - \frac{mS}{\omega}}\right)^2\right]G_j = 0 \quad . \tag{3.105}$$

With the relation (3.102), it is straightforward to show that

$$\begin{aligned} \frac{d}{du_*}\left(\frac{\tilde{j}}{r}\frac{S}{1 - \frac{eV}{\omega} + \frac{mS}{\omega}}\right) = &-\frac{\tilde{j}}{r^2}\frac{S^3}{\left(1 - \frac{eV}{\omega} + \frac{mS}{\omega}\right)^2} + \frac{\tilde{j}}{r}\frac{S^2}{\left(1 - \frac{eV}{\omega} + \frac{mS}{\omega}\right)^2}\left[\frac{S^{-1}}{r^2}\left(M - \frac{Q^2}{r}\right) - \right. \\ &\left. - \frac{S}{1 - \frac{eV}{\omega} + \frac{mS}{\omega}}\left(\frac{eV}{\omega r} + \frac{mS^{-1}}{\omega r^2}\left(M - \frac{Q^2}{r}\right)\right)\right] \quad . \end{aligned} \tag{3.106}$$

In the same way, with the relation (3.104), we can show that

$$\begin{aligned}\frac{d}{dv_*}\left(\frac{\widetilde{j}}{r}\frac{S}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\right) = -\frac{\widetilde{j}}{r^2}\frac{S^3}{\left(1-\frac{eV}{\omega}-\frac{mS}{\omega}\right)^2} + \frac{\widetilde{j}}{r}\frac{S^2}{\left(1-\frac{eV}{\omega}-\frac{mS}{\omega}\right)^2}\left[\frac{S^{-1}}{r^2}\left(M-\frac{Q^2}{r}\right) - \right. \\ \left. -\frac{S}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\left(\frac{eV}{\omega r}-\frac{mS^{-1}}{\omega r^2}\left(M-\frac{Q^2}{r}\right)\right)\right] \quad .\end{aligned} \tag{3.107}$$

It is important to point out that when we decouple Equations (3.100) and (3.101) we lose some information. So, in addition to Equations (3.103) and (3.105), the F_j's and G_j's must also satisfy Equations (3.100) and (3.101).

Now, we turn our attention to the asymptotic behaviour of F_j and G_j at region I ($r = +\infty$) and their behaviour near the event horizon at region II ($r = r_+$).

Region I: $r = +\infty$

By the fact that

$$\begin{aligned}\lim_{r\to+\infty} S = 1 \quad , \\ \lim_{r\to+\infty} V = 0 \quad ,\end{aligned} \tag{3.108}$$

one can show that the asymptotic limit of Equation (3.103) is

$$\frac{d^2}{du_*^2}(F_j)_I + \omega^2\left(\frac{\omega-m}{\omega+m}\right)(F_j)_I = 0 \quad , \tag{3.109}$$

and that $u_*(r)$ is given by

$$u_*(r) = \frac{\omega+m}{\omega}r \quad . \tag{3.110}$$

The general solution of Equation (3.109) is of the form of

$$\begin{aligned}(F_j)_I = \tilde{e}\,e^{i\omega\sqrt{\frac{\omega-m}{\omega+m}}u_*} + \tilde{f}\,e^{-i\omega\sqrt{\frac{\omega-m}{\omega+m}}u_*} = \tilde{e}\,\exp\left[i\,\mathrm{sign}(\omega+m)\sqrt{\omega^2-m^2}r\right] + \\ + \tilde{f}\,\exp\left[-i\,\mathrm{sign}(\omega+m)\sqrt{\omega^2-m^2}r\right] \quad .\end{aligned} \tag{3.111}$$

Similarly, we get for the limit of Equation (3.105):

$$\frac{d^2}{dv_*^2}(G_j)_I + \omega^2\left(\frac{\omega+m}{\omega-m}\right)(G_j)_I = 0 \quad , \tag{3.112}$$

and for $v_*(r)$:

$$v_*(r) = \frac{\omega-m}{\omega}r \quad . \tag{3.113}$$

The general solution of Equation (3.112) is

$$
\begin{aligned}
(G_j)_I = \tilde{g}\, e^{i\omega\sqrt{\frac{\omega+m}{\omega-m}}v_*} + \tilde{h}\, e^{-i\omega\sqrt{\frac{\omega+m}{\omega-m}}v_*} &= \tilde{g}\,\exp\left[i\,\mathrm{sign}(\omega-m)\sqrt{\omega^2-m^2}r\right] + \\
&+ \tilde{h}\,\exp\left[-i\,\mathrm{sign}(\omega-m)\sqrt{\omega^2-m^2}r\right] \quad .
\end{aligned}
\tag{3.114}
$$

The $\tilde{e}$, $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ appearing in the above solutions are complex-valued constants. In fact, these four constants are not all independent. Since, as we've remarked in the end of the previous section, Equations (3.100) and (3.101) must be satisfied by the solutions (3.111) and (3.114), we can express $\tilde{g}$ and $\tilde{h}$ as functions of $\tilde{e}$ and $\tilde{f}$. So, substituting $(F_j)_I$ and $(G_j)_I$ in these equations, we obtain

$$
\begin{aligned}
&\tilde{e}\,\exp\left[i\,\mathrm{sign}(\omega+m)\sqrt{\omega^2-m^2}r\right] - \tilde{f}\,\exp\left[-i\,\mathrm{sign}(\omega+m)\sqrt{\omega^2-m^2}r\right] \\
&= i\sqrt{\frac{\omega+m}{\omega-m}}\left(\tilde{g}\,\exp\left[i\,\mathrm{sign}(\omega-m)\sqrt{\omega^2-m^2}r\right] + \tilde{h}\,\exp\left[-i\,\mathrm{sign}(\omega-m)\sqrt{\omega^2-m^2}r\right]\right) \quad ,
\end{aligned}
\tag{3.115}
$$

and

$$
\begin{aligned}
&\tilde{e}\,\exp\left[i\,\mathrm{sign}(\omega+m)\sqrt{\omega^2-m^2}r\right] + \tilde{f}\,\exp\left[-i\,\mathrm{sign}(\omega+m)\sqrt{\omega^2-m^2}r\right] \\
&= i\sqrt{\frac{\omega+m}{\omega-m}}\left(\tilde{g}\,\exp\left[i\,\mathrm{sign}(\omega-m)\sqrt{\omega^2-m^2}r\right] - \tilde{h}\,\exp\left[-i\,\mathrm{sign}(\omega-m)\sqrt{\omega^2-m^2}r\right]\right) \quad .
\end{aligned}
\tag{3.116}
$$

Then, we have that

$$
\tilde{g} = -i\sqrt{\frac{\omega-m}{\omega+m}}\,\tilde{e} \quad , \tag{3.117}
$$

$$
\tilde{h} = i\sqrt{\frac{\omega-m}{\omega+m}}\,\tilde{f} \quad . \tag{3.118}
$$

