Department of Mathematics, IST — Probability and Statistics Unit

Introduction to Stochastic Processes

"Exame de Época Especial"	2nd. Semester — 2013/14
Duration: 3h00m	**2014/07/21** — **9AM**, Room **C01**

- Please justify all your answers.
- This exam has THREE PAGES and SIX GROUPS. The total of points is 40.0.

Group 1 — Introduction to Stochastic Processes **2.5 points**

Consider a stochastic process $\{X(t) : t \geq 0\}$ — with stationary and independent increments — and assume that $X(0) = 0$.

(a) Show that the mean function is equal to $E[X(t)] = \mu\, t$, where $\mu = E[X(1)]$. **(1.0)**

Hint: The only solution to the functional equation $f(t+s) = f(t) + f(s)$ is $f(t) = c\,t$, where $c = f(1)$. This result is relevant to solve lines (a) and (b).

- **Stochastic process**
  $\{X(t) : t \geq 0\}$ with stationary and independent increments
- **Initial condition**
  $X(0) = 0$
- **A property of the mean function**
  Let
  $$\begin{aligned} f(t) &= E[X(t)] \\ &\stackrel{X(0)=0}{=} E[X(t) - X(0)]. \end{aligned}$$
  Then, by capitalizing on the stationary increments of this process, we get
  $$\begin{aligned} f(t+s) &= E[X(t+s)] \\ &= E[X(t+s) - X(0)] \\ &= E\{[X(t+s) - X(s)] + [X(s) - X(0)]\} \\ &= E[X(t+s) - X(s)] + E[X(s) - X(0)] \\ &\stackrel{stat.\,inc.}{=} E[X(t) - X(0)] + E[X(s) - X(0)] \\ &= f(t) + f(s), \end{aligned}$$
  for $t, s \geq 0$.
- **Deriving the mean function**
  As mentioned in the hint, the only solution to the functional equation $f(t+s) = f(t) + f(s)$ is
  $$f(t) = \mu\, t,$$
  where $\mu = f(1) = E[X(1)]$. QED

(b) Prove that the variance function is given by $V[X(t)] = \sigma^2\, t$, where $\sigma^2 = V[X(1)]$. **(1.5)**

- **A property of the variance function**
  Let
  $$\begin{aligned} g(t) &= V[X(t)] \\ &\stackrel{X(0)=0}{=} V[X(t) - X(0)]. \end{aligned}$$
  Then, by capitalizing on both the stationary and independent increments of this process, we obtain
  $$\begin{aligned} g(t+s) &= V[X(t+s)] \\ &= V[X(t+s) - X(0)] \\ &= V\{[X(t+s) - X(s)] + [X(s) - X(0)]\} \\ &\stackrel{indep.\,inc.}{=} V[X(t+s) - X(s)] + V[X(s) - X(0)] \\ &\stackrel{stat.\,inc.}{=} V[X(t) - X(0)] + V[X(s) - X(0)] \\ &= g(t) + g(s), \end{aligned}$$
  for $t, s \geq 0$.
- **Deriving the variance function**
  Once again, the only solution to the functional equation $g(t+s) = g(t) + g(s)$ is
  $$g(t) = \sigma^2\, t,$$
  where $\sigma^2 = g(1) = V[X(1)]$. QED

Group 2 — Poisson Processes **9.5 points**

1. Arrivals of customers at a supermarket are modeled by a Poisson process with a rate of $\lambda = 10$ customers per minute.

   (a) Let M (resp. N) be the number of customers arriving between 9:00 and 9:10 (resp. 9:30 and 9:35). **(1.0)**

   What is the distribution of $M + N$?

   - **Stochastic process**
     $\{N(t) : t \geq 0\} \sim PP(\lambda = 10)$
     $N(t)$ = number of arrivals by time t (time in minutes)
   - **Relevant facts**
     $N(t) \sim \text{Poisson}(\lambda\, t)$
     $\{N(t) : t \geq 0\}$ has stationary and independent increments
   - **R.v.**
     M = number of customers arriving between 9:00 and 9:10
     N = number of customers arriving between 9:30 and 9:35
   - **Distributions of M and N**
     Due to the stationary increments of the process $\{N(t) : t \geq 0\}$ and the fact that $N(t) \sim \text{Poisson}(10\, t)$, we can add that:

$$\begin{aligned} M &= N(9\times 60+10)-N(9\times 60)\\ &\sim N(9\times 60+10-9\times 60)\\ &\sim N(10)\\ &\sim \text{Poisson}(10\times 10=100)\\ N &= N(9\times 60+35)-N(9\times 60+30)\\ &\sim N(9\times 60+35-9\times 60-30)\\ &\sim N(5)\\ &\sim \text{Poisson}(10\times 5=50). \end{aligned}$$

- **Distribution of $M+N$**
  Since M and N refer to the number of arrivals in two non-overlapping time intervals, we can invoke the fact that the process has independent increments to conclude that M and N are independent r.v.
  Moreover, since the sum of two independent Poisson r.v. with parameters λ_i, $i=1,2$, has a Poisson distribution with parameter $(\lambda_1+\lambda_2)$, we get

$$M+N\sim \text{Poisson}(100+50=150).$$

(b) Suppose that 300 customers arrived during the first 30 minutes. **(1.0)**

Obtain an approximate value to the probability that at most 200 customers arrived during the first 20 minutes?

- **R.v.**
  $(N(s)\mid N(t)=n)\overset{form.}{\sim}\text{Binomial}(n,\,s/t)$, $0<s<t$
- **Requested probability** (approximate value)
  Using the normal approximation to the binomial c.d.f., we obtain

$$\begin{aligned} P[N(20)\le 200\mid N(30)=300] &= F_{Binomial(n=300,\,s/t=20/30)}(200)\\ &\simeq \Phi\left[\frac{200-300\times\frac{2}{3}}{\sqrt{300\times\frac{2}{3}\times\left(1-\frac{2}{3}\right)}}\right]\\ &= \Phi(0)\\ &= 0.5. \end{aligned}$$

[According to *Mathematica*, $F_{Binomial(n=300,\,s/t=20/30)}(200)=0.521703$.]

(c) Admit any customer spends a random time (in minutes) in the supermarket with a Weibull distribution with scale parameter $\alpha=5\sqrt{2}$ (resp. shape parameter $\beta=2$). **(2.0)**

Find the probability that there are at least 50 customers still in the supermarket 5 minutes after it opened.

