DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Introduction to Stochastic Processes

MMA, LMAC, MECD

2nd. Semester – 2019/2020
2021/09/10 – 9AM

Duration: 180 minutes

Exam (*Época Extraordinária*)

- Please justify all your answers.
- This test has THREE PAGES and SIX GROUPS. The total of points is 40.0.

Group 1 — Introduction to Stochastic Processes — 2.5 points

Consider a stochastic process $\{X(t): t \in \mathbb{R}\}$ where $X(t) = U\cos(t) + V\sin(t)$ and U and V are two r.v.

(a) Show that the condition $E(U) = E(V) = 0$ is necessary for $\{X(t): t \in \mathbb{R}\}$ to be a second order weakly stationary process. (1.0)

- **Stochastic process**
  $\{X(t): t \geq 0\}$
  $X(t) = U\cos(t) + V\sin(t)$
  U, V r.v.
- **Requested proof**
  In case $\{X(t): t \in \mathbb{R}\}$ is a second order weakly stationary process, then its mean function,
  $$\begin{aligned} E[X(t)] &= E[U\cos(t) + V\sin(t)] \\ &= E(U) \times \cos(t) + E(V)\sin(t), \end{aligned}$$
  cannot depend on t, i.e., has to be constant. This is possible only if $E[X(t)] = 0$, for all $t \geq 0$, which in turn implies that
  $$\begin{aligned} E(U) &= E(V) \\ &= 0. \quad \checkmark \end{aligned}$$

(b) Prove that if $\{X(t): t \in \mathbb{R}\}$ is a second order weakly stationary process then U and V are uncorrelated r.v. with equal variance. (1.5)

Hint: Recall that $X(0) = U$; $X\left(\frac{\pi}{2}\right) = V$; $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$; and $\sin(x)\cos(y) + \cos(x)\sin(y) = \sin(x+y)$.

- **Requested proof**
  We just proved that if $\{X(t): t \in \mathbb{R}\}$ is a second order weakly stationary process then $E(U) = E(V) = 0$. Moreover, $V[X(t)]$ has to be constant, namely
  $$\begin{aligned} V[X(0)] &= V\left[X\left(\frac{\pi}{2}\right)\right] \\ E[X^2(0)] - E^2[X(0)] &= E\left[X^2\left(\frac{\pi}{2}\right)\right] - E^2\left[X\left(\frac{\pi}{2}\right)\right] \\ E[X^2(0)] &= E\left[X^2\left(\frac{\pi}{2}\right)\right] \\ E(U^2) &= E(V^2) \\ V(U) &= V(V). \quad \checkmark \end{aligned}$$
  In addition, the autocovariance function,
  $$\begin{aligned} Cov(X(t), X(t+s)) &= E[X(t)\,X(t+s)] - E[X(t)]\,E[X(t+s)] \\ &\overset{E[X(t)]=E[X(t+s)]=0}{=} E[X(t)\,X(t+s)] \\ &= E\{[U\cos(t) + V\sin(t)] \times [U\cos(t+s) + V\sin(t+s)]\} \\ &= E[U^2\cos(t)\cos(t+s) + UV\cos(t)\sin(t+s) \\ &\quad + UV\sin(t)\cos(t+s) + V^2\sin(t)\sin(t+s)] \\ &\overset{E(U^2)=E(V^2)}{=} E(U^2) \times [\cos(t)\cos(t+s) + \sin(t)\sin(t+s)] \\ &\quad E(UV) \times [\cos(t)\sin(t+s) + \sin(t)\cos(t+s)] \end{aligned}$$

$$\begin{aligned} Cov(X(t), X(t+s)) &= E(U^2) \times \cos(t-t-s) + E(UV) \times \sin(t+t+s) \\ &= E(U^2) \times \cos(-s) + E(UV) \times \sin(2t+s) \\ &= E(U^2) \times \cos(s) + E(UV) \times \sin(2t+s), \end{aligned}$$
cannot not depend on t, it has to depend exclusively on the time lag s. This implies that $E(UV) = 0$.
If we combine this result and the fact that $E(U) = E(V) = 0$, we conclude that
$$\begin{aligned} E(UV) &= E(UV) - E(U) \times E(V) \\ &= Cov(U, V) \\ &= 0, \end{aligned}$$
and this means that U and V are two uncorrelated r.v. $\checkmark$

Group 2 — Poisson Processes — 9.5 points

1. One estimates that meteors enter the atmosphere in a specific region of the globe according to a Poisson process with rate λ equal to 100 meteors per hour and that 1% of those meteors are visible to the *naked eye* as shooting stars.

(a) What is the probability that an observer sees at least two shooting stars in 30 minutes? (1.0)

- **Stochastic process**
  $\{N(t): t \geq 0\} \sim PP(\lambda = 100)$
  $N(t)$ = number of meteors entering the atmosphere by time (in hours) t
- **Split process**
  $\{N_{ss}(t): t \geq 0\} \sim PP(\lambda_{ss} = \lambda\, p = 100 \times 0.01 = 1)$
  $N_{ss}(t)$ = number of meteors seen as shooting stars by time (in hours) t
  $N_{ss}(t) \sim \text{Poisson}(\lambda\, p\, t = t)$
- **Requested probability**
  $$\begin{aligned} P[N_{ss}(t) \geq 2] &= 1 - P[N_{ss}(t) \leq 1] \\ &= 1 - F_{Poisson(t)}(1) \\ &= 1 - F_{Poisson(0.5)}(1) \\ &\overset{tables}{=} 1 - 0.9098 \\ &= 0.0902. \end{aligned}$$

(b) Find the expected waiting time until a fifth meteor is visible to the *naked eye* as a shooting star. What is the probability that this waiting time does not exceed its expected value? (1.0)

- **Relevant r.v. and its distribution**
  S_n = arrival time of the nth meteor seen as a shooting star
  $S_n \overset{form.}{\sim} \text{gamma}(n, \lambda_{ss})$
  $n = 5$
  $\lambda_{ss} = \lambda\, p = 1$
- **Requested expected value and probability**
  $$E(S_n) \overset{form.}{=} \frac{n}{\lambda_{ss}} = 5$$
  $$\begin{aligned} P[S_n \leq E(S_n)] &= F_{Erlang(n,\lambda_{ss})}(E(S_n)) \\ &= [1 - F_{Poisson(\lambda_{ss} \times E(S_n))}(n-1)] \\ &= 1 - F_{Poisson(n)}(n-1) \\ &\overset{n=5}{=} 1 - F_{Poisson(5)}(4) \\ &\overset{tables}{=} 1 - 0.4405 = 0.5595. \end{aligned}$$

(c) Find the joint probability that the cumulative number of meteors seen as shooting stars is equal to 6 at time 6, 15 at time 15, and 27 at time 27. (1.0)

- **Relevant fact**
  $\{N_{ss}(t): t \geq 0\}$ has independent and stationary increments
- **Requested probability**
  $P[N_{ss}(6) = 6, N_{ss}(15) = 15, N_{ss}(27) = 27]$

$$
\begin{aligned}
&= P[N_{ss}(6) = 6, N_{ss}(15) - N_{ss}(6) = 15-6, N_{ss}(27) - N_{ss}(15) = 27-15] \\
&\overset{indep.incr.}{=} P[N_{ss}(6) = 6] \times P[N_{ss}(15) - N_{ss}(6) = 15-6] \times P[N_{ss}(27) - N_{ss}(15) = 27-15] \\
&\overset{station.incr.}{=} P[N_{ss}(6) = 6] \times P[N_{ss}(15-6) = 15-6] \times P[N_{ss}(27-15) = 27-15] \\
&= P[N_{ss}(6) = 6] \times P[N_{ss}(9) = 9] \times P[N_{ss}(12) = 12] \\
&\overset{N(t)\sim Poisson(\lambda t)}{=} e^{-6}\frac{6^6}{6!} \times e^{-9}\frac{9^9}{9!} \times e^{-12}\frac{12^{12}}{12!} \\
&\simeq 0.002420.
\end{aligned}
$$

[Using the tables of the Poisson c.d.f. we would get $(0.7440 - 0.6063) \times (0.5874 - 0.4557) \times (0.5760 - 0.4616) \simeq 0.002075$.]

