Critical behavior in gravitational collapse: Analytical configurations of collapsing massless scalar wave packets

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In this thesis, we review the Roberts spacetime in the context of critical behavior in gravitational collapse, and perform the same analysis on its recent generalization to an asymptotically anti-de Sitter background. We evaluate how the introduction of a non-vanishing cosmological constant affects the black hole's scaling laws for nearly critical configurations, and find that it has no effect on the leading scaling exponent. This result is in agreement with numerical evidence from a previous work by Husain and collaborators, where Choptuik's first work on this subject was adapted to an anti-de Sitter background. Addressing the issue that the Roberts solutions yielding black hole formation lead to an indefinitely growing black hole, we perform the matching of the Roberts spacetime to an exterior radiating Vaidya solution, along an ingoing null hypersurface. A smooth matching is possible without a massive thin shell on the hypersurface, and the configuration represents a radiating massless scalar wave packet collapsing at the speed of light. The scaling laws for the now finite black hole remain unaltered. The most particular cases include a singular hypersurface sweeping the whole spacetime. Lastly, we demonstrate how the Roberts spacetime cannot be smoothly matched to an exterior Schwarzschild solution.

I. INTRODUCTION

Whilst studying the spherically symmetric scalar field in Einstein's theory [1–4], Christodoulou asked Choptuik (who was working numerically on the same subject for a massless scalar field) if a black hole resulting from the scalar field's collapse could be formed with infinitesimal mass by tuning initial parameters close to the threshold of black hole formation. Taking up on this suggestion, after developing advanced numerical methods for this purpose, Choptuik presented strong evidence that the resulting black hole can indeed form with infinitesimally small mass [5]. However, he was also able to identify two unexpected phenomena. First, the famous scaling law for the black hole mass

$$M_{\rm BH} \propto (p - p^*)^\gamma, \tag{1}$$

for $p \simeq p^*$, where $p < p^*$ yields subcritical solutions in the sense that the collapse does not result in black hole formation, $p > p^*$ yields supercritical solutions resulting in black hole formation. Choptuik found $\gamma \simeq 0.37$. The critical parameter $p = p^*$ is the critical solution representing the black hole formation threshold. The second observed phenomena was the discrete self-similar solution found, again, for $p \simeq p^*$, with a logarithmic scale period $\Delta \simeq 3.44$, which Choptuik would refer to as an "echoing". From his results, Choptuik conjectured that both the scaling exponent γ and the echoing period Δ were universal with respect to the initial data: suppose one could parameterize the collapse through various one-parameter families of initial data so that each parameterization would hold a black hole formation threshold. Then, by fine-tuning each of these families near the critical solution, one will always get the same γ and Δ. Ensuing works by several authors came to reveal the same phenomena in different systems, proving that Choptuik's findings are not limited to massless scalar field collapse, nor to spherical symmetry.

The critical phenomena identified by Choptuik is referred to as Type II critical phenomena. In some instances, the critical behavior near the same threshold can result in a stationary or time-periodic critical solution, rather then a self-similar one. These are known as Type I critical phenomena, in which the black hole forms with finite mass. Some systems might however exhibit both types of critical behavior in different parameter spaces, such as the case of a massive scalar field in [6]. Some more detailed analysis can be found in a number of reviews on this subject, from which we list a few [7–12].

For the reader's convenience, we specify here the basic notation used throughout the work. Regarding the indices of tensorial quantities, latin indices run through all the coordinates in an n-dimensional spacetime, while greek indices run through all $n-1$ coordinates. Capital latin indices will cover only $n-2$ angular coordinates. The current work is limited to the analysis of a four dimensional spacetime $n = 4$. The convention for the Riemann curvature tensor is $[\nabla_c, \nabla_d]X^a = R^a{}_{bcd}X^b$, and the Ricci tensor $R_{ab} = R^c{}_{acb}$, where repeated indices correspond to Einstein's notation for the sum through all coordinates. We use the spacelike convention for the metric as $(-,+,+,+)$. In analogy with the d'Alembert's operator in classical mechanics, we define the covariant form of the operator as $\Box \equiv -g^{ab}\nabla_a\nabla_b$. Any relevant notation specifications will be addressed whenever deemed necessary.

II. THE ROBERTS SPACETIME

In 1989, Roberts obtained an interesting exact solution to Einstein's field equations with a massless scalar field [13]. This solution admitted a homothetic Killing vector field, and thus represented a massless scalar field in a continuously self-similar spacetime. With Choptuik's discovery of critical phenomena in gravitational collapse, the Roberts solution started being used as a model for this subject, due to its self-similar properties.

The general, (3+1)-dimensional, spherically symmetric metric in a double-null coordinate system $x^a = (u, v, \theta, \phi)$ is given by the line element

$$ds^2 = -2e^{2\sigma} du dv + r^2 \, d\Omega^2_{(2)}, \tag{2}$$

where, σ and r are functions of u and v only, and $r^2 d\Omega^2_{(2)}$ is the line element of the spherically symmetric two-dimensional spheres of areal radius r, where $d\Omega^2_{(2)} = d\theta^2 + \sin^2(\theta) d\phi^2$. Null coordinates u and v are defined so that u is constant along outgoing lightlike geodesics, and v is constant along ingoing lightlike geodesics. θ and ϕ are the usual angular coordinates.

The geometry of the spacetime and the scalar field are correlated by the Einstein field equations

$$G_{ab} = 8\pi \, T_{ab}, \tag{3}$$

where the left hand side of the equation describes the geometry of the spacetime. The Einstein tensor is defined

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \tag{4}$$

where $R \equiv g^{ab} \, R_{ab}$. The energy-momentum tensor T_{ab} for the massless scalar field Φ is

$$T_{ab} = \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} g^{cd} \nabla_c \Phi \nabla_d \Phi. \tag{5}$$

The wave equation for the scalar field is given by the Klein-Gordon equation for a massless scalar field, $\Box \Phi = 0$.

A continuously self-similar spacetime is defined[14] as a spacetime that admits a homothetic vector field ξ, satisfying

$$L_\xi \, g_{ab} = 2 g_{ab}, \tag{6}$$

where L_X represents Lie differentiation along a vector field X. In our spherically symmetric coordinate system, the homothetic vector field can be defined $\xi = u\partial_u + v\partial_v$, and a series of self-similar constraints arise on σ, r and Φ, in the form of

$$\sigma(u, v) = \sigma_{\rm ss}(v/u) \,, \tag{7}$$
$$r(u, v) = -u \, r_{\rm ss}(v/u) \,, \tag{8}$$
$$\Phi(u, v) = k \, \log(-u/l) + \Phi_{\rm ss}(v/u) \,, \tag{9}$$

where k is an integration constant, l is an arbitrary length scale, and the 'ss' subscripts denote the dimensionless self-similar part of each original function. A particular solution to Eqs. (3,7-9) comes from setting $\sigma = 0$, which ultimately leads to the vanishing of k. This is known as the Roberts spacetime, and can be obtained as a two-parameter family of solutions, in

$$e^{2\sigma} = 1, \tag{10}$$
$$r\,(u, v) = \sqrt{\alpha v^2 + \beta u^2 - uv}, \tag{11}$$
$$\Phi\,(u, v) = \begin{cases} \pm \dfrac{1}{4\sqrt{\pi}} \log \left| \dfrac{2\alpha v - u\left(1 + \sqrt{1 - 4\alpha\beta}\right)}{2\alpha v - u\left(1 - \sqrt{1 - 4\alpha\beta}\right)} \right| & \text{for } \alpha \neq 0, \\ \pm \dfrac{1}{4\sqrt{\pi}} \log \left| \beta - \dfrac{v}{u} \right| & \text{for } \alpha = 0, \end{cases} \tag{12}$$

where α and β are integration constants, and parameterize the different solutions.

