# Chapter 2

# The Laplace transform

> "And, if we transmit through a wormhole, the person is always reconstituted at the other end. We can count on that happening, too."
> There was a pause.
> Stern frowned.
> "Wait a minute," he said. "Are you saying that when you transmit, the person is being reconstituted by another universe?"
> "In effect, yes. I mean, it has to be. We can't very well reconstitute them, because we're not there. We're in this universe."
>
> Michael CRICHTON (1942 — †2008), *Timeline*, Black rock

The **Laplace transform** is a very important tool for the resolution of differential equations. In this chapter we will study its definition, its properties, its application to differential equations (which is the reason we are studying this subject), and the related Fourier transform, that we will also need. *Laplace transform*

## 2.1 Definition

**Definition 2.1.** Let $t \in \mathbb{R}$ be a real variable, and $f(t) \in \mathbb{R}$ a real-valued function. The Laplace transform of function $f$, denoted by $\mathscr{L}[f(t)]$ or by $F(s)$, is a complex-valued function $F(s) \in \mathbb{C}$ of complex variable $s \in \mathbb{C}$, given by

$$\mathscr{L}\left[f(t)\right] = \int_0^{+\infty} f(t)e^{-st}\,\mathrm{d}t \quad \square \tag{2.1}$$

**Remark 2.1.** Strictly speaking, operation $\mathscr{L}$ is the Laplace transformation, and the result of applying $\mathscr{L}$ to a function gives us its Laplace transform. But it is common to call the operation itself Laplace transform as well. □

**Remark 2.2.** In (2.1), function $f(t)$ only has to be defined for $t \geq 0$. This would not be so if we were using the **bilateral Laplace transform**, which is an alternative definition given by *Bilateral Laplace transform*

$$\mathscr{L}\left[f(t)\right] = \int_{-\infty}^{+\infty} f(t)e^{-st}\,\mathrm{d}t \tag{2.2}$$

This bilateral Laplace transform is seldom used; we will use (2.1) instead, as is common. The price to pay for being able to work with functions defined in $\mathbb{R}^+$ only will be addressed below in section 2.4. □

**Remark 2.3.** The Laplace transform is part of a group of transforms known as integral transforms, given by

$$\mathscr{T}[f(t)] = \int_0^{+\infty} f(t)K(s,t)\,\mathrm{d}t \tag{2.3}$$

where $\mathscr{T}$ is a generic transform and $K(s,t)$ is a function called kernel. In the case of the Laplace transform, the kernel is $K(s,t) = e^{-st}$. □

*Existence of the Laplace transform*

The Laplace transform of function $f(t)$ will only exist if the improper integral in (2.1) converges. This will happen in one of two cases:

- If $f(t)$ is bounded in its domain $\mathbb{R}^+$, the integrand $f(t)e^{-st}$ will obviously tend to 0 as $t \to +\infty$.
- If $f(t)$ tends to infinity as $t \to +\infty$, but does so slower than $e^{-st}$ tends to 0, the integrand will still tend to 0. More rigorously, $f(t)$ must be of exponential order, i.e. there must be positive real constants $M, c \in \mathbb{R}$ such that

$$|f(t)| \le M\,e^{ct}, \quad 0 \le t \le \infty. \tag{2.4}$$

Otherwise, the integrand of (2.1) does not tend to 0 and it is obvious that the improper integral will be infinite. For complete rigour we also have to require $f(t)$ to be piecewise continuous for $F(s)$ to exist; we will not prove here that this is indeed so.

**Remark 2.4.** In fact (2.1) may converge only for some values of $s$, and thus have a region of convergence which is smaller than $\mathbb{C}$; but then it can be analytically extended to the rest of the complex plane. This is a question we will not worry about. □

## 2.2 Finding Laplace transforms

*Heaviside function*

**Example 2.1.** Let $f(t)$ be function

$$H(t) = \begin{cases} 1, & \text{if } t \ge 0 \\ 0, & \text{if } t < 0 \end{cases}, \tag{2.5}$$

*$\mathscr{L}[H(t)]$*

known as the Heaviside function. Then

$$\mathscr{L}[H(t)] = \int_0^{+\infty} 1 \times e^{-st}\,\mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^{+\infty} = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = \frac{1}{s}. \quad \square \tag{2.6}$$

*$\mathscr{L}[e^{-at}]$*

**Example 2.2.** Let $f(t)$ be a negative exponential, $f(t) = e^{-at}$. Then

$$\begin{aligned} \mathscr{L}\left[e^{-at}\right] &= \int_0^{+\infty} e^{-at}e^{-st}\,\mathrm{d}t \\ &= \left[\frac{e^{-(a+s)t}}{-a-s}\right]_0^{+\infty} = -\frac{e^{-\infty}}{s+a} - \left(-\frac{e^0}{s+a}\right) = \frac{1}{s+a}. \quad \square \end{aligned} \tag{2.7}$$

Table 2.1: Table of Laplace transforms

| | $x(t)$ | $X(s)$ |
|---|---|---|
| 1 | $\delta(t)$ | $1$ |
| 2 | $H(t)$ | $\frac{1}{s}$ |
| 3 | $t$ | $\frac{1}{s^2}$ |
| 4 | $t^2$ | $\frac{2}{s^3}$ |
| 5 | $e^{-at}$ | $\frac{1}{s+a}$ |
| 6 | $1-e^{-at}$ | $\frac{a}{s(s+a)}$ |
| 7 | $te^{-at}$ | $\frac{1}{(s+a)^2}$ |
| 8 | $t^n e^{-at},\ n \in \mathbb{N}$ | $\frac{n!}{(s+a)^{n+1}}$ |
| 9 | $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| 10 | $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}$ |
| 11 | $e^{-at}\sin(\omega t)$ | $\frac{\omega}{(s+a)^2+\omega^2}$ |
| 12 | $e^{-at}\cos(\omega t)$ | $\frac{s+a}{(s+a)^2+\omega^2}$ |
| 13 | $\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$ | $\frac{1}{(s+a)(s+b)}$ |
| 14 | $\frac{1}{ab}\left(1+\frac{1}{a-b}\left(b\,e^{-at}-a\,e^{-bt}\right)\right)$ | $\frac{1}{s(s+a)(s+b)}$ |
| 15 | $\frac{\omega}{\Xi}\,e^{-\xi\omega t}\sin(\omega\Xi t)$ | $\frac{\omega^2}{s^2+2\xi\omega s+\omega^2}$ |
| 16 | $-\frac{1}{\Xi}\,e^{-\xi\omega t}\sin(\omega\Xi t-\phi)$ | $\frac{s}{s^2+2\xi\omega s+\omega^2}$ |
| 17 | $1-\frac{1}{\Xi}\,e^{-\xi\omega t}\sin(\omega\Xi t+\phi)$ | $\frac{\omega^2}{s(s^2+2\xi\omega s+\omega^2)}$ |

In this table: $\Xi = \sqrt{1-\xi^2}$; $\phi = \arctan\frac{\Xi}{\xi}$

While Laplace transforms can be found from definition as in the two examples above, in practice they are found from tables, such as the one in Table 2.1.

