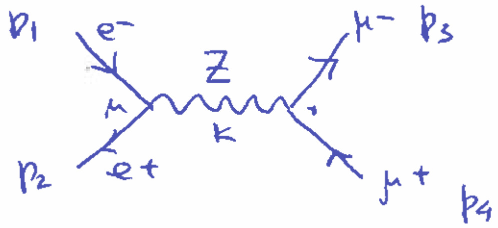
1) a) At these energies the dominant diagram is the Z exchange diagram



$k = p_1 + p_2$

because at $\sqrt{s} = M_Z$ there is a ressonant effect due to Z propagator denominator

$$\frac{1}{(s - M_Z^2) + i M_Z \Gamma_Z}$$

b)
$$iM = \left(-\frac{ig}{c_W}\right)^2 \bar{v}(p_2)\gamma^\mu (g_V^e - g_A^e \gamma_5) u(p_1) \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_Z^2}\right)}{k^2 - M_Z^2 + i M_Z \Gamma_Z} \cdot$$

$$\bar{u}(p_3)\gamma^\nu (g_V^\mu - g_A^\mu \gamma_5) v(p_4)$$

Now neglecting the fermion mass we can neglect the momenta in the Z propagator numerator. In fact

$$\bar{v}(p_2)\gamma^\mu (g_V^e - g_A^e\gamma_5) u(p_1) k_\mu =$$

$$= \bar{v}(p_2) \not{p}_1 (g_V^e - g_A^e \gamma_5) u(p_1) + \bar{v}(p_2) \not{p}_2 (g_V^e - g_A^e \gamma_5) u(p_1)$$

$$= \bar{v}(p_2) (g_V^e + g_A^e \gamma_5) \not{p}_1 u(p_1) + \bar{v}(p_2) \not{p}_2 (g_V^e - g_A^e \gamma_5) u(p_1)$$

$$= 0$$

because for massless fermions the Dirac equation gives

$$\not{p}_1 u(p_1) = 0 \quad ; \quad \bar{v}(p_2) \not{p}_2 = 0$$

then

$$M = \frac{g^2}{c_W^2 M_Z^2} \frac{M_Z^2}{(s-M_Z^2)+i M_Z \Gamma_Z} \bar{v}(p_2)\gamma^\mu(g_V - g_A\gamma_5)u(p_1)\,\bar{u}(p_3)\gamma_\mu(g_V' - g_A'\gamma_5)\,v(p_4)$$

Defining

$$\frac{g^2}{c_W^2 M_Z^2} = \frac{g^2}{M_W^2} = \frac{8G_F}{\sqrt{2}} \; ; \quad F(s) = \frac{M_Z^2}{(s-M_Z^2)+i M_Z \Gamma_Z}$$

we get finally

$$\boxed{M = \frac{8G_F}{\sqrt{2}} F(s)\, \bar{v}(p_2)\gamma^\mu(g_V^e - g_A^e\gamma_5)u(p_1)\,\bar{u}(p_3)\gamma_\mu(g_V^\mu - g_A^\mu\gamma_5)v(p_4)}$$

c) To use the helicity amplitude method we have to write the amplitude in terms of the P_L, P_R projectors.

we have

$$(g_V^f - g_A^f\gamma_5) = (g_V^f - g_A^f\gamma_5)(P_L + P_R) =$$

$$= \underbrace{(g_V^f + g_A^f)}_{g_L^f} P_L + \underbrace{(g_V^f - g_A^f)}_{g_R^f} P_R$$