(d) Suppose 4 meteors were seen as shooting stars during the first 4 hours. (0.5)
Obtain the probability that at most 2 arrived during the first hour?

- **Relevant r.v.**
  $(N(s) \mid N(t) = n) \overset{form.}{\sim} \text{binomial}(n, s/t),\ 0 < s < t$
- **Requested probability**

$$
\begin{aligned}
P[N(s) \leq 2 \mid N(4) = 4] &= F_{binomial(n=4,s/t=1/4)}(2) \\
&\overset{tables}{=} 0.9492.
\end{aligned}
$$

2. Let $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ be two independent nonhomogeneous Poisson processes, with intensity functions $\lambda_1(t)$ and $\lambda_2(t)$, respectively. (2.5)

Prove that merging these two stochastic processes results in a nonhomogeneous Poisson process with intensity function $\lambda_1(t) + \lambda_2(t)$.

Hint: $\{N(t): t \geq 0\} \sim NHPP(\lambda(t))$ iff: i) $N(0) = 0$; ii) $\{N(t): t \geq 0\}$ has independent increments; iii) $P[N(t+h) - N(t) = 1] = \lambda(t) \times h + o(h)$, $t \geq 0$; iv) $P[N(t+h) - N(t) \geq 2] = o(h)$, $t \geq 0$. Moreover, a function f is said to be $o(h)$ iff $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

- **Stochastic processes**
  $\{N_i(t): t \geq 0\} \sim NHPP(\lambda_i(t)), \quad i = 1,2$
  $\{N_1(t): t \geq 0\} \perp\!\!\!\perp \{N_2(t): t \geq 0\}$
- **Merged stochastic process**
  $\{N(t) = N_1(t) + N_2(t): t \geq 0\}$
- **Important result**
  $N_i(t+s) - N_i(s) \overset{indep.}{\sim} \text{Poisson}(m_i(t+s) - m_i(s)), \quad i = 1,2$
  $m_i(t) = \int_0^t \lambda_i(z)\,dz, \quad i = 1,2$
- **Requested proof**
  Let us take advantage of the hint and capitalize on the properties of $\{N_i(t): t \geq 0\}$, $i = 1,2$.
  i) Is $N(0) = 0$?
  $N(0) = N_1(0) + N_2(0) = 0.$ ✓
  ii) Has $\{N(t): t \geq 0\}$ independent increments?
  Let $0 < t_1 < t_2 < \cdots < t_n$ $(n = 2,3,\ldots)$ and $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$.

Since $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ are two independent stochastic processes, in any case with independent increments, the joint probability

$$P[N(t_1) = i_1, N(t_2) - N(t_1) = i_2 - i_1, \ldots, N(t_n) - N(t_{n-1}) = i_n - i_{n-1}] = \star$$

is equal to

$$
\begin{aligned}
\star &= P[N_1(t_1) + N_2(t_1) = i_1, N_1(t_2) + N_2(t_2) - N_1(t_1) - N_2(t_1) = i_2 - i_1, \ldots, \\
&\quad N_1(t_n) + N_2(t_n) - N_1(t_{n-1}) - N_2(t_{n-1}) = i_n - i_{n-1}] \\
&= P\{N_1(t_1) + N_2(t_1) = i_1, [N_1(t_2) - N_1(t_1)] + [N_2(t_2) - N_2(t_1)] = i_2 - i_1, \ldots, \\
&\quad [N_1(t_n) - N_1(t_{n-1})] + [N_2(t_n) - N_2(t_{n-1})] = i_n - i_{n-1}\} \\
&= P[N_1(t_1) + N_2(t_1) = i_1] \times P[N_1(t_2) + N_2(t_2) - N_1(t_1) - N_2(t_1) = i_2 - i_1] \times \ldots \\
&\quad \times P[N_1(t_n) + N_2(t_n) - N_1(t_{n-1}) - N_2(t_{n-1}) = i_n - i_{n-1}] \\
&= P[N(t_1) = i_1] \times P[N(t_2) - N(t_1) = i_2 - i_1] \times \ldots \\
&\quad \times P[N(t_n) - N(t_{n-1}) = i_n - i_{n-1}],
\end{aligned}
$$

that is, the r.v. $N(t_1)$, $N(t_2) - N(t_1)$, …, $N(t_n) - N(t_{n-1})$ are independent, thus $\{N(t): t \geq 0\}$ also has independent increments. ✓

iii) $P[N(t+h) - N(t) = 1] = [\lambda_1(t) + \lambda_2(t)] \times h + o(h)$, $t \geq 0$?
In other words, we have to verify that $\lim_{h\to 0} \frac{P[N(t+h)-N(t)=1]}{h} = \lambda_1(t) + \lambda_2(t)$.
There will be exactly one event in the merged stochastic process in the interval $(t, t+h]$ if there is:

- exactly one event of the N-process and 0 events of the N-process, with probability

$$
\begin{aligned}
&P[N_1(t+h) - N_1(t) = 1, \quad N_2(t+h) - N_2(t) = 0] \\
&= P[N_1(t+h) - N_1(t) = 1] \times P[N_2(t+h) - N_2(t) = 0];
\end{aligned}
$$

- or exactly one event of the N-process and 0 events of the N-process, with probability

$$
\begin{aligned}
&P[N_1(t+h) - N_1(t) = 0, \quad N_2(t+h) - N_2(t) = 1] \\
&= P[N_1(t+h) - N_1(t) = 0] \times P[N_2(t+h) - N_2(t) = 1];
\end{aligned}
$$

