Finite element dynamic analysis of beams on nonlinear foundations under a moving oscillator

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Abstract

This paper presents a study on the dynamic response of beams on elastic foundations, subjected to uniformly moving oscillators. With a finite element model programmed in MatLab environment the response of the system is studied for three different types of mechanical behaviour of the foundation: (a) linear elastic (classical Winkler model), (b) nonlinear elastic (in which the foundation reaction has a cubic dependence on the beam displacement) and (c) bilinear elastic (with different compressive and tensile stiffness). The effects of the oscillator's natural frequency and velocity and of the foundation's stiffness and damping are investigated. In particular, critical velocities of the oscillator and ranges of velocities for which the system is dynamically unstable are determined and the results are validated by comparison with the literature.
Keywords: Nonlinear elastic foundation; Moving oscillator; Dynamic amplification; Critical velocity; Instability

1. Introduction

The interaction between elastic structures and moving mechanical systems has been a topic of interest for well over a century and, nowadays, such interest has been even more stimulated by the progresses in transportation systems. For some types of soils, modern high-speed trains are able to move with velocities comparable with the minimal phase velocity of wave propagation in the elastic supporting structure (Metrikine and Verichev, 2001), causing vibrations whose amplitudes may be significantly higher than the deflections due to static loads. These vibrations may damage the supporting structures and seriously influence the comfort and safety of the passengers. Therefore, it is interesting to study the dynamic response of structures supporting moving mechanical systems in order to mitigate the above mentioned effects.

The elastic structures are commonly represented by a finite or infinite beam supported by a uniform or non-uniform viscoelastic linear or nonlinear foundation. Various foundation models such as Winkler, Pasternak, Vlasov or Reissner and either Euler-Bernoulli or Timoshenko beam models have been used. Concerning the moving system, three types of models have been mainly employed in the literature thus defining three different problems: (i) the moving oscillator (spring-mass-dashpot) problem, which is considered when the stiffness of the moving subsystem is finite and its inertial effects are not negligible; (ii) the moving mass problem, which may be conceived as a subcase of the moving oscillator problem when the stiffness of the moving subsystem approaches infinity; (iii) the moving load problem, which also neglects the inertia of the moving subsystem. The moving oscillator problem has been object of many works. Pesterev and Bergman (1997a) and Pesterev and Bergman (1997b) considered a linear conservative finite beam carrying a moving undamped oscillator and proposed a mathematical formulation that allowed the solution of the interaction problem in a series of the eigenfunctions of the elastic system. Then, the time-dependent coefficients of the expansion were obtained by solving a set of linear ordinary differential equations. A later extension of the method incorporated proportional damping (Pesterev and Bergman, 1998). Based on the previous approach, Omenzetter and Fujino (2001) examined the vibrations of a proportionally damped linear moving multi-degree-of-freedom oscillator interacting with the beam at several contact points. Similarly to the approach of Sadiku and Leipholz (1987), Yang et al. (2000) analysed a spring-mass moving oscillator, solving by numerical integration the final integral equation for the beam displacement. Fourier transforms, for both space and time variables, were used by Bitzenbauer and Dinkel (2002) to find the dynamic response of a linear multi-degree-of-freedom system moving along an infinite beam; the system was excited by the vertical imperfections of the track and its initial conditions were neglected. Muscolino and Palmeri (2007) studied the response of beams resting on viscoelastic foundations and subjected to moving oscillators. Metrikine and Verichev (2001) investigated the stability of an oscillator moving at constant velocity along an infinite Timoshenko beam on a foundation and determined the instability domains in the space of the system parameters by employing the D-decomposition method. Later they also studied the stability of a moving bogie (Verichev and Metrikine, 2002). Galerkin′s method has also been applied to reduce the partial differential equations of motion to a set of coupled ordinary differential equations containing periodic coefficients that is numerically solved. This approach was followed by Yoshimura et al. (1986) for a simply supported beam subjected to a moving oscillator including the effects of geometric nonlinearity, by Katz et al. (1987) for a simply supported beam subjected to a moving load whose amplitude is deflection dependent, and by Ding et al. (2014) in the study of the dynamic response of the oscillator-pavement coupled system by modelling the pavement as a Timoshenko beam resting on a six-parameter nonlinear foundation. The finite element method (FEM) was also used to obtain the response of beams resting on elastic foundations and subjected to moving oscillators. Hino et al. (1985) studied the vibration of finite nonlinear beams subjected to a moving oscillator by using the FEM and Newmark's implicit time integration algorithm. Lin and Trethewey (1990) also presented a FEM formulation for the dynamic analysis of elastic beams subjected to a one-foot and a two-foot spring-mass-damper moving systems. A FEM approach was also employed by Chang and Liu (1996), who analysed the vibration of a nonlinear beam on elastic foundation subjected to an oscillator moving on a randomly varying in space beam profile.

In the present paper the finite element method is used in the study of the transverse transient response of a simply supported Euler-Bernoulli beam resting on a viscoelastic foundation and interacting with a moving oscillator. The oscillator moves at constant velocity along the longitudinal beam axis. The response of the system is studied for three types of mechanical behaviour of the foundation: (a) linear elastic (classical Winkler model), (b) nonlinear elastic (in which the foundation reaction has a cubic dependence on the beam displacement) and (c) bilinear elastic (with different compressive and tensile stiffness). The use of the finite element method is a simpler and more practical alternative to the analytical methods used in many of the works mentioned above. It also has the advantage of solving nonlinear problems for which analytical solutions are not available. The effects of the oscillator's natural frequency and velocity and of the foundation's stiffness and damping are investigated; the results obtained are validated by comparison with the literature. In particular, critical velocities of the oscillator and ranges of velocities for which the system is dynamically unstable are determined.