$$\equiv g_L^f P_L + g_R^f P_R$$

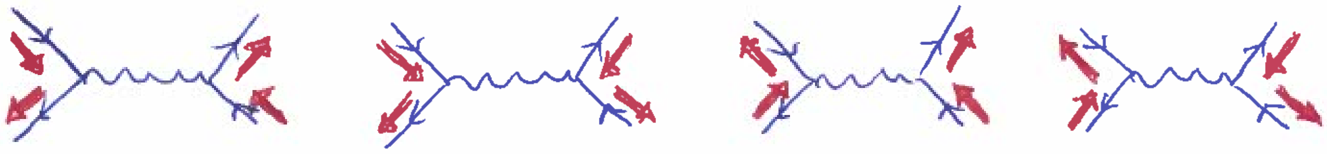
then

$$\boxed{M = \frac{8G_F}{\sqrt{2}} F(s)\, \bar{v}(p_2)\gamma^\mu(g_L^e P_L + g_R^e P_R)u(p_1)\,\bar{u}(p_3)\gamma_\mu(g_L^\mu P_L + g_R^\mu P_R)v(p_4)}$$

using $\quad P_L u(p_1) = u(p_1,\downarrow) \qquad P_R u(p_1) = u(p_1,\uparrow)$

$\qquad P_L v(p_4) = v(p_4,\uparrow) \qquad P_R v(p_4) = v(p_4,\downarrow)$

we have 4 non zero amplitudes



$M(\uparrow\downarrow;\uparrow\downarrow)$ $\quad$ $M(\uparrow\downarrow;\downarrow\uparrow)$ $\quad$ $M(\downarrow\uparrow;\uparrow\downarrow)$ $\quad$ $M(\downarrow\uparrow;\downarrow\uparrow)$

We have

$$M(\uparrow\downarrow;\uparrow\downarrow) = \frac{8 G_F F(s)}{\sqrt{2}} g_R^e g_R^\mu \quad J_{u_1 v_2}(\uparrow\downarrow) \cdot J_{u_3 v_4}(\uparrow\downarrow)$$

$$= -\frac{8 G_F F(s)}{\sqrt{2}} g_R^e g_R^\mu \; s\,(1+\cos\theta)$$

$$M(\uparrow\downarrow;\downarrow\uparrow) = \frac{8 G_F F(s)}{\sqrt{2}} g_R^e g_L^\mu \quad J_{u_1 v_2}(\uparrow\downarrow) \cdot J_{u_3 v_4}(\downarrow\uparrow)$$

$$= \frac{8 G_F F(s)}{\sqrt{2}} g_R^e g_L^\mu \; s\,(1-\cos\theta)$$

$$M(\downarrow\uparrow;\uparrow\downarrow) = \frac{8 G_F F(s)}{\sqrt{2}} g_L^e g_R^\mu \quad J_{u_1 v_2}(\downarrow\uparrow) \cdot J_{u_3 v_4}(\uparrow\downarrow)$$

$$= \frac{8 G_F F(s)}{\sqrt{2}} g_L^e g_R^\mu \; s\,(1-\cos\theta)$$

$$M(\downarrow\uparrow;\downarrow\uparrow) = \frac{8 G_F F(s)}{\sqrt{2}} g_L^e g_L^\mu \quad J_{u_1 v_2}(\downarrow\uparrow) \cdot J_{u_3 v_4}(\downarrow\uparrow)$$

$$= -\frac{8 G_F F(s)}{\sqrt{2}} g_L^e g_L^\mu \, s\,(1+\cos\theta)$$

therefore

$$\langle |M|^2 \rangle = \frac{32\, G_F^2\, s^2\, |F(s)|^2}{4} \Big[(1+\cos\theta)^2 \big[(g_R^e g_R^\mu)^2 + (g_L^e g_L^\mu)^2 \big] + (1-\cos\theta)^2 \big[(g_R^e g_L^\mu)^2 + (g_L^e g_R^\mu)^2 \big] \Big]$$

d) $$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \langle |M|^2 \rangle$$ (massless particle $|\vec{p}_i| = |\vec{p}_f|$)

then

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 s}{8\pi^2} |F(s)|^2 \Big[(1+\cos\theta)^2 \left[(g_R^e g_R^\mu)^2 + (g_L^e g_L^\mu)^2 \right] + (1-\cos\theta)^2 \left[(g_R^e g_L^\mu)^2 + (g_L^e g_R^\mu)^2 \right] \Big]$$

