Pricing real options with a hybrid genetic algorithm

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Abstract

This thesis proposes the implementation of a hybrid genetic algorithm that aims to produce random mathematic expressions for the pricing of real options. The training datasets were historic daily stock close prices and related parameters and the parameters derived from two usual option pricing models – the Black-Scholes model and the GARCH (1,1) model (Generalized Auto-Regressive Conditional Heteroscedasticity) – hence the algorithm being a hybrid. These two models were also used as benchmark models to compare the ability of the hybrid genetic algorithm to yield option values closer to the real ones. Tests made to this algorithm allowed to conclude that it is able to return closed mathematical expressions that can yield option prices that are up to 14% closer to real values than the prices determined by the Black-Scholes equation or the GARCH (1,1) model.

Keywords: hybrid genetic algorithm, option pricing, GARCH (1,1), Black-Scholes model, non-parametrized option pricing methods, metaheuristic algorithm.

Introduction

An option is a security giving the right but not the obligation to buy or sell an underlying asset under contractually stated conditions within a specified period of time. The exercise price of an option, the value at which the underlying asset is exchanged, is called the strike price (K). The time limit at which the option can be exercised is the maturity (T). An American option can be exercised at any moment until maturity is reached; a European option can only be exercised at maturity. The moneyness (M) is a measure of how much in-the-money is an option. For a call option, If the exercise price is under the stock trading price (S) the call is said to be in-the-money; otherwise it would be called out-of-the-money.

The value of an option depends on the market behaviour of the underlying asset, mainly, its volatility (σ), the risk free interest rate (r) and of the inherent dividends (D).

The purpose of an option is to hedge the price fluctuations of the underlying asset. As it depends directly on the stock price and the time left to maturity, given its lower face value, a given percentage change in the stock price will result in a larger percentage change in the option value.

Most methods used to price options are deeply established in the theoretical principles of the stock's stochastic processes' behaviour.

The most well-known model for pricing options is the Black-Scholes model (v. (& Scholes, 1973). This is a very useful method since it provides a closed form formula that yields an approximate enough value for the derivative. Nonetheless, such methods as the ones the like of the Black-Scholes model fall on assumptions that do not hold on the reality of the options' markets. The Black-Scholes model assumes ideal market conditions for the stock and the option: the risk-free interest rate is constant through time; the stock price follows a random walk in continuous time with a distribution that is log-normal and with constant variance of the rate of return; the option is of the European type; there are no transaction costs and it is possible to borrow any fraction of the price of a security; and there are no penalties to short selling.

Other models evolving from the modelling of Wiener processes and regarding the volatility stochastic nature of the returns are, for example, the Heston model (Heston, 1993), the Hull-White model (Hull & White, 1990), the CEV model (Constant Elasticity of Volatility) and the SABR model (Stochastic Alpha Beta Rho). The great advantage of the stochastic models is that they provide closed-form expressions easy to implement and yielding approximate enough results for the value of an option.

The auto-regressive models, such as the ARMA (Auto-Regressive Moving Average) and the GARCH (Generalized Auto-Regressive Conditional Heteroscedasticity), may also be fitted to the stochastic process of the volatility. These models are of particular interest to describe the auto-correlation of the returns or the volatility. Also, they allow for the introduction of such effects as the heteroscedasticity and the volatility clustering.

Many recent developments were made referring to the mathematical description of volatility and stock returns. For example, Appadoo & Thavaneswaran, 2013 propose the treatment of the stock prices as fuzzy numbers prior to the option valuation through methods such as the Black-Scholes equation. The analysis of the volatility over different time scales was made by Andersen et al, 2015, who also included the modelling of volatility time-diffusion jumps. Asai & McAleer, 2015, finding a way to describe the volatility's asymmetric, retroactive and leverage effects, proposed a model referred to as AMWSV (Asymmetric Multifactor Wishart Stochastic Volatility).

Oher side of the volatility modelling is the forecasting of events that escape the normal behaviour of the underlying assets and the related markets. Bollerslev & Todorov, 2014 relied on the EVT (Extreme Value Theory) to describe the heavy-tailless of the distribution of the volatility values. Different approaches on the development of models for the jump-diffusion dynamics were made by Byun et al, 2015 and Gong & Zhuang, 2016.

A generalization of the Black-Scholes equation was developed by Kleinert & Korbel, 2016 by applying a fractional double diffusion equation as proposed by Benoit Mandelbrot in 1982 under the designation Truncated Lévy Flight.