- **R.v.**
  $S=$ time spent in the supermarket by a customer
  $S\sim\text{Weibull}(\alpha=5\sqrt{2},\,\beta=2)$

- **Non-homogenous Bernoulli splitting**
  A customer, who arrived at time s $(0<s<t)$, will be still in the supermarket at time t with probability

$$\begin{aligned} p(s) &= P(S>t-s)\\ &= \int_{t-s}^{+\infty} f_S(u)\,du\\ &= \int_{t-s}^{+\infty}\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\,dx\\ &= -e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\Big|_{t-s}^{+\infty}\\ &= e^{-\left(\frac{t-s}{\alpha}\right)^{\beta}}. \end{aligned}$$

  Furthermore, the number of customers in this supermarket at time t, $N_{sup}(t)$, results from a non-homogenous Bernoulli splitting of $\{N(t):t\ge 0\}$. Consequently,

$$N_{sup}(t)\overset{form.}{\sim}\text{Poisson}\left(\lambda\int_0^t p(s)\,ds\right),$$

  where

$$\begin{aligned} \int_0^t p(s)\,ds &\overset{\beta=2}{=} \int_0^t e^{-\left(\frac{t-s}{\alpha}\right)^2}\,ds\\ &= \sqrt{2\pi}\times\alpha/\sqrt{2}\times\int_0^t\frac{1}{\sqrt{2\pi}\times\alpha/\sqrt{2}}\,e^{-\frac{(s-t)^2}{2\times(\alpha/\sqrt{2})^2}}\,ds\\ &= \sqrt{2\pi}\times\alpha/\sqrt{2}\times\left[F_{N(0,(\alpha/\sqrt{2})^2)}(t)-F_{N(0,(\alpha/\sqrt{2})^2)}(0)\right]\\ &= \sqrt{2\pi}\times\alpha/\sqrt{2}\times\left[\Phi\left(\frac{t-t}{\alpha/\sqrt{2}}\right)-\Phi\left(\frac{0-t}{\alpha/\sqrt{2}}\right)\right]\\ &= \sqrt{2\pi}\times\alpha/\sqrt{2}\times\left[\Phi(0)-\Phi\left(-\frac{t}{\alpha/\sqrt{2}}\right)\right]\\ &\overset{\alpha=5\sqrt{2},\,t=5}{=} \sqrt{2\pi}\times 5\times[0.5-\Phi(-1)]\\ &= \sqrt{2\pi}\times 5\times[0.5-1+\Phi(1)]\\ &\overset{tables}{=} \sqrt{2\pi}\times 5\times[0.5-1+0.8413]\\ &\simeq 4.277561. \end{aligned}$$

- **Requested probability**
  Using the normal approximation to the Poisson c.d.f., we obtain

$$\begin{aligned} P[N_{sup}(t)\ge 50] &\simeq 1-F_{Poisson(10\times 4.277561)}(50-1)\\ &\simeq 1-\Phi\left[\frac{(50-1)-10\times 4.277561}{\sqrt{10\times 4.277561}}\right]\\ &\simeq 1-\Phi(0.95)\\ &\overset{tables}{=} 1-0.8289\\ &= 0.1711. \end{aligned}$$

2. Suppose that the emissions of very rare particles are governed by a non-homogeneous Poisson process with intensity function $\lambda(t)=e^{-t}$, $t\ge 0$.

(a) Find the probability that no particles were emitted in the first 2 hours and exactly one particle was emitted after those first 2 hours. (**1.5**)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim NHPP$
  $N(t) =$ number of particle emissions by time t
- **Intensity function**
  $\lambda(t) = e^{-t}$, $t \geq 0$
- **Mean value function**

$$\begin{aligned} m(t) &= E[N(t)] \\ &= \int_0^t \lambda(s)\, ds \\ &= \int_0^t e^{-s}\, ds \\ &= -e^{-s}\big|_0^t \\ &= 1 - e^{-t},\ t \geq 0 \end{aligned}$$

- **Relevant facts**
  $N(t) \sim \text{Poisson}(m(t))$
  $N(t+s) - N(s) \sim \text{Poisson}(m(t+s) - m(s))$
  $\{N(t) : t \geq 0\}$ has independent increments
- **Requested probability**
  Since $m(t) = 1 - e^{-t} \in [0, 1]$, for $t \geq 0$, we can devise the distribution of $N(+\infty) - N(2)$, the total number of particles emitted after the first 2 hours:

$$N(+\infty) - N(2) \sim \text{Poisson}(m(+\infty) - m(2) = (1 - e^{-\infty}) - (1 - e^{-2}) = e^{-2}).$$

Thus, the requested probability:

$$\begin{aligned} P[N(2) = 0, N(\infty) - N(2) = 1] &\overset{indep.\,inc.}{=} P[N(2) = 0] \times P[N(\infty) - N(2) = 1] \\ &= \frac{e^{-m(2)}\,[m(2)]^0}{0!} \\ &\quad \times \frac{e^{-[m(+\infty)-m(2)]}\,[m(+\infty) - m(2)]^1}{1!} \\ &= e^{-(1-e^{-2})} \times e^{-e^{-2}}\, e^{-2} \\ &= e^{-3}. \end{aligned}$$

(b) Obtain $E[S_1 \mid N(2) = 0, N(\infty) - N(2) = 1]$. (**2.0**)
Hint: Recall that $E(X) = \int_0^{+\infty} [1 - F_X(x)]\, dx$ for any non-negative r.v. X.