2. Finite element method formulation

A single-degree-of-freedom oscillator with mass m_1 and unsprung mass m_2, which is always in contact with the support beam, stiffness k and damping coefficient c moves at constant velocity v along the longitudinal beam axis (Figure 1). The set of equations that govern the motion of the system is composed by the equation of the oscillator and by the system of equations that results from the finite element discretization of the beam. Denoting by w_0 the transverse displacement of the beam cross section in contact with the oscillator and by y the upward displacement of mass m_1 measured from its the static equilibrium position when $w_0 = 0$, the equation of motion of the oscillator and the contact reaction force r between the moving oscillator and the beam are, respectively,

$$m_1\ddot{y} + c\dot{y} + ky = c\dot{w}_0 + kw_0, \tag{1}$$

and

$$r = -(m_1+m_2)g - m_2\ddot{w}_0 - c(\dot{w}_0-\dot{y}) - k(w_0-y), \tag{2}$$

where $\dot{()} = \frac{\mathrm{d}()}{\mathrm{d}t}$ and g is the acceleration of gravity.

The equations governing the motion of the beam can be obtained using the finite element formulation as

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{C}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{P} + r\mathbf{N}^{\mathrm{T}}(x_0), \tag{3}$$

where $\mathbf{P}$ represents the beam self-weight action. Viscous damping is taken into account by adding the term $\mathbf{C}\dot{\mathbf{d}}$ to the left hand side of (3), where $\mathbf{C}$ is the damping matrix of the structure. Rayleigh damping is assumed with

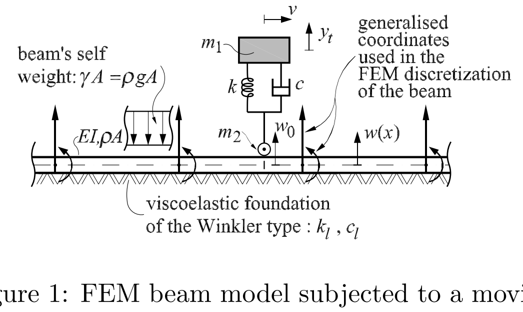

Figure 1: FEM beam model subjected to a moving oscillator.

$$\mathbf{C} = a_0\mathbf{M} \tag{4}$$

and

$$a_0 = 2\zeta\sqrt{\frac{2k_l}{\rho A}}, \tag{5}$$

where k_l is the stiffness of the linear elastic foundation. In (5), ζ_f is the damping factor, and according to Dimitrovová and Rodrigues (2012) it should be lower than 8% in the case of a linear foundation. The governing system of equations of motion can be written in matrix form as

$$\begin{aligned}
&\begin{bmatrix} \mathbf{M} + m_2\mathbf{N}^{\mathrm{T}}(x_0)\mathbf{N}(x_0) & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{y} \end{Bmatrix} \\
&+ \begin{bmatrix} \mathbf{C} + c\mathbf{N}^{\mathrm{T}}(x_0)\mathbf{N}(x_0) + 2m_2 v\mathbf{N}^{\mathrm{T}}(x_0)\mathbf{N}'(x_0) & -c\mathbf{N}^{\mathrm{T}}(x_0) \\ -c\mathbf{N}(x_0) & c \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}} \\ \dot{y} \end{Bmatrix} \\
&+ \begin{bmatrix} \mathbf{K} + k\mathbf{N}^{\mathrm{T}}(x_0)\mathbf{N}(x_0) + cv\mathbf{N}^{\mathrm{T}}(x_0)\mathbf{N}'(x_0) + m_2v^2\mathbf{N}^{\mathrm{T}}(x_0)\mathbf{N}''(x_0) & -k\mathbf{N}^{\mathrm{T}}(x_0) \\ -k\mathbf{N}(x_0) - cv\mathbf{N}'(x_0) & k \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ y \end{Bmatrix} \\
&= \begin{Bmatrix} \mathbf{P} - (m_1 + m_2)g\mathbf{N}^{\mathrm{T}}(x_0) \\ 0 \end{Bmatrix}.
\end{aligned} \tag{6}$$

3. Linear elastic foundation

Let ρ and A be the beam's density and transverse cross-sectional area, respectively. The kinetic energy of the beam finite element neglecting its rotational part, which is a good approximation in the case of a slender beam, is given by

$$\begin{aligned} T &= \int_0^l dT = \frac{1}{2}\int_0^l \dot{w}^{\mathrm{T}}\rho A\dot{w}dx = \\ &= \frac{1}{2}\dot{\mathbf{d}}^{eT}\mathbf{M}_t^e\dot{\mathbf{d}}^e, \end{aligned} \tag{7}$$

where $\mathbf{M}_t^e$ is the elementary consistent beam's mass matrix:

$$\mathbf{M}_t^e = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}. \tag{8}$$