The development of hybrid models among the several stochastic models was followed by such papers as the one written by Rombouts & Stentoft, 2015 which mingled two heteroscedastic models to produce an MN-GARCH model (GARCH in-mean and the nonlinear GARCH)

With the objective of better explaining the dependency of the price of an option a multivariate approach of the autoregressive models was proposed by Baldovin et al, 2015. A multi-factor methodology was developed under the principle of parsimony by Calvet et al, 2015 to model the volatility jumps. Still on the side of the factor analysis, Gourieroux & Monfort, 2015 produced a model for multi-dimensionally comparing the relation between several factors and variables, turning the model able for the study of portfolios of several stocks and derivatives. Another kind of multi-component analysis is the one developed by Majewski et al, 2015, where a multi-factor analysis of the volatility was made under a discontinuous framework by generalising two models – the multi-component GARCH and the HARG (Heterogeneous Autoregressive Gamma). Arismendi & De Genaro, 2016, through a Monte Carlo simulation studied a model known as MGEE (Multivariate Generalised Edgeworth) which incorporated the information of the moments of a well-known distribution of several assets in order to find an analytical solution. Still on the side of multivariate analysis, Paolella & Polak, 2015 suggested to model a set of return data through the multivariate conditional distribution of a process with the characteristics of a GARCH stochastic process with parameters optimized by an expectation-maximization algorithm.

Finally, an approach relying on robust optimization instead of assuming a probabilistic model was suggested in Bandi & Bertsimas, 2014, thus providing better computational tractability and modelling flexibility.

Although the models referred above rely in the deep and intricate theoretical behaviour of the markets, stocks and options exchange, they depend on heavily parametrized methods which narrows their application and often need to be derived to high orders in order to accommodate different types of options or underlying assets. Moreover, provided the computational tractability of a method, it should be noted that the application of a model also depends on its goodness-of-fit. As stated by Jarrow & Kwok, 2015, an appropriate model should be fitted relying in classic test-statistics such as Ljung-Box, MvLeod-Li or Hong-Li tests.

It is because of the need for flexibility and tractability that the metaheuristic methods should be considered. Although computational demanding they may be, more approximate results to the real ones might be achieved by implementing models that are entirely free of any parametrization or hampered by theoretical assumptions.

Methods such as the artificial neural networks and the genetic algorithms have proven to be able to produce reliable results (v. Chidambaran, 2010 and Gradojevic et al, 2009).

In this thesis we present the implementation of a hybrid genetic algorithm (hGA) to determine random closed-form mathematical expressions for pricing options. Hybrid in the sense that, among all the variables and parameters that can be introduced in randomly generated mathematical expressions, the parameters belonging to the GARCH (1,1) model and to the Black-Scholes equation, or from it derived, are also introduced in those expressions. This hybridization of the Genetic Algorithm is expected to enhance its performance by introducing in the global genepool individuals, such as the Black-Scholes formula, which can contribute to proven good genes for the further generation of fitter individuals. As a result, a method less computationally demanding and abler to find a good solution is expected to arise.

The remainder of this paper is organized as follows. Section 1 describes the Black-Scholes model and section 2 the GARCH (1,1) model. Section 3 introduces the Genetic Algorithm (GA). In section 4 we describe the data, the implementation of the hGA and provide several test results. Section 5 concludes.

1 Black-Scholes model

Born of the seminal work of Fischer Black and Myron Scholes in 1973, the Black-Scholes model grew to become the most widely used formula to price options. Its derivation depends on some assumptions based on the ideal behaviour of the markets, such as: known and constant risk-free interest rate; the price of the underlying stock abides the one of a Wiener process, time continuous, with variance rate proportional to the square of the stock price, thus the distribution of possible stock prices at the end of any finite interval is log-normal; the stock pays no dividends; the option is European; there are no transaction costs in buying or selling the stock or the option; the price of the stock is continuous, it is possible to borrow any fraction of the price of a security; there are no penalties to short selling.