- **R.v.**
  $S_1 =$ time of the emission of the first particle
- **C.d.f. of** $[S_1 \mid N(2) = 0, N(\infty) - N(2) = 1)$
  For $0 < t < 2$,

$$\begin{aligned} F_{S_1|N(2)=0,N(\infty)-N(2)=1}(t) &= P[S_1 \leq t \mid N(2) = 0, N(\infty) - N(2) = 1] \\ &= 0. \end{aligned}$$

Moreover, for $t \geq 2$,

$$\begin{aligned} F_{S_1|N(2)=0,N(\infty)-N(2)=1}(t) &= P[S_1 \leq t \mid N(2) = 0, N(\infty) - N(2) = 1] \\ &= P[N(t) \geq 1 \mid N(2) = 0, N(\infty) - N(2) = 1] \\ &= \frac{P[N(t) \geq 1, N(2) = 0, N(\infty) - N(2) = 1]}{P[N(2) = 0, N(\infty) - N(2) = 1]} \\ &\overset{indep.\,inc.}{=} \{P[N(2) = 0] \times P[N(t) - N(2) = 1] \\ &\qquad \times P[N(\infty) - N(t) = 0]\} \\ &\qquad \div \{P[N(2) = 0] \times P[N(\infty) - N(2) = 1]\} \\ &= \frac{P[N(t) - N(2) = 1] \times P[N(\infty) - N(t) = 0]}{P[N(\infty) - N(2) = 1]}, \end{aligned}$$

where

$$\begin{aligned} P[N(t) - N(2) = 1] &= \frac{e^{-[m(t)-m(2)]}\,[m(t) - m(2))]^1}{1!} \\ &= e^{-[(1-e^{-t})-(1-e^{-2})]} \times (e^{-2} - e^{-t}) \\ &= e^{-(e^{-2}-e^{-t})} \times (e^{-2} - e^{-t}) \\ P[N(\infty) - N(t) = 0] &= \frac{e^{-[m(+\infty)-m(t)]}\,[m(+\infty) - m(t)]^0}{0!} \\ &= e^{-[1-(1-e^{-t})]} \\ &= e^{-e^{-t}} \\ P[N(\infty) - N(2) = 1] &= \frac{e^{-[m(+\infty)-m(2)]}\,[m(+\infty) - m(2)]^1}{1!} \\ &= e^{-[1-(1-e^{-2})]}\, e^{-2} \\ &= e^{-2-e^{-2}}. \end{aligned}$$

Consequently, for $t \geq 2$,

$$\begin{aligned} F_{S_1|N(2)=0,N(\infty)-N(2)=1}(t) &= \frac{e^{-(e^{-2}-e^{-t})}\,(e^{-2} - e^{-t}) \times e^{-e^{-t}}}{e^{-2-e^{-2}}} \\ &= 1 - e^{2-t}. \end{aligned}$$

- **Requested conditional expected value**
  Since we are dealing with a non-negative r.v.,

$$\begin{aligned} E[S_1 \mid N(2) = 0, N(\infty) - N(2) = 1] &= \int_0^{+\infty} [1 - F_{S_1|N(2)=0,N(\infty)-N(2)=1}(t)]\, dt \\ &= \int_0^2 dt + \int_2^{+\infty} e^{2-t}\, dt \\ &= 2 + \left(-e^{2-t}\right)\Big|_2^{+\infty} \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

3. Suppose the number of claims generated by a portfolio of insurance policies is governed by a conditional Poisson process with random rate Λ (claims per month).

   Obtain the autocovariance function of this stochastic process. (**2.0**)

- **Relevant stochastic process**
  $\{N(t) : t \geq 0\} \sim ConditionalPP(\Lambda)$
  $N(t) =$ number of claims up to month t
- **Important**
  $\{N(t) : t \geq 0\}$ has stationary increments.
  $\{(N(t) \mid \Lambda = \lambda) : t \geq 0\} \sim PP(\lambda)$ and therefore, conditionally on $\{\Lambda = \lambda\}$, we deal with stationary and independent increments. Furthermore,

$$\begin{aligned} (N(t) \mid \Lambda = \lambda) &\sim \text{Poisson}(\lambda\, t) \\ E\,[N(t) \mid \Lambda = \lambda] &= \lambda\, t \\ V\,[N(t) \mid \Lambda = \lambda] &= \lambda\, t. \end{aligned}$$

- **Mean value function**

$$\begin{aligned} E[N(t)] &= E\,\{E\,[N(t) \mid \Lambda = \lambda]\} \\ &= E(\Lambda\, t) \\ &= E(\Lambda) \times t \end{aligned}$$

- **Variance function**

$$\begin{aligned} V[N(t)] &= V\,\{E\,[N(t) \mid \Lambda = \lambda]\} + E\,\{V\,[N(t) \mid \Lambda = \lambda]\} \\ &= V(\Lambda\, t) + E(\Lambda\, t) \\ &= V(\Lambda) \times t^2 + E(\Lambda) \times t \end{aligned}$$

- **Autocovariance function**
  Please note that, for $0 \leq s < t$,

$$\begin{aligned} E[N(s) \times N(t)] &= E\{N(s) \times [N(t) - N(s) + N(s)]\} \\ &= E\{N(s) \times [N(t) - N(s)]\} + E[N^2(s)] \\ &= E\{N(s) \times [N(t) - N(s)]\} + E[N^2(s)] \\ &= E\,(E\{N(s) \times [N(t) - N(s)] \mid \Lambda\}) + E[N^2(s)], \end{aligned}$$

where

$$\begin{aligned} E[N^2(s)] &= V[N(s)] + E^2[N(s)] \\ &= V(\Lambda)\, s^2 + [E(\Lambda)\, s]^2 \\ &= E(\Lambda^2)\, s^2 \end{aligned}$$

and the r.v. $E\{N(s) \times [N(t) - N(s)] \mid \Lambda\}$ takes value

$$\begin{aligned} E\{N(s) \times [N(t) - N(s)] \mid \Lambda = \lambda\} &\overset{cond.\,indep.\,inc}{=} E[N(s) \mid \Lambda = \lambda] \times E[N(t) - N(s) \mid \Lambda = \lambda] \\ &\overset{cond.\,stat.\,inc}{=} E[N(s) \mid \Lambda = \lambda] \times E[N(t - s) \mid \Lambda = \lambda] \\ &= \lambda\, s \times \lambda\,(t - s) \\ &= \lambda^2\, s(t - s), \end{aligned}$$

with associated p.(d.)f. $f_\Lambda(\lambda)$. Therefore

$$\begin{aligned} E\{N(s) \times [N(t) - N(s)]\} &= E\,(E\{N(s) \times [N(t) - N(s)] \mid \Lambda\}) \\ &= E[\Lambda^2\, s(t - s)] \\ &= E(\Lambda^2)\, s(t - s) \\ E[N(s) \times N(t)] &= E(\Lambda^2)\, s(t - s) + E(\Lambda^2)\, s^2 \\ &= E(\Lambda^2)\, s\, t. \end{aligned}$$

Finally, for $0 \leq s < t$,

$$\begin{aligned} cov(N(s), N(t)) &= E[N(s) \times N(t)] - E[N(s)] \times E[N(t)] \\ &= E(\Lambda^2)\, s\, t - E^2(\Lambda)\, s\, t \\ &= V(\Lambda)\, s\, t. \end{aligned}$$

Group 3 — Renewal Processes **8.0 points**

1. Airplanes land at a small airport according to a Poisson process with rate λ (airplanes per hour).

   (a) Derive the renewal function $m(t)$ of the renewal process consisting of counting just EVEN landings (i.e., the 2nd., 4th., 6th., etc. landings). **(2.5)**
   Hint: Capitalize on the fact that $\frac{\lambda^2}{s\,(s+2\lambda)} = \frac{\lambda}{2s} - \frac{\lambda}{2(s+2\lambda)}$.