Neglecting the axial and shear strains and denoting by EI the beam's bending stiffness, the elastic strain energy of the beam element is given by

$$\begin{aligned} U_b &= \frac{1}{2}\int_0^l (\mathbf{N}^{e\prime\prime}(x)\mathbf{d}^e)^{\mathrm{T}}\, EI\, \mathbf{N}^{e\prime\prime}(x)\mathbf{d}^e dx = \\ &= \frac{1}{2}(\mathbf{d}^e)^{\mathrm{T}}\mathbf{K}_b^e\mathbf{d}^e, \end{aligned} \tag{9}$$

where $\mathbf{K}_b^e$ is the elementary beam's stiffness matrix:

$$\mathbf{K}_b^e = \begin{bmatrix} \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}. \tag{10}$$

The total elastic strain energy of the finite element is given by

$$U = U_b + U_f, \tag{11}$$

where U_b is given by (9) and U_f is the elastic strain energy of the foundation underneath the finite element.

The elastic strain energy of a linear foundation underneath the finite element is given by

$$\begin{aligned} U_f &= \int_0^l dU_f = \frac{1}{2}\int_0^l w(x)^{\mathrm{T}}k_l w(x)dx = \\ &= \frac{1}{2}\int_0^l (\mathbf{N}^e(x)\mathbf{d}^e)^{\mathrm{T}}\, k_l\, \mathbf{N}^e(x)\mathbf{d}^e dx = \\ &= \frac{1}{2}(\mathbf{d}^e)^{\mathrm{T}}\mathbf{K}_f^e\mathbf{d}^e, \end{aligned} \tag{12}$$

where k_l is the stiffness of the foundation and $\mathbf{K}_f^e$ is the elementary stiffness matrix of a linear foundation:

$$\mathbf{K}_f^e = k_l\begin{bmatrix} \frac{13}{35}l & \frac{11}{210}l^2 & \frac{9}{70}l & -\frac{13}{420}l^2 \\ \frac{11}{210}l^2 & \frac{1}{105}l^3 & \frac{13}{420}l^2 & -\frac{1}{140}l^3 \\ \frac{9}{70}l & \frac{13}{420}l^2 & \frac{13}{35}l & -\frac{11}{210}l^2 \\ -\frac{13}{420}l^2 & -\frac{1}{140}l^3 & -\frac{11}{210}l^2 & \frac{1}{105}l^3 \end{bmatrix}. \tag{13}$$

The elementary vector of internal forces of the foundation is linearly related to the nodal elementary displacements

$$\frac{\partial U_f}{\partial \mathbf{q}^e} = \mathbf{K}_l^e\mathbf{q}^e \tag{14}$$

A uniform linear elastic foundation with stiffness per unit length equal to 250 kN/m^2 was considered. Extreme values (a) of positive (upward) and negative (downward) displacements of the beam and (b) of the mass m_1, (c) accelerations of the mass m_1

and (d) contact reaction force r, were determined in a parametric analysis with respect to the oscillator velocity. The considered velocity range is from 50 to 300 m/s. The finite element mesh used is uniform and is composed of 200 elements in the case of the more compliant oscillator and of 400 elements in the case of the stiffer oscillators. More refined meshes were also used in this study but we found no significant differences. The time increment was chosen in order to obtain an increment of the oscillator's displacement in one time step equal to 20% of the length of the finite element. The stiffness values used are the same of those in (Dimitrovová and Rodrigues, 2012) and (Jorge, 2013) in the case of the moving load problem.

Results for the three different oscillators moving on a simply supported beam with length L = 200 m on a uniform undamped foundation with linear stiffness $k_l =$ 250 kN/m^2 are presented in Figure 2. The curves of the beam maximum displacements (Figure 2 (a)) present a peak for a velocity of the oscillators in the range 204 to 211 m/s practically independent of the value of the stiffness of the oscillator. The occurrence of this peak also practically coincides with the critical velocity of the moving load problem in (Dimitrovová and Rodrigues, 2012) and (Jorge, 2013); thus it corresponds to the occurrence of a resonance effect when the velocity of the moving mass becomes equal to the minimum phase velocity of waves in the corresponding ideal beam-foundation system with a beam of infinite length (Metrikine and Verichev, 2001) (Metrikine and Dieterman, 1997). The same curves present a second peak for a range of velocities higher than the critical velocity. This second peak is related to the occurrence of a dynamic instability induced by anomalous Doppler waves radiated by the moving object (Metrikine and Verichev, 2001) (Metrikine and Dieterman, 1997). The range of velocities for which this dynamic instability occurs increases with the stiffness of the oscillator: the second peak is nearly imperceptible for the more compliant oscillator, it occurs for a range between 232 and 238 m/s in the case of the oscillator with intermediate stiffness and it occurs for a large range of velocities in the case of the stiffer oscillator.

The curves of the mass maximum displacements (Figure 2 (b)) also present two peaks and positive displacements may occur for supercritical velocities. The curves of the maximum mass accelerations and maximum contact reaction forces (Figure 2 (c, d)), in the case of the two more compliant oscillators, present high values only for velocities near the critical velocity and in the dynamic instability region. However the stiffer oscillator presents very high accelerations for velocities much smaller than the critical velocity. For real vehicles, these high accelerations damage the supporting structures and cause discomfort of the passengers. Positive values of the contact reaction force that indicate a derailment of the oscillator are also obtained in the three cases for velocities near the critical velocity and in the dynamic instability region.