Since we must have waves coming from $r = +\infty$, the solutions of Equation (3.95) satisfy the condition $\omega \in X$ with

$$
X = \{\omega \,|\, \omega^2 - m^2 > 0\} \quad . \tag{3.119}
$$

Now, as we have done in the other cases, we use the group velocity to obtain the incident and reflected solutions of Equation (3.95). The incident wave solution is given by

$$
(\Psi)^i_I = \sum_{j,\,k}\left[I^+\,(\Psi^+_{j\,k})^i + I^-\,(\Psi^-_{j\,k})^i\right] \quad , \tag{3.120}
$$

with

$$
(\Psi^+_{j\,k})^i = \frac{1}{r}\exp\left[-i\left(\omega\,t + \epsilon\sqrt{\omega^2-m^2}\,r\right)\right]\begin{pmatrix} \chi^k_{j-1/2} \\ -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2} \end{pmatrix} \quad , \tag{3.121}
$$

$$
(\Psi^-_{j\,k})^i = \frac{1}{r}\exp\left[-i\left(\omega\,t + \epsilon\sqrt{\omega^2-m^2}\,r\right)\right]\begin{pmatrix} \chi^k_{j+1/2} \\ -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j-1/2} \end{pmatrix} \quad , \tag{3.122}
$$

where $\epsilon = \text{sign}(\omega + m)$. In the same way, the reflected wave solution is given by

$$(\Psi)_I^r = \sum_{j,\,k} \left[R^+ \, (\Psi_{j\,k}^+)^r + R^- \, (\Psi_{j\,k}^-)^r \right] \quad , \tag{3.123}$$

with

$$(\Psi_{j\,k}^+)^r = \frac{1}{r} \exp\left[-i \left(\omega \, t - \epsilon \sqrt{\omega^2 - m^2} \, r \right) \right] \begin{pmatrix} \chi_{j-1/2}^k \\ \sqrt{\frac{\omega - m}{\omega + m}} \, \chi_{j+1/2}^k \end{pmatrix} \quad , \tag{3.124}$$

$$(\Psi_{j\,k}^-)^r = \frac{1}{r} \exp\left[-i \left(\omega \, t - \epsilon \sqrt{\omega^2 - m^2} \, r \right) \right] \begin{pmatrix} \chi_{j+1/2}^k \\ \sqrt{\frac{\omega - m}{\omega + m}} \, \chi_{j-1/2}^k \end{pmatrix} \quad . \tag{3.125}$$

The I^+, I^-, R^+ and R^- in the above solutions are complex-valued function of j and k, but to simplify the notation we do not represent this dependence explicitly.

It is very easy to see that, with the sign function ϵ, the incident and reflected waves have negative and positive group velocities along r, respectively, satisfying the boundary conditions of the problem.

Region II: $r = r_+$

Let us now turn to the behaviour of the Dirac fields near the event horizon $r = r_+$.

Using that

$$\begin{aligned} &\lim_{r \to r_+} S = 0 \quad , \\ &\lim_{r \to r_+} V = V_+ = \frac{Q}{r_+} \quad . \end{aligned} \tag{3.126}$$

we obtain the limit of Equation (3.103):

$$\frac{d^2}{du_*^2} (F_j)_{II} + \omega^2 (F_j)_{II} = 0 \quad , \tag{3.127}$$

and if we integrate relation (3.102) we easily see that

$$u_*(r) = \frac{r_+^2}{\omega \, (r_+ - r_-)} \, (\omega - e \, V_+) \log(r - r_+) \quad . \tag{3.128}$$

The general solution of Equation (3.127) is

$$(F_j)_{II} = \tilde{a} \, e^{i\omega u_*} + \tilde{b} \, e^{-i\omega u_*} = \tilde{a} \, (r - r_+)^{i r_+^2 \left(\frac{\omega - eV_+}{r_+ - r_-} \right)} + \tilde{b} \, (r - r_+)^{-i r_+^2 \left(\frac{\omega - eV_+}{r_+ - r_-} \right)} \quad . \tag{3.129}$$

In the same way, the limit of Equation (3.105) is

$$\frac{d^2}{dv_*^2} (G_j)_{II} + \omega^2 (G_j)_{II} = 0 \quad , \tag{3.130}$$

and, integrating relation (3.104), we obtain

$$v_*(r) = \frac{r_+^2}{\omega\,(r_+ - r_-)}\,(\omega - eV_+)\log(r - r_+) \quad . \tag{3.131}$$

Again, the general solution of (3.130) are of the form

$$(G_j)_{II} = \tilde{c}\,e^{i\omega u_*} + \tilde{d}\,e^{-i\omega u_*} = \tilde{c}\,(r - r_+)^{ir_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} + \tilde{d}\,(r - r_+)^{-ir_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} \quad . \tag{3.132}$$

Now, we can use Equations (3.100) and (3.101) to write $\tilde{c}$ and $\tilde{d}$ as functions of $\tilde{a}$ and $\tilde{b}$. So, substituting the solutions in Equations (3.100) and (3.101), in the limit of $r \to r_+$, we obtain the system of equations:

$$\begin{aligned} -i\left[\tilde{a}\,(r - r_+)^{i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} - \tilde{b}\,(r - r_+)^{-i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)}\right] &= \tilde{c}\,(r - r_+)^{i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} + \\ &\quad + \tilde{d}\,(r - r_+)^{-i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} \quad , \end{aligned} \tag{3.133}$$

$$\begin{aligned} -i\left[\tilde{a}\,(r - r_+)^{i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} + \tilde{b}\,(r - r_+)^{-i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)}\right] &= \tilde{c}\,(r - r_+)^{i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} - \\ &\quad - \tilde{d}\,(r - r_+)^{-i\,r_+^2\left(\frac{\omega - eV_+}{r_+ - r_-}\right)} \quad . \end{aligned} \tag{3.134}$$

This system implies that

$$\tilde{c} = -i\,\tilde{a} \quad , \tag{3.135}$$

$$\tilde{d} = i\,\tilde{b} \quad . \tag{3.136}$$

Let us define the coordinate $r_*(r)$ as

$$r_* = \frac{r_+^2}{r_+ - r_-}\log(r - r_+) \quad . \tag{3.137}$$

Using this radial coordinate r_*, we see that the general solution of Equation (3.95), in this region, is the sum of two travelling waves with symmetric group velocities. Since the transmitted wave must be entering the event horizon, they must have negative group velocity along r_*. Then, the transmitted wave solution is

$$(\Psi)^t_{II} = \sum_{j,\,k}\left[T^+\,(\Psi^+_{j\,k})^t + T^-\,(\Psi^-_{j\,k})^t\right] \quad , \tag{3.138}$$

with

$$(\Psi^+_{j\,k})^t = \frac{1}{\sqrt{r_+}\,(r_+ - r_-)^{\frac{1}{4}}\,(r - r_+)^{\frac{1}{4}}}\exp\left[-i\Big(\omega\,t + (\omega - eV_+)r_*\Big)\right]\begin{pmatrix} \chi^k_{j-1/2} \\ -\chi^k_{j+1/2} \end{pmatrix} \quad , \tag{3.139}$$

$$(\Psi^-_{j\,k})^t = \frac{1}{\sqrt{r_+}\,(r_+ - r_-)^{\frac{1}{4}}\,(r - r_+)^{\frac{1}{4}}}\exp\left[-i\Big(\omega\,t + (\omega - eV_+)r_*\Big)\right]\begin{pmatrix} \chi^k_{j+1/2} \\ -\chi^k_{j-1/2} \end{pmatrix} \quad . \tag{3.140}$$

Again, the T^{+} and T^{-} are complex-valued functions of j and k, but to simplify the notation we do not represent this dependence explicitly. It is easy to see that, with the chosen sign for the wave number, the transmitted wave has negative group velocity along r_* as we wanted.