   - **Original stochastic process**
     $\{N^\star(t) : t \geq 0\} \sim PP(\lambda)$
     $N^\star(t) =$ number of landings until time t
   - **Original inter-renewal times**
     $X_i^\star \overset{i.i.d.}{\sim} X^\star,\ i \in \mathbb{N}$
     $X^\star \sim \text{Exponential}(\lambda)$
   - **Renewal process**
     $\{N(t) : t \geq 0\}$
     $N(t) =$ number of EVEN landings until time t
   - **Inter-renewal times**
     $X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$
     $X \sim \text{Gamma}(2, \lambda)$ (convolution of two indep. exponentially distibuted r.v.)
   - **Deriving the renewal function**
     Since the $X \sim \text{Gamma}(2, \lambda)$, its LST is given by

$$\begin{aligned} \tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &= M_X(-s) \\ &\overset{form.}{=} \left(\frac{\lambda}{\lambda + s}\right)^2. \end{aligned}$$

Moreover, the LST of the renewal function can be obtained in terms of $\tilde{F}$:

$$\begin{aligned}
\tilde{m}(s) &\overset{form.}{=} \frac{\tilde{F}(s)}{1-\tilde{F}(s)} \\
&= \frac{\left(\frac{\lambda}{\lambda+s}\right)^2}{1-\left(\frac{\lambda}{\lambda+s}\right)^2} \\
&= \frac{\lambda^2}{s\,(s+2\lambda)} \\
&= \frac{\lambda}{2s} - \frac{\lambda}{2(s+2\lambda)}.
\end{aligned}$$

Taking advantage of the LT in the formulae, we obtain

$$\begin{aligned}
\frac{dm(t)}{dt} &= LT^{-1}\left[\tilde{m}(s),\, t\right] \\
&= LT^{-1}\left[\frac{\lambda}{2s} - \frac{\lambda}{2(s+2\lambda)},\, t\right] \\
&= \frac{\lambda}{2} \times LT^{-1}\left[\frac{1}{s},\, t\right] + \frac{\lambda}{2} \times LT^{-1}\left[\frac{1}{(s+2\lambda)^1},\, t\right] \\
&= \frac{\lambda}{2} \times 1 + \frac{\lambda}{2} \times \frac{t^{1-1}\, e^{-2\lambda t}}{(1-1)!} \\
&= \frac{\lambda}{2} + \frac{\lambda\, e^{-2\lambda t}}{2} \\
m(t) &= \int_0^t \left(\frac{\lambda}{2} + \frac{\lambda\, e^{-2\lambda x}}{2}\right) dx \\
&= \left.\left(\frac{\lambda\, x}{2} - \frac{e^{-2\lambda x}}{4}\right)\right|_0^t \\
&= \frac{\lambda\, t}{2} - \frac{1-e^{-2\lambda t}}{4},\ t \geq 0.
\end{aligned}$$

(b) Show that the renewal function obtained in (a) verifies the elementary renewal theorem. (**1.0**)

- **Verification of the elementary renewal theorem (ERT)**

  Let $\mu = E(X) = E[\text{Gamma}(2,\lambda)] = \frac{2}{\lambda}$. Then

$$\begin{aligned}
\lim_{t\to+\infty} \frac{m(t)}{t} &= \lim_{t\to+\infty} \left(\frac{\frac{\lambda t}{2} - \frac{1-e^{-2\lambda t}}{4}}{t}\right) \\
&= \frac{\lambda}{2} - \frac{1}{+\infty} \\
&= \frac{1}{\mu},
\end{aligned}$$

  hence the ERT is verified.

(c) Obtain an approximate value to the probability that the number of EVEN landings exceeds 10 in the first day, when $\lambda = 1$. (**1.5**)

- **Inter-renewal times**

  $X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$

  $X \sim \text{Gamma}(2, \lambda = 1)$

  $\mu = E(X) \overset{form.}{=} 2$

  $\sigma^2 = V(X) \overset{form.}{=} 2$

- **Requested approximate probability**

$$\begin{aligned}
P\left[N(t) > n\right] &= 1 - P\left[N(t) < n+1\right] \\
&\overset{form}{\simeq} 1 - \Phi\left[\frac{(n+1) - t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right] \\
&\overset{t=24h,\, n=10}{=} 1 - \Phi\left[\frac{(10+1) - 24/2}{\sqrt{24 \times 2/2^3}}\right] \\
&\simeq 1 - \Phi(-0.29) \\
&= \Phi(0.29) \\
&\overset{tables}{=} 0.6141.
\end{aligned}$$

- **Obs.** — This a rough approximation of the exact value of the requested probability: $1 - P[N(t) \leq 9] = 1 - P[N^\star(t) \leq 2 \times 9 + 1] \overset{table}{=} 1 - 0.1803 = 0.8197$.

2. The time (in minutes) Clotilde takes to get to the top of a sky piste is a r.v. with c.d.f. F and expected value μ_F, whereas the duration of the descent (also in minutes) is uniformly distributed in the interval (0,7). Admit she decides to rent a new pair of skis for c monetary units whenever the descent lasts more than τ minutes. (**3.0**)

   Find the value of τ that minimizes the money spent in ski rentals per time unit in the long-run. Comment this result.