Results for the three different oscillators moving on a simply supported beam with length $L =$ 200 m on a uniform damped foundation with linear stiffness of 250 kN/m^2 and damping factor ζ equal to 2% are presented in Figure 3. For all the oscillators, the introduction of viscous damping reduces the maximum beam displacements without practically changing the critical velocity (Figure 3 (a)). The maximum displacements and accelerations of the mass m and the maximum values of the contact reaction force are also considerably reduced in the damped case (Figure 3 (b, c, d)). However, positive values of the contact reaction force that would lead to the derailment of the oscillators are still obtained.

4. Nonlinear elastic foundation

The *force per unit length - displacement* relation of the soil under the beam is

$$F_f = F_l + F_{nl} = k_l w + k_{nl} w^3. \tag{15}$$

The elastic strain energy of the foundation underneath the finite element is given by

$$\begin{aligned} U_f &= \int_0^l \frac{1}{2} k_l w^2 + \frac{1}{4} k_{nl} w^4 dx = \\ &= \int_0^l \left\{ \frac{1}{2} k_l [\mathbf{N}^e(x)\mathbf{d}^e]^2 + \frac{1}{4} k_{nl} [\mathbf{N}^e(x)\mathbf{d}^e]^4 \right\} dx. \end{aligned} \tag{16}$$

The elementary vector of internal forces is given by

$$\begin{aligned} \frac{\partial U_f}{\partial \mathbf{d}^e} = \int_0^l \Big\{ & \mathbf{N}^{e\mathrm{T}}(x) k_l \mathbf{N}^e(x)\mathbf{d}^e + \\ & + \mathbf{N}^{e\mathrm{T}}(x) k_{nl} [\mathbf{N}^e(x)\mathbf{d}^e]^3 \Big\} dx. \end{aligned} \tag{17}$$

where there are two terms; one is linear and the other depends non-linearly on $\mathbf{d}^e$, that we designated as $\mathbf{Q}_{nl}^e(\mathbf{d}^e)$. The governing system of equations of motion is obtained by adding the first part of (6) to $\{\mathbf{Q}_{nl}(\mathbf{d})\ 0\}^\mathrm{T}$ where $\mathbf{Q}_{nl}(\mathbf{d})$ and $\mathbf{d}$ are the global vectors of nonlinear terms, obtained by assemblage of the elementary vectors $\mathbf{Q}_{nl}^e(\mathbf{d}^e)$ and $\mathbf{d}^e$. The effect of the nonlinear part of the foundation's stiffness (k_{nl}) in the case of a uniform foundation with $k_l =$ 250 kN/m^2 (linear part) and without damping is presented in Figures 4-5 for the three different oscillators moving on a simply supported beam. As in the linear case, the curves of the beam maximum displacements (Figures 4-5 (a)) present two peaks. A first peak, corresponding to the critical velocity, that is independent of the value of the stiffness of the oscillator. A second peak, for a range of velocities higher than the critical velocity and depending on the stiffness of the oscillator, related to the occurrence of a dynamic instability. It can be seen that the increase of k_{nl} results in (i) higher critical velocities and higher ranges of velocities for which dynamic instability occurs as well as (ii) a significant reduction of both upward and downward beam and mass displacements, mass accelerations and contact reaction force. As k_{nl} increases, the peaks corresponding to the critical velocity and to the region of dynamic instability become less pronounced (for $k_{nl} = 2.50 \times 10^4$ kN/m^4, the second peak disappears in the case of the more compliant oscillator). However the stiffer oscillator presents always high accelerations for velocities much smaller than the critical velocity. Positive values of the contact reaction force are also obtained in the three oscillators for velocities near the critical velocity.

5. Bilinear elastic foundation

A more realistic force-displacement relation (with different compressive and tensile stiffness) of the soil underneath the beam used in the bilinear case may be defined by

$$F_f = \begin{cases} k_{l-}w, \text{ if } w < 0 \\ k_{l+}w, \text{ if } w > 0 \end{cases} .$$

The elastic strain energy of the foundation underneath the finite element, the elementary vector of the foundation internal forces and the elementary tangent stiffness matrix of the foundation depend on six beam-foundation interaction patterns (Rodrigues, 2016). The number of patterns is limited by the fact that the shape of the beam axis is assumed to be a cubic polynomial, which has no more than one inflection point. The treatment of the cases where there is overall upward motion or overall downward motion in the element is very simple because they correspond to the case of a classical Winkler foundation element where the elementary internal force vector and tangent stiffness matrix (foundation's contribution) are given by their homologous of the linear foundation with $k_l = k_{l+}$ and $k_l = k_{l-}$, respectively. In the following subsections we derive the exact expressions of the elementary internal force vector and tangent stiffness matrix (foundation's contribution) for the other four interaction patterns between the beam and the foundation considered in this study.