To use with profit Laplace transform tables, it is necessary to prove first the following result.

*$\mathscr{L}$ is linear*

**Theorem 2.1.** The Laplace transform is a linear operator:

$$\mathscr{L}\left[k\, f(t)\right] = k\, F(s), \quad k \in \mathbb{R} \tag{2.8}$$

$$\mathscr{L}\left[f(t) + g(t)\right] = F(s) + G(s) \tag{2.9}$$

*Proof.* Both (2.8) and (2.9) are proved from the linearity of the integration operator:

$$\mathscr{L}\left[k\, f(t)\right] = \int_0^{+\infty} k\, f(t) e^{-st}\, \mathrm{d}t = k \int_0^{+\infty} f(t) e^{-st}\, \mathrm{d}t = k\, F(s) \tag{2.10}$$

$$\begin{aligned} \mathscr{L}\left[f(t) + g(t)\right] &= \int_0^{+\infty} \left(f(t) + g(t)\right) e^{-st}\, \mathrm{d}t && (2.11) \\ &= \int_0^{+\infty} f(t) e^{-st}\, \mathrm{d}t + \int_0^{+\infty} g(t) e^{-st}\, \mathrm{d}t = F(s) + G(s) \; \square \end{aligned}$$

**Example 2.3.** The Laplace transform of $f(t) = 5t$ is obtained from line 3 of Table 2.1 together with (2.8):

$$\mathscr{L}\left[5t\right] = 5 \mathscr{L}\left[t\right] = \frac{5}{s^2} \quad \square \tag{2.12}$$

**Example 2.4.** The Laplace transform of $f(t) = 1 - (1+t)e^{-3t}$ is obtained from lines 6 and 7 of Table 2.1 together with (2.9):

$$\begin{aligned} \mathscr{L}\left[f(t)\right] &= \mathscr{L}\left[1 - e^{-3t} - te^{-3t}\right] = \mathscr{L}\left[1 - e^{-3t}\right] - \mathscr{L}\left[te^{-3t}\right] \\ &= \frac{3}{s\,(s+3)} + \frac{1}{(s+3)^2} = \frac{3s + 3^2 + s}{(s+3)^2} = \frac{4s+9}{s^2+6s+9} \quad \square && (2.13) \end{aligned}$$

## 2.3 Finding inverse Laplace transforms

*Inverse Laplace transform*

Laplace transform tables can also be used to find inverse Laplace transforms, i.e. finding the $f(t)$ corresponding to a given $F(s) = \mathscr{L}\left[f(t)\right]$. This operation is denoted by $f(t) = \mathscr{L}^{-1}\left[F(s)\right]$.

**Example 2.5.** The inverse Laplace transform of $F(s) = \frac{10}{s+10}$ is obtained from line 5 of Table 2.1 together with (2.8):

$$\mathscr{L}^{-1}\left[\frac{10}{s+10}\right] = 10\, \mathscr{L}^{-1}\left[\frac{1}{s+10}\right] = 10\, e^{-10t} \quad \square \tag{2.14}$$

*Partial fraction expansion*

**Example 2.6.** The inverse Laplace transform of $F(s) = \frac{s+2}{s^2+13s+30}$ is obtained from line 5 of Table 2.1 together with (2.8). But for that it is necessary to develop $F(s)$ in a **partial fraction expansion**. First we find the roots of the polynomial in the denominator, which are $-3$ and $-10$. So $s^2 + 13s + 30 = (s+3)(s+10)$, and we can write

$$\frac{s+2}{s^2+13s+30} = \frac{A}{s+3} + \frac{B}{s+10} \tag{2.15}$$

where $A$ and $B$ still have to determined:

$$\frac{A}{s+3}+\frac{B}{s+10}=\frac{As+10A+Bs+3B}{(s+3)(s+10)}=\frac{s(A+B)+(10A+3B)}{s^2+13s+30} \tag{2.16}$$

Obviously we want that

$$\begin{cases} A+B=1 \\ 10A+3B=2 \end{cases} \Leftrightarrow \begin{cases} B=1-A \\ 10A+3-3A=2 \end{cases} \Leftrightarrow \begin{cases} B=\frac{8}{7} \\ A=-\frac{1}{7} \end{cases} \tag{2.17}$$

So $\dfrac{s+2}{s^2+13s+30}=\dfrac{-\frac{1}{7}}{s+3}+\dfrac{\frac{8}{7}}{s+10}$, and finally

$$\mathscr{L}^{-1}\left[\frac{s+2}{s^2+13s+30}\right]=\mathscr{L}^{-1}\left[\frac{-\frac{1}{7}}{s+3}+\frac{\frac{8}{7}}{s+10}\right] \tag{2.18}$$

$$=\mathscr{L}^{-1}\left[\frac{-\frac{1}{7}}{s+3}\right]+\mathscr{L}^{-1}\left[\frac{\frac{8}{7}}{s+10}\right]=-\frac{1}{7}e^{-3t}+\frac{8}{7}e^{-10t} \quad \square$$

**Remark 2.5.** Notice that the result in line 13 of Table 2.1 can be obtained from line 5 also using a partial fraction expansion:

$$\frac{1}{(s+a)(s+b)}=\frac{A}{s+a}+\frac{B}{s+b}=\frac{As+Ab+Bs+aB}{(s+a)(s+b)}=\frac{s(A+B)+(Ab+aB)}{(s+a)(s+b)} \tag{2.19}$$

We want

$$\begin{cases} A+B=0 \\ Ab+aB=1 \end{cases} \Leftrightarrow \begin{cases} B=-A \\ Ab-aA=1 \end{cases} \Leftrightarrow \begin{cases} B=\frac{-1}{b-a} \\ A=\frac{1}{b-a} \end{cases} \tag{2.20}$$

and thus

$$\mathscr{L}^{-1}\left[\frac{1}{(s+a)(s+b)}\right]=\mathscr{L}^{-1}\left[\frac{\frac{1}{b-a}}{s+a}\right]+\mathscr{L}^{-1}\left[\frac{\frac{-1}{b-a}}{s+b}\right] \tag{2.21}$$

$$=\frac{1}{b-a}e^{-at}+\frac{-1}{b-a}e^{-bt}=\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right) \quad \square$$

*Partial fraction expansion with complex roots*

**Example 2.7.** The inverse Laplace transform of $F(s)=\frac{4s^2+13s-2}{(s^2+2s+2)(s+4)}$ is obtained from lines 5, 15 and 16 of Table 2.1 together with 2.8 and 2.9. The transforms in lines 14 and 15 are used because the roots of $4s^2+13s-2$ are complex and not real ($-1\pm j$, to be precise). So we will leave that second order term intact and we make

$$\frac{4s^2+13s-2}{(s^2+2s+2)(s+4)}=\frac{As+B}{s^2+2s+2}+\frac{C}{s+4}=\frac{As^2+4As+Bs+4B+Cs^2+2Cs+2C}{(s^2+2s+2)(s+4)}$$

$$=\frac{s^2(A+C)+s(4A+B+2C)+(4B+2C)}{(s^2+2s+2)(s+4)} \tag{2.22}$$

Hence

$$\begin{cases} A+C=4 \\ 4A+B+2C=13 \\ 4B+2C=-2 \end{cases} \Leftrightarrow \begin{cases} C=4-A \\ 4A+B+8-2A=13 \\ 4A-3B=15 \end{cases} \Leftrightarrow \begin{cases} C=4-A \\ 2A+B=5 \\ 4A-3B=15 \end{cases} \Leftrightarrow \begin{cases} C=1 \\ A=3 \\ B=-1 \end{cases} \tag{2.23}$$