Conserved currents

Let us introduce the currents

$$J_{F_j} = -\frac{i}{2}\left((F_j)^* \frac{d}{du_*} F_j - F_j \frac{d}{du_*}(F_j)^*\right) \quad , \tag{3.141}$$

and

$$J_{G_j} = -\frac{i}{2}\left((G_j)^* \frac{d}{dv_*} G_j - G_j \frac{d}{dv_*}(G_j)^*\right) \quad , \tag{3.142}$$

for each j and k.

It is very easy to show that these currents are conserved along u_* and v_*, respectively. So:

$$\begin{aligned}
\frac{d}{du_*} J_{F_j} &= -\frac{i}{2}\left((F_j)^* \frac{d^2}{du_*^2} F_j - F_j \frac{d^2}{du_*^2}(F_j)^*\right) \\
&= -\frac{i}{2}\left\{\left[\frac{d}{du_*}\left(\frac{\widetilde{j}}{r}\frac{S}{1-\frac{eV}{\omega}+\frac{mS}{\omega}}\right) - \omega^2\left(\frac{1-\frac{eV}{\omega}-\frac{mS}{\omega}}{1-\frac{eV}{\omega}+\frac{mS}{\omega}}\right) + \frac{\widetilde{j}^2}{r^2}\left(\frac{S}{1-\frac{eV}{\omega}+\frac{mS}{\omega}}\right)^2\right] - \right. \\
&\left. - \left[\frac{d}{du_*}\left(\frac{\widetilde{j}}{r}\frac{S}{1-\frac{eV}{\omega}+\frac{mS}{\omega}}\right) - \omega^2\left(\frac{1-\frac{eV}{\omega}-\frac{mS}{\omega}}{1-\frac{eV}{\omega}+\frac{mS}{\omega}}\right) + \frac{\widetilde{j}^2}{r^2}\left(\frac{S}{1-\frac{eV}{\omega}+\frac{mS}{\omega}}\right)^2\right]\right\}|F_j|^2 = 0 \quad ,
\end{aligned} \tag{3.143}$$

and

$$\begin{aligned}
\frac{d}{dv_*} J_{G_j} &= -\frac{i}{2}\left((G_j)^* \frac{d^2}{dv_*^2} G_j - G_j \frac{d^2}{dv_*^2}(G_j)^*\right) \\
&= -\frac{i}{2}\left\{-\left[\frac{d}{dv_*}\left(\frac{\widetilde{j}}{r}\frac{S}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\right) + \omega^2\left(\frac{1-\frac{eV}{\omega}+\frac{mS}{\omega}}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\right) - \frac{\widetilde{j}^2}{r^2}\left(\frac{S}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\right)^2\right] + \right. \\
&\left. + \left[\frac{d}{dv_*}\left(\frac{\widetilde{j}}{r}\frac{S}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\right) + \omega^2\left(\frac{1-\frac{eV}{\omega}+\frac{mS}{\omega}}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\right) - \frac{\widetilde{j}^2}{r^2}\left(\frac{S}{1-\frac{eV}{\omega}-\frac{mS}{\omega}}\right)^2\right]\right\}|G_j|^2 = 0 \quad ,
\end{aligned} \tag{3.144}$$

where we used Equations (3.103) and (3.105) and their complex conjugates.

We shall point out that these currents are not physical currents. They are mathematical conserved quantities which allow us to relate the asymptotic solutions of Equations (3.103) and (3.105) in regions I and II.

With $(F_j)_I = F_j^i + F_j^r$, where

$$F_j^i = I^\eta \exp\left[-i\,\omega\sqrt{\frac{\omega-m}{\omega+m}}\,u_*\right] \quad , \tag{3.145}$$

$$F_j^r = R^\eta \exp\left[i\,\omega\sqrt{\frac{\omega-m}{\omega+m}}\,u_*\right] \quad , \tag{3.146}$$

and $\eta = \mathrm{sign}(j)$, it is easy to show that

$$(J_{F_j})_I = J^i_{F_j} + J^r_{F_j} \quad . \tag{3.147}$$

Moreover, using $(F_j)_{II} = F^t_j$, with

$$F^t_j = T^\eta e^{-i\tilde{\epsilon}\omega u_*} \quad , \tag{3.148}$$

it is obvious that

$$(J_{F_j})^t = J^t_{F_j} \quad . \tag{3.149}$$

By the fact that this current is conserved along u_*, the relation

$$J^i_{F_j} + J^r_{F_j} = J^t_{F_j} \quad , \tag{3.150}$$

is satisfied.

Then, using that the incident current is

$$J^i_{F_j} = -\frac{i}{2}\left((F^i_j)^* \frac{d}{du_*} F^i_j - F^i_j \frac{d}{du_*}(F^i_j)^*\right) = -\omega\sqrt{\frac{\omega - m}{\omega + m}}\,|I^\eta|^2 \quad , \tag{3.151}$$

the reflected current is

$$J^r_{F_j} = -\frac{i}{2}\left((F^r_j)^* \frac{d}{du_*} F^r_j - F^r_j \frac{d}{du_*}(F^r_j)^*\right) = \omega\sqrt{\frac{\omega - m}{\omega + m}}\,|R^\eta|^2 \quad , \tag{3.152}$$

and the transmitted one is

$$J^t_{F_j} = -\frac{i}{2}\left((F^t_j)^* \frac{d}{du_*} F^t_j - F^t_j \frac{d}{du_*}(F^t_j)^*\right) = -\omega\,|T^\eta|^2 \quad , \tag{3.153}$$

from Equation (3.150), we obtain

$$|I^\eta|^2 - |R^\eta|^2 = \sqrt{\frac{\omega + m}{\omega - m}}\,|T^\eta|^2 \quad . \tag{3.154}$$

In exactly the same way, with $(G_j)_I = G^i_j + G^r_j$, where

$$G^i_j = -i\sqrt{\frac{\omega - m}{\omega + m}}\,I^\eta \exp\left[-i\,\omega\sqrt{\frac{\omega + m}{\omega - m}}\,u_*\right] \quad , \tag{3.155}$$

$$G^r_j = i\sqrt{\frac{\omega - m}{\omega + m}}\,R^\eta \exp\left[i\,\omega\sqrt{\frac{\omega + m}{\omega - m}}\,u_*\right] \quad , \tag{3.156}$$

it is easily shown that

$$(J_{G_j})_I = J^i_{G_j} + J^r_{G_j} \quad . \tag{3.157}$$

Furthermore, using $(G_j)_{II} = G^t_j$, with

$$G^t_j = i\,T^\eta e^{-i\omega u_*} \quad , \tag{3.158}$$

it is very easy to see that

$$(J_{G_j})_{II} = J^t_{G_j} \quad . \tag{3.159}$$

Since this current is conserved along v_*, the relation

$$J^i_{G_j} + J^r_{G_j} = J^t_{G_j} \quad , \tag{3.160}$$

holds.