Since these two events are mutually exclusive, we get

$$
\begin{aligned}
\lim_{h\to 0} \frac{P[N(t+h) - N(t) = 1]}{h} &= \lim_{h\to 0} \frac{P[N_1(t+h) - N_1(t) = 1] \times P[N_2(t+h) - N_2(t) = 0]}{h} \\
&\quad + \lim_{h\to 0} \frac{P[N_1(t+h) - N_1(t) = 0] \times P[N_2(t+h) - N_2(t) = 1]}{h} \\
&= \lim_{h\to 0} \frac{e^{-[m_1(t+h)-m_1(t)]}[m_1(t+h) - m_1(t)] \times e^{-[m_2(t+h)-m_2(t)]}}{h} \\
&\quad + \lim_{h\to 0} \frac{e^{-[m_1(t+h)-m_1(t)]} \times e^{-[m_2(t+h)-m_2(t)]}[m_2(t+h) - m_2(t)]}{h} \\
&= \lim_{h\to 0} \frac{m_1(t+h) - m_1(t)}{h} + \lim_{h\to 0} \frac{m_2(t+h) - m_2(t)}{h} \\
&= \lambda_1(t) + \lambda_2(t). \quad ✓
\end{aligned}
$$

iv) $P[N(t+h) - N(t) \geq 2] = o(h)$, $t \geq 0$?
Equivalently, we have to verify that $\lim_{h\to 0} \frac{P[N(t+h)-N(t)\geq 2]}{h} = 0$. Similarly and invoking L'Hôpital rule,

$$
\begin{aligned}
\lim_{h\to 0} \frac{P[N(t+h) - N(t) \geq 2]}{h} &= \lim_{h\to 0} \frac{1 - P[N(t+h) - N(t) = 0,1]}{h} \\
&\overset{iii)}{=} \lim_{h\to 0} \frac{1 - P[N(t+h) - N(t) = 0]}{h} - [\lambda_1(t) + \lambda_2(t)] \\
&= \lim_{h\to 0} \frac{1 - P[N_1(t+h) - N_1(t) = 0] \times P[N_2(t+h) - N_2(t) = 0]}{h} \\
&\quad -[\lambda(t) + \mu(t)] \\
&= \lim_{h\to 0} \frac{1 - e^{-[m_1(t+h)-m_1(t)]} \times e^{-[m_2(t+h)-m_2(t)]}}{h} - [\lambda(t) + \mu(t)]
\end{aligned}
$$

$$\begin{aligned}\lim_{h\to 0}\frac{P[N(t+h)-N(t)\geq 2]}{h} &\stackrel{Hrule}{=} \lim_{h\to 0}\frac{\frac{d[1-e^{-[m_1(t+h)-m_1(t)]}\times e^{-[m_2(t+h)-m_2(t)]}]}{dh}}{1}-[\lambda(t)+\mu(t)] \\ &= \lim_{h\to 0}\lambda_1(t+h)\,e^{-[m_1(t+h)-m_1(t)]-[m_2(t+h)-m_2(t)]} \\ &\quad +\lim_{h\to 0}\lambda_2(t+h)\,e^{-[m_1(t+h)-m_1(t)]-[m_2(t+h)-m_2(t)]} \\ &\quad -[\lambda_1(t)+\lambda_2(t)] \\ &= 0. \quad \checkmark\end{aligned}$$

Thus, $\{N^\star(t): t\geq 0\}$ is indeed a NHPP with intensity function $\lambda_1(t)+\lambda_2(t)$. ✓

3. Admit that: buses arrive at a music festival in accordance with a Poisson process $\{N(t): t\geq 0\}$ with rate λ; the numbers of *festivalgoers* in each bus are independent and identically distributed to $Y \sim \text{geometric}(p)$; these r.v. are independent of $N(t)$.

Let $X(t)$ represent the number of *festivalgoers* who have arrived by time t.

(a) Obtain the mean and variance of $X(t)$. (1.0)

- **Compound PP**
  $\left\{X(t)=\sum_{i=1}^{N(t)} Y_i : t>0\right\} \sim CpPP(\lambda, Y)$
  $\{N(t): t>0\} \sim PP(\lambda)$
  $Y_i =$ number of festivalgoers in bus i
  $Y_i \stackrel{i.i.d.}{\sim} Y, \quad i \in \mathbb{N}$
  $Y \sim \text{geometric}(p)$
- **Requested mean and variance**

$$\begin{aligned}E[X(t)] &\stackrel{form.}{=} \lambda\, t \times E(Y) \\ &\stackrel{form.}{=} \frac{\lambda\, t}{p}\end{aligned}$$

$$\begin{aligned}V[X(t)] &\stackrel{form.}{=} \lambda\, t \times E(Y^2) \\ &= \lambda\, t \times [V(Y)+E^2(Y)] \\ &\stackrel{form.}{=} \lambda\, t \times \left(\frac{1-p}{p^2}+\frac{1}{p^2}\right) \\ &= \frac{\lambda\, t\,(2-p)}{p^2}.\end{aligned}$$

(b) Derive $P[X(t)=0]$ and a simplified expression for $P[X(t)=x]$, $x \in \mathbb{N}$. (1.5)

- **Requested probability**
  Since Y is a positive integer r.v.,

$$\begin{aligned}P[X(t)=0] &= P[N(t)=0] \\ &= e^{-\lambda\, t}.\end{aligned}$$

- **Requested p.f.**
  Recall that $(X(t) \mid N(t)=n) \sim \text{negBinomial}(n,p)$ and the number of buses cannot exceed the number of festivalgoers, hence the application of the total probability law leads to

$$\begin{aligned}P[X(t)=x] &= \sum_{n=1}^{x} P[X(t)=x \mid N(t)=n] \times P[N(t)=n] \\ &\stackrel{form.}{=} \sum_{n=1}^{x}\binom{x-1}{n-1} p^n\,(1-p)^{x-n} \times \frac{e^{-\lambda\, t}\,(\lambda\, t)^n}{n!} \\ &= e^{-\lambda\, t}\,(1-p)^x \sum_{n=1}^{x}\binom{x-1}{n-1}\frac{\left(\frac{\lambda\, t\, p}{1-p}\right)^n}{n!}, \quad x \in \mathbb{N}.\end{aligned}$$

- **[Comment**
  $X(t)$ is said to have a *geometric Poisson distribution* or a *Pólya-Aeppli distribution*. This distribution is defined in *Mathematica* with a slightly different parametrization: $p' = 1-p$.]

(c) Consider $\lambda = 1$ (bus per minute), $p = 0.025$, $t = 720$ minutes, and $x = 28\,000$. (1.0)
Use the central limit theorem to compute an approximate value to $P\{X(t)\leq x\}$.

- **Requested probability**
  We can apply the CLT to provide the following approximate value:

$$\begin{aligned}P[X(t)\leq x] &\simeq \Phi\left(\frac{x-E[X(t)]}{\sqrt{V[X(t)]}}\right) \\ &= \Phi\left(\frac{x-\frac{\lambda\, t}{p}}{\sqrt{\frac{\lambda\, t\,(2-p)}{p^2}}}\right) \\ &= \Phi\left(\frac{28\,000-\frac{1\times 720}{0.025}}{\sqrt{\frac{1\times 720\times(2-0.025)}{0.025^2}}}\right) \\ &= \Phi\left(\frac{28\,000-28\,800}{\sqrt{2\,275\,200}}\right) \\ &\simeq \Phi(-0.53) \\ &\simeq 1-\Phi(0.53) \\ &\stackrel{tables}{=} 1-0.7019 \\ &= 0.2981.\end{aligned}$$

Group 3 — Renewal Processes	8.0 points

1. Admit that: $N(t)$ represents the cumulative number of covid-19 cases in a region in the interval $(0,t]$; $\{N(t): t\geq 0\}$ is a renewal process with renewal function

$$m(t) = \frac{t}{3} + \frac{e^{-\frac{3t}{2}}\left[\sin\left(\frac{\sqrt{3}t}{2}\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]}{3\sqrt{3}} - \frac{1}{3}, \quad t\geq 0.$$

(a) Derive the inter-renewal distribution of $\{N(t): t\geq 0\}$. (1.5)
Hint: $\tilde{m}(s) = \int_{0^-}^{+\infty} e^{-st}\, dm(t) = \frac{1}{s^3+3s^2+3s}$, $s>0$.

