DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

# Introduction to Stochastic Processes

**MMA, LMAC**

2nd. Semester – 2018/2019
2019/07/10 – 8AM, Room P12

Duration: 90 minutes **Test 1** (*Recurso*)

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

## Group 0 — Introduction to Stochastic Processes 2.5 points

Consider a stochastic process $\{X(t): t \in \mathbb{R}\}$ where: $X(t) = U\cos(t) + V\sin(t)$;[1] $U$ and $V$ are i.i.d. r.v. with common p.f. $P(U=-2) = \frac{1}{3}$ and $P(U=1) = \frac{2}{3}$.

(a) Obtain the mean function and the autocovariance function of this stochastic process. (1.5)

**Note**: $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$.

- **Stochastic process**
  $\{X(t): t \in \mathbb{R}\}$
  $X(t) = U\cos(t) + V\sin(t)$
  $U$ and $V$ two i.i.d. r.v. with common: p.f. $P(U=-2) = \frac{1}{3}$ and $P(U=1) = \frac{2}{3}$; and mean and variance
  $E(U) = -2 \times \frac{1}{3} + 1 \times \frac{2}{3} = 0$
  $V(U) = E(U^2) - E(U^2) = (-2)^2 \times \frac{1}{3} + 1^2 \times \frac{2}{3} = 2$
- **Mean function**

$$\begin{aligned} E[X(t)] &= E[U\cos(t) + V\sin(t)] \\ &= E(U)\cos(t) + E(V)\sin(t) \\ &\overset{E(U)=E(V)=0}{=} 0,\ t \in \mathbb{R}. \end{aligned}$$

- **Autocovariance function**
  Taking advantage of the properties of the covariance operator (it is symmetric, bilinear, etc.) and of the independence between $U$ and $V$, we get:

$$\begin{aligned} cov(X(t), X(t+h)) &= cov(U\cos(t) + V\sin(t), U\cos(t+h) + V\sin(t+h)) \\ &\overset{U \perp\!\!\!\perp V}{=} \cos(t)\cos(t+h) \times V(U) + \sin(t)\sin(t+h) \times V(V) \\ &= [\cos(t)\cos(t+h) + \sin(t)\sin(t+h)] \times V(U) \\ &= \cos(t-t-h) \times 2 \\ &= 2\cos(h). \end{aligned}$$

(b) Show that $\{X(t): t \in \mathbb{R}\}$ is a second order weakly stationary process but not a strictly stationary process. (1.0)

**Note**: All moments of a strictly stationary process $\{Y(t): t \in \mathbb{R}\}$, e.g. $E[Y^n(t)]$, must be independent of time.

- **Checking whether the process is (second order weakly) stationary**
  The mean function $E[X(t)]$ does not depend on $t$ and the autocovariance $cov(X(t), X(t+h))$ only depends on the time lag $h$, hence $\{X(t): t \in \mathbb{R}\}$ is a second order weakly stationary process.

[1] $X(t) = U\cos(t) + V\sin(t) = \sqrt{U^2+V^2} \times \sin\left[t + \arctan\left(\frac{U}{V}\right)\right]$, for $V \neq 0$ and $-\frac{\pi}{2} < \arctan\left(\frac{U}{V}\right) < \frac{\pi}{2}$, thus $X(t)$ could represent for instance the cash flow of a company (measured as a percentage of total assets).



- **Checking whether the process is strictly stationary**
  Capitalizing on the note and taking $n = 3$...

$$\begin{aligned} E[X^3(t)] &= E\{[U\cos(t) + V\sin(t)]^3\} \\ &= E(U^3)\cos^3(t) + 3E(U^2V)\cos^2(t)\sin(t) \\ &\quad +3E(UV^2)\cos(t)\sin^2(t) + E(V^3)\sin^3(t) \\ &\overset{U \perp\!\!\!\perp V}{=} E(U^3)\cos^3(t) + 3E(U^2)E(V)\cos^2(t)\sin(t) \\ &\quad +3E(U)E(V^2)\cos(t)\sin^2(t) + E(V^3)\sin^3(t) \\ &\overset{E(U)=E(V)=0}{=} E(U^3)\cos^3(t) + E(V^3)\sin^3(t) \\ &\overset{E(U^3)=E(V^3)=(-2)^3 \times \frac{1}{3} + 1^3 \times \frac{2}{3} = -2}{=} -2[\cos^3(t) + \sin^3(t)] \end{aligned}$$

  depends on time $t$, thus $\{X(t): t \in \mathbb{R}\}$ cannot be a strictly stationary process.