The incident current is

$$J^i_{G_j} = -\frac{i}{2}\left((G^i_j)^* \frac{d}{dv_*} G^i_j - G^i_j \frac{d}{dv_*}(G^i_j)^*\right) = -\omega\sqrt{\frac{\omega - m}{\omega + m}}\,|I^\eta|^2 \quad , \tag{3.161}$$

the reflected current is

$$J^r_{G_j} = -\frac{i}{2}\left((G^r_j)^* \frac{d}{dv_*} G^r_j - G^r_j \frac{d}{dv_*}(G^r_j)^*\right) = \omega\sqrt{\frac{\omega - m}{\omega + m}}\,|R^\eta|^2 \quad , \tag{3.162}$$

and the transmitted one is

$$J^t_{G_j} = -\frac{i}{2}\left((G^t_j)^* \frac{d}{dv_*} G^t_j - G^t_j \frac{d}{dv_*}(G^t_j)^*\right) = -\omega\,|T^\eta|^2 \quad . \tag{3.163}$$

So, by Equation (3.160):

$$|I^\eta|^2 - |R^\eta|^2 = \sqrt{\frac{\omega + m}{\omega - m}}\,|T^\eta|^2 \quad . \tag{3.164}$$

Since we obtained the same relation as in (3.154), we see that our result is consistent.

Now, as in the scalar field case, to analyse the existence of superradiance we need a physical current. We use the particle-number current which, in this case, is given by

$$J^\mu = \frac{1}{2}\bar{\Psi} G^\mu \Psi \quad . \tag{3.165}$$

This current is the Noether's conserved current associated with the U(1) symmetry of the scalar field Ψ.

One can show that this current is covariantly conserved:

$$\begin{aligned}\nabla_\mu J^\mu = \frac{1}{2}\left[\left(\mathcal{D}_\mu \bar{\Psi}\right) G^\mu \Psi + \bar{\Psi}\, G^\mu \left(\mathcal{D}_\mu \Psi\right)\right] &= \frac{1}{2}\left[\left(\widetilde{\mathcal{D}}^*_\mu \bar{\Psi}\right) G^\mu \Psi + \bar{\Psi}\, G^\mu \left(\widetilde{\mathcal{D}}_\mu \Psi\right)\right] \\ &= \frac{i}{2}(m - m)\bar{\Psi}\Psi = 0 \quad ,\end{aligned} \tag{3.166}$$

where we used the generalization of Equation (3.64) to the case where an electromagnetic field is present and the hermitian conjugate of that equation right multiplied by γ^0. The derivative operator $\widetilde{\mathcal{D}}_\mu$ includes the electromagnetic minimal coupling term. We used also that

$$\nabla_\mu G^\mu = \gamma^a \mathcal{D}_\mu\, {e_a}^\mu = 0 \quad , \tag{3.167}$$

because it can be shown (see Ref. [44] for details) that the compatibility of the affine connection with the metric implies that

$$\mathcal{D}_\mu \, e_a{}^\mu = 0 \quad . \tag{3.168}$$

Now, we proceed in exactly the same way as we did in the scalar field case. So, again, the flux $\mathcal{F}$ of the particle-number current flowing out of a spherical surface of radius r with $r \to \infty$ is

$$\mathcal{F} = \lim_{r\to+\infty} \int_{S_r} d\Omega \, r^2 J^r \quad . \tag{3.169}$$

The incident particle-number current is

$$\begin{aligned}(J^r)^i &= \frac{1}{2}\bar{\Psi}^i_I G^r (\Psi)^i_I = \frac{1}{2}(\Psi^i_I)^\dagger \gamma^0 \gamma^r \Psi^i_I \\ &= \frac{1}{2\,r^2}\sum_{j,\,k}\sum_{j',\,k'}\Bigg[(I^+)^* \left(\begin{array}{cc} (\chi^k_{j-1/2})^\dagger & -\sqrt{\frac{\omega-m}{\omega+m}}\,(\chi^k_{j+1/2})^\dagger \end{array} \right) + \\ &+ (I^-)^* \left(\begin{array}{cc} (\chi^k_{j+1/2})^\dagger & -\sqrt{\frac{\omega-m}{\omega+m}}\,(\chi^k_{j-1/2})^\dagger \end{array} \right)\Bigg]\Bigg[I'^+ \left(\begin{array}{c} -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^{k'}_{j'-1/2} \\ \chi^{k'}_{j'+1/2} \end{array} \right) + \\ &+ I'^- \left(\begin{array}{c} -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^{k'}_{j'+1/2} \\ \chi^{k'}_{j'-1/2} \end{array} \right)\Bigg] \quad ,\end{aligned} \tag{3.170}$$

and the reflected current is given by

$$\begin{aligned}(J^r)^r &= \frac{1}{2}\bar{\Psi}^r_I G^r (\Psi)^r_I = \frac{1}{2}(\Psi^r_I)^\dagger \gamma^0 \gamma^r \Psi^r_I \\ &= \frac{1}{2\,r^2}\sum_{j,\,k}\sum_{j',\,k'}\Bigg[(R^+)^* \left(\begin{array}{cc} (\chi^k_{j-1/2})^\dagger & \sqrt{\frac{\omega-m}{\omega+m}}\,(\chi^k_{j+1/2})^\dagger \end{array} \right) + \\ &+ (R^-)^* \left(\begin{array}{cc} (\chi^k_{j+1/2})^\dagger & \sqrt{\frac{\omega-m}{\omega+m}}\,(\chi^k_{j-1/2})^\dagger \end{array} \right)\Bigg]\Bigg[R'^+ \left(\begin{array}{c} \sqrt{\frac{\omega-m}{\omega+m}}\,\chi^{k'}_{j'-1/2} \\ \chi^{k'}_{j'+1/2} \end{array} \right) + \\ &+ R'^- \left(\begin{array}{c} \sqrt{\frac{\omega-m}{\omega+m}}\,\chi^{k'}_{j'+1/2} \\ \chi^{k'}_{j'-1/2} \end{array} \right)\Bigg] \quad ,\end{aligned} \tag{3.171}$$

where we used that $G^r = \gamma^r$ in region I, because the RN metric is asymptotically flat.

By the orthonormality relation of the spinor spherical harmonics, the flux of the incident current ($\mathcal{F}^i$) is

$$\mathcal{F}^i = -\sqrt{\frac{\omega-m}{\omega+m}}\sum_{j,\,k}(|I^+|^2 + |I^-|^2) \quad , \tag{3.172}$$

and the flux of the reflected current ($\mathcal{F}^r$) reads

$$\mathcal{F}^r = \sqrt{\frac{\omega-m}{\omega+m}}\sum_{j,\,k}(|R^+|^2 + |R^-|^2) \quad . \tag{3.173}$$

Let us consider the quantity

$$\left|\frac{\mathcal{F}^r}{\mathcal{F}^i}\right| = \frac{|R^+|^2 + |R^-|^2}{|I^+|^2 + |I^-|^2} = 1 - \sqrt{\frac{\omega+m}{\omega-m}}\,\frac{|T^+|^2 + |T^-|^2}{|I^+|^2 + |I^-|^2} \quad , \tag{3.174}$$

where we used the relation (3.150) in the last equality.