- **Renewal process**
  $\{N(t): t\geq 0\}$
  $N(t) =$ cumulative number of covid-19 cases until time t
- **Inter-renewal times**
  $X_i \stackrel{i.i.d.}{\sim} X, \quad i\in\mathbb{N}$
- **LST of the renewal function**
  $\tilde{m}(s) = \int_{0^-}^{+\infty} e^{-st}\, dm(t) = \frac{1}{s^3+3s^2+3s}$
- **Deriving the inter-renewal distribution**
  Since $\tilde{m}(s) \stackrel{form.}{=} \frac{\tilde{F}(s)}{1-\tilde{F}(s)}$ the LST of the inter-renewal distribution can be obtained in terms of the one of m:

$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &\stackrel{form.}{=} \frac{\tilde{m}(s)}{1+\tilde{m}(s)}\end{aligned}$$

$$\begin{aligned}
\tilde{F}(s) &= \frac{\frac{1}{s^3+3s^2+3s}}{1+\frac{1}{s^3+3s^2+3s}} \\
&= \frac{\frac{1}{s^3+3s^2+3s}}{\frac{s^3+3s^2+3s+1}{s^3+3s^2+3s}} \\
&= \frac{1}{s^3+3s^2+3s+1} \\
&= \left(\frac{1}{1+s}\right)^3 \\
&\equiv M_{gamma(3,1)}(-s),
\end{aligned}$$

i.e., the IRT are $X_i \overset{i.i.d.}{\sim} X \sim \text{gamma}(3,1)$, $i \in \mathbb{N}$.

(b) Compute the expected number of covid-19 cases in the interval $(t, 2t]$, where $t = \frac{\pi}{\sqrt{3}}$. (1.0)

- **Requested value**

$$\begin{aligned}
m(2t) - m(t) &= \left\{\frac{2t}{3} + \frac{e^{-3t}\left[\sin\left(\sqrt{3}t\right)+\sqrt{3}\cos\left(\sqrt{3}t\right)\right]}{3\sqrt{3}} - \frac{1}{3}\right\} \\
&\quad - \left\{\frac{t}{3} + \frac{e^{-\frac{3t}{2}}\left[\sin\left(\frac{\sqrt{3}t}{2}\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]}{3\sqrt{3}} - \frac{1}{3}\right\} \\
&= \frac{t}{3} + \frac{e^{-3t}\left[\sin\left(\sqrt{3}t\right)+\sqrt{3}\cos\left(\sqrt{3}t\right)\right] - e^{-\frac{3t}{2}}\left[\sin\left(\frac{\sqrt{3}t}{2}\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]}{3\sqrt{3}} \\
&\overset{t=\frac{\pi}{\sqrt{3}}}{=} \frac{\pi}{3\sqrt{3}} + \frac{e^{-\sqrt{3}\pi}\left[\sin(\pi)+\sqrt{3}\cos(\pi)\right] - e^{-\frac{\sqrt{3}\pi}{2}}\left[\sin\left(\frac{\pi}{2}\right)+\sqrt{3}\cos\left(\frac{\pi}{2}\right)\right]}{3\sqrt{3}} \\
&= \frac{\pi - \sqrt{3}e^{-\sqrt{3}\pi} - e^{-\frac{\sqrt{3}\pi}{2}}}{3\sqrt{3}} \\
&\simeq 0.590487.
\end{aligned}$$

(c) Consider time in hours and the time origin the beginning of 2020. (1.5)
Provide an approximate value to the expected number of covid-19 cases bound to occur on the last day of the second week.

- **Expected IRT**
  $\mu = E(X) = E[\text{gamma}(3,1)] = \frac{3}{1} = 3$
- **Requested expected value**
  Since the inter-renewal distribution is non-lattice,we can apply Blackwell's theorem and state that, for very large t (such as $t = 13 \times 24 = 576$ hours), the expected number of covid-19 cases in the interval $(t, t+a]$ (for $a = 24$ hours), $E[N(t+a) - N(t)]$, can be approximated as follows:

$$\begin{aligned}
E[N(t+a)-N(t)] &\simeq \lim_{z\to+\infty}[m(z+a)-m(z)] \\
&\equiv \frac{a}{\mu} \\
&= \frac{24}{3} \\
&= 8.
\end{aligned}$$

(d) Admit an inspection was made on January 1, 2021. (2.0)
Obtain an approximate value to the probability that the first covid-19 case after this inspection occurred after January 2, 2021.

- **C.d.f. of the IRT**

$$\begin{aligned}
F(u) &= F_{gamma(3,1)}(u) \\
&\overset{form.}{=} 1 - F_{Poisson(u)}(3-1)
\end{aligned}$$

$$\begin{aligned}
F(u) &= 1 - \sum_{i=0}^{2} \frac{e^{-u}u^i}{i!} \\
&= 1 - e^{-u}\left(1 + u + \frac{u^2}{2}\right)
\end{aligned}$$

- **Recurrence time**
  $Y(t) \overset{form}{=} S_{N(t)+1} - t =$ time until the first covid-19 case after the inspection at time t
- **Requested probability** (approximate value)
  Since the value of $t = 365 \times 24 = 8760$ hours is rather large, we can provide the following approximate value to $P[S_{N(t)+1} > t + x] = \star$

$$\begin{aligned}
\star &= 1 - P[S_{N(t)+1} - t \le x] \\
&= 1 - F_{Y(t)}(x) \\
&\simeq 1 - \lim_{z\to+\infty} P[Y(z) \le x] \\
&\overset{form.}{=} 1 - \frac{\int_0^x \{1 - F(u)\}\, du}{E(X)} \\
&\quad 1 - \frac{1}{E(X)} \times \int_0^x e^{-u}\left(1 + u + \frac{u^2}{2}\right) du \\
&= 1 - \frac{1}{E(X)} \times \int_0^x \left[f_{gamma(1,1)}(u) + f_{gamma(2,1)}(u) + f_{gamma(3,1)}(u)\right] du \\
&= 1 - \frac{1}{E(X)} \times \left[F_{gamma(1,1)}(x) + F_{gamma(2,1)}(x) + F_{gamma(3,1)}(x)\right] \\
&\overset{form.}{=} 1 - \frac{[1 - F_{Poisson(x)}(1-1)] + [1 - F_{Poisson(x)}(2-1)] + [1 - F_{Poisson(x)}(3-1)]}{E(X)} \\
&\overset{x=1}{=} 1 - \frac{3 - F_{Poisson(1)}(0) - F_{Poisson(1)}(1)] - F_{Poisson(1)}(2)}{E(X)} \\
&\overset{tables}{=} 1 - \frac{3 - 0.3679 - 0.7358 - 0.9197}{3} \\
&\simeq 0.674467.
\end{aligned}$$

[The result obtained using Mathematica is 0.674446.]