## Group 1 — Poisson Processes 9.0 points

**1.** Assume that migrants apprehensions at the US-Mexico border occur according to a Poisson process with rate $\lambda = 40\,000$ (migrants per month) in 2018.

(a) Admit that unaccompanied children account for about 12% of all border apprehensions in 2018.[2] Obtain an approximate value to the probability that more than 28 800 unaccompanied children are apprehended in the first semester of 2018. (1.5)

- **Stochastic process**
  $\{N(t): t \geq 0\} \sim PP(\lambda)$
  $N(t)$ = number of border apprehensions by time $t$ (time in month)
  $\lambda = 40\,000$
  $N(t) \sim \text{Poisson}(\lambda\, t)$
- **Split process**
  $N_{unacc}(t)$ = number of border apprehensions of unaccompanied children by time $t$
  $p = P$(apprehension of an unaccompanied children) $= 0.12$
  $\{N_{unacc}(t): t \geq 0\} \sim PP(\lambda p = 40\,000 \times 0.12 = 4\,800)$
  $N_{unacc}(t) \sim \text{Poisson}(\lambda p \times t = 4\,800 \times t)$
- **Requested probability**
  For $t = 6$, we have

$$\begin{aligned} P[N_{unacc}(6) > 28\,800] &= 1 - F_{Poisson(4800\times 6)}(28\,800) \\ &\simeq 1 - \Phi\left(\frac{28\,800 - 4\,800 \times 6}{\sqrt{4\,800 \times 6}}\right) \\ &= 1 - \Phi(0) \\ &= 0.5. \end{aligned}$$

(b) Border agents apprehended 54 000 unaccompanied children in 2018. How many unaccompanied children would you expect to have been apprehended in the first semester of 2018? (0.5)

- **Conditional distribution**
  $(N_{unacc}(6) \mid N_{unacc}(12) = 54\,000) \sim \text{Binomial}\left(54\,000, \frac{6}{12}\right)$ (see formulae)
- **Requested expected value**

$$\begin{aligned} E[N_{unacc}(6) \mid N_{unacc}(12) = 54\,000] &= 54\,000 \times \frac{6}{12} \\ &= 27\,000. \end{aligned}$$

**2.** Latecomers arrive according to a non-homogeneous Poisson process $\{N(t): t \geq 0\}$ with intensity function (2.0)

[2] Source: *Border apprehensions increased in 2018 – especially for migrant families.*

$\lambda(t) = (t+1)^{-2}$, $t > 0$.

Suppose that exactly one latecomer arrived in the first hour. Obtain the expected value of his/her arrival time.

- **Stochastic process**
  $\{N(t): t \geq 0\} \sim NHPP$
  $N(t)$ = number of latecomer by time $t$ (in hours)
- **Intensity function**
  $\lambda(t) = (t+1)^{-2}$, $t \geq 0$
- **Mean value function**

$$\begin{aligned} m(t) &= \int_0^t \lambda(z)\, dz \\ &= \int_0^t (z+1)^{-2}\, dz \\ &= -\left.\frac{1}{z+1}\right|_0^t \\ &= \frac{t}{t+1}, \quad t \geq 0 \end{aligned}$$