Superradiant amplification exists when the absolute value of the flux of the reflected current at $r = +\infty$ is larger than the incident one, or, in mathematical terms, $|\mathcal{F}^r| > |\mathcal{F}^i|$. But, as we see from the above quantity that we evaluated, we always have $|\mathcal{F}^r| \leq |\mathcal{F}^i|$. Then, we showed that Dirac fields do not exhibit superradiance in a RN background. This result was know as a limit of the more general Kerr-Newman background [8]. Nevertheless, here we show it in a simpler way, making use of the spherically symmetry of RN to separate the Dirac equation and decoupling the field equations with an appropriate change of variables.

3.3 Scattering of non-linear Dirac fields

In this section we want to consider the scattering of a fermion condensate in RN and show that there are solutions which exhibit superradiance. But, first, we shall point out that the Dirac equation in curved spacetime can be obtained from the action

$$S_{\text{Dirac}} = \int dx^4 \sqrt{-g}\,[\, i\,\bar{\Psi} G^\mu \widetilde{\mathcal{D}}_\mu \Psi - m\bar{\Psi}\Psi\,] \quad , \tag{3.175}$$

with m the mass of the Dirac field and $\widetilde{\mathcal{D}}_\mu = \partial_\mu + ieA_\mu - \Gamma_\mu$, where $e > 0$ is the electric charge of the field and Γ_μ is the spin connection introduced in the last section. So, Equation (3.64) and its hermitian conjugate right multiplied by γ^0 can be obtained by varying the action in $\bar{\Psi}$ and Ψ, respectively.

Now, we want to consider a non-linear Dirac theory of the same kind of the one we considered in the Klein paradox chapter. But, now since we are in curved spacetime, we need to use our generalised derivative operator. Then, we want to study the theory described by the action

$$S = \int dx^4 \sqrt{-g}\,[\, i\,\bar{\Psi} G^\mu \widetilde{\mathcal{D}}_\mu \Psi - m\bar{\Psi}\Psi - \frac{\lambda}{2}(\bar{\Psi}\Psi)^2] \quad , \tag{3.176}$$

with the coupling

$$\lambda(r) = \widetilde{\lambda}\, e^2 A_\mu A^\mu = \widetilde{\lambda}\, e^2\, \frac{Q^2}{r^2} > 0 \quad , \tag{3.177}$$

where $\widetilde{\lambda} > 0$ is a real constant.

With this action and using the same tetrad of RN and constant covariant vector a_α of the last section, we obtain the Dirac equation:

$$\begin{aligned}\Bigg[\gamma^0 \left(\frac{i}{S(r)}\frac{\partial}{\partial t} - \frac{e}{S(r)}V(r)\right) + \gamma^r \left(iS(r)\frac{\partial}{\partial r} + \frac{i}{r}(S(r)-1) + \frac{i}{2}S'(r)\right) + \\ + i\gamma^\theta \frac{\partial}{\partial \theta} + i\gamma^\varphi \frac{\partial}{\partial \varphi} - m - \lambda\bar{\Psi}\Psi\Bigg]\,\Psi = 0 \quad .\end{aligned} \tag{3.178}$$

Since, here, we are more concerned about existence rather than generality, let us consider the ansatz

$$\Psi(t, r, \theta, \varphi) = N e^{-i\omega t} \frac{S^{-1/2}}{r} \begin{pmatrix} \chi^k_{j-1/2}(\theta,\varphi)\ F(r) \\ i\,\chi^k_{j+1/2}(\theta,\varphi)\ G(r) \end{pmatrix} \quad , \tag{3.179}$$

with $j = k = +\frac{1}{2}$. Substituting this ansatz in the Dirac equation (3.178), we obtain

$$S\frac{d}{dr}F - \frac{1}{r}F = -\left(\frac{\omega - eV}{S} + m + \frac{\lambda}{4\pi r^2 S}|N^2|\left(|F|^2 - |G|^2\right)\right)G \quad , \tag{3.180}$$

$$S\frac{d}{dr}G + \frac{1}{r}G = \left(\frac{\omega - eV}{S} - m - \frac{\lambda}{4\pi r^2 S}|N^2|\left(|F|^2 - |G|^2\right)\right)F \quad , \tag{3.181}$$

where we used exactly the same procedure of the last section with the substitution $m \to m + \lambda\bar{\Psi}\Psi$ and the fact that, for $j = k = +\frac{1}{2}$:

$$(\chi^k_{j-1/2})^\dagger \chi^k_{j-1/2} = \frac{1}{4\pi} \quad , \tag{3.182}$$

$$(\chi^k_{j+1/2})^\dagger \chi^k_{j+1/2} = \frac{1}{4\pi} \quad , \tag{3.183}$$

and, so,

$$\bar{\Psi}\Psi = \frac{1}{4\pi r^2 S}|N^2|\left(|F|^2 - |G|^2\right) \quad , \tag{3.184}$$

for the ansatz (3.179).

By changing the variable to $u_*(r)$, with

$$\frac{du_*}{dr} = \frac{1 - \frac{eV}{\omega} + \frac{\widetilde{m}S}{\omega}}{S^2} \quad , \tag{3.185}$$

where

$$\widetilde{m} = m + \frac{\lambda}{4\pi r^2 S}|N^2|\left(|F|^2 - |G|^2\right) \quad , \tag{3.186}$$

we obtain

$$\frac{d^2}{du_*^2}F - \left[\frac{d}{du_*}\left(\frac{1}{r}\frac{S}{1 - \frac{eV}{\omega} + \frac{\widetilde{m}S}{\omega}}\right) - \omega^2\left(\frac{1 - \frac{eV}{\omega} - \frac{\widetilde{m}S}{\omega}}{1 - \frac{eV}{\omega} + \frac{\widetilde{m}S}{\omega}}\right) + \frac{1}{r^2}\left(\frac{S}{1 - \frac{eV}{\omega} + \frac{\widetilde{m}S}{\omega}}\right)^2\right]F = 0 \quad . \tag{3.187}$$

In the same way, if we change the variable to $v_*(r)$, with

$$\frac{dv_*}{dr} = \frac{1 - \frac{eV}{\omega} - \frac{\widetilde{m}S}{\omega}}{S^2} \quad , \tag{3.188}$$

we obtain

$$\frac{d^2}{dv_*^2}G + \left[\frac{d}{dv_*}\left(\frac{1}{r}\frac{S}{1 - \frac{eV}{\omega} - \frac{\widetilde{m}S}{\omega}}\right) + \omega^2\left(\frac{1 - \frac{eV}{\omega} + \frac{\widetilde{m}S}{\omega}}{1 - \frac{eV}{\omega} - \frac{\widetilde{m}S}{\omega}}\right) - \frac{1}{r^2}\left(\frac{S}{1 - \frac{eV}{\omega} - \frac{\widetilde{m}S}{\omega}}\right)^2\right]G = 0 \quad . \tag{3.189}$$

As in the last section, we point out that we loose information when decoupling Equations (3.180) and (3.181). Then, F and G satisfy Equations (3.187) and (3.189), but they must also satisfy Equations (3.180) and (3.181).