e) Nothing depends on ϕ so

$$\int d\Omega = 2\pi \int_{-1}^{1} d\cos\theta$$

$$\int_{-1}^{1} d(\cos\theta)\, (1+\cos\theta)^2 = \int_{-1}^{1} dx\, (1+x)^2 = 2 + \frac{2}{3} = \frac{8}{3}$$

$$\int_{-1}^{1} d\cos\theta\, (1-\cos\theta)^2 = \int_{-1}^{1} dx\, (1-x)^2 = 2 + \frac{2}{3} = \frac{8}{3}$$

so

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{G_F^2 s}{8\pi^2} |F(s)|^2 \times 2\pi \times \frac{8}{3} \left[(g_R^e g_R^\mu)^2 + (g_L^e g_L^\mu)^2 + (g_R^e g_L^\mu)^2 + (g_L^e g_R^\mu)^2 \right]$$

$$= \frac{2 G_F^2 s |F(s)|^2}{3\pi} \left[(g_R^e g_R^\mu)^2 + (g_L^e g_L^\mu)^2 + (g_R^e g_L^\mu)^2 + (g_L^e g_R^\mu)^2 \right]$$

at $\sqrt{s} = M_Z$

$$\sigma(\sqrt{s} = M_Z) = \frac{2}{3} \frac{G_F^2}{\pi} \frac{M_Z^4}{\Gamma_Z^2} \left[\dots \right]$$

We have $g_L^e = g_L^\mu$; $g_R^e = g_R^\mu$ and

$$g_L^e = g_V^e + g_A^e = T_3^e + s_W^2 \simeq -0.27$$

$$g_R^e = g_V^e - g_A^e = s_W^2 = 0.23$$

$$\Gamma_Z \simeq 2.5 \text{ GeV}$$

$$M_Z \simeq 91.2 \text{ GeV}$$

$$\sigma(\sqrt{s} = M_Z) = 1969 \text{ pb}$$

f) The Branching Ratio $BR(Z \to e e^-) = 3.36\%$ (PDG). At $\sqrt{s} = M_Z$

$$\sigma(e^+e^- \to All) = \frac{\sigma(e^+e^- \to \mu^+\mu^-)}{BR(Z \to \mu^+\mu^-)} = 58.6 \text{ nb}$$

2) a) Using the results of Problem 1 we have

$$\langle |M_L|^2 \rangle = \frac{1}{2}\left[|M(\downarrow\uparrow;\uparrow\downarrow)|^2 + |M(\downarrow\uparrow;\downarrow\uparrow)|^2 \right]$$

$$= 16 G_F^2 s^2 |F(s)|^2 \left[(1-\cos\theta)^2 (g_L^e g_R^\mu)^2 + (1+\cos\theta)^2 (g_L^e g_L^\mu)^2 \right]$$

$$= 16 G_F^2 s^2 |F(s)|^2 (g_L^e)^2 \left[(1-\cos\theta)^2 (g_R^\mu)^2 + (1+\cos\theta)^2 (g_L^\mu)^2 \right]$$

$$\langle |M_R|^2 \rangle = \frac{1}{2}\left[|M(\uparrow\downarrow;\uparrow\downarrow)|^2 + |M(\uparrow\downarrow;\downarrow\uparrow)|^2 \right]$$

$$= 16 G_F^2 s^2 |F(s)|^2 {g_R^e}^2 \left[(1+\cos\theta)^2 (g_R^\mu)^2 + (1-\cos\theta)^2 (g_L^\mu)^2 \right]$$

b) using 6

$$\int d\Omega\,(1\pm\cos\theta)^2 = 2\pi \times \frac{8}{3} = \frac{16\pi}{3}$$

we have

$$\sigma_L = \int d\Omega\,\frac{d\sigma_L}{d\Omega} = \frac{16\,G_F^2 s^2}{64\pi^2 s}|F(s)|^2 (g_L^e)^2\,\frac{16\pi}{3}\left((g_R^\mu)^2 + (g_L^\mu)^2\right)$$