- **Relevant fact**
  $(S_1, \ldots, S_n \mid N(t) = n) \sim (Y_{(1)}, \ldots, Y_{(n)})$, where $Y_i \overset{i.i.d.}{\sim} Y$ with $P(Y \leq u) = \frac{m(u)}{m(t)}$, $0 \leq u \leq t$.
- **Requested expected value**
  For $n = 1$ and $t = 1$, we get $(S_1 \mid N(1) = 1) \sim Y$ and

$$\begin{aligned} E(S_1 \mid N(1) = 1) &= E(Y) \\ &\overset{Y \geq 0}{=} \int_0^{+\infty} P(Y > u)\, du \\ &= \int_0^1 \left[1 - \frac{m(u)}{m(1)}\right] du \\ &= \int_0^1 \left[1 - \frac{\frac{u}{u+1}}{\frac{1}{1+1}}\right] du \\ &= \int_0^1 \left(1 - \frac{2u}{u+1}\right) du \\ &= \int_0^1 \frac{1-u}{u+1}\, du \\ &= \int_0^1 \left(\frac{2}{u+1} - 1\right) du \\ &= [2\ln(u+1) - u]|_0^1 \\ &= 2\ln(2) - 1 \\ &\simeq 0.386294. \end{aligned}$$

**3.** Let $N(t)$ represent the number of initiated data transmissions up to time $t$ $(t \geq 0)$. Admit that $\{N(t): t \geq 0\}$ forms a conditional Poisson process with a random rate $\Lambda$ taking values in $\{r, r+1, \ldots\}$ $(r \in \mathbb{N})$ and with a negative binomial distribution with parameters $r \in \mathbb{N}$ and $p \in (0,1)$.

(a) Obtain the probability that at least one data transmission was initiated in a time unit. (1.5)

- **Stochastic process**
  $\{N(t): t \geq 0\} \sim ConditionalPP(\Lambda)$
  $N(t)$ = number of initiated data transmissions up to time $t$
- **Random arrival rate**
  $\Lambda \sim \text{NegativeBin}(r, p)$, $r \in \mathbb{N}$, $p \in (0,1)$
  $P(\Lambda = \lambda) = \binom{\lambda-1}{r-1} p^r (1-p)^{\lambda-r}$, $\lambda = r, r+1, \ldots$
- **Requested probability**
  Since

$$P[N(t+s) - N(s) = n] \overset{form}{=} \int_0^{+\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, dG(\lambda),$$

where $G$ represents the c.d.f. of $\Lambda$, we get

$$\begin{aligned} P[N(1) \geq 1] &= 1 - P[N(1) = 0] \\ &= 1 - \int_0^{+\infty} e^{-\lambda}\, dG(\lambda) \\ &= 1 - E\left(e^{-\Lambda}\right) \\ &= 1 - M_{NegativeBin(r,p)}(-1)] \\ &\overset{form.}{=} 1 - \left[\frac{pe^{-1}}{1-(1-p)e^{-1}}\right]^r. \end{aligned}$$

(b) Calculate $P[\Lambda = r \mid N(1) = 0]$. (1.5)

- **Requested probability**
  Since

$$\begin{aligned} P[N(t) = n \mid \Lambda = \lambda] &= e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0 \\ P(\Lambda = \lambda) &= \binom{\lambda-1}{r-1} p^r (1-p)^{\lambda-r}, \quad \lambda = r, r+1, \ldots \\ P[N(t) = n] &= \int_0^{+\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, dG(\lambda), \end{aligned}$$

we obtain

$$\begin{aligned} P[\Lambda = r \mid N(1) = 0] &\overset{T.Bayes}{=} \frac{P[N(1) = 0 \mid \Lambda = r] \times P(\Lambda = r)}{P[N(1) = 0]} \\ &\overset{(a)}{=} \frac{e^{-r} \times \binom{r-1}{r-1} p^r (1-p)^{r-r}}{\left[\frac{pe^{-1}}{1-(1-p)e^{-1}}\right]^r} \\ &= \frac{e^{-r} \times p^r}{\left[\frac{pe^{-1}}{1-(1-p)e^{-1}}\right]^r} \\ &= \left[1 - (1-p)e^{-1}\right]^r. \end{aligned}$$

**4.** Orders arrive in batches at a depot. Successive batch sizes are i.i.d. r.v. $Y_i$, $i \in \mathbb{N}$. The batches themselves arrive according to a non-homogeneous Poisson process $\{N(t): t \geq 0\}$ with mean value function $m(t)$, $t > 0$. (2.0)

Derive the moment generating function and the expected value of the total number of orders that arrived up to time $t$.