The formation of an apparent horizon can be tracked through the surfaces whose outward normals are null [15]

$$g^{ab} r_{,a} r_{,b} = 0, \tag{13}$$

which, as the boundary of outer trapped surfaces, corresponds to writing coordinate-wise,

$$u = 2\alpha v, \tag{14}$$

for $\alpha \neq 0$. For the case $\alpha = 0$, one actually finds that there is no apparent horizon anywhere on the spacetime. A local mass function for the spacetime can be defined as [16]

$$1 - \frac{2m}{r} = g^{ab} r_{,a} r_{,b}, \tag{15}$$

which coordinate-wise yields.

$$m(u, v) = \frac{-uv(1 - 4\alpha\beta)}{4r}. \tag{16}$$

Some regions of the spacetime (namely those satisfying $uv > 0$) have negative values for the mass function, and these correspond to a scalar field flowing faster than the speed of light. In order to study the Roberts solutions in the context of gravitational collapse, this issue must be addressed first.

III. CRITICAL BEHAVIOR IN THE ROBERTS SOLUTION

Due to the lightlike nature of the scalar field flux along the null hypersurfaces $v = 0$ and $u = 0$, one can prove that, using the spacetime junction formalism of Israel [17, 18], the Roberts spacetime in region $uv < 0$ can be matched to an interior Minkowski spacetime along the null hypersurfaces $v = 0$ and $u = 0$. This can be interpreted as the scalar field being turned on at $v = 0$, and if no spacelike singularity is formed prior to $u = 0$, then the scalar field will disperse back to infinity along the hypersurface $u = 0$.

Consider the one-parameter family of solutions in α by setting $\beta = 0$. In this parameterization one can find a critical solution in $\alpha = 0$ separating those that result in the dispersion of the field ($0 < \alpha < 1/4$), from those that lead to black hole formation ($\alpha < 0$). In the threshold of black hole formation $\alpha = 0$ lies a solution with a central lightlike singularity. For the supercritical solutions, the mass function on the apparent horizon is given by

$$m_{\rm BH} = \frac{v}{2}\sqrt{-\alpha}\sqrt{1-4\alpha}, \tag{17}$$

which can be evaluated near the critical solution in the limit $\alpha \to 0^-$ to yield a power law

$$m_{\rm BH} \simeq \frac{v}{2}\left|\alpha\right|^{1/2}, \tag{18}$$

in resemblance to the one conjectured by Choptuik. However, this only holds for initial values of v, since the black hole is indefinitely growing, due to the scalar field influx never being turned off, instead of leading to an infinitesimally small black hole, which would better represent Choptuik's observations.

A. The problem of addressing a critical solution through a vanishing critical parameter

As first suggested first by Choptuik (and later evidenced in countless related works), the black hole mass scaling exponent is independent of how we choose to parameterize the collapsing solutions, due to the conjectured universality of the critical solution. If we parameterize the collapse through a parameter p, then the scaling exponent should also be invariant under any transformation in p, so long as the newly transformed parameter can also yield the same critical solution. However, this does not hold true for the parameterization we are currently using in α. Under a transformation of the form

$$\alpha \to \tilde{\alpha} = \alpha^{-b}, \tag{19}$$

for an odd positive integer b, the reparameterization leads to an exponent of $b/2$ for the black hole mass, rather than the supposedly universal $1/2$ of Eq. (18) . The problem is easily identified as a consequence of the value corresponding to the critical solution $\alpha^* = 0$ not being well defined under this particular transformation. Thus, the scaling exponent is only invariant under a transformation of the form

$$p \to \tilde{p} = p^{-b}, \begin{cases} b \in \mathbb{R} & \text{if } p \geq 0, \\ b = 2n+1 & \text{otherwise,} \end{cases} \tag{20}$$

for positive integers n, if $p^* \neq 0$. As an example, let us first reparameterize α (with $\beta = 1$) according to

$$\alpha \to \tilde{\alpha} = \frac{1}{4} - \alpha \tag{21}$$

so that it has a non-zero critical value. Then, a transformation of the same type as Eq. (19) into $\hat{\alpha} = \tilde{\alpha}^{-b}$, leads to the newly transformed scaling law for the black hole mass

$$m_{\rm BH} = \frac{v\sqrt{b}}{2^b}\left(\hat{\alpha} - \hat{\alpha}^*\right)^{1/2}, \tag{22}$$

close to $\hat{\alpha}^* = (1/4)^{-b}$. This problem could have been an issue in the work of Ref. [19], had the integration constants arisen in some different power. This was not an issue in the work of Ref. [20], as they used a parameterization where the critical parameter was non-vanishing.

Unbeknownst to this, we will keep using the same parameterization in α with $\alpha^* = 0$ in the coming chapters, mainly for historical reason. This should not be a problem as the exponent's validity can always be easily checked.

IV. THE ROBERTS-ANTI-DE SITTER SPACETIME

The study of critical phenomena occurring in gravitational collapse, is not limited to asymptotically flat spacetimes, and, in that regard, it is only natural to look for a generalization of the Roberts spacetime for a non-vanishing cosmological constant. The solution was recently generalized by Roberts himself [21], and later analyzed in greater detail by Maeda [22]. We will be focusing in particular, on the anti-de Sitter generalization, with a negative cosmological constant.