Finally,

$$\mathscr{L}^{-1}\left[\frac{4s^2+13s-2}{(s^2+2s+2)(s+4)}\right] = \mathscr{L}^{-1}\left[\frac{3s-1}{s^2+2s+2}+\frac{1}{s+4}\right] \tag{2.24}$$
$$= 3\mathscr{L}^{-1}\left[\frac{s}{s^2+2s+2}\right]-\frac{1}{2}\mathscr{L}^{-1}\left[\frac{2}{s^2+2s+2}\right]+\mathscr{L}^{-1}\left[\frac{1}{s+4}\right]$$

and since for the first two terms we have

$$\omega = \sqrt{2} \tag{2.25}$$
$$\xi\omega = 1 \tag{2.26}$$
$$\xi = \frac{1}{\sqrt{2}} \tag{2.27}$$
$$\Xi = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}} \tag{2.28}$$
$$\omega\Xi = 1 \tag{2.29}$$
$$\varphi = \arctan\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{\pi}{4} \tag{2.30}$$

we arrive at

$$\mathscr{L}^{-1}\left[\frac{4s^2+13s-2}{(s^2+2s+2)(s+4)}\right] = -3\sqrt{2}e^{-t}\sin\left(t-\frac{\pi}{4}\right)-\frac{1}{2}2e^{-t}\sin(t)+e^{-4t}$$
$$= e^{-4t}+e^{-t}\left[-3\sqrt{2}\left(\sin t\cos\frac{\pi}{4}-\cos t\sin\frac{\pi}{4}\right)-\sin t\right]$$
$$= e^{-4t}+e^{-t}\left[-3\sqrt{2}\left(\sin t\frac{1}{\sqrt{2}}-\cos t\frac{1}{\sqrt{2}}\right)-\sin t\right]$$
$$= e^{-4t}+e^{-t}\left(-4\sin t+3\cos t\right) \quad \square \tag{2.31}$$

**Remark 2.6.** If in the example above we had decided to expand the second order term and use only line 5 of Table 2.1, we would have arrived at the very same result, albeit with more lengthy and tedious calculations involving complex numbers. We would have to separate $\frac{3s-1}{s^2+2s+2}$ in two as follows:

$$\frac{3s-1}{s^2+2s+2} = \frac{A+Bj}{s+1+j}+\frac{C+Dj}{s+1-j} \tag{2.32}$$
$$= \frac{As+A-Aj+Bjs+Bj+B+Cs+C+Cj+Djs+Dj-D}{s^2+s-js+s+1-j+js+j+1}$$
$$= \frac{s(A+C)+js(B+D)+(A+B+C-D)+j(-A+B+C+D)}{s^2+2s+2}$$

Then

$$\begin{cases} A+C=3 \\ B+D=0 \\ A+B+C-D=-1 \\ -A+B+C+D=0 \end{cases} \Leftrightarrow \begin{cases} C=3-A \\ D=-B \\ A+B+3-A+B=-1 \\ -A+B+3-A-B=0 \end{cases} \Leftrightarrow \begin{cases} C=\frac{3}{2} \\ D=2 \\ B=-2 \\ A=\frac{3}{2} \end{cases} \tag{2.33}$$

Consequently

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{4s^2+13s-2}{(s^2+2s+2)(s+4)}\right] &= \mathscr{L}^{-1}\left[\frac{\frac{3}{2}-2j}{s+1+j}+\frac{\frac{3}{2}+2j}{s+1-j}+\frac{1}{s+4}\right] \\
&= \left(\frac{3}{2}-2j\right)\mathscr{L}^{-1}\left[\frac{1}{s+1+j}\right]+\left(\frac{3}{2}+2j\right)\mathscr{L}^{-1}\left[\frac{1}{s+1-j}\right]+\mathscr{L}^{-1}\left[\frac{1}{s+4}\right] \\
&= \left(\frac{3}{2}-2j\right)e^{-(1+j)t}+\left(\frac{3}{2}+2j\right)e^{-(1-j)t}+e^{-4t} \\
&= e^{-4t}+\left(\frac{3}{2}-2j\right)e^{-t}\left(\cos(-t)+j\sin(-t)\right)+\left(\frac{3}{2}+2j\right)e^{-t}\left(\cos t+j\sin t\right) \\
&= e^{-4t}+e^{-t}\left(\frac{3}{2}\cos t-\frac{3}{2}j\sin t-2j\cos t-2\sin t+\right. \\
&\quad \left.+\frac{3}{2}\cos t+\frac{3}{2}j\sin t+2j\cos t-2\sin t\right) \\
&= e^{-4t}+e^{-t}(3\cos t-4\sin t) \qquad (2.34)
\end{aligned}
$$

Notice how all the complex terms appear in complex conjugates, so that the imaginary parts cancel out. This has to be the case, since $f(t)$ is a real-valued function. □

**Example 2.8.** The inverse Laplace transform of $F(s)=\frac{s^2+22s+119}{(s+10)^3}$ is obtained from lines 5, 7 and 8 of Table 2.1 together with (2.8) and (2.9):

*Partial fraction expansion with multiple roots*

$$
\begin{aligned}
\frac{s^2+22s+119}{(s+10)^3} &= \frac{A}{s+10}+\frac{B}{(s+10)^2}+\frac{C}{(s+10)^3} \\
&= \frac{As^2+20As+100A+Bs+10B+C}{(s+10)^3} \qquad (2.35)
\end{aligned}
$$

Hence

$$
\begin{cases} A=1 \\ 20A+B=22 \\ 100A+10B+C=119 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=2 \\ C=-1 \end{cases} \qquad (2.36)
$$

Finally,

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s^2+22s+119}{(s+10)^3}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s+10}+\frac{2}{(s+10)^2}+\frac{-1}{(s+10)^3}\right] \qquad (2.37) \\
&= \mathscr{L}^{-1}\left[\frac{1}{s+10}\right]+2\mathscr{L}^{-1}\left[\frac{2}{(s+10)^2}\right]-\frac{1}{2}\mathscr{L}^{-1}\left[\frac{2}{(s+10)^3}\right] \\
&= e^{-10t}+2t\,e^{-10t}-\frac{1}{2}t^2\,e^{-10t}=e^{-10t}\left(1+2t-\frac{1}{2}t^2\right)
\end{aligned}
$$

□

**Example 2.9.** The inverse Laplace transform of $F(s)=\frac{2s+145}{s+70}$ is obtained from lines 1 and 5 of Table 2.1, but for that it is necessary to begin by dividing the numerator of $F(s)$ by the denominator. Because the denominator is of first order, in this case polynomial division can be carried out with Ruffini's rule

*Division of polynomials*

(otherwise a long division would be necessary):

$$\begin{array}{r|cc} & 2 & 145 \\ -70 & & -140 \\ \hline & 2 & \boxed{5} \end{array} \tag{2.38}$$

So $\dfrac{2s+145}{s+70} = 2 + \dfrac{5}{s+70}$, and finally

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{2s+145}{s+70}\right] &= 2\mathscr{L}^{-1}[1] + 5\mathscr{L}^{-1}\left[\frac{1}{s+70}\right] \\ &= 2\delta(t) + e^{-70t} \quad \square \end{aligned} \tag{2.39}$$

All polynomial operations mentioned in this sections can be performed with Matlab using the following commands:

- `roots` finds the roots of a polynomial, represented by a vector with its coefficients (in decreasing order of the exponent);
- `conv` multiplies two polynomials, represented by two vectors as above;
- `residue` performs polynomial division and partial fraction expansion, as needed, for a rational function, given the numerator and denominator polynomials represented by two vectors as above.

