DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Introduction to Stochastic Processes

LEIC-A, LMAC, MECD, MEIC-A, MMA

2nd. Semester – 2020/2021
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Test 1

Duration: 90 minutes

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 0 — Introduction to Stochastic Processes — 2.5 points

A system comprises two computers which operate independently and in discrete time slots with constant size. A priority task arises with probability p_1 (resp. p_2) at the beginning of each time slot in computer 1 (resp. computer 2), independently of other time slots, and requires one full time slot to be complete. Moreover, a time slot is said to be *busy* (resp. *idle*) if within this time slot at least one of the two computers executes (resp. does not execute) a priority task.

(a) Identify the p.f. and expected value of the time index of the second *busy* slot of this system. (1.0)

- **Stochastic processes**
  $X_i^{(j)} = \begin{cases} 1, & \text{if the computer } j \text{ is executing a priority task within the time slot } i \\ 0, & \text{otherwise} \end{cases}$
  $\{X_i^{(j)} : i \in \mathbb{N}\} \overset{indep}{\sim} BP(p_j), \quad j = 1,2$
- **Merged stochastic process**
  $X_i = \begin{cases} 1, & \text{if the computer is } busy \text{ within the time slot } i \\ 0, & \text{otherwise} \end{cases}$
  $\left\{X_i = \max\{X_i^{(1)}, X_i^{(2)}\} : i \in \mathbb{N}\right\} \sim BP(p)$, where $p = p_1 + p_2 - p_1 \times p_2$, because we are merging two independent Bernoulli processes.
- **R.v.**
  T_2 = time index of the second *busy* slot
  $T_2 \overset{form.}{\sim} \text{negativeBin}(2, p)$
- **Requested p.f. and expected value**
  $P(T_2 = x) = \binom{x-1}{2-1} (1-p)^{x-2} p^2, \quad x = 2,3,\ldots$
  $E(T_2) \overset{form.}{=} \frac{2}{p}$.

(b) Calculate the probability that 2 out of the first 5 time slots were *busy*, given that 10 out of the first 15 time slots were also *busy*. (1.5)

- **New r.v.**
  S_n = number of *busy* time slots out of the first n time slots
- **Conditional distribution**
  $S_m \mid S_n = k \overset{form.}{\sim} \text{hypergeometric}(n, m, k), \quad 0 \le m \le n, 0 \le k \le n$
- **P.f.**
  $P(S_m = x \mid S_n = k) = \frac{\binom{m}{x}\binom{n-m}{k-x}}{\binom{n}{k}}$, $x = \max\{0, k-n+m\}, \ldots, \min\{k, m\}$
- **Requested probability**
  Considering $n = 15$, $m = 5$, $k = 10$ and $x = 2$ yields

$$
\begin{aligned}
P(S_5 = 2 \mid S_{15} = 10) &= \frac{\binom{5}{2}\binom{15-5}{10-2}}{\binom{15}{10}} \\
&= \frac{\frac{5!}{2!3!}\,\frac{10!}{8!2!}}{\frac{15!}{10!5!}} \\
&= \frac{\frac{5\times4}{2} \times \frac{10\times9}{2}}{\frac{15\times14\times13\times12\times11}{120}} \\
&= \frac{20\times10\times9}{15\times14\times13\times12\times11} \times \frac{120}{4} \\
&= \frac{150}{1001} \\
&\simeq 0.149850.
\end{aligned}
$$

Group 1 — Poisson Processes — 9.5 points

1. Pulses in a Geiger-Müller counter occur in accordance with a Poisson process at a rate of three pulses per minute.

(a) Obtain the probability that no pulse occurs in the interval [8, 9]. (0.5)

- **Stochastic process**
  $\{N(t) : t \ge 0\} \sim PP(\lambda = 3)$
  $N(t)$ = number of arrivals of passengers to the train station by time t (time in hours)
- **Relevant distribution**
  $N(t) \sim \text{Poisson}(\lambda\, t = 3\, t), \quad t > 0$
  $P[N(t) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0$
- **Requested probability**

$$
\begin{aligned}
P[N(9) - N(8) = 0] &\overset{stationary\ inc.}{=} P[N(1) = 0] \\
&\overset{N(1)\sim Poisson(\lambda)}{=} e^{-3} \\
&\simeq 0.049787.
\end{aligned}
$$