Based on the principle of hedging securities, on the risk-free limit, auto-replicating portfolios, and the conditions stated above, the Black-Scholes formula is as follows.

$$W(S,t) = \varepsilon S\Phi(\varepsilon d_1) - \varepsilon K e^{-rT^*}\Phi(\varepsilon d_2)\ ,\ \varepsilon = \begin{cases} 1, & for\ a\ call\ option \\ -1, & for\ a\ put\ option \end{cases}$$

$$d_1 = \frac{ln(S/K) + T^*(r + \sigma^2/2)}{\sigma\sqrt{T^*}} \qquad (1)$$

$$d_2 = \frac{ln(S/K) + T^*(r - \sigma^2/2)}{\sigma\sqrt{T^*}} = d_1 - \sigma\sqrt{T^*}$$

Where W is the option price, S is the stock price at time t, K is the strike price, T^* is the time left to maturity, σ is the volatility and r is the risk free interest rate.

2 GARCH model

The Generalized Auto-Regressive Conditional Autoregressive model is very appropriated to describe the volatility path of a stock. The fact that it is auto-regressive is useful to accommodate volatility clusters created by volatility spikes that decrease through time. Moreover, the heteroscedasticity element of this model mimics the increased uncertainty for longer periods of time ahead.

The GARCH model is stated on the definition of the log-stock returns over time (r_t) as a function of the average (μ_t) and the innovation (a_t): $r_t = \mu_t + a_t$. The innovation is said to follow a GARCH (m, s) process if:

$$a_t = \sigma_t \, \epsilon_t \tag{2}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \, a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \, \sigma_{t-j}^2 \tag{3}$$

Where σ_t is the conditional volatility, α and β are the conditional volatility parameters with $\alpha_0 > 0$, $\alpha_i, \beta_j > 0$ and $\sum_{i=1}^{max(m,s)}(\alpha_i + \beta_i) < 1$, to ensure the condition for stationary volatility. $\{\epsilon_t\}$ is a series of iid variables.

The average is usually modelled according to an ARMA (p, q) model:

$$\mu_t = \sum_{i=1}^{p} \phi_i \, r_{t-i} - \sum_{j=1}^{q} \theta_j \, a_{t-j} \tag{4}$$

After fitting the GARCH parameters through a likelihood function, the volatility foreseeable values are determined through a Monte Carlo simulation.

3 Genetic Algorithm

The aim of the genetic algorithm applied to pricing options is to generate a random closed-form mathematical expression comprising the usual mathematical operations and functions, including probability distribution functions, and several parameters that might be useful, such as the stock price, interest rate, moneyness, strike price or time left to maturity.

The genetic algorithm works mimicking the genetic evolution. Starting from a randomly generated population, each individual is genetically different, constituted by genes that characterize it. Here, the genes are each of the constituents of the mathematical expression. The individuals are classified according a fitness value that dictates how fit they are into providing a solution close to the real values. The fitness value determines the probability an individual has of being chosen to pass on its genes. After that selection step, one or two individuals are subject to genetic operations. The usual genetic operations are the reproduction, when an individual is simply passed on from a population into the population of the next generation, the mutation, that consists in the change of one or more of the individual's genes, and the cross-over, which happens when two individuals are spliced and a segment of one individual is exchanged with a segment of the other individual. In a genetic algorithm, the individuals chosen to reproduce and produce the individuals of the next generation might or might not be themselves passed on the next generation. Either way, each generation ends when a maximum number of new individuals is achieved. After a defined number of generations, or when the objective function is realised, the algorithm is terminated.

In the case presented here, the individuals were coded as numeric vectors, each position, each gene, coding for one element of the mathematical function. The other elements of the individual are: the corresponding mathematical expression, as a string; the fitness value; and the generation in which it was produced. As the individuals chosen to generate new individuals, the sons, were kept from one generation to the next, the three genetic operations introduced in the algorithm were the mutation of a single gene coding for a parameter, the mutation of an aggregated segment of the individual and the cross-over of two individuals.

The hybrid characteristic of the algorithm is due to the inclusion of the Black-Scholes model parameters, and its derived parameters known as the Greeks, and the parameters of the GARCH (1,1) model. A list of the parameters used in the development of the hGA can be seen in *Table 1*.

Table 1: Parameters and constants used in building the hGA expressions.