2. The duration of the consecutive phone calls made by a seller are independent r.v. with common c.d.f. (2.0)

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ x^2(3-2x), & 0 < x < \tau \\ 1, & x \ge \tau, \end{cases}$$

where τ is a constant in $(0, \frac{1}{2}]$. If the seller manages to persuade a customer to acquire a product before time τ, then the seller gets a reward of one monetary unit.

Derive the expected reward per time unit in the long-run.

Hint: Recall that for any non-negative r.v. X, $E(X) = \int_0^{+\infty}[1 - F_X(x)]\, dx$.

- **Renewal process**
  $\{N(t) : t \ge 0\}$
  $N(t) =$ number of phone calls by time t
- **IRT**
  $X_n \overset{i.i.d.}{\sim} X, \quad n \in \mathbb{N}$

$$F(x) = F_X(x) = \begin{cases} 0, & x \le 0 \\ x^2(3-2x), & 0 < x < \tau \\ 1, & x \ge \tau \end{cases}$$

- **Reward renewal process**
  $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \ge 0\}$

$R(t)$ = total amount of rewards got by the seller until time t

$$R_n = \begin{cases} 1, & \text{if } X_n < \tau \text{ (i.e., during the } n^{th} \text{ phone call the seller persuaded the customer} \\ & \text{to acquire a product before time } \tau) \\ 0, & \text{otherwise} \end{cases}$$

$(X_n, R_n) \overset{i.i.d.}{\sim} (X, R),\ n \in \mathbb{N}$

- **Expected IRT**

$$\begin{aligned} E(X) &\overset{X\geq 0}{=} \int_0^{+\infty} [1 - F_X(x)]\,dx \\ &= \int_0^{\tau} [1 - x^2(3-2x)]\,dx \\ &= \int_0^{\tau} (1 - 3x^2 + 2x^3)\,dx \\ &= \left(x - x^3 + \frac{x^4}{2}\right)\Big|_0^{\tau} \\ &= \tau - \tau^3 + \frac{\tau^4}{2} \\ &= \tau\left(1 - \tau^2 + \frac{\tau^3}{2}\right) \end{aligned}$$

- **Expected reward per phone call**

$$\begin{aligned} E(R) &= 1 \times P(X < \tau) + 0 \times P(X \geq \tau) \\ &= \lim_{h \to 0^+} F_X(\tau - h) \\ &= \tau^2(3 - 2\tau) \end{aligned}$$

- **Expected reward per time unit in the long-run**

  Since $E(X), E(R) < +\infty$, we can invoke the ERT for renewal reward processes and add that the expected reward per time unit in the long-run is given by

$$\begin{aligned} \lim_{t\to+\infty} \frac{E[R(t)]}{t} &= \frac{E(R)}{E(X)} \\ &= \frac{\tau^2(3-2\tau)}{\tau\left(1-\tau^2+\frac{\tau^3}{2}\right)} \\ &= \frac{2(3-2\tau)\tau}{\tau^3 - 2\tau^2 + 2}. \end{aligned}$$

- [**Note**: the value of τ ($\tau \in (0, \frac{1}{2}]$) that maximizes the expected reward per time unit in the long-run, $h(\tau) = \frac{E(R)}{E(X)}$, is $\tau = \frac{1}{2}$.]

Group 4 — Renewal Processes (cont'd)	2.5 points

Let $\{X_n : n \in \mathbb{N}_0\}$ be an irreducible, aperiodic, positive recurrent DTMC with limiting distribution $\{\pi_j : j \in \mathcal{S}\}$, where $\mathcal{S}$ represents the state space of this DTMC. (2.5)

Use regenerative processes to show that the mean inter-visit time to state j is given by $\frac{1}{\pi_j}$.

- **Proof**

  Consider an irreducible, aperiodic, positive recurrent DTMC that is initially in state i.

  By the Markovian property, each time the process reenters state i the DTMC restarts probabilistically. Thus returns to state i are renewals and constitute the beginnings of new cycles.

  The expected length of each cycle is μ_{ii} and the IRT are constant an equal to 1.

  When we are dealing with a regenerative renewal process $\{X(t) : t \geq 0\}$, we have

$$\begin{aligned} P_j &= \lim_{t\to+\infty} P[X(t) = j] \\ \lim_{t\to+\infty} P[X(t) = j] &= \frac{E[\text{amount of time in } j \text{ during a cycle}]}{E[\text{time of a cycle}]} \\ \lim_{n\to+\infty} P_{ij}^n &= \frac{1}{\mu_{ii}} \\ \pi_j &= \frac{1}{\mu_{ii}} \end{aligned}$$

If we take j to equal i, then we obtain

$$\begin{aligned} \pi_j &= \frac{1}{\mu_{jj}} \\ \mu_{jj} &= \frac{1}{\pi_j}. \quad \checkmark \end{aligned}$$

Group 5 — Discrete time Markov chains	9.5 points

1. The patients in a hospital are classified as belonging to the following states: (1) coronary care unit; (2) intensive care unit; (3) ambulatory unit; (4) extended care unit; (5) home or dead. As soon as a patient goes home or dies, a new patient is admitted to the coronary unit. The successive states the patient belongs to form a DTMC $\{X_n : n \in \mathbb{N}_0\}$ with TPM

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 0.9 & 0 \\ 0.1 & 0.1 & 0.1 & 0.5 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Draw the associated transition diagram, identify and classify the communicating classes of this DTMC. (1.5)

- **DTMC**

  $\{X_n : n \in \mathbb{N}_0\}$

- **State space**

  $\mathcal{S} = \{1, 2, 3, 4, 5\}$

- **TPM**

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & \frac{9}{10} & 0 \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{2} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

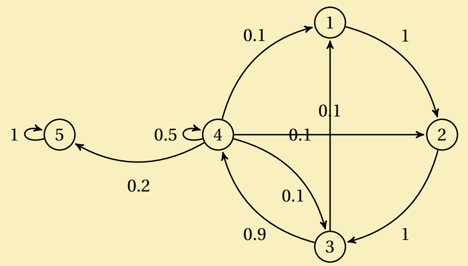
- **Communicating classes and classification of the states of the DTMC**

  Judging by the transition diagram, the communicating classes of this Markov chain are $\{1, 2, 3, 4\}$, and $\{5\}$. Hence, the state space is not a single communicating class therefore we are dealing with a reducible DTMC.

  Since state 5 is an absorbing state, that is, $P_{55} = 1$, no other state is accessible from it. Needless to say that $\{5\}$ is a closed communicating class with a recurrent state.

  Note that while state 5 is accessible from states 1, 2, 3, 4, the reverse is not true. $\{1, 2, 3, 4\}$ is not a closed communicating class. Moreover, we can also add these 4 states are transient, because given that the DTMC starts in any of them, there is a non-zero probability that it will never return to them.

- **Transition diagram**



(b) Admit that the initial state X_0 has p.f. $\underline{\alpha} = [0.4 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0]$. (1.5)
Compute $P(X_1 = 1, X_2 = 2, X_4 = 4)$.