(b) Find the joint probability that the cumulative number of pulses is equal to 6 at time 2, 15 at time 5, and 27 at time 9. (1.5)

- **Relevant fact**
  $\{N(t) : t \ge 0\}$ has independent and stationary increments
- **Requested probability**
  $P[N(2) = 6, N(5) = 15, N(9) = 27]$

$$
\begin{aligned}
&= P[N(2) = 6, N(5) - N(2) = 15 - 6, N(9) - N(5) = 27 - 15] \\
&\overset{indep.incr.}{=} P[N(2) = 6] \times P[N(5) - N(2) = 15 - 6] \times P[N(9) - N(5) = 27 - 15] \\
&\overset{station.incr.}{=} P[N(2) = 6] \times P[N(5-2) = 15 - 6] \times P[N(9-5) = 27 - 15] \\
&= P[N(2) = 6] \times P[N(3) = 9] \times P[N(4) = 12] \\
&\overset{N(t)\sim Poisson(\lambda t)}{=} e^{-6}\frac{6^6}{6!} \times e^{-9}\frac{9^9}{9!} \times e^{-12}\frac{12^{12}}{12!} \\
&\simeq 0.002420.
\end{aligned}
$$

  [Using the tables of the Poisson c.d.f. we would get $(0.7440 - 0.6063) \times (0.5874 - 0.4557) \times (0.5760 - 0.4616) \simeq 0.002074654296$.]

(c) Calculate the probability that the time at which the fourth pulse occurs exceeds its expected value. (1.0)

- **Relevant r.v.**
  S_n = time at which the nth pulse occurs
  $S_n \sim \text{Erlang}(n, \lambda)$
  $n = 4$

- **Requested expected value and probability**

$$
\begin{aligned}
E(S_n) &\overset{form.}{=} \frac{n}{\lambda} \\
&\overset{n=4,\lambda=3}{=} \frac{4}{3} \\
P[S_n > E(S_n)] &= 1 - F_{Erlang(n,\lambda)}(E(S_n)) \\
&= 1 - [1 - F_{Poisson(\lambda \times E(S_n))}(n-1)] \\
&= F_{Poisson(n)}(n-1) \\
&\overset{n=4}{=} F_{Poisson(4)}(3) \\
&\overset{tables}{=} 0.4335.
\end{aligned}
$$

[Interestingly, this probability does not depend on the rate of the Poisson process.]

2. Admit that, since the beginning of year 2000 (the time origin), people pass the bar exam in a certain country according to a Poisson process, with rate equal to λ (people per year). After passing this exam, each lawyer immediately starts to practise law for a random time period (L, in years) with a gamma($\alpha = 2, \beta$) distribution, independently of other lawyers.

(a) Derive a simplified expression for the probability that a lawyer, who passed her/his bar exam at time s (in this century), is still practising law at time t ($t > s$). (1.0)

- **Stochastic process**

  $\{N(t) : t \geq 0\} \sim PP(\lambda)$ ($t = 0 \equiv$ beginning of this century; t in years)

  $N(t) =$ lawyers who pass the bar exam until time t

- **Requested probability**

  A lawyer who passed her/his bar exam (in this century) at time s is still practising law at time t with probability

$$
\begin{aligned}
p(s) &= P(L > t-s) \\
&= 1 - F_L(t-s) \\
&= 1 - F_{gamma(\alpha,\beta)}(t-s) \\
&\overset{\alpha\in\mathbb{N},\, form.}{=} F_{Poisson(\beta(t-s))}(\alpha - 1) \\
&\overset{\alpha=2}{=} e^{-\beta(t-s)} + e^{-\beta(t-s)}[\beta(t-s)], \quad 0 < s < t.
\end{aligned}
$$

(b) Obtain the distribution and the expected value of the number of lawyers who pass the bar exam in this century and who are still practising law at the end of year 2049, when $\lambda = 500$ and $\beta = 1/15$. (1.5)

- **Relevant r.v. / non-homogenous Bernoulli splitting**

  The number of lawyers (who pass the bar exam in this century and) who are still practising law at time t, $N_{law}(t)$, results from a non-homogenous Bernoulli splitting of $\{N(t) : t \geq 0\}$.