   - **Renewal process**

     $\{N(t) : t \geq 0\}$

     $N(t) =$ number of completed cycles of ascent/descent by time t

   - **R.v.**

     $A_n =$ duration of the n^{th} ascent to the top of the piste

     $A_n \overset{i.i.d.}{\sim} A \sim F,\ n \in \mathbb{N}$

     $D_n =$ duration of the n^{th} descent from the top of the piste

     $D_n \overset{i.i.d.}{\sim} D \sim \text{Uniform}(0,7),\ n \in \mathbb{N}$

   - **Inter-renewal times**

     $X_n \overset{i.i.d.}{\sim} X,\ n \in \mathbb{N}$

     $X = A + D$

   - **Reward renewal process**

     $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \geq 0\}$

     $R(t) =$ total spent in ski rentals until time t

     $$R_n = \begin{cases} c, & \text{if } D_n > \tau \text{ (i.e., if the } n^{th} \text{ descent lasted more than } \tau \text{ minutes)} \\ 0, & \text{otherwise} \end{cases}$$

     $(X_n, R_n) \overset{i.i.d.}{\sim} (X, R),\ n \in \mathbb{N}$

- **Expected inter-renewal time**
  $E(X) = E(A) + E(D) \stackrel{form.}{=} \mu_F + \frac{0+7}{2} = \mu_F + \frac{7}{2}$
- **Expected amount spent (per completed cycle of ascent/descent)**
  For $0 < \tau \le 7$, we have

$$\begin{aligned} E(R) &= c \times P(D > \tau) + 0 \times P(D \le \tau) \\ &= c \times \int_\tau^{+\infty} f_D(u)\, du \\ &= c \times \int_\tau^7 \frac{1}{7}\, du \\ &= c \times \frac{7-\tau}{7} \end{aligned}$$

- **Amount spent (per completed cycle of ascent/descent) per time unit in the long-run**
  Since $E(X), E(R) < +\infty$, we can add that

$$\frac{R(t)}{t} \stackrel{w.p.1}{\rightarrow} \frac{E(R)}{E(X)},$$

where

$$\begin{aligned} \frac{E(R)}{E(X)} &= h(\tau) \\ &= \frac{\frac{c(7-\tau)}{7}}{\mu_F + \frac{7}{2}} \\ &= \frac{2c(7-\tau)}{14\,\mu_F + 49},\ 0 < \tau \le 7. \end{aligned}$$

- **Minimizing the amount spent per time unit in the long-run**
  Since $h(\tau)$, $0 < \tau \le 7$, is a decreasing function of τ in the interval $(0, 7]$, the value of τ that minimizes $h(\tau)$ is equal to $\tau^\star = 7$.
- **Comment**
  Not only $\tau^\star$ does not depend on the ski rental (c) or on μ_F, but also its value means that Clotilde should never replace her skis if she is willing to minimize the amount spent per time unit in the long-run in ski rentals.

Group 4 — Renewal Processes (cont'd) **1.5 points**

The number of inspections by a supervisor to an industrial plant is governed by a delayed renewal process such that: **(1.5)**

- the first inspection time (in years) follows an exponential distribution with unit mean;
- the subsequent inter-inspection times follow an exponential distribution with expected value equal to 0.5.

Derive the c.d.f. of S_2, the time of the second inspection.

- **Delayed renewal process**
  $\{N_D(t) : t \ge 0\}$
  $N_D(t) =$ number of inspections done by time t
- **Inter-renewal times**
  X_i independent r.v., $i \in \mathbb{N}$
  $X_1 \sim \text{Exponential}(1)$
  $X_i \stackrel{i.i.d.}{\sim} \text{Exponential}(0.5^{-1} = 2)$
- **Important**
  $G(x) = P(X_1 \le x) = 1 - e^{-x}$, for $x \ge 0$ (0, otherwise)
  $F(x) = P(X_i \le x) = 1 - e^{-2x}$, for $x \ge 0$ and $i \in \mathbb{N}\backslash\{1\}$ (0, otherwise)
- **R.v.**
  $S_2 =$ time of the second inspection
- **C.d.f. of S_2**

$$\begin{aligned} P(S_n \le t) &\stackrel{form.}{=} (G \star F_{n-1})(t) \\ &\stackrel{n=2}{=} \int_0^t G(t-x)\, dF(x) \\ &= \int_0^t \left[1 - e^{-(t-x)}\right] \times 2\,e^{-2x}\, dx \\ &= \int_0^t 2\,e^{-2x}\, dx - 2\,e^{-t} \int_0^t e^{-x}\, dx \\ &= -e^{-2x}\Big|_0^t + 2\,e^{-t} \times \left(-e^{-x}\right)\Big|_0^t \\ &= (1 - e^{-2t}) + 2\,e^{-t}\,(e^{-t} - 1) \\ &= 1 + e^{-2t} - 2\,e^{-t}. \end{aligned}$$

Group 5 — Discrete time Markov chains **9.5 points**

1. A study of occupational mobility of families across generations was conducted after WWII. Three occupation levels were identified:
   - *upper* level (executive, managerial, high administrative, professional) — state 1;
   - *middle* level (high grade supervisor, non-manual, skilled manual) — state 2;
   - *lower* level (semi-skilled or unskilled) — state 3.

   Transition probabilities from generation to generation were estimated to be

$$\mathbf{P} = \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}.$$

   (a) Determine $f_{ij}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$, for $i, n = 1, 2, 3$ and $j = 1$. **(2.0)**

- **DTMC**
$\{X_n : n \in \mathbb{N}_0\}$
X_n = level of the family at the n^{th} generation
- **State space**
$\mathcal{S} = \{1, 2, 3\}$
$1 =$ *upper* level
$2 =$ *middle* level
$3 =$ *lower* level
- **TPM**

$$\mathbf{P} = \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}$$

- **Requested probabilities**
Let:
(i) $f_{ij}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$ be the probability of reaching state j for the first time starting from state i, for $i, j \in \mathcal{S}$ and $n \in \mathbb{N}$;
(ii) $\underline{f}_j^n = [f_{ij}^n]_{i \in \mathcal{S}}$ be the associated vector for fixed $j \in \mathcal{S}$ and $n \in \mathbb{N}$.
According to the formulae,

$$\underline{f}_j^n = \begin{cases} \underline{f}_j^1 = [P_{ij}]_{i\in\mathcal{S}}, & n = 1 \\ {}^{(j)}\mathbf{P} \times \underline{f}_j^{n-1} = \left[{}^{(j)}\mathbf{P}\right]^{n-1} \times \underline{f}_j^1, & n = 2, 3, \ldots, \end{cases}$$