- **Initial state**
  X_0
  $\underline{\alpha} = [P(X_0 = i)]_{i\in\mathscr{S}} = [0.4 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0]$
- **Requested probability**
  By applying the multiplication rule and the Markov property, we get

$$\begin{aligned} P(X_1 = 1, X_2 = 2, X_4 = 4) &= P(X_1 = 1) \times P(X_2 = 2 \mid X_1 = 1) \times P(X_4 = 4 \mid X_2 = 2) \\ &\overset{form.}{=} (\underline{\alpha} \times \mathbf{P})_1 \times P_{12} \times P^2_{24} \end{aligned}$$

  where:

$$\begin{aligned} P(X_1 = 1) &= \underline{\alpha} \times 1\text{st. column of } \mathbf{P} \\ &= [0.4 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0] \times [0 \quad 0 \quad 0.1 \quad 0.1 \quad 0]^\top \\ &= 0.04; \end{aligned}$$

$$\begin{aligned} P(X_2 = 2 \mid X_1 = 2) &= P_{12} \\ &= 1; \end{aligned}$$

$$\begin{aligned} P(X_4 = 4 \mid X_2 = 2) &= P^2_{24} \\ &= 2\text{nd. row of } \mathbf{P} \times 4\text{th. column of } \mathbf{P} \\ &= [0 \quad 0 \quad 1 \quad 0 \quad 0] \times [0 \quad 0 \quad 0.9 \quad 0.5 \quad 0]^\top \\ &= 0.9. \end{aligned}$$

  Thus, $P(X_1 = 1, X_2 = 2, X_4 = 4) = 0.04 \times 1 \times 0.9 = 0.036$.

(c) Determine the expected number of transitions until a patient reaches state 5, given that he/she started in the coronary care unit ($X_0 = 1$). (2.0)
Hint: Check the footnote.[1]

- **Initial state**
  $X_0 = 1$

[1]The following results may come handy:
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -\frac{1}{10} & 0 & 1 & -\frac{9}{10} \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & \frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{16}{9} & \frac{41}{18} & \frac{25}{9} & 5 \\ \frac{7}{9} & \frac{41}{18} & \frac{25}{9} & 5 \\ \frac{7}{9} & \frac{23}{18} & \frac{25}{9} & 5 \\ \frac{2}{3} & \frac{7}{6} & \frac{5}{3} & 5 \end{bmatrix};$$
$$\begin{bmatrix} 1 & -\frac{5}{6} & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{6} & 0 & 0 \\ -\frac{1}{12} & 0 & 1 & -\frac{3}{4} & 0 \\ -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{7}{12} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{243}{203} & \frac{225}{203} & \frac{30}{29} & \frac{270}{203} & \frac{270}{203} \\ \frac{48}{203} & \frac{270}{203} & \frac{36}{29} & \frac{324}{203} & \frac{324}{203} \\ \frac{288}{1015} & \frac{402}{1015} & \frac{216}{145} & \frac{1944}{1015} & \frac{1944}{1015} \\ \frac{249}{1015} & \frac{411}{1015} & \frac{78}{145} & \frac{2442}{1015} & \frac{2442}{1015} \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

- **Important**
  We are dealing with an absorbing DTMC. The sub-stochastic matrix governing the transitions between its transient states ($T = \{1, 2, 3, 4\}$) is given by

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & \frac{9}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{2} \end{bmatrix}$$

- **Requested expected value**
  Let $\tau = \inf\{n \in \mathbb{N}_0 : X_n \not\in T\}$ be the number of transitions until a patient reaches the absorbing state 5.
  The result in the footnote yields

$$\begin{aligned} [E(\tau \mid X_1 = i)]_{i\in T} &\overset{form.}{=} (\mathbf{I} - \mathbf{Q})^{-1} \times \underline{1} \\ &= \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & \frac{9}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{2} \end{bmatrix} \right)^{-1} \times \underline{1} \\ &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -\frac{1}{10} & 0 & 1 & -\frac{9}{10} \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & \frac{1}{2} \end{bmatrix}^{-1} \times \underline{1} \\ &= \begin{bmatrix} \frac{16}{9} & \frac{41}{18} & \frac{25}{9} & 5 \\ \frac{7}{9} & \frac{41}{18} & \frac{25}{9} & 5 \\ \frac{7}{9} & \frac{23}{18} & \frac{25}{9} & 5 \\ \frac{2}{3} & \frac{7}{6} & \frac{5}{3} & 5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ [&= \begin{bmatrix} \frac{71}{6} \\ \frac{65}{6} \\ \frac{59}{6} \\ \frac{17}{2} \end{bmatrix}]. \end{aligned}$$

  Thus, the requested expected value is $E(\tau \mid X_1 = 1) = \frac{16}{9} + \frac{41}{18} + \frac{25}{9} + 5 = \frac{71}{6} = 11.8(3)$.

(d) Suppose being in states 1, 2, 3, 4, 5 cost 250, 300, 200, 150, 100 euro (respectively). (2.0)
Assume a discount factor $\alpha = \frac{5}{6}$ and compute the expected total discounted expenditure of a patient starting in the coronary care unit.
Hint: Check the footnote.

- **Vector of costs**

$$\underline{c} = [c(j)]_{j\in\mathscr{S}} = \begin{bmatrix} 250 \\ 300 \\ 200 \\ 150 \\ 100 \end{bmatrix}$$

- **Discount factor**
  $\alpha = \frac{5}{6}$
- **Vector of the expected total discounted expenditures**
  The expected total discounted cost incurred, starting at state i, is equal to $\phi(i) = E\left[\sum_{n=0}^{+\infty} \alpha^n c(X_n) \mid X_0 = i\right]$. Equivalently,

$$\underline{\phi} = [\phi(i)]_{i\in\mathscr{S}} = (\mathbf{I} - \alpha\mathbf{P})^{-1} \times \underline{c}.$$

We get

$$\begin{aligned}
\underset{\sim}{\phi} &= (\mathbf{I}-\alpha\mathbf{P})^{-1}\times \underline{c}\\
&= \left(\begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\end{bmatrix} - \frac{5}{6}\times\begin{bmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ \frac{1}{10}&0&0&\frac{9}{10}&0\\ \frac{1}{10}&\frac{1}{10}&\frac{1}{10}&\frac{1}{2}&\frac{1}{5}\\ 0&0&0&0&1\end{bmatrix}\right)^{-1}\times\underline{c}\\
&= \begin{bmatrix} 1&-\frac{5}{6}&0&0&0\\ 0&1&-\frac{5}{6}&0&0\\ -\frac{1}{12}&0&1&-\frac{3}{4}&0\\ -\frac{1}{12}&-\frac{1}{12}&-\frac{1}{12}&\frac{7}{12}&-\frac{1}{6}\\ 0&0&0&0&\frac{1}{6}\end{bmatrix}^{-1}\times\underline{c}\\
&= \begin{bmatrix} \frac{243}{203}&\frac{225}{203}&\frac{30}{29}&\frac{270}{203}&\frac{270}{203}\\ \frac{48}{203}&\frac{270}{203}&\frac{36}{29}&\frac{324}{203}&\frac{324}{203}\\ \frac{288}{1015}&\frac{402}{1015}&\frac{216}{145}&\frac{1944}{1015}&\frac{1944}{1015}\\ \frac{249}{1015}&\frac{411}{1015}&\frac{78}{145}&\frac{2442}{1015}&\frac{2442}{1015}\\ 0&0&0&0&6\end{bmatrix}\times\begin{bmatrix}250\\300\\200\\150\\100\end{bmatrix}
\end{aligned}$$