5.1. Case: "upward motion on the left of the element"; Case: "upward motion on the right of the element"

We begin by considering the case in which the beam element moves upward in the segment of length s on the left side and moves downward in the segment of length $l-s$ on the right side. The cross section in the transition between the sections moving upward and the sections moving downward has abscissa s: the elastic strain energy of the foundation underneath the finite element is

$$\begin{aligned} U_f = {} & \frac{1}{2}k_{l+}\mathbf{d}^{e\mathrm{T}} \int_0^{s(\mathbf{d}^e)} \mathbf{N}^{e\mathrm{T}}(x)\mathbf{N}^e(x)dx\mathbf{d}^e \\ & + \frac{1}{2}k_{l-}\mathbf{d}^{e\mathrm{T}} \int_{s(\mathbf{d}^e)}^{l} \mathbf{N}^{e\mathrm{T}}(x)\mathbf{N}^e(x)dx\mathbf{d}^e. \end{aligned} \tag{18}$$

The case in which the beam element moves downward in a segment of length s on the left side and moves upward in the segment of length $l-s$ on the right side can be very simply obtained from the previous case by interchanging k_{l+} with k_{l-}. The cross section in the transition between the sections moving upward and the sections moving downward has abscissa s, which depends on the generalized coordinates $\mathbf{q}^e$: $s = s(\mathbf{q}^e)$.

5.2. Case: "upward motion on the left and on the right of the element"; Case: "upward motion in the middle of the element"

We begin by considering the case in which the beam element moves upward in the segment of length s_1 on the left side and in the segment of length s_2 on the right side and moves downward in the segment of length $l - s_2 - s_1$ in the middle. In this case, the elastic strain energy of the foundation underneath the finite element is

$$\begin{aligned} U_f = {} & \frac{1}{2}k_{l+}\mathbf{d}^{e\mathrm{T}} \int_0^{s_1(\mathbf{d}^e)} \mathbf{N}^{e\mathrm{T}}(x)\mathbf{N}^e(x)dx\mathbf{d}^e \\ & + \frac{1}{2}k_{l-}\mathbf{d}^{e\mathrm{T}} \int_{s_1(\mathbf{d}^e)}^{l-s_2(\mathbf{d}^e)} \mathbf{N}^{e\mathrm{T}}(x)\mathbf{N}^e(x)dx\mathbf{d}^e \\ & + \frac{1}{2}k_{l+}\mathbf{d}^{e\mathrm{T}} \int_{l-s_2(\mathbf{d}^e)}^{l} \mathbf{N}^{e\mathrm{T}}(x)\mathbf{N}^e(x)dx\mathbf{d}^e. \end{aligned} \tag{19}$$

The case in which the beam element moves downward in a segment of length s_1 on the left side and in a segment of length s_2 on the right side and moves upward in the segment of length $l - s_2 - s_1$ in the middle can be obtained from the previous case by interchanging k_{l+} with k_{l-}.

5.3. Results and discussion

The effect of a bilinear foundation is analysed next. In the analyses, the stiffness k_{l+} (for upward motions) decreases from 150 kN/m^2 to zero, while the stiffness k_{l-} (for downward motions) is kept equal to 250 kN/m^2 in all the cases. The rail used and the velocity range are kept the same as in the previous sections. The finite element meshes used in these analyses are composed of 400 elements. More refined meshes were also used in this study but, again, we found no significant differences.

From the observation of Figures 6-7 we conclude that the decrease of the tensile stiffness is accompanied by (i) a decrease of the critical velocity, that is a decrease of the velocity corresponding to the first peak in the curves of the beam and mass maximum displacements (as in the linear case, the critical velocity is the same for the three oscillators), (ii) an increase in the range of velocities for which dynamic instability occurs (as in the linear case, this range of velocities increases with the stiffness of the oscillator), (iii) an increase of the maximum displacements of the beam, (iv) an increase of the maximum displacements and accelerations of the mass and of the contact reaction force (as in the linear case, the maximum displacements and accelerations of the mass and the maximum values of the contact reaction force increase with the stiffness of the oscillator) and (v) an increase of the irregularity of the curves. For a vanishing tensile stiffness (tensionless foundation) (Figure 7) the beam maximum displacements become extremely large (well above the domain of validity of the geometrical linearity assumption). For velocities near the critical velocity, the maximum upward displacements of the beam are amplified when the tensile stiffness of the foundation decreases. On the other hand, contrarily to what happens in the moving load problem (Jorge, 2013), for supercritical velocities (that is in the region of dynamic instability) both the beam maximum downward and upward displacements are amplified when the tensile stiffness of the foundation decreases.

6. Conclusions

In this work, a finite element program was developed in MatLab environment to analyse the dynamic response of a simply supported beam on an elastic foundation, subjected to a uniformly moving oscillator. Using this program, critical velocities of three different oscillators and ranges of velocities for which dynamic instability occurs are determined for three different types of foundations: (a) linear elastic, (b) nonlinear elastic and (c) bilinear elastic, with or without viscous damping. The most important conclusions taken from the results presented in the previous sections are summarized next.