where ${}^{(j)}\mathbf{P}$ is obtained by setting all the entries of the j^{th} column of $\mathbf{P}$ equal to 0.
When $j = 1$, we successively get

$$\begin{aligned} {}^{(1)}\mathbf{P} &= \begin{bmatrix} 0 & 0.48 & 0.07 \\ 0 & 0.70 & 0.25 \\ 0 & 0.50 & 0.49 \end{bmatrix} \\ \underline{f}_1^1 &= [P_{i1}]_{i\in\mathcal{S}} \\ &= \begin{bmatrix} 0.45 \\ 0.05 \\ 0.01 \end{bmatrix} \\ \underline{f}_1^2 &= {}^{(1)}\mathbf{P} \times \underline{f}_1^1 \\ &= \begin{bmatrix} 0 & 0.48 & 0.07 \\ 0 & 0.70 & 0.25 \\ 0 & 0.50 & 0.49 \end{bmatrix} \times \begin{bmatrix} 0.45 \\ 0.05 \\ 0.01 \end{bmatrix} \\ &= \begin{bmatrix} 0.0247 \\ 0.0375 \\ 0.0299 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \underline{f}_1^3 &= {}^{(1)}\mathbf{P} \times \underline{f}_1^2 \\ &= \begin{bmatrix} 0 & 0.48 & 0.07 \\ 0 & 0.70 & 0.25 \\ 0 & 0.50 & 0.49 \end{bmatrix} \times \begin{bmatrix} 0.0247 \\ 0.0375 \\ 0.0299 \end{bmatrix} \\ &= \begin{bmatrix} 0.020093 \\ 0.033725 \\ 0.033401 \end{bmatrix}. \end{aligned}$$

(b) What is the long-run percentage of generations that a family spends in state 3?[1] **(2.0)**

- **Important**
We are dealing with an irreducible DTMC with finite state space. Hence, all states are positive recurrent[, by Prop. 3.55]. Furthermore, the DTMC seems aperiodic.
- **Stationary distribution**
Since the DTMC is irreducible positive recurrent and aperiodic we can add that

$$\lim_{n\to+\infty} P_{ij}^n = \pi_j > 0,\ i, j \in \mathcal{S},$$

where $\{\pi_j : j \in \mathcal{S}\}$ is the unique stationary distribution and satisfies the following system of equations:

$$\begin{cases} \pi_j = \sum_{i\in\mathcal{S}} \pi_i P_{ij},\ j \in \mathcal{S} \\ \sum_{j\in\mathcal{S}} \pi_j = 1. \end{cases}$$

Equivalently [(see Prop. 3.68)], the row vector denoting the stationary distribution, $\underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$, is given by

$$\underline{\pi} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1},$$

where:
$\underline{1} = [1 \quad \cdots \quad 1]$ a row vector with $\#\mathcal{S}$ ones;
$\mathbf{I}$ = identity matrix with rank $\#\mathcal{S}$;
$\mathbf{P} = [P_{ij}]_{i,j\in\mathcal{S}}$ is the TPM;
ONE is the $\#\mathcal{S} \times \#\mathcal{S}$ matrix all of whose entries are equal to 1.
By capitalizing on the inverse in the footnote, we obtain

$$\begin{aligned} \underline{\pi} &= \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1} \\ &= \underline{1} \times \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right)^{-1} \\ &= \underline{1} \times \begin{bmatrix} 1.55 & 0.52 & 0.93 \\ 0.95 & 1.3 & 0.75 \\ 0.99 & 0.5 & 1.51 \end{bmatrix}^{-1} \end{aligned}$$

[1]The following result may be useful: $\begin{bmatrix} 1.55 & 0.52 & 0.93 \\ 0.95 & 1.3 & 0.75 \\ 0.99 & 0.5 & 1.51 \end{bmatrix}^{-1} \simeq \begin{bmatrix} 1.179441 & -0.237819 & -0.608289 \\ -0.513963 & 1.054516 & -0.207219 \\ -0.603090 & -0.193256 & 1.129679 \end{bmatrix}$.

$$\underline{\pi} \simeq [1 \quad 1 \quad 1] \times \begin{bmatrix} 1.179441 & -0.237819 & -0.608289 \\ -0.513963 & 1.054516 & -0.207219 \\ -0.603090 & -0.193256 & 1.129679 \end{bmatrix}$$
$$= [0.062389 \quad 0.623440 \quad 0.314171].$$

Thus, the long-run percentage of generations that a family spends in state 3 is equal to [the sum of the entries of the 3rd. column of $(\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1}$]:

$$\pi_3 \simeq 0.314171.$$

(c) Determine the expected number of generations it takes a family to reach state 1, starting from state 3. **(2.0)**

- **Initial/present state**
  $X_0 = i$
- **Important**
  To obtain the expected number of generations until a family to reaches state 1, given $X_0 = i$, we have to consider another DTMC where state 1 is absorbing. The associated TPM is

$$\mathbf{P}' = \begin{bmatrix} 1 & 0 & 0 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}.$$

- **Requested expected value**
  Let

$$\mathbf{Q} = \begin{bmatrix} 0.70 & 0.25 \\ 0.50 & 0.49 \end{bmatrix}$$

be the substochastic matrix governing the transitions between the states in $T = \{2, 3\}$, the class of transient states of this new DTMC, and

$$\tau = \inf\{n \in \mathbb{N}_0 : X_n \notin T\}$$

be the number of generations until a family to reaches state 1. Then, by capitalizing on the fact that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

we obtain

$$\begin{aligned}
[E(\tau \mid X_0 = i)]_{i \in T} &= (\mathbf{I} - \mathbf{Q})^{-1} \times \underline{1} \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.70 & 0.25 \\ 0.50 & 0.49 \end{bmatrix} \right)^{-1} \times \underline{1} \\
&= \begin{bmatrix} 0.3 & -0.25 \\ -0.5 & 0.51 \end{bmatrix}^{-1} \times \underline{1}
\end{aligned}$$

$$\begin{aligned}
[E(\tau \mid X_0 = i)]_{i \in T} &= \frac{1}{0.3 \times 0.51 - (-0.25) \times (-0.5)} \begin{bmatrix} 0.51 & 0.25 \\ 0.5 & 0.3 \end{bmatrix} \times \underline{1} \\
&= \frac{1}{0.028} \begin{bmatrix} 0.76 \\ 0.8 \end{bmatrix} \\
&= \begin{bmatrix} 27.142857 \\ 28.571429 \end{bmatrix}.
\end{aligned}$$

Thus, the requested expected value equals

$$E(\tau \mid X_0 = 3) = 28.571429.$$

2. The following model can be used to describe the number of women (mothers and daughters) in a given area. The number of mothers is a r.v. $X \sim \text{Poisson}(\lambda)$. Independently of the others, every mother gives birth to a Poisson(μ)−distributed number of daughters.