*Matlab's command `roots`*

**Example 2.10.** The roots of $s^2+3s+2$ are $-2$ and $-1$:

```
>> roots([1 3 2])
ans =
    -2
    -1
```

$\square$

**Example 2.11.** The roots of $4s^3+3s^2+2s+1$ are $-0.6058$, $-0.0721+0.6383j$ and $-0.0721-0.6383j$:

```
>> roots([4 3 2 1])
ans =
  -0.6058 + 0.0000i
  -0.0721 + 0.6383i
  -0.0721 - 0.6383i
```

$\square$

*Matlab's command `conv`*

**Example 2.12.** The product of $s^2+2s+3$ and $4s^3+5s^2+6s+7$ is $4s^5+13s^4+28s^3+34s^2+32s+21$:

```
>> conv([1 2 3],[4 5 6 7])
ans =
     4    13    28    34    32    21
```

$\square$

*Matlab's command `residue`*

**Example 2.13.** The partial fraction expansion (2.18) from Example 2.6 is obtained as

```
>> [r,p,k] = residue([1 2],[1 13 30])
r =
    1.1429
   -0.1429
p =
   -10
    -3
k =
     []
```

Vector r contains the **residues** or numerators of the fractions in the partial fraction expansion. Vector p contains the **poles** or roots of the denominator of the original expression. Vector k contains (the coefficients of the polynomial which is) the integer part of the polynomial division, which in this case is 0 because the order of the denominator is higher than the order of the numerator. *Residues* *Poles*

The polynomials of the original rational function can be recovered feeding this function back vectors r, p and k:

```
>> [num,den] = residue(r,p,k)
num =
     1     2
den =
     1    13    30
```

□

**Example 2.14.** The partial fraction expansion (2.34) from Example 2.7 and Remark 2.6 is obtained as

```
>> [r,p,k] = residue([4 13 -2],conv([1 2 2],[1 4]))
r =
   1.0000 + 0.0000i
   1.5000 + 2.0000i
   1.5000 - 2.0000i
p =
  -4.0000 + 0.0000i
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
k =
     []
```

□

**Example 2.15.** The partial fraction expansion from Example 2.9 is obtained as

```
>> [r,p,k] = residue([2 145],[1 70])
r =
     5
p =
   -70
k =
     2
```

Notice how this time there is an integer part of the polynomial division, since the order of the numerator is not lower than the order of the denominator. □

**Example 2.16.** From

```
>> [r,p,k] = residue([1 2 3 4 5 6],[7 8 9 10])
r =
   0.1451 + 0.0000i
  -0.0276 - 0.2064i
  -0.0276 + 0.2064i
p =
  -1.1269 + 0.0000i
  -0.0080 + 1.1259i
  -0.0080 - 1.1259i
k =
    0.1429    0.1224    0.1050
```

we learn that

$$\begin{aligned}&\frac{s^5+2s^4+3s^3+4s^2+5s+6}{7s^3+8s^2+9s+10} \qquad (2.40)\\&=0.1429s^2+0.1224s+0.1050+\frac{0.1451}{s+1.1269}+\frac{-0.0276-0.2064j}{s+0.0080-1.1259j}+\frac{-0.0276+0.2064j}{s+0.0080+1.1259j}\end{aligned}$$

□

## 2.4 Important properties: derivatives and integrals

Now that we know how to find Laplace transforms, it is time to wonder why we are studying them. To answer this, we have to first establish some very important results.

*$\mathscr{L}$ of the derivative*

**Theorem 2.2.** If $\mathscr{L}[f(t)] = F(s)$, then

$$\mathscr{L}[f'(t)] = s\,F(s) - f(0) \qquad (2.41)$$

*Proof.* Apply integration by parts $\int uv' = uv - \int u'v$ to definition (2.1):

$$\begin{aligned}\mathscr{L}[f(t)] &= \int_0^{+\infty} \overbrace{f(t)}^{u}\, \overbrace{e^{-st}}^{v'}\, \mathrm{d}t\\&= \left[f(t)\frac{e^{-st}}{-s}\right]_0^{+\infty} - \int_0^{+\infty} f'(t)\frac{e^{-st}}{-s}\,\mathrm{d}t\\&= \lim_{t\to+\infty}\left(f(t)\frac{e^{-st}}{-s}\right) - f(0)\frac{e^0}{-s} + \frac{1}{s}\int_0^{+\infty} f'(t)e^{-st}\,\mathrm{d}t \qquad (2.42)\end{aligned}$$

The limit has to be 0, otherwise $F(s)$ would not exist. The integral is, by definition, $\mathscr{L}[f'(t)]$. From here (2.41) is obtained rearranging terms. □

**Corollary 2.1.** If $\mathscr{L}[f(t)] = F(s)$, then

$$\mathscr{L}[f''(t)] = s^2\,F(s) - s\,f(0) - f'(0) \qquad (2.43)$$

*Proof.* Apply (2.41) to itself:

$$\mathscr{L}[f''(t)] = s\,\mathscr{L}[f'(t)] - f'(0) = s\,(s\,F(s) - f(0)) - f'(0) \tag{2.44}$$

Then rearrange terms. ☐

**Corollary 2.2.** If $\mathscr{L}[f(t)] = F(s)$, then

$$\begin{aligned}\mathscr{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right] &= s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \ldots - \left.\frac{\mathrm{d}^{n-1} f(t)}{\mathrm{d}t}\right|_{t=0} \\ &= s^n\,F(s) - \sum_{k=1}^{n} s^{n-k} \left.\frac{\mathrm{d}^{k-1} f(t)}{\mathrm{d}t^{k-1}}\right|_{t=0} \end{aligned} \tag{2.45}$$