The generalization of the Roberts spacetime for a non-vanishing cosmological constant Λ can be obtained through a conformal transformation. Consider the metric $\bar{g}_{ab}$ conformally related to g_{ab} of Eq. (2) as

$$\bar{g}_{ab}(x^a) = \Omega^2(x^a) g_{ab}(x^a), \tag{23}$$

satisfying the conformally transformed field equations

$$\bar{G}_{ab} + \Lambda \bar{g}_{ab} = 8\pi \bar{T}_{ab}, \tag{24}$$
$$\bar{\Box}\Phi = 0. \tag{25}$$

The conformally related energy-momentum tensor remains unaltered, as well as the wave equation, since $\bar{\nabla}_a\Phi = \nabla_a\Phi$ and $\bar{\Box}\Phi = \Omega^{-2}\Box\Phi$. So as to preserve spherical symmetry, we assume $\Omega = \Omega(u,v)$, and are able to find a solution of Ω to Eq. (24) as

$$\Omega(u,v) = \left(1 - \frac{\Lambda uv}{6}\right)^{-1}. \tag{26}$$

This allows us to write the Λ-generalized Roberts spacetime metric through

$$ds^2 = -\left(1 - \frac{\Lambda uv}{6}\right)^{-2} 2du\,dv + r^2 d\Omega^2_{(2)}, \tag{27}$$

where the generalized areal radius is now written as

$$r = \left(1 - \frac{\Lambda uv}{6}\right)^{-1} \sqrt{\alpha v^2 + \beta u^2 - uv}. \quad (28)$$

The new conformal spacetime now admits ξ as a conformal Killing vector satisfying the conformal Killing equations

$$L_\xi g_{ab} = \frac{6 + \Lambda uv}{6 - \Lambda uv} 2g_{ab}. \quad (29)$$

Then, there is a conserved quantity $C = g_{ab}k^a\xi^b$, where $k^a = dx^a/d\lambda$ is tangent to a radial null geodesic parameterized by λ. Along an outgoing null geodesic at $u = u_0$, we can write

$$C = -\left(1 - \frac{\Lambda u_0 v}{6}\right)^{-2} u_0 \frac{dv}{d\lambda}, \quad (30)$$

which we can integrate to get

$$\left(1 - \frac{\Lambda u_0 v}{6}\right)^{-1} = -\frac{\Lambda}{6} C\left(\lambda - \lambda_0\right), \quad (31)$$

for some integration constant λ_0. In the same manner, along an ingoing null geodesic at $v = v_0$, one gets

$$\left(1 - \frac{\Lambda u v_0}{6}\right)^{-1} = -\frac{\Lambda}{6} C\left(\lambda - \lambda_0\right). \quad (32)$$

From these two equations, one concludes that null infinity $|\lambda| \to \infty$ corresponds to the hypersurface described by

$$1 - \frac{\Lambda uv}{6} = 0, \quad (33)$$

which is a timelike hypersurface for $\Lambda < 0$, or a spacelike hypersurface if $\Lambda > 0$. Another important property that distinguishes the Roberts generalization from the original asymptotically flat solution is the extendability of the spacetime through the coordinate infinities $u, v = \pm\infty$, as these correspond to finite values of λ in Eqs. (31,32). So long as the hypersurfaces $u \to \pm\infty$ and $v \to \pm\infty$ correspond to physical regions (i.e. $r^2 > 0$) and are regular, we have to extend the spacetime through them for a complete description of the Roberts-anti-de Sitter universe.

Consider the coordinate transformation $u \to \bar{u} = 1/u$ and $v \to \bar{v} = 1/v$. Although these are not well defined at $u = 0$ or $v = 0$, we will look the other way since we mean to study them only at the infinities of u and v. The resulting metric yields

$$ds^2 = \left(\frac{\Lambda}{6} - \bar{u}\bar{v}\right)^{-2} \left(-2d\bar{u}d\bar{v} + \bar{r}^2 d\Omega^2_{(2)}\right), \quad (34)$$

$$\bar{r}^2(\bar{u}, \bar{v}) = \alpha\bar{u}^2 + \beta\bar{v}^2 - \bar{u}\bar{v}, \quad (35)$$

which is not singular at $\bar{u} = 0$ or $\bar{v} = 0$. Since, along some $\bar{v} =$ cte, the transition from $\bar{u} = 0^-$ to $\bar{u} = 0^+$ happens smoothly (and the same goes for $\bar{v}$ along some $\bar{u} = cte$), going back to the spacetime in coordinates of Eqs. (27,28), we attach patches of the same spacetime on the hypersurfaces $u, v \to \pm\infty$, such that, on one side $u, v \to +\infty$, and, on the other side $u, v = -\infty$, emphasizing that only when these are regular.

The apparent horizon can again be tracked through Eq. (13), which, for the boundary of the outer trapped surfaces yields, coordinate-wise,

$$v = 2u\left(\frac{1 - \beta\,\Lambda u^2/3}{4\alpha - \Lambda u^2/3}\right), \quad (36)$$

for $\alpha \neq 0$. Unlike in Sec. II, there is now a solution for the $\alpha = 0$ case, however, it is immediately discarded as it is located in the unphysical region $r^2 < 0$. The generalized local mass can also be written [23]

$$1 - \frac{\Lambda r^2}{3} - \frac{2m}{r} := g^{ab} r_{,a} r_{,b}, \quad (37)$$

which, coordinate-wise, can be written as

$$m = \left(1 - \frac{\Lambda uv}{6}\right)^{-1} \frac{-uv(1 - 4\alpha\beta)}{\sqrt{\alpha v^2 + \beta u^2 - uv}}. \quad (38)$$

V. CRITICAL BEHAVIOR IN THE ROBERTS-ANTI-DE SITTER SOLUTION

Naturally, we will want to investigate how the introduction of a non-vanishing cosmological constant will affect the scaling laws obtained through Brady's approach [19], discussed in Sec. III. We find that the same parameterization in α with $\beta = 1$ yields the same solutions: subcritical solutions for $0 < \alpha < 1/4$, the critical solution at $\alpha = 0$, and supercritical solutions for $\alpha < 0$. Due to the timelike nature of the null infinity, one can also identify an event horizon, and although some light rays can escape to infinity, the black hole solutions still represent an indefinitely growing black hole.

Unlike in Sec. III, the spacetime no longer admits ξ as a homothetic Killing vector field, but rather a confomal one. Since one of the main reasons the Roberts solution was used for studying critical behavior in gravitational collapse was its self-similar properties, one can ask if studying critical behavior on the Roberts-anti-de Sitter solution is of any less interest, as it is no longer globally self-similar. However, since we intend on studying the properties pf the black hole, one can take the strong field limit by taking $|\Lambda uv| \ll 1$ close to the apparent horizon. Then, in the aforementioned limit, the solution is approximately self-similar.