- **Auxiliary stochastic processes**
  $\{N(t): t \geq 0\} \sim NHPP$ with mean value function $m(t)$, $t > 0$
  $N(t)$ = number of batches of orders arrived up to time $t$
  $N(t) \sim \text{Poisson}(m(t))$
  $\{Y_i : i \in \mathbb{N}\} \overset{i.i.d.}{\sim} Y$
  $Y_i$ = number of orders in batch $i$
  $\{N(t): t \geq 0\} \perp\!\!\!\perp \{Y_i : i \in \mathbb{N}\}$

- **Relevant stochastic process**
  $\left\{X(t)=\sum_{i=1}^{N(t)} Y_i : t\geq 0\right\}$
  $X(t)$ = total number of orders that arrived up to time $t$

- **Requested m.g.f.**
  Note that
  $$\begin{aligned} M_{X(t)}(s) &= E\left[e^{s\,X(t)}\right] \\ &= E\left\{E\left[e^{s\sum_{i=1}^{N(t)} Y_i} \mid N(t)\right]\right\}, \end{aligned}$$
  where the r.v. $E\left[e^{s\sum_{i=1}^{N(t)} Y_i} \mid N(t)\right]$ takes the value
  $$\begin{aligned} E\left[e^{s\sum_{i=1}^{N(t)} Y_i} \mid N(t)=n\right] &\overset{N(t)\perp\!\!\!\perp Y_i}{=} E\left[e^{s\sum_{i=1}^{n} Y_i}\right] \\ &\overset{Y_i \overset{i.i.d.}{\sim} Y}{=} E\left[e^{s\sum_{i=1}^{n} Y_i}\right] \\ &= \prod_{i=1}^{n} E\left(e^{s\,Y}\right) \\ &= [M_Y(s)]^n \end{aligned}$$
  with probability $P[N(t)=n]$. Consequently,
  $$\begin{aligned} M_{X(t)}(s) &= E\left\{[M_Y(s)]^{N(t)}\right\} \\ &= P_{Poisson(m(t))}[M_Y(s)] \qquad \text{(p.g.f. of a Poisson r.v.)} \\ &= e^{m(t)\times[M_Y(s)-1]}. \end{aligned}$$

- **Requested expected value**
  $$\begin{aligned} E[X(t)] &= \left.\frac{d\,M_{X(t)}(s)}{ds}\right|_{s=0} \\ &= \left.\frac{d\,e^{m(t)\times[M_Y(s)-1]}}{ds}\right|_{s=0} \\ &= \left. m(t)\frac{d\,M_Y(s)}{ds}\times e^{m(t)\times[M_Y(s)-1]}\right|_{s=0} \\ &= m(t)\times E(Y). \end{aligned}$$

| **Group 2 — Renewal Processes** | 8.5 points |
|---|---|

**1.** Let $\{N(t): t\geq 0\}$ be a renewal process with i.i.d. inter-renewal times (in hours) with common p.d.f. $f(t)=\lambda^2\,t\,e^{-\lambda t}$, $t\geq 0$. Assume $N(t)$ represents the number of hospital admissions up to time $t$.

(a) Compute $P[N(t)=n]$, $n\in\mathbb{N}_0$. (2.0)

- **Renewal process**
  $\{N(t): t\geq 0\}$
  $N(t)$ = number of hospital admissions until $t$

- **Inter-renewal times**
  $X_i \overset{i.i.d.}{\sim} X$, $i\in\mathbb{N}$
  $X\sim \text{Gamma}(2,\lambda)$ because $f(t)=\lambda^2\,t\,e^{-\lambda t}\equiv f_{Gamma(2,\lambda)}(t)$, $t\geq 0$.