Symbol	Deffinition	Source
$0, 1, 2$	Constants, integers	
pi	Constant, pi	
S	Stock price over time t (S_t)	Market
T	Time left to maturity ($T^* = T - t$)	Option contract
K	Strike price	Option contract
r	Risk-free interest rate	Market
M	Moneyness ($M = S_t/K$)	
$logR$	Log-return	
$sigma$	Volatility	
$d1$	Black-Scholes parameter	BS
$d2$	Black-Scholes parameter	BS
$delta$	Black-Scholes derived parameter (Greek)	BS
$vega$	Black-Scholes derived parameter (Greek)	BS
$teta$	Black-Scholes derived parameter (Greek)	BS
ro	Black-Scholes derived parameter (Greek)	BS
$a0$	GARCH (1, 1) parameter	GARCH
$a1$	GARCH (1, 1) parameter	GARCH
$b1$	GARCH (1, 1) parameter	GARCH
$cond.sigma$	GARCH (1, 1) parameter	GARCH

The hGA is trained using a dataset of historic stock values. The determination of each individual's fitness is computed through a measure of the residues between the calculated values and the real ones.

4 Implementation of the hybrid Genetic Algorithm

The programming of the hGA was made on the R platform on RStudio.

The data used as training dataset was the one provided through the Quandl API of American stocks traded on NASDAQ. The values of the risk-free interest rates were the ones got on the (Board of Governors of the Federal Reserve System, 2016).

To construct the rest of the training dataset, for each of the observations on the stock price, to which a strike price and a maturity was attributed, the annualised volatility, moneyness and log-return value were determined. The Black-Scholes parameters and the Greeks and the GARCH (1,1) parameters were also added to the training dataset. For the tests done over the hGA, it was only determined the price of European call options. For its redundancy, tests to determine the put option price were not realised.

A scheme of the computational implementation of the hGA is presented in *Figure 1*.

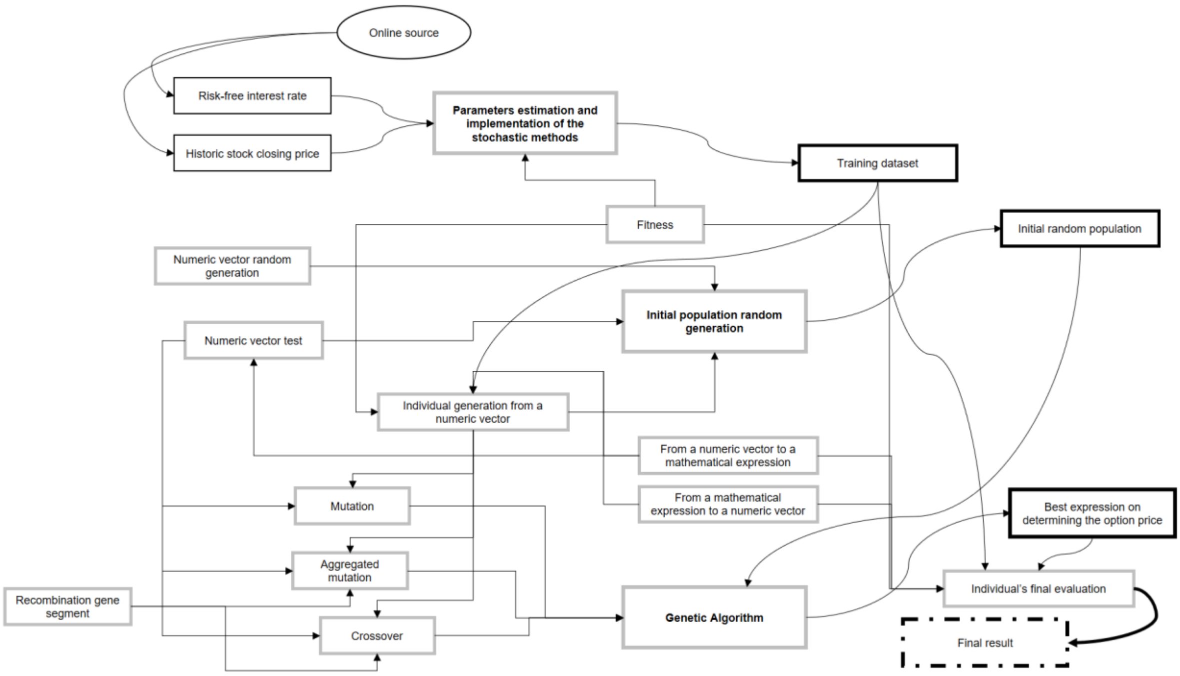

Figure 1: scheme for the implementation of the hybrid Genetic Algorithm. The boxes with grey outline are hGA related functions, the ones with black outline are the outcome of functions. The curve arrows stand for information flux, while the right angled refer to the order in which the functions are connected.

The three central components of the hGA are the function for the generation of the initial population, the fitness function and the genetic algorithm function *per se*. The other functions shown in *Figure 1* are secondary functions which purpose is not dwelt upon in this abridged text.