To evaluate how the cosmological constant affects the black hole's mass scaling law obtained in Eq. (18), we would like to be able to write Eq. (36) with u as a function of v. However, this is far from trivial due to the terms introduced with $\Lambda \neq 0$, and we would like to avoid bringing up the limit $|\Lambda u^2| << 1$ so soon in the analysis. To solve this issue, we will choose to take the limit $\alpha \to 0^-$ before evaluating the mass function on the apparent horizon, as this is really the situation we ultimately want to study. We start by trying to find the minimum value $u_{\rm AH}$ reaches. The apparent horizon of Eq. 36 for

$\beta = 1$ becomes

$$v = 2u\left(\frac{1-\Lambda u^2/3}{4\alpha - \Lambda u^2/3}\right), \tag{39}$$

Since the apparent horizon covering the singularity will be spacelike everywhere, then for increasing v, $u_{\rm AH}$ must always be decreasing, reaching a minimum when it reaches null infinity. Lets address this minimum as $u_{\rm EH}$, as it describes the event horizon, satisfying both Eq. (33) and Eq. (39),

$$u_{\rm EH}^2 = -\frac{3}{\Lambda}\left(\sqrt{1-4\alpha}-1\right). \tag{40}$$

From Eq. (40), if we bring the supercritical solution near criticality by taking the limit $\alpha \to 0^-$, it is obvious that $|\Lambda u_{\rm EH}^2| << 1$, and if this is true on the event horizon, than so is it for any u on the apparent horizon. Then, in the limit $\alpha \to 0^-$, the apparent horizon equation can be reduced to

$$u = v\,\frac{3}{\Lambda v^2}\left(\sqrt{1+4\alpha\frac{\Lambda v^2}{3}}-1\right). \tag{41}$$

We are now in a position to evaluate the mass function on the apparent horizon for nearly critical solutions. Inserting Eq. (41) into the mass function of Eq. (38), yields a power law for the black hole mass of the form

$$m_{\rm BH} = \frac{v}{2}|\alpha|^{1/2} + \left(1-\frac{\Lambda v^2}{3}\right)v|\alpha|^{3/2} + \mathcal{O}(|\alpha|^{5/2}), \tag{42}$$

where we introduce terms up to the order where Λ manifests itself. As we can see, Λ only shows up in higher order terms in α, and the scaling with the leading exponent is actually unaffected by the cosmological constant.

VI. RADIATING SCALAR WAVE PACKETS

One of the most fascinating consequences of the scaling law that arises for the black hole's mass (and radius), is the possibility of forming infinitesimally small black holes near the critical solution. However, the analysis performed in Secs. III and V does not result in a finite mass/radius black hole. As addressed in previous chapters, this is due to the scalar field influx never being turned off in the supercritical solutions. In the following chapter, we will take up on Brady's suggestion [19], and will turn off the scalar influx at some finite $v = v_1$. In order to study the gravitational collapse of a Roberts scalar wave packet, we will turn off the scalar influx, by matching it to an exterior radiating Vaidya spacetime [24] along a radially ingoing null hypersurface.

We start by adapting the interior Roberts spacetime in Eqs. (27,28) to an ingoing Eddington-Finkelstein coordinate system $x_-^a = (v, r, \theta, \phi)$ by writing

$$ds_-^2 = -e^{\psi_-}\left(f_- dv^2 - 2e^{\eta_-}dvdr\right) + r^2 d\Omega_{(2)}^2, \tag{43}$$

where ψ_-, f_-, and η_- are functions of v and r only, and the minus sign is used to identify that this is the spacetime we want to match to a different exterior. It will prove useful to simplify these functions by defining the quantity $\hat{u}(v,r) > 0$ as

$$\hat{u} \equiv \sqrt{v^2(1-4\alpha) + 4r^2\left(1-\frac{\Lambda v^2}{6}\left[1-\alpha\frac{\Lambda v^2}{6}\right]\right)}, \tag{44}$$

despite the fact that it will not have any relevant physical meaning in our work. The metric functions are found to be given by

$$e^{\psi_-} = \frac{2r}{\hat{u}}\left(1-\frac{\Lambda v^2}{6}\,\frac{v-\hat{u}}{2v}\right)^{-2}, \tag{45}$$

$$f_- = \left(1+\frac{\Lambda v^2}{6}\frac{\Lambda r^2}{6}\right)\frac{\left(1-\Lambda r^2/3\right)\hat{u} - v(1-4\alpha)}{2r} + \frac{\Lambda r v}{3}\left(1+\frac{\Lambda v^2}{6}\frac{\Lambda r^2}{6} - \frac{\Lambda(\alpha v^2 + r^2)}{3}\right), \tag{46}$$

$$e^{\eta_-} = \left(1+\frac{\Lambda v^2}{6}\frac{\Lambda r^2}{6}\right)\left[1-\frac{\Lambda v^2}{6}\left(1-\alpha\frac{\Lambda v^2}{6}\right)\right] + \frac{\Lambda v r}{6}\left(\frac{\hat{u}}{r} - \frac{\Lambda v^2}{6}\,\frac{\hat{u}-v(1-4\alpha)}{2r}\right). \tag{47}$$

The generalized Misner-Sharp mass function is written as

$$m = \frac{r}{2}\left(1-\frac{\Lambda r^2}{3} - f_- e^{-\psi_- - 2\eta_-}\right). \tag{48}$$

The exterior Vaidya solution can be written in outgoing Eddington-Finkelstein coordianates $x_+^a = (U, r, \theta, \phi)$ as

$$ds_+^2 = -e^{\psi_+}\left(f_+ dU^2 + 2e^{\eta_+}dUdr\right) + r^2 d\Omega_{(2)}^2, \tag{49}$$

where ψ_+, f_+, and η_+ are functions of U and r only, and are given by

$$e^{\psi_+} = 1, \tag{50}$$

$$f_+ = 1 - \frac{\Lambda r^2}{3} - \frac{2M(U)}{r}, \tag{51}$$

$$e^{\eta_+} = 1. \tag{52}$$

This solution represents a purely emitting null dust field (or pure radiation) spacetime, with an energy-momentum tensor given by

$$T^+_{ab} = -\frac{dM/dU}{4\pi r^2}\delta^U_a \delta^U_b, \tag{53}$$

satisfying the field equations in Eq. (24). In our configuration, this means that whatever energy the scalar field loses during the collapse inside the matching hypersurface, will be converted into pure radiation outside the hypersurface. The Kretschmann scalar in the Vaidya spacetime is given by

$$K = \frac{48M^2(U)}{r^6}. \tag{54}$$

A. Matching on a null hypersurface

Having described both the interior and the exterior spacetimes, we will now proceed to matching them along the null hypersurface Σ. Since we are working with different coordinate systems on the inside and outside, Σ will need a description as seen from each side, which we address as $\Psi_\pm(x^a_\pm)$,

$$\Sigma : \begin{cases} \Psi_+(x^a_+) = U - U_1(r) = 0, \\ \Psi_-(x^a_-) = v - v_1 = 0, \end{cases} \tag{55}$$

where v_1 is constant, and U_1 satisfies an ingoing null geodesic in the Vaidya spacetime $f_+ dU_1 + 2e^{\eta_+}dr = 0$, or equivalently,

$$\frac{dU_1}{dr} = -\frac{2\,e^{\eta_+}}{f_+}. \tag{56}$$

The induced metric on Σ with coordinates $y^\alpha = (r, \theta, \phi)$

$$h_{\alpha\beta} = r^2 \left(\delta^\theta_\alpha \delta^\theta_\beta + \sin^2(\theta)\delta^\phi_\alpha \delta^\phi_\beta\right). \tag{57}$$

In the coordinate system we are using, the metric on the hypersurface is seen from both sides to be the same, and so the first junction condition $[h_{\alpha\beta}] = 0$ is automatically verified.