or

$$\sigma_L = \frac{4G_F^2 s}{3\pi}|F(s)|^2 (g_L^e)^2\left[(g_R^\mu)^2 + (g_L^\mu)^2\right]$$

for σ_R

$$\sigma_R = \int d\Omega\,\frac{d\sigma_R}{d\Omega} = \frac{16\,G_F^2 s^2}{64\pi^2 s}|F(s)|^2 (g_R^e)^2\,\frac{16\pi}{3}\left[(g_R^\mu)^2 + (g_L^\mu)^2\right]$$

c)

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{(g_L^e)^2 - (g_R^e)^2}{(g_L^e)^2 + (g_R^e)^2}$$

Now

$$g_L^e = g_V^e + g_A^e \quad ; \quad g_R^e = g_V^e - g_A^e$$

$$(g_L^e)^2 - (g_R^e)^2 = 4\,g_V^e g_A^e$$

$$(g_R^e)^2 + (g_L^e)^2 = 2(g_V^e)^2 + 2(g_A^e)^2$$

therefore

$$\boxed{A_{LR} = \frac{2 g_V^e g_A^e}{(g_V^e)^2 + (g_A^e)^2}}$$

As we are summing over the polarizations of the μ^+ the factor $(g_R^\mu)^2 + (g_L^\mu)^2$ cancels in the ratio.

d) $P_e = 0.224$, $N_L = 5226$; $N_R = 4998$

$$A_{LR}(\text{measured}) = \frac{1}{P_e} \frac{N_L - N_R}{N_L + N_R} \simeq 0.099 \ (\pm 0.044)$$

e) $$g_V^e g_A^e = (-\tfrac{1}{4})\left[-\tfrac{1}{4} + s_W^2\right] = \frac{1}{16}\left[1 - 4 s_W^2\right]$$

$$(g_V^e)^2 + (g_A^e)^2 = \frac{1}{16} + \left(-\frac{1}{4} + s_W^2\right)^2 = \frac{1}{16}\left[1 + (1 - 4 s_W^2)^2\right]$$

$$= \frac{1}{16}\left[2 - 8 s_W^2 + 16 s_W^4\right] = \frac{1}{8}\left[1 - 4 s_W^2 + 8 s_W^4\right]$$

So

$$A_{LR} = \frac{2 g_V^e g_A^e}{(g_V^e)^2 + (g_A^e)^2} = \frac{1 - 4 s_W^2}{1 - 4 s_W^2 + 8 s_W^4}$$

From here we have

$$8 s_W^4 A_{LR} + 4 s_W^2 (1 - A_{LR}) - (1 - A_{LR}) = 0$$

$$\boxed{s_W^2 = \frac{-(1 - A_{LR}) + \sqrt{(1 - A_{LR})^2 + 2(1 - A_{LR}) A_{LR}}}{4 A_{LR}}}$$

1

f) If $A_{LR} = 0.099$ we get 0.2376 which is well above the current value. However taking in account the given error we get

$$s_w^2(0.099+0.044) = 0.2320$$

$$s_w^2(0.099-0.044) = 0.2431$$

which is closer to the present value even at 1σ. If we take 2σ in the experimental value of 1992 we get

$$\boxed{0.2264 \leq s_w^2(1992) \leq 0.2486} \quad \text{at } 2\sigma$$

that includes the present measured value.

<u>Note</u>: this is just a quick way to find if the result is reasonable. A more correct way would be to propagate the error in ①. One would get

$$\delta s_w^2 = \left| \frac{d s_w^2}{d A_{LR}} \right| \delta A_{LR} = 0.0055$$

and at 2σ

$$\boxed{0.2266 \leq s_w^2(1992) \leq 0.2486}$$

in agreement with the above estimative.