- **Requested expected total discounted expenditure (customer starting with brand 1)**

$$\begin{aligned}
\phi(1) &= \text{1st. row of } (\mathbf{I}-\alpha\mathbf{P})^{-1}\times\underline{c}\\
&= \begin{bmatrix} \frac{243}{203} & \frac{225}{203} & \frac{30}{29} & \frac{270}{203} & \frac{270}{203}\end{bmatrix}\times\begin{bmatrix}250\\300\\200\\150\\100\end{bmatrix}\\
&= \frac{237750}{203}\\
&\simeq 1171.18.
\end{aligned}$$

2. Let $\{X_n : n\in\mathbb{N}_0\}$ be a branching process such that: the number of offspring per individual has a binomial$(2,p)$ distribution, where $p\in\left(\frac{1}{2},1\right]$; the number of initial individuals is a r.v. $X_0\sim$ geometric(p). (2.5)

Derive the extinction probability of this branching process.

- **Simple branching process**

  $\{X_n : n\in\mathbb{N}_0\}$

  $X_0 = 1$

  $X_n =$ size of generation n

  $X_n = \sum_{l=1}^{X_{n-1}} Z_l, \quad n\in\mathbb{N}$

- **Number of offspring per individual and its p.g.f.**

  $Z_l \equiv Z_{l,n} =$ number of offspring of the l^{th} individual of generation n

  $Z_l \overset{i.i.d.}{\sim} Z\sim$ binomial$(2,p), \quad l\in\mathbb{N}$

  $E(Z)\overset{form.}{=} 2p>1$, when $p\in\left(\frac{1}{2},1\right]$

  $P_Z(s) = E(s^Z)\overset{form.}{=}(1-p+ps)^2$

- **Probability of extinction** (with one single initial individual)

  $E(Z)>1$, therefore the probability of extinction, $\pi\overset{form.}{=}\lim_{n\to+\infty}P(X_n=0)$, is the smallest positive number satisfying

$$\begin{aligned}
s &\overset{form.}{=} P_Z(s)\\
s &= (1-p+ps)^2\\
p^2s^2-2p^2s+p^2+2ps-2p+1-s &= 0\\
p^2s^2-(2p^2-2p+1)s+(p^2-2p+1) &= 0\\
s &= \frac{(2p^2-2p+1)\pm\sqrt{(2p^2-2p+1)^2-4p^2(1-p)^2}}{p^2}\\
s &= \ldots\\
s &= \frac{(2p^2-2p+1)\pm\sqrt{1-4p-4p^2}}{p^2}\\
s &= \frac{(2p^2-2p+1)\pm\sqrt{(1-2p)^2}}{p^2}\\
s &= \frac{(2p^2-2p+1)\pm(1-2p)}{p^2}\\
s &= 1 \quad\text{or}\quad \frac{(1-p)^2}{p^2}.
\end{aligned}$$

Since $p\in\left(\frac{1}{2},1\right]$, the smallest positive root is $\pi=\frac{(1-p)^2}{p^2}$.

- **New initial state**

  $X_0\sim$ geometric$(p), \quad p\in\left(\frac{1}{2},1\right]$

  $P_{X_0}(s) = E(s^{X_0})\overset{form.}{=}\frac{ps}{1-(1-p)s}$

- **New probability of extinction**

  Using the total probability law, the fact that the offspring are produced independently and $X_0\sim Z$, we get

$$\begin{aligned}
P(\text{extinction}) &= \sum_{j=0}^{+\infty} P(\text{extinction}\mid X_0=j)\times P(X_0=j)\\
&= \sum_{j=0}^{+\infty}\pi^j\times P(X_0=j)\\
&= P_{X_0}(\pi)\\
&= \frac{p\pi}{1-(1-p)\pi}\\
&= \frac{p\,\frac{(1-p)^2}{p^2}}{1-(1-p)\,\frac{(1-p)^2}{p^2}}\\
&= \frac{(p-1)^2p}{p^3-2p^2+3p-1}.
\end{aligned}$$

Group 6 — Continuous time Markov chains	8.0 points

1. Admit that a company uses m ($m\in\mathbb{N}$) robots. Furthermore: each robot breaks down after an exponentially distributed time with parameter λ; the company has m repair people to do service when robots fail (one repair person per robot); the repair time for each robot is exponentially distributed with mean μ^{-1}.

Let $X(t)$ represent the number of robots being repaired at time t and admit that $\{X(t): t\geq 0\}$ is a CTMC.

(a) Identify the state space and the infinitesimal generator $\mathbf{R}$ of this birth and death process. (1.5)

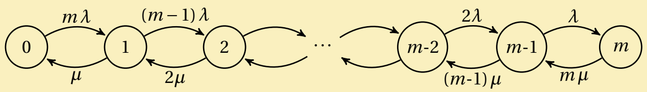
Draw the associated rate diagram.

- **CTMC**
  $\{X(t): t \geq 0\}$
  $X(t)$ = number of robots being repaired at time t

- **State space**
  $\mathscr{S} = \{0, 1, 2, \ldots, m\}$

- **Birth/death rates**
  We can interpret a break down of a robot as an arrival (to the repair shop) or a *birth*. Also note that as soon as a reparation of a robot is concluded, we can say that a departure (from the repair shop) or *death* has occurred. Moreover, there are m repair people or servers.
  [We are dealing with an $M/M/m$ queueing system with a finite customer population, namely m.]
  Thus,
  $$\begin{aligned} \lambda_i &= (m-i)\lambda, \quad i = 0, 1, \ldots, m-1 \\ \mu_i &= i\mu, \quad i = 1, 2, \ldots, m. \end{aligned}$$

- **Rate diagram**
  [Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node i to node j (if $q_{ij} > 0$) with q_{ij} written on it. These rates coincide with the birth and death rates...]



- **Infinitesimal generator**
  This matrix has entries
  $$r_{ij} = \begin{cases} q_{ij}, & i \neq j \\ -\nu_i = -\sum_{m \in \mathscr{S}} q_{im}, & j = i \end{cases}$$
  and in this case $\mathbf{R} = [r_{ij}]_{i,j \in \mathscr{S}}$ is equal to

$$\begin{bmatrix} -m\lambda & m\lambda & 0 & 0 & 0 & 0 & 0 \\ \mu & -[(m-1)\lambda+\mu] & (m-1)\lambda & 0 & 0 & 0 & 0 \\ 0 & 2\mu & -[(m-1)\lambda+2\mu] & (m-2)\lambda & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & (m-2)\mu & -[2\lambda+(m-2)\mu] & 2\lambda & 0 \\ 0 & 0 & 0 & 0 & (m-1)\mu & -[\lambda+(m-1)\mu] & \lambda \\ 0 & 0 & 0 & 0 & 0 & m\mu & -m\mu \end{bmatrix}.$$

(b) Obtain the equilibrium probabilities $P_j = \lim_{t\to+\infty} P[X(t) = j \mid X(0) = 0]$. (2.5)
Show that L_s, the number of robots being repaired upon arrival to the repair shop in the long-run, has a binomial distribution and compute its expected value.
Hint: $(a+b)^m = \sum_{n=0}^{m} \binom{m}{n} a^n b^{m-n}$, for $m \in \mathbb{N}$.