- **Relevant facts**
  $$\begin{aligned} S_n &= \text{time of the } n^{th} \text{ event} \\ &\sim \text{Gamma}(2n,\lambda), \quad n\in\mathbb{N} \end{aligned}$$
  $$\begin{aligned} P[N(t)\geq n] &= P(S_n\leq t) \\ &= F_{S_n}(t). \end{aligned}$$

- **Requested p.f.**
  $$\begin{aligned} P[N(t)=n] &= P[N(t)\geq n]-P[N(t)\geq n+1] \\ &= F_{S_n}(t)-F_{S_{n+1}}(t) \\ &= F_{Gamma(2n,\lambda)}(t)-F_{Gamma(2(n+1),\lambda)}(t) \\ &\overset{form.}{=} [1-F_{Poisson(\lambda t)}(2n-1)]-[1-F_{Poisson(\lambda t)}(2(n+1)-1)] \\ &= F_{Poisson(\lambda t)}(2n+1)-F_{Poisson(\lambda t)}(2n-1) \\ &= \sum_{i=2n}^{2n+1} e^{-\lambda t}\frac{(\lambda t)^i}{i!}, \quad n\in\mathbb{N}_0. \end{aligned}$$

(b) Admit that $\lambda=5$ and an officer inspected the hospital on February 1. Obtain an approximation to the probability that the last hospital admission before this inspection occurred at most 6 hours ago. (2.0)

- **C.d.f. of the inter-renewal times**
  $X\sim\text{Gamma}(2,\lambda)$
  $f_{Gamma(2,\lambda)}(x)\overset{form.}{=}\lambda^2\,x\,e^{-\lambda x}$, $x\geq 0$
  $F_{Gamma(2,\lambda)}(x)=1-F_{Poisson(\lambda x)}(1)=1-e^{-\lambda x}-\lambda\,x\,e^{-\lambda x}$, $x\geq 0$
  $E(X)=\frac{2}{\lambda}$

- **Recurrence time**
  $A(t)\overset{form}{=}t-S_{N(t)}$ = time until the last arrival before the inspection at time $t$

- **Requested probability** (approximate value)
  We can once again invoke that $t=24\times 31=744$ hours is sufficiently large and provide the following approximate value
  $$\begin{aligned} P[A(t)\leq x] &\simeq \lim_{z\to+\infty} P[A(z)\leq x] \\ &\overset{form.}{=} \frac{\int_0^x\{1-F(u)\}\,du}{E(X)} \\ &= \frac{\int_0^x\{1-[1-e^{-\lambda u}-\lambda\,u\,e^{-\lambda u}]\}\,du}{\frac{\lambda}{2}} \\ &= \frac{\int_0^x\left(\lambda e^{-\lambda u}+\lambda^2\,u\,e^{-\lambda u}\right)du}{2} \\ &= \frac{1}{2}\times\left[1-e^{-\lambda x}+F_{Gamma(2,\lambda)}(x)\right] \\ &= \frac{1}{2}\times\left[1-e^{-\lambda x}+1-e^{-\lambda x}(1+\lambda x)\right] \\ &= 1-e^{-\lambda x}-\frac{\lambda x\,e^{-\lambda x}}{2} \\ &\overset{\lambda=5,x=6/24}{=} 1-e^{-5\times 0.25}-\frac{5\times 0.25\,e^{-5\times 0.25}}{2} \\ &\simeq 0.534430. \end{aligned}$$

**2.** Consider a renewal process $\{N(t): t\geq 0\}$ with i.i.d. inter-renewal times with common hypo-exponential distribution with parameters $\lambda$ and $\mu$ $(\lambda\neq\mu)$. (2.5)

Derive the renewal function $m(t)$ by using the Laplace-Stieltjes transform method and capitalizing on the table of important Laplace transforms in the formulae.

- **Renewal process**

  $\{N(t): t \geq 0\}$

  $N(t)$ = number of events until time $t$

- **Inter-renewal times**

  $X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$

  $X \sim$ Hypo-exp.$(\lambda, \mu)$, with $\lambda \neq \mu$ is a sum of two independent exponentially distributed r.v. with means $\lambda^{-1}$ and $\mu^{-1}$.