The vectors normal to Σ as seen from both sides $n^\pm_a := \mu^{-1}_\pm \partial_a \Psi_\pm$, for some negative functions $\mu_\pm$, yield

$$n^-_a = \mu^{-1}_- \delta^v_a, \tag{58}$$

$$n^+_a = \mu^{-1}_+ \left(\delta^U_a + \frac{2e^{\eta_+}}{f_+}\delta^r_a\right). \tag{59}$$

The vectors tangent to the hypersurface, $e^a_{\pm(\alpha)} := \partial x^a/\partial y^\alpha$, as seen from both sides, are

$$e^a_{-(r)} = \delta^a{}_r \tag{60}$$

$$e^a_{+(r)} = -\frac{2e^{\eta_+}}{f_+}\delta^a{}_U + \delta^a{}_r, \tag{61}$$

$$e^a_{\pm(A)} = \delta^a_A, \tag{62}$$

where A runs through (θ, ϕ). Lastly, the "transverse"null vectors $N^\pm_a$, satisfying

$$N^\pm_a N^a_\pm = 0, \quad N^\pm_a n^a_\pm = -1, \quad N^\pm_a e^a_{\pm(A)} = 0, \tag{63}$$

can be computed to yield

$$N^-_a = \mu_- \left(\frac{e^{\psi_-} f_-}{2}\delta^v_a - e^{\psi_- + \eta_-}\delta^r_a\right), \tag{64}$$

$$N^+_a = \mu_+ \frac{e^{\psi_+} f_+}{2}\delta^U_a. \tag{65}$$

In order to make that both $N^a_\pm$ represent the same vector, we project them from both sides on the hypersurface Σ

$$N^+_a e^a_{+(\alpha)}\Big|_\Sigma = N^-_a e^a_{-(\alpha)}\Big|_\Sigma, \tag{66}$$

and find that

$$\mu_\pm = -e^{-\psi_\pm - \eta_\pm}, \tag{67}$$

with which we can now rewrite the "transverse"vectors as

$$N^-_a = -\frac{f_- e^{-\eta_-}}{2}\delta^v_a + \delta^r_a, \tag{68}$$

$$N^+_a = -\frac{f_+ e^{-\eta_+}}{2}\delta^U_a. \tag{69}$$

The "transverse"extrinsic curvature can be calculated through $C^\pm_{\alpha\beta} := -N^\pm_a e^b_{\pm(\beta)} \nabla_b e^a_{\pm(\alpha)}$, from which we directly compute its jump across Σ, as

$$[C_{\alpha\beta}] = -8\pi P \delta^r_\alpha \delta^r_\beta - 4\pi\sigma h_{\alpha\beta}, \tag{70}$$

where σ and P arise from the definition of the surface energy-momentym tensor $\tau^{\alpha\beta} := -S^{\alpha\beta}$ with

$$\begin{aligned} -S^{\alpha\beta} &= \frac{\gamma_{\gamma\delta}}{16\pi}\left(h^{\alpha\gamma}\delta^\beta_r \delta^\delta_r + h^{\beta\delta}\delta^\alpha_r \delta^\gamma_r - h^{\alpha\beta}\delta^\gamma_r \delta^\delta_r - h^{\gamma\delta}\delta^\alpha_r \delta^\beta_r\right) \\ &= \sigma \delta^\alpha_r \delta^\beta_r + P h^{\alpha\beta}. \end{aligned} \tag{71}$$

Although σ and P cannot give an absolute physical description as the surface energy density or pressure, respectively, since the hypersurface is lightlike and hence there is no rest frame, they are still valid for determining results from measurements by any specific observer[18], and so we will keep addressing σ as the surface energy density, and P.

We have obtained the junction conditions for both spacetimes with (or without) a thin shell described by σ and P. A smooth matching requires that $[C_{\alpha\beta}]$ vanish, and so both σ and P must vanish. Writing the exterior metric functions ψ_+, f_+, and η_+, and replacing for σ and

P lets us rewrite them with the Vaidya mass function $M\left(U_1(r)\right)$ as seen from the exterior of Σ,

$$\sigma = \frac{1}{4\pi r^2}\left[M - m|_\Sigma\right], \tag{72}$$

$$P = \frac{1}{8\pi}\left[\frac{dM}{dr} - \frac{r}{2}\left(1 - \frac{\Lambda r^2}{3} - \frac{2M}{r}\right)(\psi_- + \eta_-)_{,r}\right], \tag{73}$$

If we take the thin shell's mass to be zero by making $\sigma = 0$, we find that the mass along Σ, M is given by

$$M = m|_\Sigma\,, \tag{74}$$

which corresponds the mass function of Eq. (48) with $v = v_1$. Then, it will prove useful to decompose the total mass function outside Σ as $M = m|_\Sigma + M_{\rm shell}$, with $M_{\rm shell}$ such that

$$\sigma = \frac{M_{\rm shell}}{4\pi r^2}. \tag{75}$$

Since, by setting $M_{\rm shell} = 0$, the total mass function $M = m|_\Sigma$ immediately satisfies $P = 0$, it is possible to match both spacetimes along Σ without a massive thin shell.

Since Eqs. (72,73) cover a vast range of thin shells that allow the junction of both spacetimes, we will focus only on the case where P vanishes. Then, by decomposing the total mass function and inserting it in Eq. (73), we get a differential equation for the thin shell mass,

$$\frac{dM_{\rm shell}}{dr} = -M_{\rm shell}\left(\psi_- + \eta_-\right), \tag{76}$$

to which a general solution proves to be

$$M_{\rm shell}(r) = p\,\frac{v_1\sqrt{v_1^2(1-4\alpha) + 4r^2\left(1 - \frac{\Lambda v_1^2}{6}\left[1 - \alpha\frac{\Lambda v_1^2}{6}\right]\right)}}{2r}, \tag{77}$$

for some (dimensionless) integration constant p which defines the shell's mass sign. In the following sections, we proceed to study the spacetimes that result for $p = 0$ (no thin shell), $p > 0$ (massive thin shell), and $p < 0$ (negative mass thin shell).

B. Matching without a thin shell

In the case $P = 0$ and $\sigma = 0$, there is no massive thin shell on the hypersurface Σ, and the solutions represent a radiating Roberts scalar wave packet collapsing at the speed of light. In this case, the same parameterization as in Secs. III and V can be made.

The subcritical configurations are given by $0 < \alpha < 1/4$, where, by the time the scalar field reaches $u = 0$, the energy contained inside Σ has been completely radiated, at which point the spacetime can be matched to an interior Minkowski (anti-de Sitter) spacetime for $\Lambda = 0$ ($\Lambda < 0$). This is also possible on the Vaidya exterior: since the solution is characterized by radially outgoing null dust, there is no flux of energy across outgoing null hypersurfaces, and so, as long as the mass along said hypersurface vanishes, the Vaidya spacetime can be matched to a Minkowski (or anti-de Sitter) spacetime.