- **Requested distribution and expected value**

  We are dealing

$$
N_{law}(t) \overset{form.}{\sim} \text{Poisson}\left(\lambda \int_0^t p(s)\, ds\right),
$$

  where the expected value equals

$$
\begin{aligned}
\lambda \int_0^t p(s)\, ds &= \lambda \int_0^t P(L > t-s)\, ds \\
&= \lambda \int_0^t P(L > u)\, du \quad \text{[change of variable: } u = t-s] \\
&= \lambda \int_0^t \left[e^{-\beta u} + e^{-\beta u}(\beta u)\right] du \\
[&= \lambda \left[\frac{1}{\beta}\int_0^t \beta e^{-\beta u}\, du + \frac{\Gamma(2)}{\beta}\int_0^t \frac{\beta^2}{\Gamma(2)} u^{2-1} e^{-\beta u}\, du\right]
\end{aligned}
$$

$$
\begin{aligned}
\lambda \int_0^t p(s)\, ds &= \frac{\lambda}{\beta}\left[F_{exponential(\beta)}(t) + F_{gamma(2,\beta)}(t)\right) \\
&= \frac{\lambda}{\beta}\left[F_{exponential(\beta)}(t) + 1 - F_{Poisson(\beta t)}(2-1)\right] \\
&= \frac{\lambda}{\beta}\left[1 - e^{-\beta t} + 1 - e^{-\beta t} - e^{-\beta t}(\beta t)\right] \\
&= \frac{\lambda}{\beta}\left[2 - e^{-\beta t}(2 + \beta t)\right] \\
&\overset{\lambda=500,\beta=1/15,t=50}{=} 13573.040266.
\end{aligned}
$$

3. Admit that jobs arrive to a workstation according to a non-homogeneous Poisson process with intensity function $\lambda(t) = 1 + e^{-t}$, $t \geq 0$ (time in hours).

(a) Suppose two jobs arrived during the first hour. What is the probability that both jobs arrived during the first 20 minutes? (1.5)

- **Stochastic process**

  $\{N(t) : t > 0\} \sim NHPP(\lambda(t))$

  $N(t) =$ number of jobs arrived to the workstation until time t

- **Intensity and mean value functions**

$$
\begin{aligned}
\lambda(t) &= 1 + e^{-t}, \quad t \geq 0 \\
m(t) &= \int_0^t \lambda(s)\, ds \\
&= \int_0^t (1 + e^{-s})\, ds \\
&= t + 1 - e^{-t}, \quad t \geq 0
\end{aligned}
$$

  Requested probability

  Since

$$
(N(s) \mid N(t) = n) \sim \text{binomial}(n, m(s)/m(t)), \quad 0 < s < t,
$$

  $s = 1/3$, $t = 1$, $n = 2$, and

$$
\begin{aligned}
\frac{m(s)}{m(t)} &= \frac{1/3 + 1 - e^{-1/3}}{1 + 1 - e^{-1}} \\
&\simeq 0.377914,
\end{aligned}
$$

  we get

$$
\begin{aligned}
P[N(1/3) = 2 \mid N(1) = 2] &\simeq P_{binomial(2,0.377914)}(2) \\
&= \binom{2}{2} \times (0.377914)^2 \times (1 - 0.377914)^{2-2} \\
&= (0.377914)^2 \\
&\simeq 0.142819.
\end{aligned}
$$