Now, we need to study the behaviour of F and G at regions I and II.

Region I: $r = +\infty$

Since in this region the coupling λ vanishes, the treatment of this region follows exactly the same way as the linear Dirac theory case of section (3.70). Then, we have the incident wave solution

$$\Psi^i = \frac{I}{r}\exp\left[-i\left(\omega\, t + \epsilon\sqrt{\omega^2 - m^2}\, r\right)\right]\begin{pmatrix} \chi^k_{j-1/2} \\ -\sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2}\end{pmatrix} \quad , \tag{3.190}$$

and the reflected solution given by

$$\Psi^r = \frac{R}{r}\exp\left[-i\left(\omega\, t - \epsilon\sqrt{\omega^2 - m^2}\, r\right)\right]\begin{pmatrix} \chi^k_{j-1/2} \\ \sqrt{\frac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2}\end{pmatrix} \quad , \tag{3.191}$$

with $\epsilon = \text{sign}(\omega + m)$ and $j = k = +\frac{1}{2}$. The I and R are complex constants.

Region II: $r = r_+$

In this region, we want to search for solutions of the form

$$\Psi^t = T e^{-i\omega t}\frac{S^{-1/2}}{r}\begin{pmatrix} \chi^k_{j-1/2}(\theta,\varphi)\ F^t \\ i\,\chi^k_{j+1/2}(\theta,\varphi)\ G^t\end{pmatrix} \quad , \tag{3.192}$$

with $j = k = +\frac{1}{2}$ and T a complex constant. Furthermore, let us assume that our solution have $|F^t| = 1$ and $G^t = i\,\eta F^t$, with η a real constant. At the end we can check if these assumptions hold. Using that

$$\begin{aligned} &\lim_{r\to r_+} S = \lim_{r\to r_+}\frac{\sqrt{r_+ - r_-}}{r_+}\sqrt{r - r_+} = 0 \quad , \\ &\lim_{r\to r_+} V = V_+ = \frac{Q}{r_+} \quad , \end{aligned} \tag{3.193}$$

we obtain that Equation (3.187), in this region, reads

$$\frac{d^2}{du_*^2}F^t + \omega^2\left(\frac{\omega - e\,V_+ - \bar{\lambda}|T|^2(1-\eta^2)}{\omega - e\,V_+ + \bar{\lambda}|T|^2(1-\eta^2)}\right)F^t = 0 \quad , \tag{3.194}$$

with

$$\bar{\lambda} = \frac{\lambda}{4\pi r_+^2} \quad , \tag{3.195}$$

and if we integrate relation (3.185) we have

$$u_*(r) = \frac{r_+^2}{\omega\,(r_+ - r_-)}\left[\omega - e\,V_+ + \bar{\lambda}|T|^2(1-\eta^2)\right]\log(r - r_+) \quad . \tag{3.196}$$

If we change the coordinate to $r_*(r)$ given by

$$r_*(r) = \frac{r_+^2}{r_+ - r_-}\log(r - r_+) \quad , \tag{3.197}$$

we see that Equation (3.194) reads

$$\frac{d^2}{dr_*^2}F^t + \left[(\omega - e\,V_+)^2 - \bar{\lambda}^2|T|^4(1-\eta^2)^2\right]F^t = 0 \quad . \tag{3.198}$$

The general solution of the above equation is

$$F^t = \widetilde{a}\,e^{isr_*} + \widetilde{b}\,e^{-isr_*} \quad , \tag{3.199}$$

with s satisfying

$$s^2 = (\omega - e\,V_+)^2 - \bar{\lambda}^2|T|^4(1-\eta^2)^2 \quad . \tag{3.200}$$

In the same way, in this region, Equation (3.189) reads

$$\frac{d^2}{du_*^2}G^t + \omega^2\left(\frac{\omega - e\,V_+ + \bar{\lambda}|T|^2(1-\eta^2)}{\omega - e\,V_+ - \bar{\lambda}|T|^2(1-\eta^2)}\right)G^t = 0 \quad , \tag{3.201}$$

and integrating the relation (3.188) we obtain

$$v_*(r) = \frac{r_+^2}{\omega\,(r_+ - r_-)}\left[\omega - e\,V_+ - \bar{\lambda}|T|^2(1-\eta^2)\right]\log(r - r_+) \quad . \tag{3.202}$$

Using the coordinate $r_*(r)$, Equation (3.201) reads

$$\frac{d^2}{dr_*^2}G^t + \left[(\omega - e\,V_+)^2 - \bar{\lambda}^2|T|^4(1-\eta^2)^2\right]G^t = 0 \quad . \tag{3.203}$$

The general solution of this equation is

$$G^t = \widetilde{c}\,e^{isr_*} + \widetilde{d}\,e^{-isr_*} \quad . \tag{3.204}$$

Now, we must use Equations (3.180) and (3.181) to write $\widetilde{c}$ and $\widetilde{d}$ as functions of $\widetilde{a}$ and $\widetilde{b}$. So, substituting the solutions in Equations (3.180) and (3.181), we obtain the system of equations:

$$-i\,\frac{s}{\omega - e\,V_+ + \bar{\lambda}|T|^2(1-\eta^2)}\left(\widetilde{a}\,e^{isr_*} - \widetilde{b}\,e^{-isr_*}\right) = \widetilde{c}\,e^{isr_*} + \widetilde{d}\,e^{-isr_*} \quad , \tag{3.205}$$

$$-i\,\frac{\omega - e\,V_+ - \bar{\lambda}|T|^2(1-\eta^2)}{s}\left(\widetilde{a}\,e^{isr_*} - \widetilde{b}\,e^{-isr_*}\right) = \widetilde{c}\,e^{isr_*} + \widetilde{d}\,e^{-isr_*} \quad . \tag{3.206}$$

Let us assume that s is real. Then, it is easy to see that the two above equations are equivalent and they both say that

$$\widetilde{c} = -i\,\frac{\omega - e\,V_+ - \bar{\lambda}|T|^2(1-\eta^2)}{s}\widetilde{a} \quad , \tag{3.207}$$

$$\widetilde{d} = i\,\frac{\omega - e\,V_+ - \bar{\lambda}|T|^2(1-\eta^2)}{s}\widetilde{b} \quad . \tag{3.208}$$

So, the transmitted solution is

$$\Psi^t = Te^{-i(\omega\, t+s\, r_*)}\frac{1}{\sqrt{r_+}\,(r-r_+)^{\frac{1}{4}}(r_+-r_-)^{\frac{1}{4}}}\begin{pmatrix} \chi^k_{j-1/2}(\theta,\varphi) \\ -\eta\,\chi^k_{j+1/2}(\theta,\varphi)\end{pmatrix} \quad , \tag{3.209}$$

with $j=k=+\frac{1}{2}$. Where η and s satisfy Equation (3.200) and

$$(\omega - e\,V_+) - \bar{\lambda}|T|^2(1-\eta^2) - \eta\, s = 0 \quad . \tag{3.210}$$