*Proof.* This is proved by mathematical induction. The first case is (2.41). The inductive step is proved applying (2.41) to (2.45) as follows:

$$\begin{aligned}\mathscr{L}\left[\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}} f(t)\right] &= s\,\mathscr{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right] - \left.\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right|_{t=0} \qquad (2.46)\\ &= s\left(s^n\,F(s) - \sum_{k=1}^{n} s^{n-k} \left.\frac{\mathrm{d}^{k-1} f(t)}{\mathrm{d}t^{k-1}}\right|_{t=0}\right) - \left.\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right|_{t=0} \\ &= s^{n+1}\,F(s) - \left(\sum_{k=1}^{n} s^{n-k+1} \left.\frac{\mathrm{d}^{k-1} f(t)}{\mathrm{d}t^{k-1}}\right|_{t=0}\right) - \left.\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right|_{t=0} \\ &= s^{n+1}\,F(s) - \left(\sum_{k=1}^{n} s^{n+1-k} \left.\frac{\mathrm{d}^{k-1} f(t)}{\mathrm{d}t^{k-1}}\right|_{t=0}\right) - \sum_{k=n+1} s^{n+1-k} \left.\frac{\mathrm{d}^{k-1} f(t)}{\mathrm{d}t^{k-1}}\right|_{t=0}\end{aligned}$$ ☐

**Theorem 2.3.** If $\mathscr{L}[f(t)] = F(s)$, then

*$\mathscr{L}$ of the integral*

$$\mathscr{L}\left[\int_0^t f(t)\right] = \frac{1}{s}\,F(s) \tag{2.47}$$

*Proof.* In (2.42), make

$$f(t) = \int_0^t g(t)\,\mathrm{d}t, \tag{2.48}$$

whence $f'(t) = g(t)$. Then

$$\mathscr{L}\left[\int_0^t g(t)\,\mathrm{d}t\right] = -\int_0^0 g(t)\,\mathrm{d}t\frac{1}{-s} + \frac{1}{s}\int_0^{+\infty} g(t)e^{-st}\,\mathrm{d}t \tag{2.49}$$

The first integral is 0, the second is $\mathscr{L}[g(t)]$. ☐

**Remark 2.7.** Notice that the Laplace transform of a derivative (2.41) involves $f(0)$, the value of the function itself at $t = 0$. This is because we are using the Laplace transform as defined by (2.1), rather than the bilateral Laplace transform (2.2).

## 2.5 What do we need this for?

We are now in position of answering the question above: we need Laplace transforms as a very useful tool to solve differential equations.

*Use $\mathscr{L}$ to solve differe equations*

**Example 2.17.** Solve the following differential equation, assuming that $y(0) = 0$:

$$y'(t) + y(t) = e^{-t} \tag{2.50}$$

Apply the Laplace transform to obtain

$$\begin{aligned} &\mathscr{L}\left[y'(t) + y(t)\right] = \mathscr{L}\left[e^{-t}\right] \Leftrightarrow sY(s) + Y(s) = \frac{1}{s+1} \Leftrightarrow Y(s) = \frac{1}{(s+1)^2} \Leftrightarrow \\ &\Leftrightarrow \mathscr{L}^{-1}\left[Y(s)\right] = \mathscr{L}^{-1}\left[\frac{1}{(s+1)^2}\right] \Leftrightarrow y(t) = t\,e^{-t} \end{aligned} \tag{2.51}$$

It is easy to verify that this is indeed the solution: $y'(t) = e^{-t} - t\,e^{-t}$, and thus

$$y'(t) + y(t) = e^{-t} \Leftrightarrow e^{-t} - t\,e^{-t} + t\,e^{-t} = e^{-t}, \tag{2.52}$$

as desired. □

Notice how the Laplace transform turned the differential equation in $t$ into an algebraic equation in $s$, which is trivial to solve. All that is left is to apply the inverse Laplace transform to turn the solution in $s$ into a solution in $t$.

*Take care of non-null initial conditions*

Initial conditions must be taken into account if they are not zero.

**Example 2.18.** Solve the following differential equation, assuming that $y(0) = \frac{1}{3}$ and $y'(0) = 0$:

$$y''(t) + 4y'(t) + 3y(t) = 4e^{t} \tag{2.53}$$

Using the Laplace transform, we get

$$\begin{aligned} &s^2Y(s) - \frac{1}{3}s - 0 + 4\left(sY(s) - \frac{1}{3}\right) + 3Y(s) = \frac{4}{s-1} \Leftrightarrow \\ &\Leftrightarrow Y(s)\left(s^2 + 4s + 3\right) - \frac{s}{3} - \frac{4}{3} = \frac{4}{s-1} \end{aligned} \tag{2.54}$$

Because $s^2 + 4s + 3 = (s+1)(s+3)$, we get

$$Y(s) = \frac{4}{(s-1)(s+1)(s+3)} + \frac{1}{3}\,\frac{s+4}{(s+1)(s+3)} \tag{2.55}$$

We now need two partial fraction expansions:

$$\begin{aligned} &\frac{4}{(s-1)(s+1)(s+3)} + \frac{1}{3}\,\frac{s+4}{(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3} + \frac{1}{3}\left(\frac{D}{s+1} + \frac{E}{s+3}\right) \\ &= \frac{A(s^2+4s+3) + B(s^2+2x-3) + C(s^2-1)}{(s-1)(s+1)(s+3)} + \frac{1}{3}\left(\frac{Ds + 3D + Es + E}{(s+1)(s+3)}\right) \\ &= \frac{s^2(A+B+C) + s(4A+2B) + (3A-3B-C)}{(s-1)(s+1)(s+3)} + \frac{1}{3}\left(\frac{s(D+E) + (3D+E)}{(s+1)(s+3)}\right) \end{aligned} \tag{2.56}$$

whence

$$\begin{cases} A+B+C=0 \\ 4A+B=0 \\ 3A-3B-C=4 \end{cases} \Leftrightarrow \begin{cases} 4A-2B=4 \\ 4A+B=0 \\ C=3A-3B-4 \end{cases} \Leftrightarrow \begin{cases} 8A=4 \\ B=-2A \\ C=3A-3B-4 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{2} \\ B=-1 \\ C=\frac{1}{2} \end{cases} \tag{2.57}$$

and

$$\begin{cases} D+E=1 \\ 3D+E=4 \end{cases} \Leftrightarrow \begin{cases} E=1-D \\ 2D=3 \end{cases} \Leftrightarrow \begin{cases} E=-\frac{1}{2} \\ D=\frac{3}{2} \end{cases} \tag{2.58}$$

Thus

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left[\frac{\frac{1}{2}}{s-1} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+3} + \frac{1}{3}\left(\frac{\frac{3}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}\right)\right] \\ &= \frac{1}{2}\,e^{t} - e^{-t} + \frac{1}{2}\,e^{-3t} + \frac{1}{3}\left(\frac{3}{2}\,e^{-t} - \frac{1}{2}\,e^{-3t}\right) = \frac{1}{2}\,e^{t} - \frac{1}{2}e^{-t} + \frac{1}{3}\,e^{-3t} \end{aligned} \tag{2.59}$$