The fitness function is probably one of the most important subjects for the proper function of the hGA. This measure is usually applied differently in each problem. In this case it was tailored to be suitable to the particularities of option pricing. The next equation was the one devised for this purpose:

$$fitness = \frac{1}{1 + \frac{\sqrt{\sum_{i=1}^{N}(residue_i^2)}}{N}} \times \left[1 - \left(\frac{\sum_{i=1}^{N}(Option_{GA_i} == 0)}{N}\right)^4\right] \tag{5}$$

As stated before, the fitness is a measure of the residues between the computed values and the real ones. The first term of the fitness function gives a standardized value of this measure. The second term of the fitness function is a handicap feature introduced due to the fact that, since there are no negative valued options, just null ones, a mathematical expression that prices an abnormal amount of options as null should be put back.

Several tests were made regarding several stocks with very different amounts of observations. In detail we present the result of applying the hybrid genetic algorithm to the stocks of AmGen, comprising over 7600 observations. For these stocks, 80 simulations were run varying the number of generations (between 5 and 500), the maximum population of each generation (between 10 and 500) and the number of individuals to be kept from one generation to the next (between 5 and 30). The output of one of these simulations, the best individual found over all the 80 simulations is the mathematical expression shown in equation (6) after editing.

$$c = T^{*} + d_1 + \mathrm{P} \times \sqrt{r\,K} - d_2 + \mathrm{P} \times \sqrt{r\,S} - b_1^{\left(\nu \times \sqrt{|d_1{}^2 - \mathrm{P}|}\right)} \times b_1^{\left(4\,S\,\Theta \times \sqrt{|\Delta\,\nu - 2||\pi\,\sigma - r||b_1{}^2 - 2||\pi\,\sigma - \Delta|}\right)} + b_1 + 2^{\left(T^{*}\,d_1 \times \sqrt{|\sigma_{cond}\Theta - \nu|}\right)} + 1 + S - K - 2\,exp(-2\,a_0) \times \Phi(d_2) \quad (6)$$

Figure 2 shows a graphic representation comparing the real price of the call option, as if one possessed complete knowledge of the future stock price, and the price of the option as determined by the Black-Scholes equation, by the GARCH (1,1) model, and by the hGA generated expression presented above.

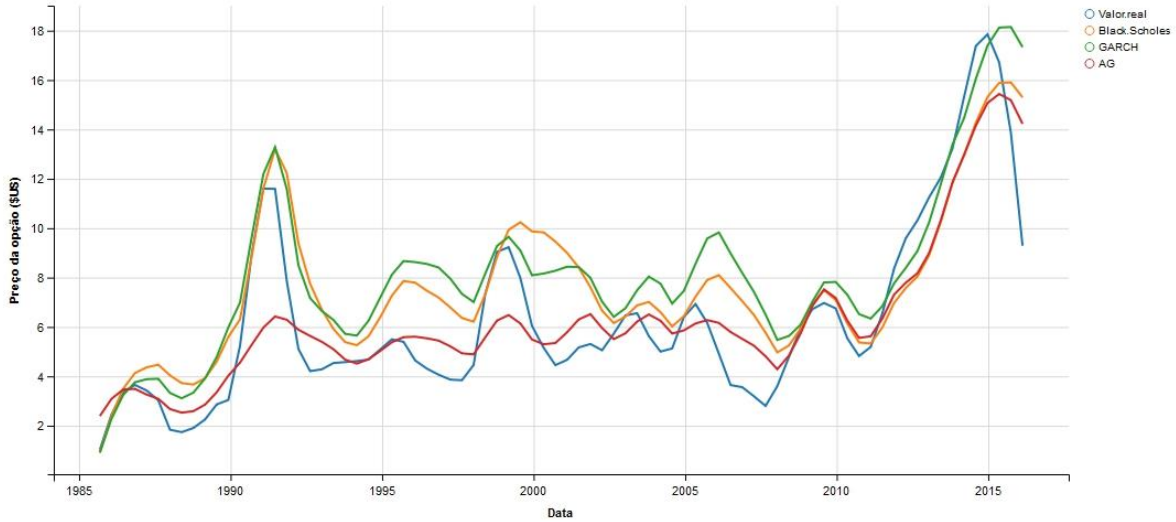

Figure 2: Graphic showing the smoothed option price time series for the AmGen stocks. The real option price is in blue. The red line is the price determined by the expression produced by the hybrid genetic algorithm. The orange and he green lines are the option prices calculated by the Black-Scholes formula and according to the GARCH (1,1) model, respectively.