In the case of a linear foundation, the curves of the beam and mass maximum displacements for the three different oscillators moving on a simply supported beam present two peaks. The first one is practically independent of the value of the stiffness of the oscillator and coincides with the critical velocity in the moving load problem (Jorge, 2013); thus it corresponds to the occurrence of a resonance effect when the velocity of the moving oscillator becomes equal to the minimum phase velocity of waves in the beam-foundation system (Metrikine and Verichev, 2001);(Metrikine and Dieterman, 1997). The second one is related to the occurrence of a dynamic instability induced by anomalous Doppler waves radiated by the moving object (Metrikine and Verichev, 2001);(Metrikine and Dieterman, 1997). The range of velocities for which this dynamic instability occurs increases with the stiffness of the oscillator. It was also observed that an increase in the foundation stiffness leads to a reduction of peak values of displacements and to higher critical velocities and ranges of velocities for which the dynamic instability occurs. The consideration of viscous damping in the foundation reduces further the peak displacements and the second peak disappears for the more compliant oscillators. However the stiffer oscillator presents always high accelerations, that may damage the supporting structures and cause discomfort of the passengers, for velocities much smaller than the critical velocity. Positive values of the contact reaction force, that in a practical situation would lead to derailment of the oscillators, are also obtained when their velocities are near the critical velocity.

As for the case of a foundation with nonlinear (cubic) behaviour, it was concluded that the increase of the nonlinear part of the foundation's stiffness results in higher critical velocities and higher ranges of velocities for which dynamic instability occurs as well as in a significant reduction of both upward and downward beam and mass displacements, mass accelerations and contact reaction force. As in the linear case, the range of velocities for which dynamic instability occurs is always higher than the critical velocity. As the nonlinearity of the foundation increases, the peaks corresponding to the critical velocity and to the region of dynamic instability become less pronounced and in some cases the second peak disappears for the more compliant oscillators.

In the case of a foundation with bilinear behaviour, we conclude that the decrease of the tensile stiffness leads to a decrease of the critical velocity and to an increase in the range of velocities for which dynamic instability occurs. It also leads to an increase of the maximum upward displacements of the beam and to an increase of the maximum displacements and accelerations of the mass and of the contact reaction force. For a vanishing tensile stiffness (tensionless foundation) the beam maximum upward displacements become extremely large (well above the domain of validity of the geometrical linearity hypothesis).

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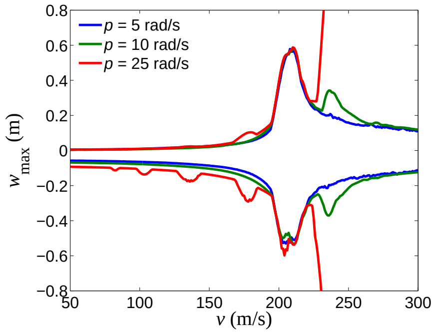
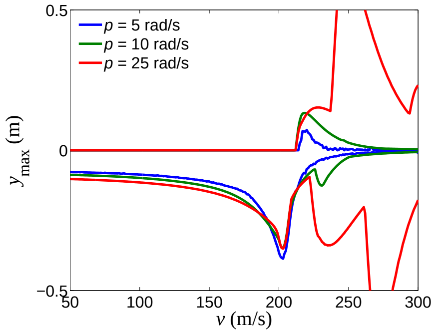
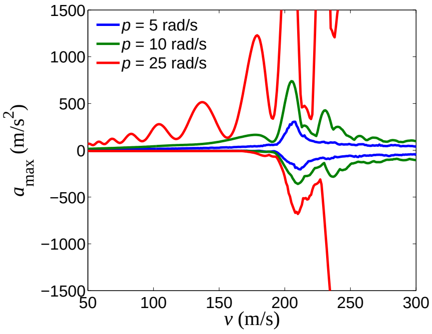


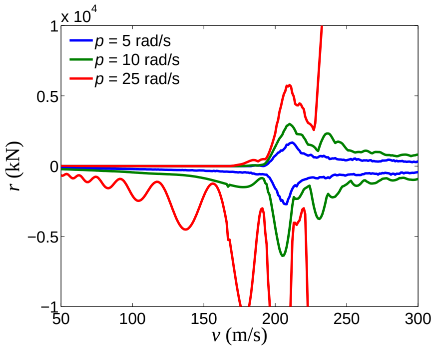
(a) Maximum displacements of the beam.



(b) Maximum displacements of the mass of the oscillator.

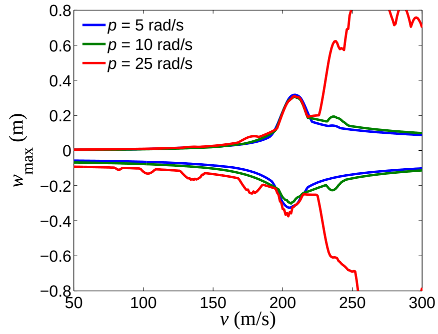


(c) Maximum accelerations of the mass of the oscillator.

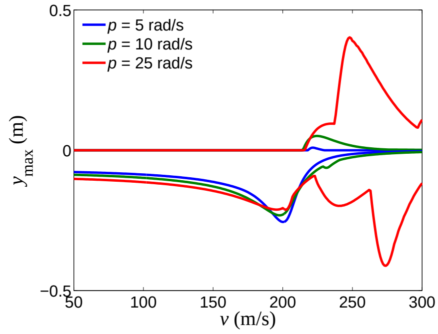


(d) Maximum values of the contact reaction force.

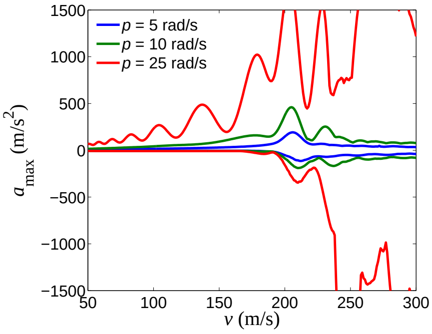
Figure 2: Uniform linear elastic foundation with $k_l = 250$ kN/m^2, $\zeta = 0\%$.