- **[Alternatively...**

  Since $N(t) \sim \text{Poisson}(m(t))$, for $t \geq 0$, and $N(t) - N(s) \sim \text{Poisson}(m(t) - m(s))$, for $0 \leq s < t$, we have

$$
\begin{aligned}
P[N(1/3) = 2 \mid N(1) = 2] &= \frac{P[N(1/3) = 2,\, N(1) = 2]}{P[N(1) = 2]} \\
&= \frac{P[N(1/3) = 2,\, N(1) - N(1/3) = 2-2]}{P[N(1) = 2]} \\
&\overset{indep.incr.}{=} \frac{P[N(1/3) = 2] \times P[N(1) - N(1/3) = 2-2]}{P[N(1) = 2]} \\
&= \frac{e^{-m(1/3)}\frac{[m(1/3)]^2}{2!} \times e^{-[m(1)-m(1/3)]}\frac{[m(1)-m(1/3)]^{2-2}}{(2-2)!}}{e^{-m(1)}\frac{[m(1)]^2}{2!}}
\end{aligned}
$$

$$\begin{aligned} P[N(1/3)=2 \mid N(1)=2] &= \left[\frac{m(1/3)}{m(1)}\right]^2 \\ &\simeq (0.377914)^2 \\ &\simeq 0.142819.] \end{aligned}$$

(b) Obtain the probability that the first job arrives to the workstation during the first hour. (0.5)

- **Relevant r.v. and its c.d.f.**
  S_n = time of the n^{th} job arrival
  $F_{S_n}(t) \stackrel{form.}{=} 1 - F_{Poisson(m(t))}(n-1)$
- **Requested probability**
  For $n=1$, $t=1$ and our particular NHPP,

$$\begin{aligned} P(S_1 \le 1) &= 1 - F_{Poisson(m(1))}(1-1) \\ &= 1 - e^{-m(1)} \\ &= 1 - e^{-(1+1-e^{-1})} \\ &\simeq 0.804485. \end{aligned}$$

4. Admit that: (2.0)

- electrical shocks with random amplitudes occur according to a Poisson process with rate λ, $\{N(t): t \ge 0\}$;
- the amplitudes A_i of the successive shocks are independent r.v. and also independent of the arrival times S_i of the associated shocks;
- the amplitudes A_i of the electrical shocks have a common c.d.f. F and mean μ;
- the amplitude of a shock decreases with time at an exponential rate α, meaning that an initial amplitude A will have value $A e^{-\alpha x}$ after an additional time x has elapsed.

Let $A(t) = \sum_{i=1}^{N(t)} A_i e^{-\alpha(t-S_i)}$ denote the sum of the amplitudes of all electrical shocks at time t.

Find $E[A(t)]$ by conditioning on $N(t)$.

Hint: $E\left[\sum_{i=1}^n e^{\alpha Y_{(i)}}\right] = E\left[\sum_{i=1}^n e^{\alpha Y_i}\right]$.

- **Auxiliary r.v. and stochastic process**
  A_i = amplitude of the i^{th} electrical shock
  $A_i \stackrel{i.i.d.}{\sim} A, \quad i \in \mathbb{N}$
  $E(A) = \mu$
  $N(t)$ = number of electrical shocks up to time t
  $N(t) \sim \text{Poisson}(\lambda t)$
  S_i = event time of the i^{th} electrical shock
  $\{S_i : i \in \mathbb{N}\} \perp\!\!\!\perp \{A_i : i \in \mathbb{N}\}$
- **Relevant r.v.**
  The sum of the amplitudes of all electrical shocks at time t is given by

$$A(t) = \sum_{i=1}^{N(t)} A_i e^{-\alpha(t-S_i)}$$

- **Requested expected value**
  By conditioning on $N(t)$, we get

$$\begin{aligned} E[A(t)] &= E\{E[A(t) \mid N(t)]\} \\ &= \sum_{n=0}^{N(t)} E[A(t) \mid N(t)=n] \times P[N(t)=n], \end{aligned}$$

where the r.v. $E[A(t) \mid N(t)]$ takes value

$$\begin{aligned} E[A(t) \mid N(t)=n] &= E\left[\sum_{i=1}^{N(t)} A_i e^{-\alpha(t-S_i)} \mid N(t)=n\right] \\ &\stackrel{A_i\, indep.\, of\, S_i}{=} \sum_{i=1}^{n} e^{-\alpha t} E(A_i) \times E\left[e^{\alpha S_i} \mid N(t)=n\right] \\ &\stackrel{E(A_i)=\mu}{=} e^{-\alpha t} E(A) \times E\left[\sum_{i=1}^{n} e^{\alpha S_i} \mid N(t)=n\right]. \end{aligned}$$

with probability $P[N(t)=n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, $n \in \mathbb{N}$.