It is easy to verify that this is indeed the solution: on the one hand,

$$y'(t) = \frac{1}{2}\,e^{t} + \frac{1}{2}e^{-t} - e^{-3t} \tag{2.60}$$

$$y''(t) = \frac{1}{2}\,e^{t} - \frac{1}{2}e^{-t} + 3e^{-3t} \tag{2.61}$$

and thus

$$\begin{aligned} y''(t)+4y'(t)+3y(t) &= \frac{1}{2}\,e^{t} - \frac{1}{2}e^{-t} + 3e^{-3t} + 2e^{t} + 2e^{-t} - 4e^{-3t} + \frac{3}{2}\,e^{t} - \frac{3}{2}e^{-t} + e^{-3t} \\ &= 4e^{t} \end{aligned} \tag{2.62}$$

as desired; on the other hand,

$$y(0) = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = \frac{1}{3} \tag{2.63}$$

$$y'(t) = \frac{1}{2} + \frac{1}{2} - 1 = 0 \tag{2.64}$$

as required. □

**Remark 2.8.** Notice what would have happened if we had forgot to include initial conditions. It would have been as if initial conditions were null, and we would have got

$$s^2Y(s) + 4sY(s) + 3Y(s) = \frac{4}{s-1} \Leftrightarrow Y(s)\left(s^2+4s+3\right) = \frac{4}{s-1} \tag{2.65}$$

and then

$$y(t) = \mathscr{L}^{-1}\left[\frac{\frac{1}{2}}{s-1} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+3}\right] = \frac{1}{2}\,e^{t} - e^{-t} + \frac{1}{2}\,e^{-3t} \tag{2.66}$$

In this case,

$$y'(t) = \frac{1}{2}e^t + e^{-t} - \frac{3}{2}e^{-3t} \tag{2.67}$$

$$y''(t) = \frac{1}{2}e^t - e^{-t} + \frac{9}{2}e^{-3t} \tag{2.68}$$

and so it remains true that

$$\begin{aligned} y''(t) + 4y'(t) + 3y(t) &= \frac{1}{2}e^t - e^{-t} + \frac{9}{2}e^{-3t} + 2e^t + 4e^{-t} - \frac{12}{2}e^{-3t} + \frac{3}{2}e^t - 3e^{-t} + \frac{3}{2}e^{-3t} \\ &= 4e^t \end{aligned} \tag{2.69}$$

but the initial conditions are indeed

$$y(0) = \frac{1}{2} - 1 + \frac{1}{2} = 0 \tag{2.70}$$

$$y'(t) = \frac{1}{2} + 1 - \frac{3}{2} = 0 \tag{2.71}$$

To conclude: if in fact initial conditions were as in Example 2.18, and if we had written (2.65) instead of (2.54), we would get a wrong result. □

## 2.6 More important properties: initial and final values, convolution

Before we are done with Laplace transforms, we must establish some additional important properties that will often be needed.

*Final value theorem*

**Theorem 2.4.** If $f(t)$ and $f'(t)$ have Laplace transforms,

$$\lim_{t\to+\infty} f(t) = \lim_{s\to 0} s\,F(s) \tag{2.72}$$

provided that $\lim_{t\to+\infty} f(t) \in \mathbb{R}$.

*Proof.* Apply a limit to (2.41) to get

$$\begin{aligned} \lim_{s\to 0} \mathscr{L}\left[f'(t)\right] &= \lim_{s\to 0} \left(s\,F(s) - f(0)\right) \\ \Leftrightarrow f(0) + \lim_{s\to 0} \int_0^{+\infty} f'(t)e^{-st}\,\mathrm{d}t &= \lim_{s\to 0} s\,F(s) \\ \Leftrightarrow f(0) + \int_0^{+\infty} \lim_{s\to 0}\left(f'(t)e^{-st}\right)\,\mathrm{d}t &= \lim_{s\to 0} s\,F(s) \\ \Leftrightarrow f(0) + \int_0^{+\infty} f'(t)\,\mathrm{d}t &= \lim_{s\to 0} s\,F(s) \\ \Leftrightarrow f(0) + \lim_{t\to+\infty} f(t) - f(0) &= \lim_{s\to 0} s\,F(s) \;\square \end{aligned} \tag{2.73}$$

**Example 2.19.** Let $f(t) = e^{-at},\ a > 0$. We know that $\lim_{t\to+\infty} f(t) = 0$. We have $F(s) = \frac{1}{s+a}$. And $\lim_{s\to 0} s\,F(s) = \lim_{s\to 0} \frac{s}{s+a} = 0$.

Notice that, when $a < 0$, it is still true that $F(s) = \frac{1}{s+a}$ and that $\lim_{s\to 0} s\,F(s) = \lim_{s\to 0} \frac{s}{s+a} = 0$. But now $\lim_{t\to+\infty} f(t) = +\infty$, which is not real. □

**Example 2.20.** Let $F(s) = \frac{1}{s(s+a)}$, $a > 0$. We have $\lim_{s\to 0} s\,F(s) = \lim_{s\to 0} \frac{1}{s+a} = \frac{1}{a}$. At the same time, $f(t) = \frac{1}{a}\left(1 - e^{-at}\right)$, and $\lim_{t\to+\infty} f(t) = \frac{1}{a}$.

When $a < 0$, we are in a situation similar to that of the former example: we still have $\lim_{s\to 0} s\,F(s) = \frac{1}{a}$, but $\lim_{t\to+\infty} f(t) = +\infty$. □

*Initial value theorem*

**Theorem 2.5.** If $f(t)$ and $f'(t)$ have Laplace transforms,

$$\lim_{t\to 0^+} f(t) = \lim_{s\to+\infty} s\,F(s) \tag{2.74}$$

provided that $\lim_{s\to+\infty} s\,F(s) \in \mathbb{R}$.

*Proof.* Apply a limit to (2.41) to get

$$\begin{aligned} \lim_{s\to+\infty} \mathscr{L}\left[f'(t)\right] &= \lim_{s\to+\infty} \left(s\,F(s) - f(0)\right) \\ \Leftrightarrow f(0) + \lim_{s\to+\infty} \int_0^{+\infty} f'(t) e^{-st}\,\mathrm{d}t &= \lim_{s\to+\infty} s\,F(s) \end{aligned} \tag{2.75}$$

In the integrand, $e^{-st}$ goes to zero as $s \to +\infty$. If $f'(t)$ has a Laplace transform, it must be of exponential order, and thus $e^{-st}$ goes to zero faster enough to ensure that $\lim_{s\to+\infty} \int_0^{+\infty} f'(t) e^{-st}\,\mathrm{d}t = 0$. Because we are assuming the unilateral Laplace transform definition, $f(0)$ is in reality $\lim_{t\to 0^+} f(t)$, as whatever may happen for $t < 0$ is not taken into account. □

**Example 2.21.** Let $f(t) = e^{-at}$. We know that $\lim_{t\to 0^+} f(t) = 1$. We have $F(s) = \frac{1}{s+a}$. And $\lim_{s\to+\infty} s\,F(s) = \lim_{s\to 0} \frac{s}{s+a} = 1$.

Notice that, unlike what happened when we applied the final value theorem in Example 2.19, there is now no need to restrict $a > 0$.

**Example 2.22.** Let $F(s) = \frac{1}{s(s+a)}$. We have $\lim_{s\to+\infty} s\,F(s) = \lim_{s\to+\infty} \frac{1}{s+a} = 0$. At the same time, $f(t) = \frac{1}{a}\left(1 - e^{-at}\right)$, and $\lim_{t\to+\infty} f(t) = 0$. There is again no need now to make $a > 0$.