- **Deriving the renewal function**

  The LST of the inter-renewal d.f. of $X$ is given by

$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &= E\left(e^{-sX}\right) \\ &= M_X(-s) \\ &= M_{Exp(\lambda)}(-s) \times M_{Exp(\mu)}(-s) \\ &\overset{form.}{=} \frac{\lambda}{\lambda+s} \times \frac{\mu}{\mu+s}\end{aligned}$$

  Moreover, the LST of the renewal function can be obtained in terms of the one of $F$:

$$\begin{aligned}\tilde{m}(s) &\overset{form.}{=} \frac{\tilde{F}(s)}{1-\tilde{F}(s)} \\ &= \frac{\frac{\lambda}{\lambda+s} \times \frac{\mu}{\mu+s}}{1 - \frac{\lambda}{\lambda+s} \times \frac{\mu}{\mu+s}} \\ &= \frac{\lambda\mu}{(\lambda+s)(\mu+s) - \lambda\mu} \\ &= \frac{\lambda\mu}{(s+0)[s+(\lambda+\mu)]}.\end{aligned}$$

  Taking advantage of the LT in the formulae, we successively get:

$$\begin{aligned}\frac{dm(t)}{dt} &= LT^{-1}\left[\tilde{m}(s),\, t\right] \\ &= LT^{-1}\left[\frac{\lambda\mu}{(s+0)[s+(\lambda+\mu)]},\, t\right] \\ &= \lambda\mu \times \frac{e^{-0\times t} - e^{-(\lambda+\mu)\times t}}{(\lambda+\mu)-0} \\ &= \lambda\mu \times \frac{1-e^{-(\lambda+\mu)\times t}}{\lambda+\mu} \\ m(t) &= \frac{\lambda\mu}{\lambda+\mu}\int_0^t \left[1-e^{-(\lambda+\mu)\times t}\right] dx \\ &= \frac{\lambda\mu}{\lambda+\mu}\left[t - \frac{1-e^{-(\lambda+\mu)\times t}}{\lambda+\mu}\right] \\ &= \frac{\lambda\mu\, t}{\lambda+\mu} - \frac{\lambda\mu\left[1-e^{-(\lambda+\mu)\times t}\right]}{(\lambda+\mu)^2}, \quad t \geq 0.\end{aligned}$$

**3.** A computer runs continuously as long as two critical parts are working. The two parts have mutually independent exponential durations with means equal to 10 and 20 weeks, for parts 1 and 2 (resp.). When a part fails, the computer is shut down, the failed part is replaced. The times to replace parts 1 and 2 have mean of 1 and 2.5 weeks (resp.).



What is the long-run proportion of time that the computer is working? (2.0)

- **State variable**

$$X(t) = \begin{cases} 0, & \text{if at least of the two critical parts is not working} \\ 1, & \text{if both critical parts are working} \end{cases}$$

- **Alternating renewal process**

  $\{X(t): t \geq 0\}$

- **Up time**

  $U$ = time system is UP/ON

  $O_i$ = operating time of critical part $i$, $i = 1, 2$

  $O_i \overset{indep}{\sim}$ Exponential$(\lambda_i)$, $i = 1, 2$

  $\lambda_1 = \frac{1}{10}$, $\quad \lambda_2 = \frac{1}{20}$

  $U = \min\{O_1, O_2\} \sim$ Exponential$(\lambda_1 + \lambda_2 = \frac{3}{20})$

  $E(U) = \frac{1}{\lambda_1+\lambda_2} = \frac{20}{3}$

- **Down time**

  $D$ = time spent replacing the failed critical part

  $R_i$ = time spent replacing critical part $i$, $i = 1, 2$

  $E(R_1) = 1, \quad E(R_2) = 2.5$

$$E(D) = \begin{cases} 1, & \text{if critical part 1 fails before critical part 2} \quad \text{with prob. } \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{\frac{1}{10}}{\frac{1}{10}+\frac{1}{20}} = \frac{2}{3} \\ 2.5, & \text{otherwise} \quad \text{with prob. } \frac{1}{3}. \end{cases}$$

  $E(D) = 1 \times \frac{2}{3} + 2.5 \times \frac{1}{3} = 1.5$

- **Long-run proportion of time system is working**

$$\begin{aligned}\lim_{t\to+\infty} P[X(t)=1] &\overset{[Prop. 2.106]}{=} \frac{E(U)}{E(U)+E(D)} \\ &= \frac{\frac{20}{3}}{\frac{20}{3}+1.5} \\ &\simeq 0.816327.\end{aligned}$$