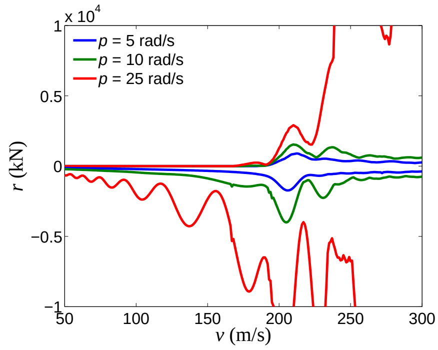
(a) Maximum displacements of the beam.



(b) Maximum displacements of the mass of the oscillator.

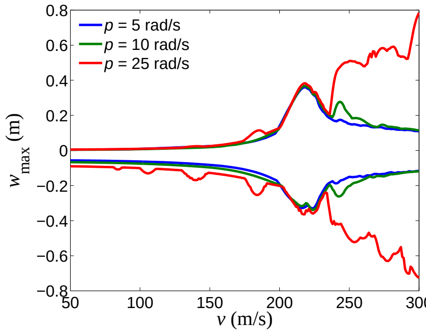


(c) Maximum accelerations of the mass of the oscillator.

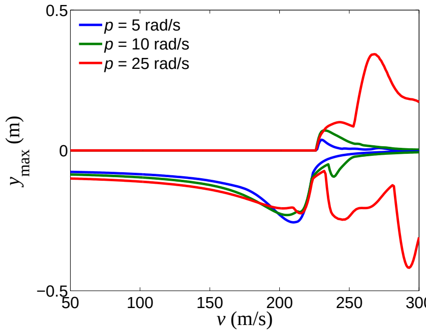


(d) Maximum values of the contact reaction force.

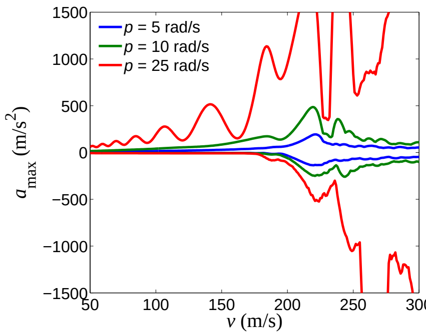
Figure 3: Uniform linear elastic foundation with $k_l = 250$ kN/m^2, $\zeta = 2\%$.



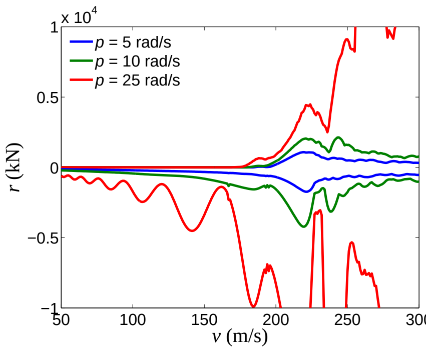
(a) Maximum displacements of the beam.



(b) Maximum displacements of the mass of the oscillator.

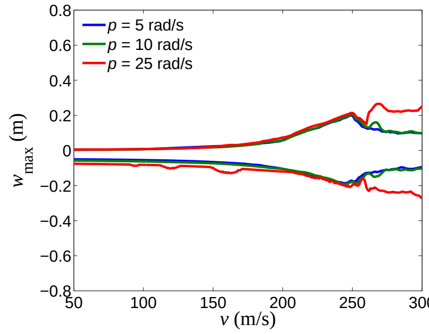


(c) Maximum accelerations of the mass of the oscillator.

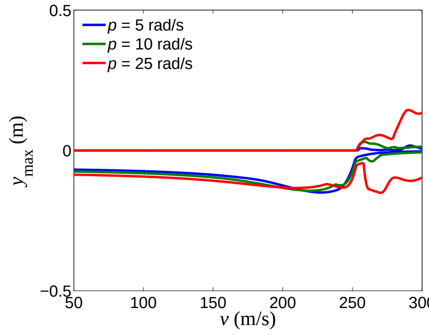


(d) Maximum values of the contact reaction force.

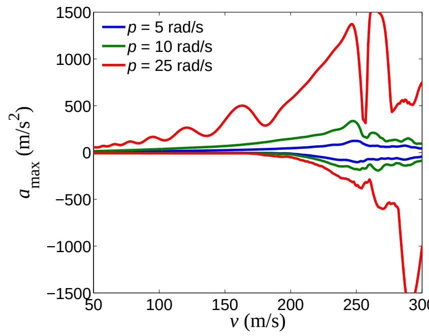
Figure 4: Effect of the nonlinear part of the foundation's stiffness on the upward and downward extreme displacements for a uniform foundation with k_{nl}= 2.5 $\times$ 10^3 kN/m^4, k_l= 250 kN/m^2, undamped.



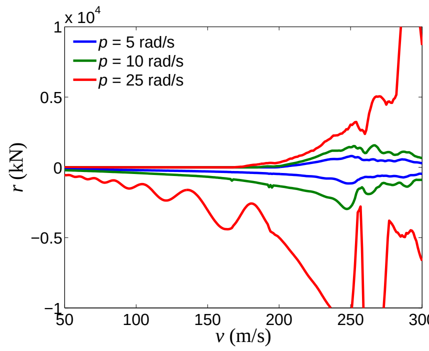
(a) Maximum displacements of the beam.