   Let W be the total number of women (mothers and daughters) in the area. Show that:

   (a) the p.g.f. of W is given by $e^{-\lambda\,[1 - s\, e^{-\mu\,(1-s)}]}$; **(2.5)**

- **Auxiliary r.v.**
  X = number of mothers
  $X \sim \text{Poisson}(\lambda)$
  Z_l = number of daughters from mother l
  $Z_l \overset{i.i.d.}{\sim} Z,\ l \in \mathbb{N}$
- **Important r.v.**
  $W = X + \sum_{l=1}^{X} Z_l$
  = total number of women (mothers and daughters)
- **Requested p.g.f.**

$$\begin{aligned}
P_W(s) &= E\left(s^W\right) \\
&= E\left(s^{X + \sum_{l=1}^{X} Z_l}\right) \\
&= E\left[E\left(s^{X + \sum_{l=1}^{X} Z_l} \mid X\right)\right],
\end{aligned}$$

where the r.v. $E\left(s^{X + \sum_{l=1}^{X} Z_l} \mid X\right)$ takes value

$$\begin{aligned}
E\left(s^{X + \sum_{l=1}^{X} Z_l} \mid X = x\right) &\overset{X \perp\!\!\!\perp Z_l}{=} s^x\, E\left(s^{\sum_{l=1}^{x} Z_l}\right) \\
&\overset{Z_l \overset{i.i.d.}{\sim} Z}{=} s^x \left[E\left(s^Z\right)\right]^x \\
&= \left[s\, P_Z(s)\right]^x,
\end{aligned}$$

with probability $P(X = x)$. Consequently,

$$\begin{aligned}
P_W(s) &= E\left\{\left[s\, P_Z(s)\right]^X\right\} \\
&\overset{form.}{=} e^{-\lambda[1 - s\, P_Z(s)]} \\
&\overset{form.}{=} e^{-\lambda[1 - s\, e^{-\mu(1-s)}]}.
\end{aligned}$$

QED

(b) $E(Z) = \lambda(1+\mu)$. **(1.0)**

- **Requested expected value**

$$\begin{aligned} E(W) &\overset{form.}{=} \left.\frac{dP_W(s)}{ds}\right|_{s=1} \\ &= \left.\frac{d\, e^{-\lambda[1-s\,e^{-\mu(1-s)}]}}{ds}\right|_{s=1} \\ &= -\left.\frac{d\,\lambda[1-s\,e^{-\mu(1-s)}]}{ds}\right|_{s=1} \times \left. e^{-\lambda[1-s\,e^{-\mu(1-s)}]}\right|_{s=1} \\ &= \lambda \times \left.\frac{d\,s\,e^{-\mu(1-s)}}{ds}\right|_{s=1} \times 1 \\ &= \lambda \times \left[e^{-\mu(1-s)} + s\,\mu\,e^{-\mu(1-s)}\right]\Big|_{s=1} \\ &= \lambda(1+\mu). \end{aligned}$$

QED

Group 6 — Continuous time Markov chains **9.0 points**

1. Passengers arrive at (resp. trains depart from) a train station according to a Poisson process with rate equal to 20 passengers per minute (resp. 12 trains per hour). Let $X(t)$ be the number of passengers at the station at time t waiting for the next train to depart.

(a) Draw the rate diagram and derive the infinitesimal generator $\mathbf{R}$ of the CTMC $\{X(t) : t \geq 0\}$. **(1.5)**

Hint: Even though $\{X(t) : t \geq 0\}$ is not a birth-death process, it might be useful to interpret an arrival of a passenger as a *birth* and note that a departure of a train implies the "*death*" of all passengers at the train station.

- **CTMC**
  $\{X(t) : t \geq 0\}$
  $X(t) =$ no. of passengers at the train station at time t waiting for the next train to depart

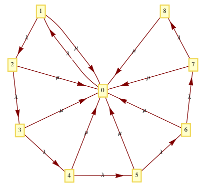
- **Auxiliary r.v.**
  $B =$ time (in minutes) until the arrival of the next passenger
  $B \sim \text{Exponential}(\lambda = 20)$
  $D =$ time (in minutes) until the departure of the next train
  $D \sim \text{Exponential}(\mu = 12/60 = 0.2)$

- **State space**
  $\mathcal{S} = \mathbb{N}_0$

- **Possible transitions (embedded DTMC)**
  If we interpret an arrival of a passenger as a *birth* and note that a departure of a train implies the "*death*" of all passengers at the train station, the embedded DTMC transitions from:
  - state i to state 0 ($i \in \mathbb{N}$) — if a train departs before the next passenger arrives;
  - state i to state $i+1$ ($i \in \mathbb{N}_0$) — if a passenger arrives before the next train departs.

- **Rate diagram**
  Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node i to node j (if $q_{ij} > 0$) with q_{ij} written on it.

...