The critical solution corresponds to $\alpha = 0$, from which results a central lightlike singularity to the past of Σ. Just outside the matching hypersurface (to the future of Σ), the Vaidya mass function near the center follows

$$\lim_{r\to 0} M(r) = \frac{r}{4}, \tag{78}$$

from which one gathers that the Kretschmann scalar from Eq. (54) diverges. Then, the singularity is extended to the future of Σ, and maintains its lightlike nature.

The supercritical solutions with $\alpha < 0$ now correspond to the formation of a finite mass black hole. By taking the limit $\alpha \to 0^-$, we are now able to represent the formation of an infinitesimally small black hole. The scaling law for the black hole mass is now given by

$$M_{\rm BH} = \frac{v_1}{2}|\alpha|^{1/2}. \tag{79}$$

C. Matching with a massive thin shell

The case $p > 0$ represents solutions where the scalar field is enveloped by a massive pressureless thin shell, and the collapse always results in black hole formation. In that regard, a critical behavior analysis is questionable. However, the solutions are interesting in the pure context of gravitational collapse.

For $0 < \alpha < 1/4$, the scalar field inside Σ has been completely radiated by the time it reaches $u = 0$, and so, in the region $u > 0$, the thin shell goes on to collapse with a Schwarzschild(-anti-de Sitter) exterior. During the collapse in region $u < 0$, the shell's mass is given by Eq. (77), but in the region $u < 0$, the thin shell has a constant mass, given by

$$M_{\rm shell} = p\,\frac{v_1}{2}\left(\frac{1 - \alpha\Lambda v_1^2/3}{\sqrt{\alpha}}\right), \tag{80}$$

which eventually collapses to form a black hole.

From this type of collapse, there are two situations worth distinguishing: (1) no energy radiated by the scalar field gets trapped by the subsequently forming event horizon; (2) the event horizon is able to trap some of the emitted radiation. Depending on whichever occurs, the black hole's quantities will be evaluated according to the Schwarzschild's exterior (1), or the Vaidya's exterior (2). To track the different situations, we need only determine whether the event horizon is formed at $u > 0$ or $u < 0$. Let us search for what values of α vs. p, the Schwarzschild black hole satisfies the event horizon equation

$$M_{\rm shell} = \frac{r}{2}\left(1 - \frac{\Lambda r^2}{3}\right)\bigg|_\Sigma, \tag{81}$$

which yields

$$p = \left(1 - \frac{u_{\rm EH}}{2\alpha v_1}\right)\left(\alpha - \frac{\Lambda u_{\rm EH}}{12}\right)\left(1 - \frac{\Lambda v_1 u_{\rm EH}}{6}\right)^{-3}. \tag{82}$$

From Eq. (82), we see the the Schwarzschild exterior is only able to describe the black hole for $p \leq \alpha$. At $p = \alpha$, the event horizon will be located at $u = 0$, and both exteriors can be used to describe the black hole. For $p > \alpha$, the this shell is massive enough to form an event horizon that can trap some of the radiation emitted by the scalar field. Summarizing, we get for $0 < \alpha < 1/4$,

$$\begin{cases} p < \alpha \Rightarrow M_{\rm BH} = \dfrac{p v_1}{2\sqrt{\alpha}} \left(1 - \alpha \dfrac{\Lambda v_1^2}{3}\right), \\ p = \alpha \Rightarrow M_{\rm BH} = \dfrac{\sqrt{\alpha} v_1}{2} \left(1 - \alpha \dfrac{\Lambda v_1^2}{3}\right), \\ p > \alpha \Rightarrow M_{\rm BH} = m|_{\rm EH} + M_{\rm shell}|_E H. \end{cases} \tag{83}$$

Solutions with $\alpha \leq 0$ yield a black with a final mass given by that of Eq. 83 for $p > \alpha$.

D. Matching with a negative mass thin shell

Solutions with $p < 0$ represent a scalar wave packet enveloped by a pressureless thin shell with negative mass. Again, no parameterization leads to a black hole formation threshold, so we limit ourselves to an analysis in the context of gravitational collapse.

Solutions with $0 < \alpha < 1/4$ lead to a Schwartzschild spacetime with negative mass after the scalar field is completely radiated. The thin shell eventually collapses to $r = 0$ with constant negative mass to form a Schwartzschild timelike singularity.

For $\alpha = 0$, the scalar field inside Σ forms a central lightlike singularity at $u = 0$. As the shell collapses to $r = 0$ (and $u = 0$), the mass will diverge as $M_{\rm shell} \propto -|p|\, r^{-1}$. As a consequence, the Kretschmann scalar will also diverge along $u = 0$, maintaining the lightlike nature of the curvature singularity, although, it will no longer be central, as the divergence of the curvature scalars is due to $M^2 \to +\infty$, rather than $r \to 0$. The non-central curvature singularity will sweep the whole spacetime, as all $r =$ cte lines end up on the singularity.

The $\alpha < 0$ solutions yield a central spacelike singularity inside Σ due to the collapsing scalar field. We are interested in seeing how $M = m|_\Sigma + M_{\rm shell}$ will behave as it reaches $r = 0$. At $v = v_1$, $r = 0$ is equally described by $u = u_1$, with

$$u_1 = \frac{v_1}{2}\left(1 - \sqrt{1 - 4\alpha}\right). \tag{84}$$

Then, as u approaches u_1 from lower values, the behavior of M can be described as

$$\lim_{u \to u_1^-} M \propto -\frac{p - p_1}{\sqrt{|u - u_1|}} = \begin{cases} +\infty & \text{if } p > p_1, \\ 0 & \text{if } p = p_1, \\ -\infty & \text{if } p < p_1, \end{cases} \tag{85}$$

where p_1 is

$$p_1 = -\frac{\sqrt{1-4\alpha}}{4}\left\{ \frac{\sqrt{1-4\alpha} - 1 + 2\alpha \frac{\Lambda v_1^2}{3}\left(1 - \frac{\Lambda v_1^2}{6}\frac{1+\sqrt{1-4\alpha}}{4}\right)}{\left(1 - \frac{\Lambda v_1^2}{6}\left[1 - \alpha \frac{\Lambda v_1^2}{6}\right]\right)^2} \right\}. \tag{86}$$

Then, depending on what value we choose for p, the spacetime will have a different behavior.

For $p > p_1$ (i.e. $|p| < |p_1|$), the total mass outside Σ will diverge to $+\infty$, and the central curvature singularity will maintain its spacelike nature. As the thin shell crosses the apparent horizon, it will introduce a discontinuity on the apparent horizon in the exterior spacetime. The final mass of the black hole will be given by $M(u_{\rm EH})$.