(b) Maximum displacements of the mass of the oscillator.

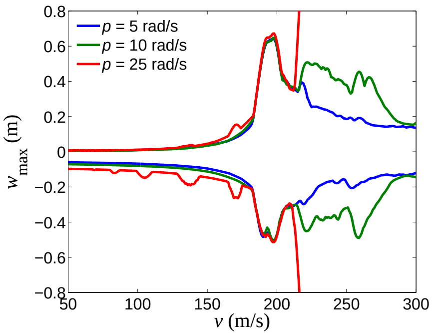


(c) AMaximum accelerations of the mass of the oscillator.

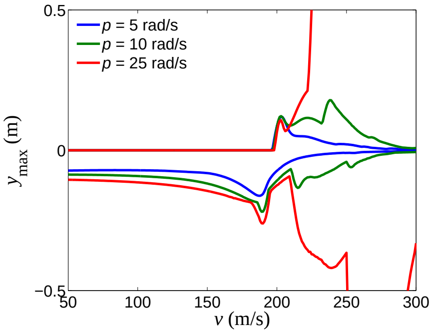


(d) Maximum values of the contact reaction force.

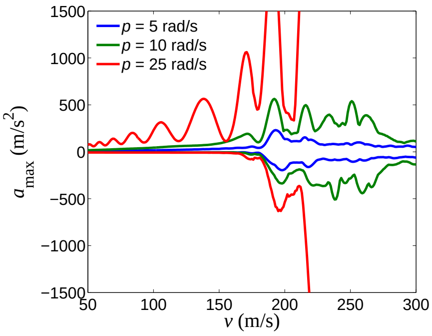
Figure 5: Effect of the nonlinear part of the foundation's stiffness on the upward and downward extreme displacements for a uniform foundation with k_{nl}= 2.5 $\times$ 10^4 kN/m^4, k_l= 250 kN/m^2, undamped.



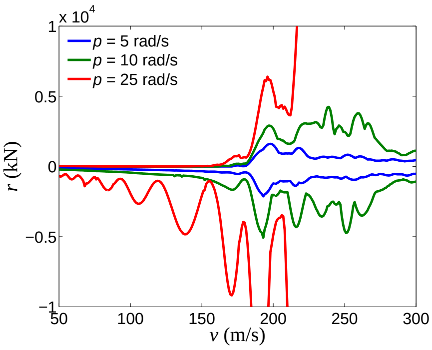
(a) Maximum displacements of the beam.



(b) Maximum displacements of the mass of the oscillator.

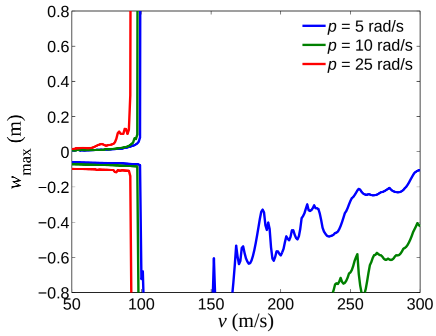


(c) Maximum accelerations of the mass of the oscillator.

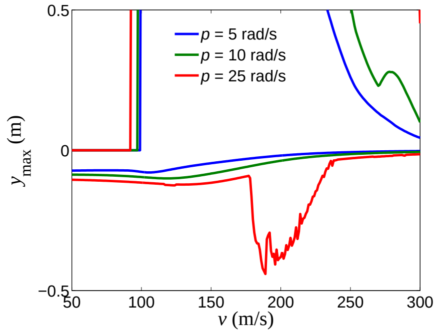


(d) Maximum values of the contact reaction force.

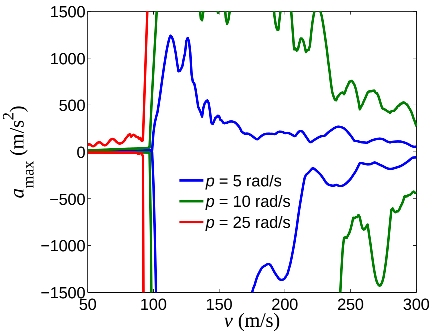
Figure 6: Bilinear foundation with $k_{l+} = 150$ kN/m^2, $k_{l-} = 250$ kN/m^2, undamped.



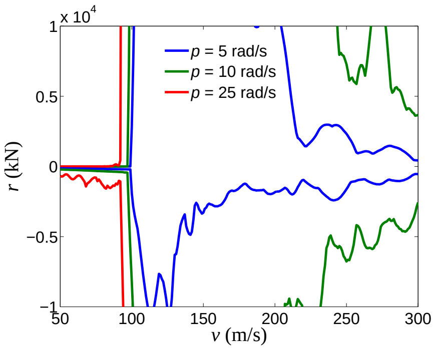
(a) Maximum displacements of the beam.



(b) Maximum displacements of the mass of the oscillator.



(c) Maximum accelerations of the mass of the oscillator.



(d) Maximum values of the contact reaction force.

Figure 7: Bilinear foundation with $k_{l+} = 0$ kN/m^2, $k_{l-} = 250$ kN/m^2, undamped.