Under the condition that n events have occurred in $(0,t]$, the event times $S_1, \ldots, S_n$ behave as the order statistics $Y_{(1)}, \ldots, Y_{(n)}$ associated with $Y_i \sim_{i.i.d.} Y \sim \text{uniform}(0,t)$, $i = 1, \ldots, n$, thus

$$E\left[\sum_{i=1}^{n} e^{\alpha S_i} \mid N(t)=n\right] = E\left[\sum_{i=1}^{n} e^{\alpha Y_{(i)}}\right]$$

Moreover, since $(Y_{(1)}, \ldots, Y_{(n)})$ is a permutation of $(Y_1, \ldots, Y_n)$, we have

$$\begin{aligned} E\left[\sum_{i=1}^{n} e^{\alpha Y_{(i)}}\right] &= E\left[\sum_{i=1}^{n} e^{\alpha Y_i}\right] \\ E[A(t) \mid N(t)=n] &= e^{-\alpha t} E(A) \times E\left[\sum_{i=1}^{n} e^{\alpha Y_i}\right] \\ &\stackrel{Y_i \sim_{i.i.d.} Y}{=} e^{-\alpha t} E(A) \times n E[e^{\alpha Y}] \\ &= n \times e^{-\alpha t} E(A) \times M_Y(\alpha) \\ &\stackrel{Y\sim \text{uniform}(0,t),\, form.}{=} n \times e^{-\alpha t} E(A) \times \frac{e^{t\times\alpha} - e^{0\times\alpha}}{(t-0)\times\alpha} \\ &= n \times \mu \times \frac{1-e^{-\alpha t}}{\alpha t} \\ E[A(t)] &= E\left[N(t) \times \mu \times \frac{1-e^{-\alpha t}}{\alpha t}\right] \\ &= (\lambda t) \times \mu \times \frac{1-e^{-\alpha t}}{\alpha t} \\ &= \lambda\mu \times \frac{1-e^{-\alpha t}}{\alpha}. \end{aligned}$$

Group 2 — Renewal Processes 8.0 points

1. Consider a renewal process $\{N(t): t \ge 0\}$, with inter-renewal times $X_i \stackrel{i.i.d.}{\sim} \text{Poisson}(1)$, for $i \in \mathbb{N}$.

(a) Verify that $P[N(0)=0] = 1 - e^{-1}$. Identify the distribution of the event time $S_n = \sum_{i=1}^n X_i$, for $n \in \mathbb{N}$, and use it to derive $P[N(0)=n]$, for $n \in \mathbb{N}$. (1.5)

- **Renewal process**
  $\{N(t): t \ge 0\}$
  $N(t)$ = number of renewals until and at time t
- **Inter-renewal times**
  $X_i \stackrel{i.i.d.}{\sim} X \sim \text{Poisson}(1), \quad i \in \mathbb{N}$ (*)

$P(X=x) \overset{form.}{=} \frac{e^{-1}1^x}{x!} = \frac{e^{-1}}{x!}, \quad x \in \mathbb{N}_0$

- **Requested probability**

$$\begin{aligned} P[N(0)=0] &= P(X_1>0) \\ &= 1-P(X_1=0) \\ &= 1-e^{-1} \end{aligned}$$

- **Requested distribution**

$S_n = \sum_{i=1}^n X_i \overset{(*)}{\sim} \text{Poisson}(n), \quad n \in \mathbb{N}$

$F_n(t) = P(S_n \le t)$

- **Requested p.f.**

For $n \in \mathbb{N}$,

$$\begin{aligned} P[N(0)=n] &\overset{form.}{=} F_n(0) - F_{n+1}(0) \\ &= F_{Poisson(n)}(0) - F_{Poisson(n+1)}(0) \\ &= e^{-n} - e^{-(n+1)} \\ &= \left(e^{-1}\right)^n \left(1-e^{-1}\right). \end{aligned}$$

[We can conclude that $N(0) \sim \text{geometric}^*(1-e^{-1})$.]