Notice that the equations satisfied by η and s are the same as the ones of section (2.3), with the substitutions

$$\begin{aligned} \lambda &\to \bar{\lambda} \quad , \\ \widetilde{V} &\to V_+ \quad . \end{aligned} \tag{3.211}$$

As in that section, it is easy to show that Equations (3.200) and (3.210) imply that

$$(\eta^2+1)s^2 - 2\eta s(\omega - e\,V_+) = 0 \quad . \tag{3.212}$$

One can show that Equations (3.210) and (3.212) admit the particular solution:

$$s = 0 \quad , \tag{3.213}$$

$$\eta = -\sqrt{1-\frac{\omega - e\,V_+}{\bar{\lambda}|T|^2}} < 0 \quad , \tag{3.214}$$

with ω satisfying

$$\omega < e\,V_+ + \bar{\lambda}|T|^2 \quad . \tag{3.215}$$

Conserved currents

Here, we use an alternative method similar to the one of Ref. [7], which makes use of Gauss's theorem to check for the existence of superradiant amplification. It is possible to show that with this method we can obtain exactly the same results that we obtained using the mathematical conserved currents. So, in some sense, it serves as a consistency check to our results.

To analyse suparradiant amplification, we use the same particle-number current of the last section. Again, in this non-linear case, this current is the Noether's conserved current associated with the U(1) symmetry of Ψ.

It is easy to show that this current is covariantly conserved:

$$\begin{aligned} \nabla_\mu J^\mu = \frac{1}{2}\left[\left(\mathcal{D}_\mu\bar{\Psi}\right)G^\mu\Psi + \bar{\Psi}\,G^\mu\left(\mathcal{D}_\mu\Psi\right)\right] &= \frac{1}{2}\left[\left(\widetilde{\mathcal{D}}^*_\mu\bar{\Psi}\right)G^\mu\Psi + \bar{\Psi}\,G^\mu\left(\widetilde{\mathcal{D}}_\mu\Psi\right)\right] \\ &= \frac{i}{2}(m+\lambda\bar{\Psi}\Psi - m - \lambda\bar{\Psi}\Psi)\bar{\Psi}\Psi = 0 \quad , \end{aligned} \tag{3.216}$$

where we used that $\mathcal{D}_\mu G^\mu = 0$ (for the same reasons that we said in the last section) and that λ and $\bar{\Psi}\Psi$

are real. We can see that $\bar{\Psi}\Psi$ is real from the relation (3.184).

Consider now the closed region of spacetime $\mathcal{U}$, delimited by the two constant time slices χ_1 at t and χ_2 at $t+\delta t$, with χ_2 obtained by a time translation of χ_1, and by the two timelike hypersurfaces $S_{r\sim r_+}$ (sphere of radius $r \sim r_+$) and S_∞ (sphere with radius $r \to +\infty$). The unit normal n_μ to the boundary $\partial\mathcal{U}$ points inwards to the BH on $S_{r\sim r_+}$ and outward to infinity on S_∞. On χ_1 the normal n_μ points to the future and on χ_2 to the past. So, we use the Gauss theorem to show

$$\begin{aligned} 0 &= \int_{\mathcal{U}} d^4x\sqrt{g}\,\nabla_\mu J^\mu = \int_{\partial\mathcal{U}} d^3x\sqrt{h}\,J^\mu n_\mu \\ &= \int_{S_{r\sim r_+}} d^3x\sqrt{h}\,J^\mu n_\mu + \int_{S_\infty} d^3x\sqrt{h}\,J^\mu n_\mu + \int_{\chi_1} d^3x\sqrt{h}\,J^\mu n_\mu + \int_{\chi_2} d^3x\sqrt{h}\,J^\mu n_\mu \quad , \end{aligned} \tag{3.217}$$

with $h_{\mu\nu}$ the induced metric on the boundary $\partial\mathcal{U}$. Since the only dependence of Ψ on t is $e^{-i\omega t}$, it is easy to see that the last two terms of the above expression are symmetric and their sum vanishes. So, the above equation implies that

$$\int_{S_{r\sim r_+}} d\Omega\, r_+^2 J^r = \int_{S_\infty} d\Omega\, r_+^2 J^r \quad , \tag{3.218}$$

where we took the derivative in t of Equation (3.217) and we used that, on $S_{r\sim r_+}$:

$$\sqrt{h} = \sqrt{r-r_+}\sqrt{r_+-r_-}\; r_+\sin\theta \quad , \tag{3.219}$$

$$n_\mu = \frac{r_+}{\sqrt{r-r_+}\sqrt{r_+-r_-}}\,\delta^r_\mu \quad , \tag{3.220}$$

and, on S_∞, we have

$$\sqrt{h} = r_+^2\sin\theta \quad , \tag{3.221}$$

$$n_\mu = \delta^r_\mu \quad . \tag{3.222}$$

By the orthonormality conditions of the spinor harmonics, it is very easy to show that

$$\mathcal{F}_\infty = \int_{S_\infty} d\Omega\, r_+^2 J^r = \int_{S_\infty} d\Omega\, r_+^2\left[(J^r)^i + (J^r)^r\right] = \mathcal{F}^i + \mathcal{F}^r \quad , \tag{3.223}$$

where $(J^r)^i$ and $(J^r)^r$ are the radial currents associated with Ψ^i and Ψ^r, respectively. We define also the transmitted flux:

$$\mathcal{F}_{r\sim r_+} = \int_{S_{r\sim r_+}} d\Omega\, r_+^2 J^r = \int_{S_{r\sim r_+}} d\Omega\, r_+^2 (J^r)^t = \mathcal{F}^t \quad , \tag{3.224}$$

where $(J^r)^t$ is the radial current associated with Ψ^t.

Then, by the relation (3.218), we have that

$$\mathcal{F}^i + \mathcal{F}^r = \mathcal{F}^t \quad . \tag{3.225}$$

The incident and reflected particle-number currents are equal to the ones of the last section, because the non-linear coupling vanishes in that region. So, these currents are given by

$$
\begin{aligned}
(J^r)^i &= \frac{1}{2}\bar{\Psi}^i G^r(\Psi)^i \\
&= \frac{1}{2\,r^2}\Bigg[(I^+)^*\left(\begin{array}{cc}(\chi^k_{j-1/2})^\dagger & -\sqrt{\tfrac{\omega-m}{\omega+m}}\,(\chi^k_{j+1/2})^\dagger\end{array}\right) + \\
&+ (I^-)^*\left(\begin{array}{cc}(\chi^k_{j+1/2})^\dagger & -\sqrt{\tfrac{\omega-m}{\omega+m}}\,(\chi^k_{j-1/2})^\dagger\end{array}\right)\Bigg]\Bigg[I^+\left(\begin{array}{c}-\sqrt{\tfrac{\omega-m}{\omega+m}}\,\chi^k_{j-1/2} \\ \chi^k_{j+1/2}\end{array}\right) + \\
&+ I^-\left(\begin{array}{c}-\sqrt{\tfrac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2} \\ \chi^k_{j-1/2}\end{array}\right)\Bigg] \quad ,
\end{aligned}
\tag{3.226}
$$