If $p = p_1$, the trapped spheres inside the apparent horizon, will become untrapped as the shell crosses the apparent horizon. Once the shell collapse to $r = 0$, the mass function will vanish at $u = u_1$, and the central singularity is extended to the future of Σ, since the Kretschmann scalar outside will diverge with r^{-4}, and will become lightlike (outgoing) in nature, satisfying $2M = r = 0$ at $u = u_1$.

If $p < p_1$ (i.e. $|p| > |p_1|$), the spacetime outside Σ will once again represent a singular lightlike hypersurface sweeping the whole spacetime.

VII. NO SMOOTH MATCHING TO A VACUUM EXTERIOR SOLUTION

Motivated by the interesting behavior of the Roberts solution along constant mass hypersurfaces, we attempt matching the Roberts spacetime to an exterior Schwarzschild solution using Israel's junction formalism for matching spacetimes along timelike, and spacelike hypersurfaces [17]. For simplicity, we shall only make the demonstrations for the asymptotically flat case. In ingoing Eddington-Finkelstein coordinates, the Roberts spacetime is given

$$ds_-^2 = -e^{\psi_-}\left(f_- dv^2 - 2dvdr\right) + r^2 d\Omega_{(2)}^2, \tag{87}$$

$$e^{\psi_-} = \frac{2r}{\sqrt{v^2(1-4\alpha) + 4r^2}}, \tag{88}$$

$$f_- = \frac{\sqrt{v^2(1-4\alpha) + 4r^2} - v(1-4\alpha)}{2r}, \tag{89}$$

where, as usual, we have set $\beta = 1$. The Misner-Sharp mass is given by

$$m = \frac{r}{2}\left(1 - f_- e^{-\psi_-}\right). \tag{90}$$

The exterior Schwarzschild spacetime can be described in ingoing Eddington-Finkelstein coordinates $x^a_+ = (V, r, \theta, \phi)$ as

$$ds^2_+ = -\left(1 - \frac{2M}{r}\right)dV^2 + 2dVdr + r^2 d\Omega^2_{(2)}, \tag{91}$$

where M is the Schwarzschild effective mass.

Consider a general hypersurface Σ parameterized by τ along which we intend to match both spacetimes. The induced metric $h_{\alpha\beta}$ on the hypersurface is given by

$$\begin{aligned} ds^2_\Sigma &= h_{\alpha\beta} dy^\alpha dy^\beta \\ &= -T(\tau)d\tau^2 + r^2(\tau)d\Omega^2_{(2)}, \end{aligned} \tag{92}$$

where T and r are functions of τ only, and greek indices run through coordinates $y^\alpha = (\tau, \theta, \phi)$. The metric function T can be either positive or negative, depending on whether the hypersurface is timelike or spacelike, since for the case $\alpha < 0$, constant m hypersurfaces go from being timelike to being spacelike when crossing the apparent horizon. The induced metric as seen from each side of Σ is given by $h^\pm_{\alpha\beta} = g^\pm_{ab} e^a_{\pm(\alpha)} e^b_{\pm(\beta)}$, and the tangent vectors $e^a_{\pm(\alpha)} := \partial x^a / \partial y^\alpha$ are

$$e^a_{+(\tau)} = \dot{V}\delta^a_V + \dot{r}\delta^a_r, \tag{93}$$
$$e^a_{-(\tau)} = \dot{v}\delta^a_v + \dot{r}\delta^a_r, \tag{94}$$
$$e^a_{\pm(A)} = \delta^a_A, \tag{95}$$

where A runs through (θ, ϕ), and an overdot denotes differentiation along τ. Continuity across Σ demands that the first junction condition $[h_{\alpha\beta}] = 0$ be satisfied. Computing $h^\pm_{\alpha\beta}$, we find that this corresponds to satisfying $h^+_{\tau\tau} = h^-_{\tau\tau} = -T(\tau)$, where

$$h^+_{\tau\tau} = \dot{V}\left(2\dot{r} - \left[1 - \frac{2M}{r}\right]\dot{V}\right), \tag{96}$$
$$h^-_{\tau\tau} = e^{\psi_-}\dot{v}\left(2\dot{r} - f_-\dot{v}\right). \tag{97}$$

To assess smoothness across Σ, we must compute the extrinsic curvatures, for which we must specify whether Σ is timelike or spacelike. The outward unit normals to Σ are given by the relations

$$n^\pm_a e^a_{\pm(\alpha)} = 0, \quad n^\pm_a n^a_\pm = 1, \quad g^{ab}_\pm n^\pm_a \, r_{,b} > 0, \tag{98}$$

which yield the vectors

$$n^+_a = \frac{1}{\sqrt{T}}\left(\dot{V}\delta^r_a - \dot{r}\delta^V_a\right), \tag{99}$$
$$n^-_a = \frac{e^{\psi_-}}{\sqrt{T}}\left(\dot{v}\delta^r_a - \dot{r}\delta^v_a\right). \tag{100}$$

The extrinsic curvatures as seen from each side of Σ are obtained through $K^\pm_{\alpha\beta} := -n^\pm_a e^b_{\pm(\beta)} \nabla_b e^a_{\pm(\alpha)}$. The interior and exterior spacetimes can only be smoothly matched if, along Σ, the second junction condition $[K_{\alpha\beta}] = 0$ is satisfied.

To check for the possibility of smoothly matching both spacetimes, we will attempt to find a suitable hypersurface. In order to satisfy $[K_{\theta\theta}] = 0$, we must have $f_-\dot{v} = (1 - 2M/r)\dot{V}$. Then, inserting in Eqs. (96,97), the first junction condition is only satisfied if $\dot{V} = e^{\psi_-}\dot{v}$, which is equivalent to saying that the mass function from the interior Roberts spacetime is constant and equal to the exterior effective Schwarzschild mass, $m = M$. Let us now denote Σ as an hypersurface of $m =$ cte. In order to evaluate if $[K_{\tau\tau}] = 0$ is satisfied along Σ, it proves useful find a possible description of $v = v(\tau)$ and $r = r(\tau)$. Because the behavior of Σ in the cases $0 \leq \alpha < 1/4$ is different from that of $\alpha < 0$, these must be interpreted seperately. A description for v in the cases $0 \leq \alpha < 0$ may be given by

$$v(\tau) = v_1 + e^\tau, \tag{101}$$

where $v_1 = 4m/(1 - 4\alpha)$, while the cases $\alpha < 0$ may be given by

$$v(\tau) = \begin{cases} v_1 + (v_{\rm AH} - v_1)e^{-\tau^2} & \text{for } (\tau \leq 0), \\ v_{\rm AH} - \tau^2 & \text{for } (\tau > 0), \end{cases} \tag{102}$$

where we chose to identify Σ crossing the apparent horizon at $\tau = 0$, and v at the apparent horizon is given by $v_{\rm AH} = v_1\sqrt{(1 - 4\alpha)/(-4\alpha)}$. Outside the apparent horizon, (which for cases $0 \leq \alpha < 1/4$ is always) the areal radius along constant mass hypersurfaces can be given as a function of v as

$$r = \frac{v^2/2}{v^2 - v_1^2}\left(\sqrt{v_1^2 + 4\alpha\left(v^2 - v_1^2\right)} + v_1\right), \tag{103}$$

while inside the apparent horizon, it is given by

$$r = \frac{v^2/2}{v^2 - v_1^2}\left(v_1^2 - \sqrt{v_1^2 + 4\alpha\left(v^2 - v_1^2\right)}\right). \tag{104}$$

Since no explicit terms in V will show up on the extrinsic curvatures, only on its first derivative, $\dot{V} = e^{\psi_-}\dot{v}$ is enough (along with the derived functions $v(\tau)$ and $r(\tau)$) to fully describe $[K_{\tau\tau}]$. However, one actually finds that the second junction condition is never satisfied, as $[K_{\tau\tau}] \neq 0$.