- **Performance measure**
  L_s = number of robots being repaired upon arrival to the repair shop in the long-run

- **Equilibrium probabilities**
  Since this CTMC has a finite state space, we only need to deal with $\rho = \frac{\lambda}{\mu} < +\infty$ to guarantee the existence of equilibrium probabilities $P_j = \lim_{t\to+\infty} P_j(t) = P(L_s = j)$.
  $$P_0 = \left[1 + \sum_{n=1}^{+\infty}\left(\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)\right]^{-1}$$

$$\begin{aligned} P_0 &= \left[1 + \sum_{n=1}^{+\infty}\left(\prod_{i=0}^{n-1}\frac{(m-i)\lambda_i}{(i+1)\mu}\right)\right]^{-1} \\ &= \left[1 + \sum_{n=1}^{m}\frac{m\times(m-1)\times\ldots 2\times 1}{1\times 2\times\ldots(m-1)\times m}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} \\ &= \left[1 + \sum_{n=1}^{m}\binom{m}{n}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} \\ &= \left[\sum_{n=0}^{m}\binom{m}{n}\left(\frac{\lambda}{\mu}\right)^n \times 1^{m-n}\right]^{-1} \\ &= \frac{1}{\left(1+\frac{\lambda}{\mu}\right)^m} \\ &= \left(\frac{\mu}{\lambda+\mu}\right)^m \end{aligned}$$

$$\begin{aligned} P_j &= P_0 \times \prod_{i=0}^{j-1}\frac{\lambda_i}{\mu_{i+1}} \\ &= P_0 \times \prod_{i=0}^{j-1}\frac{(m-i)\lambda_i}{(i+1)\mu} \\ &= \frac{\binom{m}{j}\left(\frac{\lambda}{\mu}\right)^j}{\left(1+\frac{\lambda}{\mu}\right)^m} \\ &= \binom{m}{j}\left(\frac{\frac{\lambda}{\mu}}{1+\frac{\lambda}{\mu}}\right)^j\left(\frac{1}{1+\frac{\lambda}{\mu}}\right)^{m-j} \\ &= \binom{m}{j}\left(\frac{\lambda}{\lambda+\mu}\right)^j\left(\frac{\mu}{\lambda+\mu}\right)^{m-j}, \quad j = 1, 2, \ldots, m, \end{aligned}$$

i.e., $P(L_s = j) = \binom{m}{j}\left(\frac{\lambda}{\lambda+\mu}\right)^j\left(\frac{\mu}{\lambda+\mu}\right)^{m-j}$, $j = 0, 1, 2, \ldots, m$, hence
$$L_s \sim \text{binomial}\left(m, \frac{\lambda}{\lambda+\mu}\right).$$

- **Requested expected value**
  Since $L_s \sim \text{binomial}\left(m, \frac{\lambda}{\lambda+\mu}\right)$, we get
  $$E(L_s) \stackrel{form.}{=} m \times \frac{\lambda}{\lambda+\mu}.$$

(c) This repair shop can be modelled as a $M/M/m$ queueing system with a finite customer population of size m. Let $\lambda_e = \sum_{j=0}^{m}\lambda_j P_j$ be the rate of entering customers in the repair shop, in the long-run. Use Little's law to confirm that the expected value of the time a machine spends in the repair shop is equal to $\frac{1}{\mu}$, in the long-run. (1.0)

- **Important**
  Let W_s be the time a machine spends in the repair shop, in the long-run.
  According to Little's law, $E(Ls) = \lambda_e \times E(W_s)$.

- **Rate of entering customers**
  $$\begin{aligned} \lambda_e &= \sum_{j=0}^{m}\lambda_j \times P_j \\ &= \sum_{j=0}^{m}(m-j)\lambda \times P_j \\ &= m\lambda - \lambda E(L_s) \end{aligned}$$

$$\lambda_e \overset{(b)}{=} m\lambda - \lambda \frac{m\lambda}{\lambda+\mu} = \frac{m\lambda\mu}{\lambda+\mu}$$

- **Requested expected value**

$$E(W_s) = \frac{E(L_s)}{\lambda_e} = \frac{\frac{m\lambda}{\lambda+\mu}}{\frac{m\lambda\mu}{\lambda+\mu}} = \frac{1}{\mu}. \quad \checkmark$$

2. Customers arrive to a laundromat according to a Poisson process with rate equal to 8 customers per hour. Assume the laundromat has five machines and customers are only allowed to use one machine at a time. The time a customer uses a machine is exponentially distributed with a mean of one hour. Assume that customers do not enter the laundromat if all five machines are taken.

(a) After having chosen a suitable queueing system, calculate the fraction of arriving customers who find the laundromat full and are lost, in the long-run. (2.0)

- **Birth-death queueing system**
  $M/M/m/m$
  $m = 5$
- **State space**
  $\mathscr{S} = \{0, 1, \ldots, m\}$
- **Birth/death rates**
  $\lambda_k = \lambda = 8, \quad k \in \{0, 1, \ldots, m-1\}$
  $\mu_k = k\mu = k, \quad k \in \{1, 2, \ldots, m\}$
- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{m\mu} = \frac{8}{5\times 1} = 8/5 < +\infty$
- **Performance measure (in the long-run)**
  L_s = number of customers an arriving customer sees to the laundromat

$$P(L_s = k) = \begin{cases} \frac{\frac{(m\rho)^k}{k!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}}, & k = 0, 1, \ldots, m \\ 0, & k = m+1, m+2, \ldots \end{cases}$$

- **Requested fraction**

$$P(L_s = m) = B(m, m\rho) = \frac{\frac{(m\rho)^m}{m!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}} \overset{m=5,\lambda=8,\mu=1}{=} \frac{\frac{8^5}{5!}}{\sum_{j=0}^{5} \frac{8^j}{j!}}$$

$$= \frac{8^5/5!}{1+8+8^2/2!+8^3/3!+8^4/4!+8^5/5!} = \frac{\frac{32768}{120}}{1+8+32+\frac{512}{6}+\frac{4096}{24}+\frac{32768}{120}} = \frac{4096}{8551} \simeq 0.479008.$$

(b) On average, how many servers are idle, in the long-run? (1.0)

- **Requested average**
  Since $5, 4, 3, 2, 1, 0$ machines are idle if and only if $L_s = 0, 1, 2, 4, 5$, respectively.

$$\sum_{j=0}^{m} (m-j) \times P_j = m \times \sum_{j=0}^{m} P_j - \sum_{j=0}^{m} j \times P_j = m \times 1 - \sum_{j=0}^{m} j \times P_j = m - E(L_s) \overset{form.}{=} m - m\rho\,[1 - B(m, m\rho)] \overset{(a)}{\simeq} 5 - \frac{8}{1} \times (1 - 0.479008) \simeq 0.832064.$$