In this case, concerning the average residue, the hGA generated expression was 12.23% better than the Black-Scholes equation computing the price of the call option, and 14.46% better than the GARCH (1,1) model.

Tests on other stocks are also presented in the next table.

Table 2: results on the implementation of the hGA on several stocks.

Stock	**Number of observations**	**Black-Scholes equation fitness**	**GARCH *(1, 1) fitness***	**hGA generated individual's fitness**	**Generation of the best individual**	**Total number of generations**	**Maximum population**	**Number of individuals ket from one generation to the next**	**Computation time (s)**	**hGA improvement vs Black-Scholes equation**	**hGA improvement vs GARCH (1,1) model**
AmGen	7614	0.9377428	0.9364521	0.9418412	497	500	30	15	34346	12.23%	14.46%
Apple	8565	0.8696329	0.867569	0.8716453	174	175	50	15	25688	4.68%	4.85%
Amazon	4441	0.7726863	0.7714513	0.7734449	1	175	50	15	9618	0.13%	-1.17%
Accenture	3393	0.9526715	0.9517335	0.9528280	3	300	100	20	34761	-0.05%	0.110%
Boing	13266	0.9613783	0.9609192	0.9654529	4	175	50	15	48132	14.03%	14.09%
CocaCola	7061	0.9791061	0.9763827	0.9800270	7	180	75	20	16475	4.64%	0.00%
Bank of America	7181	0.9627126	0.9529265	0.9650016	19	175	50	15	22195	11.42%	26.06%

The largest constraint on the application of this method is the amount of computation time required for a solution and to produce sufficient, statistically relevant tests. On average, to run a simulation comprising a training dataset of 8000 observations, roughly 3 seconds per individual are needed, which amounts to a several hours of running time for a larger number of generations or maximum population. The correlation between running time and number of individuals generated is approximately linear. For the case of the AmGen dataset, a stable fitness value over 0.94 (around 10% better than the Black-Scholes equation or the GARCH (1,1) model) was only reached for over 2000 generated individuals.

Not always he hGA was able to find a good enough solution. This was the case for the CocaCola, Accenture and Amazon stocks. The reason for this fact is not entirely clear, since the only clear difference between the datasets is the number of observations.

However, for the datasets that the hGA produced an expression that was clearly better than the two benchmark models, there seems to exist a correlation between the number of observations and the number of generations needed to find a good solution. The Boing dataset, with over 13000 observations, allowed for the appearance of a better expression with much less generations than the datasets half the size.

The hGA, not referring the huge amount of computation time that can be overwhelming into obtaining a good solution, will always be at least as good as the Black-Scholes equation, since this equation is included in the algorithm's population itself.

One other fact that can be noted in the results presented in *Table 2*, for the Bank of America options pricing, is the disparity between the performance of the Black-Scholes equation and the GARCH (1,1) model. The latter was twice as larger as the former, which might mean that the GARCH (1,1) model should be either implemented on higher order or that the time-series do not hold heteroscedasticity effects. However, no statistical tests were performed to determine the hypothesis of the existence of several events, such as the asymmetry, high kurtosis or heteroscedasticity of the volatility distribution, being present in the analysed time-series. As stated by Jarrow & Kwok, 2015 it is paramount to determine which model is appropriate beforehand. The hGA surpasses this necessity, as it is completely free of parametrization or theoretically imposed limits.

Nonetheless, to completely evaluate the performance of the hGA, it should be tested on a broader type of stocks time-series and against other, more complex stochastic models.

5 Conclusion

The application of a hGA on pricing options proved to be successful when compared to the two usual benchmark methods – the Black-Scholes equation and the GARCH (1,1) model. Although the possibility of using this metaheuristic method for financial time-series analysis was not explored, the algorithm developed showed it is able to randomly generate mathematical expressions that can yield option prices close enough to the real ones. We find that the most interesting future use of such method, given its high computational requirements when compared to the more recent and developed stochastic methods, is to provide a different way to study which terms might be useful to describe financial stochastic processes. We also conclude on the importance of comparing the performance of this method to the broad variety of stochastic models available. Moreover, the treatment of raw data under sampling and noise management should also be an area worth exploring on the study of the behaviour of financial time series.

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