VIII. CONCLUSIONS

We reviewed several important aspects of the Roberts asymptotically flat spacetime and Brady's approach to critical behavior in the solution. In Sec. III A, we address an issue that might arise in an analysis of this sort.

The recent generalization of the Roberts solution to an asymptotically anti-de Sitter spacetime has allowed to perform the same type of analysis of the critical behavior of the solution. More importantly, we wanted to see how the introduction of a negative cosmological constant would affect the scaling law for the black hole mass near the critical. The analysis revealed that there was no influence on the leading scaling exponent, in agreement a numeric study by Husain, Kunstatter, Preston, and Birukou [25], in which Choptuik's first work on the subject is adapted to an asymptotically anti-de Sitter spacetime. The authors found strong numerical evindence that a negative cosmological constant has no effect on the scaling exponent.

In Sec. VI, we matched the Roberts spacetime to an exterior radiating Vaidya solution along a null hypersurface. The matching can be smoothly (without a massive thin shell), and in this case we finally obtain the scaling for an infinitesimally small black hole. Some solutions where the matching is performed with a pressureless thin shell were also discussed in the context of gravitational collapse.

Lastly, in Sec. VII, we demontrate how the Roberts spacetime cannot be smoothly matched to a vacuum exterior. That is to say, the Roberts massless scalar field cannot represent a gravitational collapse with conservation of energy. The work in this chapter was motivated by the interesting behavior of the solution along hypersurfaces of constant mass-energy, as the scaling law predicted (were the smooth matching possible) would be different from that obtained in previous chapters.

[1] D. Christodoulou, "The Structure and Uniqueness of Generalized Solutions of the Spherically Symmetric Einstein Scalar Equations," *Commun. Math. Phys.*, vol. 109, pp. 591–611, 1987.
[2] D. Christodoulou, "The formation of black holes and singularities in spherically symmetric gravitational collapse," *Commun. Pure Appl. Math.*, vol. 44, no. 3, pp. 339–373, 1991.
[3] D. Christodoulou, "Bounded variation solutions of the spherically symmetric Einsteinscalar field equations," *Commun. Pure Appl. Math.*, vol. 46, pp. 1131–1220, 1993.
[4] D. Christodoulou, "Examples of naked singularity formation in the gravitational collapse of a scalar field," *Annals Math.*, vol. 140, pp. 607–653, 1994.
[5] M. W. Choptuik, "Universality and scaling in gravitational collapse of a massless scalar field," *Phys. Rev. Lett.*, vol. 70, pp. 9–12, 1993.
[6] P. R. Brady, C. M. Chambers, and S. M. C. V. Goncalves, "Phases of massive scalar field collapse," *Phys. Rev.*, vol. D56, pp. 6057–6061, 1997.
[7] P. Bizon, "How to make a tiny black hole?," *Acta Cosmologica*, vol. 22, p. 81, 1996.
[8] M. W. Choptuik, "Critical behaviour in gravitational collapse," *Prog. Theor. Phys. Suppl.*, vol. 136, pp. 353–365, 1999.
[9] C. Gundlach, "Critical phenomena in gravitational collapse," *Adv. Theor. Math. Phys.*, vol. 2, pp. 1–49, 1998.
[10] C. Gundlach, "Critical phenomena in gravitational collapse," *Living Rev. Rel.*, vol. 2, p. 4, 1999.
[11] C. Gundlach, "Critical phenomena in gravitational collapse," *Phys. Rept.*, vol. 376, pp. 339–405, 2003.
[12] C. Gundlach and J. M. Martin-Garcia, "Critical phenomena in gravitational collapse," *Living Rev. Rel.*, vol. 10, p. 5, 2007.
[13] M. D. Roberts, "Scalar Field Counterexamples to the Cosmic Censorship Hypothesis," *Gen. Rel. Grav.*, vol. 21, pp. 907–939, 1989.
[14] M. E. Cahill and A. H. Taub, "Spherically symmetric similarity solutions of the Einstein field equations for a perfect fluid," *Commun. Math. Phys.*, vol. 21, pp. 1–40, 1971.
[15] D. M. Eardley and L. Smarr, "Time function in numerical relativity. Marginally bound dust collapse," *Phys. Rev.*, vol. D19, pp. 2239–2259, 1979.
[16] C. W. Misner and D. H. Sharp, "Relativistic equations for adiabatic, spherically symmetric gravitational collapse," *Phys. Rev.*, vol. 136, pp. B571–B576, 1964.
[17] W. Israel, "Singular hypersurfaces and thin shells in general relativity," *Nuovo Cim.*, vol. B44S10, p. 1, 1966. [Nuovo Cim.B44,1(1966)].
[18] C. Barrabes and W. Israel, "Thin shells in general relativity and cosmology: The Lightlike limit," *Phys. Rev.*, vol. D43, pp. 1129–1142, 1991.
[19] P. R. Brady, "Analytic example of critical behaviour in scalar field collapse," *Class. Quant. Grav.*, vol. 11, no. 5, pp. 1255–1260, 1994.
[20] Y. Oshiro, K. Nakamura, and A. Tomimatsu, "Critical behavior of black hole formation in a scalar wave collapse," *Prog. Theor. Phys.*, vol. 91, pp. 1265–1270, 1994.
[21] M. D. Roberts, "Hybrid imploding scalar and ads spacetime," 2014.
[22] H. Maeda, "The Roberts-(A)dS spacetime," *Class. Quant. Grav.*, vol. 32, no. 13, p. 135025, 2015.
[23] K. Nakao, "On a quasilocal energy outside the cosmological horizon," 1995.
[24] P. C. Vaidya, "Radiation absorbing systems," *Proc. Nat. Acad. Sci.*, vol. 21, pp. 193–204, 1951.
[25] V. Husain, G. Kunstatter, B. Preston, and M. Birukou, "Anti-de Sitter gravitational collapse," *Class. Quant. Grav.*, vol. 20, pp. L23–L30, 2003.