(b) Calculate the exact and an approximate value to $P[N(t)<n]$, where $n=t=40$. (1.5)

- **Requested exact probability**

$$\begin{aligned} P[N(t)<n] &= 1-P[N(t)\ge n] \\ &= 1-F_n(t) \\ &\overset{t=n=40,(a)}{=} 1-F_{Poisson(40)}(40) \\ &\overset{tables}{=} 1-0.5419 \\ &= 0.4581. \end{aligned}$$

- **Requested approximate probability**

$\mu = E(X) = 1$

$\sigma^2 = V(X) = 1$ [In this problem, we do not need to know σ^2 to obtain the prob.]

Now, applying the CLT for RP, we get

$$\begin{aligned} P[N(t)<n] &\overset{form.}{\simeq} \Phi\left(\frac{n-t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right) \\ &\overset{t=n=40}{=} \Phi\left(\frac{40-40/1}{\sqrt{40\times 1/1^3}}\right) \\ &= \Phi(0) \\ &= 0.5. \end{aligned}$$

- **[Comment**

The approximate value overestimates the exact one; the relative error is $(0.5-0.4581)/0.4581 \times 100\% \simeq 9.15\%$. This slight deviation is probably due to the fact that $t=40$ is not sufficiently large to achieve a very good approximate value by applying the CLT for RP.]

(c) Derive an expression for $E[\text{number of renewals } \textbf{at } n] = m(n)-m(n-1)$, for $n \in \mathbb{N}$.[1] (1.5)

Obtain $\lim_{n\to+\infty} E[\text{number of renewals } \textbf{at } n]$.

- **Requested expression**

Since we are dealing with a lattice inter-renewal distribution with period $d=1$ (the Poisson distribution),

$$\begin{aligned} E[\text{number of renewals } \textbf{at } n] &= m(n)-m(n-1) \\ &\overset{form.}{=} \sum_{i=1}^{+\infty} F_i(n) - \sum_{i=1}^{+\infty} F_i(n-1) \\ &= \sum_{i=1}^{+\infty} [F_{Poisson(i)}(n) - F_{Poisson(i)}(n-1)] \\ &= \sum_{i=1}^{+\infty} P_{Poisson(i)}(n) \\ &= \sum_{i=1}^{+\infty} e^{-i}\frac{i^n}{n!}, \quad n \in \mathbb{N}. \end{aligned}$$

- **Requested limit**

Since the inter-renewal distribution is lattice with period $d=1$ and expected value $\mu=1$, the application of Blackwell's theorem leads to the conclusion that

$$\begin{aligned} \lim_{n\to+\infty} E[\text{number of renewals } \textbf{at } nd] \quad &[= \lim_{n\to+\infty} \sum_{i=1}^{+\infty} e^{-i}\frac{i^n}{n!}] \\ &= \frac{d}{\mu} \\ &= 1. \end{aligned}$$

2. A machine consists of two independent components set in parallel.[2] When a machine fails, a new machine is promptly put into use.

(a) Obtain the expected duration of a machine, when its component j operates for an exponential time with rate λ_j $(j=1,2)$. (1.5)

- **R.v.**

Y_j = duration of component j

$Y_j \overset{indep.}{\sim} \text{exponential}(\lambda_j), \quad j=1,2$

X = duration of a machine = $\max\{Y_1,Y_2\}$

- **Requested expected value**

$$\begin{aligned} F_X(x) &= P[\max\{Y_1,Y_2\}\le x] \\ &\overset{Y_s\, indep.r.v.}{=} \prod_{j=1}^2 P(Y_j \le x) \\ &\overset{Y_j\sim \exp(\lambda_j)}{=} \left(1-e^{-\lambda_1 x}\right)\times\left(1-e^{-\lambda_2 x}\right) \\ &= 1-e^{-\lambda_1 x}-e^{-\lambda_2 x}+e^{-(\lambda_1+\lambda_2)x} \\ E(X) &\overset{X\ge 0}{=} \int_0^{+\infty} [1-F_X(x)]\,dx \\ &= \int_0^{+\infty} \left[e^{-\lambda_1 x}+e^{-\lambda_2 x}-e^{-(\lambda_1+\lambda_2)x}\right] dx \\ &= \frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_1+\lambda_2}. \end{aligned}$$