*Convolution*

**Definition 2.2.** Given two functions $f(t)$ and $g(t)$ defined for $t \in \mathbb{R}^+$, their **convolution**, denoted by $*$, is a function of $t$ given by

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau)\,\mathrm{d}\tau \tag{2.76}$$

**Theorem 2.6.** Convolution is commutative.

*Proof.* Use the change of variables $\mathfrak{t} = t - \tau$, for which $\mathrm{d}\tau = -\mathrm{d}\mathfrak{t}$. With this change of variables, when $\tau = 0$ we have $\mathfrak{t} = t$, and when $\tau = t$ we have $\mathfrak{t} = 0$. Apply this to (2.76) to get

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(t-\tau) g(\tau)\,\mathrm{d}\tau \\ &= -\int_t^0 f(\mathfrak{t}) g(t-\mathfrak{t})\,\mathrm{d}\mathfrak{t} \\ &= \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau = g(t) * f(t) \;\square \end{aligned} \tag{2.77}$$

**Theorem 2.7.** If these Laplace transforms exist,

$$\mathscr{L}\left[f(t) * g(t)\right] = F(s)\, G(s) \tag{2.78}$$

*Proof.*

$$\mathscr{L}\left[f(t) * g(t)\right] = \int_0^{+\infty} \left( \int_0^t f(t-\tau) g(\tau)\, \mathrm{d}\tau \right) e^{-st}\, \mathrm{d}t \tag{2.79}$$

We can change the limits of integration of the inner integral by including a Heaviside function $H(t-\tau)$:

$$\mathscr{L}\left[f(t) * g(t)\right] = \int_0^{+\infty} \left( \int_0^{+\infty} f(t-\tau) H(t-\tau) g(\tau)\, \mathrm{d}\tau \right) e^{-st}\, \mathrm{d}t \tag{2.80}$$

$H(t-\tau) = 1$ if $t - \tau \geq 0 \Leftrightarrow \tau \leq t$, which is the range of values in (2.79). But $H(t-\tau) = 0$ if $t - \tau < 0 \Leftrightarrow \tau > t$, the additional range of values added in (2.79), which thus does not change the result. We can now change the order of integration:

$$\begin{aligned}
\mathscr{L}\left[f(t) * g(t)\right] &= \int_0^{+\infty} \left( \int_0^{+\infty} f(t-\tau) H(t-\tau) g(\tau)\, \mathrm{d}\tau \right) e^{-st}\, \mathrm{d}t \\
&= \int_0^{+\infty} \int_0^{+\infty} f(t-\tau) H(t-\tau) g(\tau) e^{-st}\, \mathrm{d}t\, \mathrm{d}\tau \\
&= \int_0^{+\infty} g(\tau) \int_0^{+\infty} f(t-\tau) H(t-\tau) e^{-st}\, \mathrm{d}t\, \mathrm{d}\tau
\end{aligned} \tag{2.81}$$

We now apply to the inner integral the change of variables $\mathfrak{t} = t - \tau$, for which $\mathrm{d}t = \mathrm{d}\mathfrak{t}$. With this change of variables, when $t = 0$ we have $\mathfrak{t} = -\tau$, and when $t \to +\infty$ we have $\mathfrak{t} \to +\infty$ too.

$$\mathscr{L}\left[f(t) * g(t)\right] = \int_0^{+\infty} g(\tau) \int_{-\tau}^{+\infty} f(\mathfrak{t}) H(\mathfrak{t}) e^{-s(\tau+\mathfrak{t})}\, \mathrm{d}\mathfrak{t}\, \mathrm{d}\tau \tag{2.82}$$

We have $H(\mathfrak{t}) = 1$ if $\mathfrak{t} \geq 0$ and $H(\mathfrak{t}) = 0$ if $\mathfrak{t} < 0$, so the integration limits can be changed accordingly:

$$\mathscr{L}\left[f(t) * g(t)\right] = \int_0^{+\infty} g(\tau) \int_0^{+\infty} f(\mathfrak{t}) e^{-s\tau} e^{-s\mathfrak{t}}\, \mathrm{d}\mathfrak{t}\, \mathrm{d}\tau \tag{2.83}$$

All that is left is taking outside integrals terms that do not depend on the corresponding variables:

$$\begin{aligned}
\mathscr{L}\left[f(t) * g(t)\right] &= \int_0^{+\infty} g(\tau) e^{-s\tau} \left( \int_0^{+\infty} f(\mathfrak{t}) e^{-s\mathfrak{t}}\, \mathrm{d}\mathfrak{t} \right) \mathrm{d}\tau \\
&= \int_0^{+\infty} f(\mathfrak{t}) e^{-s\mathfrak{t}}\, \mathrm{d}\mathfrak{t} \int_0^{+\infty} g(\tau) e^{-s\tau}\, \mathrm{d}\tau
\end{aligned} \tag{2.84}$$

and these integrals are the definitions of $F(s)$ and $G(s)$. □

**Example 2.23.** From $\mathscr{L}^{-1}\left[\frac{1}{s}\right] = H(t)$ we get

$$\mathscr{L}^{-1}\left[\frac{1}{s^2}\right] = \mathscr{L}^{-1}\left[\frac{1}{s}\frac{1}{s}\right] = \int_0^t H(t-\tau) H(\tau)\, \mathrm{d}\tau = \int_0^t 1\, \mathrm{d}\tau = t \quad \square \tag{2.85}$$

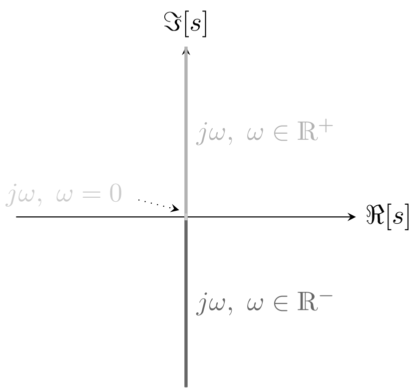


Figure 2.1: The imaginary axis in the complex plane.

*slope ramp*

**Remark 2.9.** The function obtained is known as the unit slope ramp:

$$f(t) = \begin{cases} t, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases} \quad \square \tag{2.86}$$

Table 2.2 gives a list of important properties of the Laplace transform.

## 2.7 The Fourier transform

*Fourier transform*

**Definition 2.3.** If $F(s)$ is the Laplace transform of $f(t)$, then the **Fourier transform** of $f(t)$, denoted by $\mathscr{F}[f(t)]$, is the restriction of $F(s)$ to purely imaginary values of $s$, i.e. to the imaginary axis of the complex plane, and

$$\mathscr{F}[f(t)] = \mathscr{L}[f(t)]|_{s=j\omega} = F(j\omega), \ \omega \in \mathbb{R} \quad \square \tag{2.87}$$

See Figure 2.1.