and

$$
\begin{aligned}
(J^r)^r &= \frac{1}{2}\bar{\Psi}^r G^r(\Psi)^r \\
&= \frac{1}{2\,r^2}\Bigg[(R^+)^*\left(\begin{array}{cc}(\chi^k_{j-1/2})^\dagger & \sqrt{\tfrac{\omega-m}{\omega+m}}\,(\chi^k_{j+1/2})^\dagger\end{array}\right) + \\
&+ (R^-)^*\left(\begin{array}{cc}(\chi^k_{j+1/2})^\dagger & \sqrt{\tfrac{\omega-m}{\omega+m}}\,(\chi^k_{j-1/2})^\dagger\end{array}\right)\Bigg]\Bigg[R^+\left(\begin{array}{c}\sqrt{\tfrac{\omega-m}{\omega+m}}\,\chi^k_{j-1/2} \\ \chi^k_{j+1/2}\end{array}\right) + \\
&+ R^-\left(\begin{array}{c}\sqrt{\tfrac{\omega-m}{\omega+m}}\,\chi^k_{j+1/2} \\ \chi^k_{j-1/2}\end{array}\right)\Bigg] \quad ,
\end{aligned}
\tag{3.227}
$$

where we are using $j = k = +\frac{1}{2}$. The transmitted particle-number radial current is given by

$$
\begin{aligned}
(J^r)^t &= \frac{1}{2}\bar{\Psi}^t G^r(\Psi)^t \\
&= \frac{|T|^2}{2\,r_+^2}\left(\begin{array}{cc}(\chi^k_{j-1/2})^\dagger & -\eta\,(\chi^k_{j+1/2})^\dagger\end{array}\right)\left(\begin{array}{c}-\eta\,\chi^k_{j-1/2} \\ \chi^k_{j+1/2}\end{array}\right) = -\frac{|T|^2}{4\pi\,r_+^2}\,\eta \quad ,
\end{aligned}
\tag{3.228}
$$

where we used $G^r = S\gamma^r$ and the relations (3.182) and (3.183).

So, the fluxes of the particle-number currents are given by

$$
\mathcal{F}^i = -\sqrt{\frac{\omega-m}{\omega+m}}\,\left(|I^+|^2 + |I^-|^2\right) \quad , \tag{3.229}
$$

$$
\mathcal{F}^r = \sqrt{\frac{\omega-m}{\omega+m}}\,\left(|R^+|^2 + |R^-|^2\right) \quad , \tag{3.230}
$$

and

$$
\mathcal{F}^t = -|T|^2\eta \quad , \tag{3.231}
$$

where we used the orthonormality relations of the spinor harmonics.

We obtain that

$$
\left|\frac{\mathcal{F}^r}{\mathcal{F}^i}\right| = \left|1 - \frac{\mathcal{F}^t}{\mathcal{F}^i}\right| = 1 - \eta\sqrt{\frac{\omega+m}{\omega-m}}\,\frac{|T|^2}{|I^+|^2 + |I^-|^2} > 1 \quad , \tag{3.232}
$$

where we used the relation (3.225) and that $\eta < 0$ in the regime under consideration.

Thus, we see that, in this non-linear case, there exists a solution with $|\mathcal{F}^r| > |\mathcal{F}^i|$ for

$$\omega < e\,V_+ + \bar{\lambda}|T|^2 \quad . \tag{3.233}$$

We showed, now on a RN background, that there exist fermion condensates which can exhibit superradiant amplification. Here, we point out again that, in principle, there are many other theories that also admit superradiant solutions. One example is the same theory that we used but, this time, with a negative coupling constant $\widetilde{\lambda} < 0$.

Conclusions

4

4.1 Achievements

In this thesis, we showed that a charged scalar field has superradiant modes both when scattering on a strong electrostatic potential barrier (Klein paradox) and on a RN background. This conclusion was already known and the superradiant modes that we obtained are in agreement with the ones of Refs. [3, 6]. Furthermore, we also proved that Dirac fields do not exhibit superradiant amplification both in the case of Klein paradox as well as in RN background. This fact is well known, but, from what we know, there was no direct proof of the absence of superradiant amplification for Dirac fields scattering on RN geometry [3, 4]. In fact, Lee proved the absence of superradiance for Dirac fields scattering on the more general Kerr-Newman BH [8]. However, since RN is spherically symmetric, we can prove this absence directly in a easier way. In particular, we do not need to use the Newman-Penrose formalism to separate the Dirac equation. Instead, we follow a procedure that explores the spherically symmetry of the problem as in Ref. [45].

We accomplished the main objective of this work, which was to answer the questions: Is it possible to have a fermion condensate which can exhibit superradiant amplification? Or, is there a non-linear interaction between fermions which enables them to exhibit superradiant amplification? It turns out that the answer to these two questions is yes. In fact, in this thesis, we provided a non-linear Dirac field theory which we believe to describe a fermion condensate and has solutions with superradiant amplification both in the Klein paradox and in the RN background. This serves only as a proof of concept that such a fermion condensate can exist, but this conclusion is very important. Because it gives consistency to the usual interpretation of the Penrose process as the particle analogue of superradiant amplification phenomenon. In fact, since the Penrose process is a classical phenomenon, we expect it to happens with ordinary (classical) matter. But, if the Penrose process is the particle analogous of superradiant amplification, this ordinary matter must exhibit also superradiance. Now, since ordinary (baryonic) matter is made of fermions at the fundamental level, we expect superradiance to be restored by some kind of non-linear interaction between the fermions. In this work, we saw that, in fact, this can happen.

4.2 Future Work

The most important thing that remains to be done is the numerical analysis of the non-linear Dirac field theory that we introduced in this work. In particular, it should be confirmed numerically that superradiant solutions are allowed in this theory. Moreover, it is important to do numerical studies of this kind of solutions to get a better understanding of their behaviour.

In the case of RN background, we do not need to use the non-linear Dirac theory with the coupling λ proportional to $A_\mu A^\mu$. In fact, we introduced a coupling of this kind to assure that we have a linear Dirac equation at infinity. We want to have this asymptotic linearity, because we need to identify the incident and the reflected waves. However, in RN geometry the Dirac fields fall with $1/r$ near infinity. So, even with a constant coupling λ, which is exactly the Nambu-Jona-Lasinio model [40], the Dirac equation is asymptotically linear. In principle, this model admit superradiant solutions of the same kind, but it would be interesting to confirm it and to do a numerical analysis of these solutions. It would also be interesting to analyse if these non-linear Dirac theories admit superradiant solutions on Kerr background.

From what we know, the analysis of superradiant scattering for Rarita-Schwinger (spin-3/2) fields has not been done. This study is very important to test the idea that fermionic fields do not exhibit superradiance. Furthermore, if this Rarita-Schwinger fields are prevented from being superradiantly amplified, it would be interesting to analyse if, again, there is some non-linear interaction that restores them this capability.

Finally, since Dirac fields manifest themselves quantumly in nature, it would be of interest to use quantum field theory to analyse the physical significance of our superradiant solutions.

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