(b) A cost K is incurred whenever a machine failure occurs. Moreover, operating costs at a rate of c_i euros per unit time are incurred whenever the machine in use has i working components $(i=1,2)$. Obtain the average cost incurred per machine and the long-run average cost per time unit. (2.0)

- **Cost incurred per machine**

$R = K + c_2 \times T_2 + c_1 \times T_1$

T_2 = time a machine spends with two components operating

$= \min\{Y_1,Y_2\} \overset{form.}{\sim} \text{exponential}(\lambda_1+\lambda_2)$

[1] Do not use the LST method or try to solve the renewal equation to obtain the renewal function $m(t)$, $t \ge 0$.

[2] That is, the machine operates as long as at least one of these two components function.

T_1 = time a machine spends with only one component operating

$$\sim \begin{cases} Y_2, & \text{if component 1 fails first (i.e., } Y_1 = \min\{Y_1, Y_2\}\text{), with probability } \frac{\lambda_1}{\lambda_1+\lambda_2} \\ Y_1, & \text{if component 2 fails first (i.e., } Y_2 = \min\{Y_1, Y_2\}\text{), with probability } \frac{\lambda_2}{\lambda_1+\lambda_2} \end{cases}$$

[Note that, due to the lack of memory of the exponential distribution, the residual life of the surviving component i has an exponential distribution with parameter λ_j $(j = 1, 2)$. Furthermore, T_2 is a mixture of variables Y_2 and Y_1 with associated *weights* $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{\lambda_2}{\lambda_1+\lambda_2}$ (respectively).]

- **Expected value of the cost incurred per machine**

$$\begin{aligned} E(R) &= E(K + c_2 \times T_2 + c_1 \times T_1) \\ &= K + c_2 \times E(T_2) + c_1 \times E(T_1) \\ &= K + c_2 \times \frac{1}{\lambda_1+\lambda_2} + c_1 \times \frac{\lambda_1}{\lambda_1+\lambda_2} E(Y_2) + c_1 \times \frac{\lambda_2}{\lambda_1+\lambda_2} E(Y_1) \\ &= K + c_2 \times \frac{1}{\lambda_1+\lambda_2} + c_1 \times \frac{\lambda_1}{\lambda_1+\lambda_2} \frac{1}{\lambda_2} + c_1 \times \frac{\lambda_2}{\lambda_1+\lambda_2} \frac{1}{\lambda_1} \end{aligned}$$

- **Renewal process**

  $\{N(t) : t \geq 0\}$

  $N(t)$ = number of machines that failed until time t

- **Inter-renewal times; rewards**

  $X_n \overset{i.i.d.}{\sim} X, \quad n \in \mathbb{N}$

  $R_n \overset{i.i.d.}{\sim} R, \quad n \in \mathbb{N}$

- **Reward renewal process**

  $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \geq 0\}$

  $R(t)$ = total reward gained until time t

  $(X_n, R_n) \overset{i.i.d.}{\sim} (X, R),\ n \in \mathbb{N}$

- **Long-run expected total reward per time unit**

  Since $E(X), E(R) < +\infty$, we can apply the ERT for renewal reward processes and get

$$\begin{aligned} \lim_{t \to +\infty} \frac{E[R(t)]}{t} &= \frac{E(R)}{E(X)} \\ &= \frac{K + c_2 \times \frac{1}{\lambda_1+\lambda_2} + c_1 \times \frac{\lambda_1}{\lambda_1+\lambda_2} \frac{1}{\lambda_2} + c_1 \times \frac{\lambda_2}{\lambda_1+\lambda_2} \frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}}. \end{aligned}$$

- **[Obs.**

  $X = T_2 + T_1$, thus $E(X) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2} \equiv E(T) = \frac{1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2} \frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} \frac{1}{\lambda_1}$.]