**Remark 2.10.** Notice that:

- $f(t)$ is a real-valued function that depends on a real variable: $f(t) \in \mathbb{R}$, and $t \in \mathbb{R}$;
- the Laplace transform of $f(t)$, which is $F(s) = \mathscr{L}[f(t)]$, is a complex-valued function that depends on a complex variable: $F(s) \in \mathbb{C}$, and $s \in \mathbb{C}$;
- the Fourier transform of $f(t)$, which is $F(j\omega) = \mathscr{F}[f(t)]$, is a complex-valued function that depends on a real variable, that is the coordinate along the imaginary axis: $F(j\omega) \in \mathbb{C}$, and $\omega \in \mathbb{R}$. $\square$

Table 2.2: Laplace transform properties

| | $x(t)$ | $X(s)$ |
|---|---|---|
| 1 | $Ax_1(t) + Bx_2(t)$ | $AX_1(s) + BX_2(s)$ |
| 2 | $ax(at)$ | $X\left(\frac{s}{a}\right)$ |
| 3 | $e^{at}x(t)$ | $X(s-a)$ |
| 4 | $\begin{cases} x(t-a) & t>a \\ 0 & t<a \end{cases}$ | $e^{-as}X(s)$ |
| 5 | $\frac{dx(t)}{dt}$ | $sX(s) - x(0)$ |
| 6 | $\frac{d^2x(t)}{dt^2}$ | $s^2X(s) - sx(0) - x'(0)$ |
| 7 | $\frac{d^nx(t)}{dt^n}$ | $s^nX(s) - s^{n-1}x(0) - \ldots - x^{(n-1)}(0)$ |
| 8 | $-tx(t)$ | $\frac{dX(s)}{ds}$ |
| 9 | $t^2x(t)$ | $\frac{d^2X(s)}{ds^2}$ |
| 10 | $(-1)^nt^nx(t)$ | $\frac{d^nX(s)}{ds^n}$ |
| 11 | $\int_0^t x(u)\,du$ | $\frac{1}{s}X(s)$ |
| 12 | $\int_0^t \cdots \int_0^t x(u)\,du = \int_0^t \frac{(t-u)^{(n-1)}}{(n-1)!}x(u)\,du$ | $\frac{1}{s^n}X(s)$ |
| 13 | $x_1(t) * x_2(t) = \int_0^t x_1(u)\,x_2(t-u)\,du$ | $X_1(s)\,X_2(s)$ |
| 14 | $\frac{1}{t}x(t)$ | $\int_s^\infty X(u)\,du$ |
| 15 | $x(t) = x(t+T)$ | $\frac{1}{1-e^{-sT}}\int_0^T e^{-su}X(u)\,du$ |
| 16 | $x(0)$ | $\lim_{s\to\infty} sX(s)$ |
| 17 | $x(\infty) = \lim_{t\to\infty} x(t)$ | $\lim_{s\to 0} sX(s)$ |

**Example 2.24.** Let $f(t) = e^{-t} - e^{-10t}$. Then

$$F(s) = \frac{9}{(s+1)(s+10)} \tag{2.88}$$

$$\begin{aligned} F(j\omega) &= \frac{9}{(j\omega+1)(j\omega+10)} \\ &= \frac{9}{(10-\omega^2)+j11\omega} \\ &= \frac{9\left((10-\omega^2)-j11\omega\right)}{\left((10-\omega^2)+j11\omega\right)\left((10-\omega^2)-j11\omega\right)} \\ &= \frac{9\left(10-\omega^2\right)-j99\omega}{\left(10-\omega^2\right)^2+121\omega^2} \\ &= \frac{90-9\omega^2}{\omega^4+101\omega^2+100}+j\frac{-99\omega}{\omega^4+101\omega^2+100} \quad \square \end{aligned} \tag{2.89}$$

*Inverse Fourier transform*

**Example 2.25.** Let $F(j\omega) = \frac{\omega_0}{\omega_0^2-\omega^2}$, where $\omega_0$ is a real constant. The function $f(t)$ of which $F(j\omega)$ is the Fourier transform is the **inverse Fourier transform** of $F(j\omega)$, and is given by

$$\begin{aligned} f(t) &= \mathscr{F}^{-1}\left[F(j\omega)\right] = \mathscr{F}^{-1}\left[\frac{\omega_0}{\omega_0^2-\omega^2}\right] = \mathscr{F}^{-1}\left[\frac{\omega_0}{\omega_0^2+(j\omega)^2}\right] \\ &= \mathscr{L}^{-1}\left[\frac{\omega_0}{\omega_0^2+s^2}\right] = \sin\left(\omega_0 t\right) \quad \square \end{aligned} \tag{2.90}$$

While it should now be clear what we need Laplace transforms for, we will only see what we need Fourier transforms for in chapter 10.

# Glossary

I said it in Hebrew — I said it in Dutch —
I said it in German and Greek:
But I wholly forgot (and it vexes me much)
That English is what you speak!

Lewis Carroll (1832 — †1898), *The hunting of the Snark*, 4

**bilateral Laplace transform** transformada de Laplace bilateral
**convolution** convolução
**differential equation** equação diferencial
**exponential order function** função de ordem exponencial
**Fourier transform** transformada de Fourier
**integral transform** transformada integral
**Laplace transform** transformada de Laplace
**Laplace transformation** transformação de Laplace
**partial fraction expansion** expansão em frações parciais
**pole** polo
**polynomial division** divisão de polinómios
**residue** resíduo

## Exercises

1. Find the Laplace transforms of the following functions:

   (a) $f(t) = 80e^{-0.9t}$

   (b) $f(t) = 1000 - e^{-6t}$

   (c) $f(t) = 97.5 \times 10^{-3} \sin(0.2785t) + 546.9 \times 10^{-3} e^{0.9575t} \cos(0.9649t)$

   (d) $f(t) = \sin\left(5t + \frac{\pi}{6}\right)$ *Hint:* remember that $\sin(a+b) = \sin a \cos b + \cos a \sin b$.

2. Find the inverse Laplace transforms of the following functions:

   (a) $F(s) = \frac{1}{3s^2+15s+18}$

   (b) $F(s) = \frac{1}{5s^2+6s+5}$

   (c) $F(s) = \frac{8s^2+34s-2}{s^3+3s^2-4s}$

   (d) $F(s) = \frac{s^2+2s+8}{2s+4}$

   (e) $F(s) = \frac{-s^2+5s-2}{s^3-2s^2-4s+8}$

3. Solve the following differential equations:

   (a) $y''(t) + y(t) = t\,e^{-t},\ y(0) = 0,\ y'(0) = 0$

   (b) $y''(t) + y(t) = t\,e^{-t},\ y(0) = \frac{1}{2},\ y'(0) = -\frac{1}{2}$

   (c) $y''(t) + y(t) = 10t - 20,\ y(0) = 0,\ y'(0) = 0$

   (d) $3y''(t) + 7y'(t) + 2y(t) = 0,\ y(0) = -5,\ y'(0) = 10$

4. Use the final value and initial value theorems to find the initial and final values of the inverse Laplace transforms of the functions of exercise 2.

5. Find the Fourier transforms of the functions of exercises 1 and 2, putting them in the form $F(j\omega) = \Re\left[F(j\omega)\right] + j\Im\left[F(j\omega)\right]$.

6. Prove the result in line 7 of Table 2.1. *Hint:* use (2.78) together with the result in line 5.

7. Prove the result in line 8 of Table 2.1. *Hint:* use mathematical induction.