- **Infinitesimal generator**
  This matrix has entries

$$r_{ij} = \begin{cases} q_{ij}, & i \neq j \\ -\nu_i = -\sum_{m\in\mathcal{S}} q_{im}, & j = i \end{cases}$$

and in this case it is equal to

$$\begin{aligned} \mathbf{R} &= [r_{ij}]_{i,j\in\mathcal{S}} \\ &= \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & \cdots \\ \mu & 0 & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ \mu & 0 & 0 & -(\lambda+\mu) & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \end{aligned}$$

(b) Write the Kolmogorov's forward differential equations in terms of $P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$, for $j \in \mathbb{N}_0$. **(1.0)**

- **Kolmogorov's forward differential equations**
  These can be written in matrix form:

$$\begin{aligned} \frac{d\,\mathbf{P}(t)}{dt} &= \left[\frac{d\,P_{ij}(t)}{dt}\right]_{i,j\in\mathcal{S}} \\ &\overset{form.}{=} \mathbf{P}(t) \times \mathbf{R}. \end{aligned}$$

Since $i = 0$, we are only interested in the first row of the previous matrix

$$\left[\frac{d\,P_{0j}(t)}{dt}\right]_{j\in\mathcal{S}} = [P_{0j}(t)]_{j\in\mathcal{S}} \times \mathbf{R}.$$

Hence the following Kolmogorov's forward differential equations:

$$\begin{aligned} \frac{d\,P_{00}(t)}{dt} &= -\lambda\,P_{00}(t) + \mu \sum_{m=1}^{+\infty} P_{0m}(t) \\ &= -\lambda\,P_{00}(t) + \mu\,[1 - P_{00}(t)] \\ &= -(\lambda+\mu)\,P_{00}(t) + \mu \end{aligned}$$

$$\begin{aligned}\frac{d\,P_{01}(t)}{dt} &= \lambda\,P_{00}(t) - (\lambda+\mu)\,P_{01}(t)\\ \frac{d\,P_{02}(t)}{dt} &= \lambda\,P_{01}(t) - (\lambda+\mu)\,P_{02}(t)\\ &\vdots\\ \frac{d\,P_{0j}(t)}{dt} &= \lambda\,P_{0\,j-1}(t) - (\lambda+\mu)\,P_{0j}(t),\ j\in\mathbb{N}.\end{aligned}$$

(c) Show that the equilibrium probabilities $P_j = \lim_{t\to+\infty} P_{0j}(t) = (1/101)\times(100/101)^j$, $j\in\mathbb{N}_0$. **(2.5)**

Hint: Recall that $\underline{P}\times\mathbf{R} = \underline{0}$ and $\sum_{j=0}^{+\infty} P_j = 1$, where $\underline{P} = [P_j]_{j\in\mathbb{N}_0}$ is the row vector of the equilibrium probabilities.

- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P_{0j}(t)$
  Let $\underline{P} = [P_j]_{j\in\mathbb{N}_0}$ be the row vector of the equilibrium probabilities. Then these probabilities can be obtained by solving

  $$\underline{P}\times\mathbf{R} = \underline{0},$$

  subjected to $\sum_{j=0}^{+\infty} P_j = 1$. For instance, the first set of equations (corresponding to $i=0$) lead to

$$\begin{cases} -\lambda\,P_0 + \mu\sum_{m=1}^{+\infty} P_m = 0\\ \lambda\,P_0 - (\lambda+\mu)\,P_1 = 0\\ \lambda\,P_1 - (\lambda+\mu)\,P_2 = 0\\ \vdots\\ \lambda\,P_{j-1} - (\lambda+\mu)\,P_j = 0,\ j\in\mathbb{N}\end{cases}$$

$$\begin{cases} -\lambda\,P_0 + \mu\,(1-P_0) = 0\\ P_1 = \frac{\lambda}{\lambda+\mu}\,P_0\\ P_2 = \frac{\lambda}{\lambda+\mu}\,P_1\\ \vdots\\ P_j = \frac{\lambda}{\lambda+\mu}\,P_{j-1},\ j\in\mathbb{N}.\end{cases}$$

$$\begin{cases} P_0 = \frac{\mu}{\lambda+\mu}\\ P_1 = \frac{\lambda}{\lambda+\mu}\times\frac{\mu}{\lambda+\mu}\\ P_2 = \left(\frac{\lambda}{\lambda+\mu}\right)^2\times\frac{\mu}{\lambda+\mu}\\ \vdots\\ P_j = \left(\frac{\lambda}{\lambda+\mu}\right)^j\times\frac{\mu}{\lambda+\mu},\ j\in\mathbb{N}.\end{cases}$$

  Equivalently,

  $$P_j \overset{\lambda=20,\mu=0.2}{=} (1/101)\times(100/101)^j,\ j\in\mathbb{N}_0.$$

2. An average of 80 jobs are submitted to a university computer center per hour. Assuming that the computer service is modeled as an $M/M/1$ queueing system:

(a) what should be the service rate if the average turnaround time (period from the submission a job until getting this job done) is to be smaller than 10 minutes? **(1.5)**

- **Birth-death queueing system**
  $M/M/1$ with $\lambda = 80$ (jobs per hour)
- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{\mu} = \frac{80}{\mu} < 1$
- **Performance measure (in the long-run)**
  $W_s =$ turnaround time (in hours)
  $E(W_s) \overset{form}{=} \frac{1}{\mu(1-\rho)}$
- **Requested service rate**
  We have to deal with $\mu > 80$ and

$$\begin{aligned}\mu\ :\ & E(W_s) < \frac{10}{60}\ \text{(i.e., 10 minutes)}\\ & \frac{1}{\mu\left(1-\frac{\lambda}{\mu}\right)} < \frac{1}{6}\\ & \frac{1}{\mu-\lambda} < \frac{1}{6}\\ & \mu > \frac{6}{1} + \lambda\\ & \mu > 86\ \text{(jobs per hour).}\end{aligned}$$

(b) find the average number of jobs found in the system, when the service rate is equal to $\mu = 1.5$ jobs per minute; **(1.0)**

- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{\mu} = \frac{80}{1.5\times60} = \frac{8}{9} < 1$
- **Performance measure (in the long-run)**
  $L_s =$ number of customers in the drive-in banking service
- **Requested expected value**

$$\begin{aligned}E(L_s) &\overset{form.}{=} \frac{\rho}{1-\rho}\\ &= \frac{\frac{8}{9}}{1-\frac{8}{9}}\\ &= 8.\end{aligned}$$

(c) calculate the probability that the turnaround time exceeds 10 minutes, considering the same service rate as in (b). **(1.5)**

- **Performance measure (in the long-run)**
  $W_s =$ turnaround time (in hours)
  $W_s \sim \text{Exponential}(\mu(1-\rho))$
- **Requested probability**

$$\begin{aligned}P\,(W_s > t) &= e^{-\mu(1-\rho)\,t}\\ &\overset{t=\frac{10}{60},\,etc.}{=} e^{-90\,(1-8/9)\,t}\\ &= e^{-\frac{5}{3}}.\end{aligned}$$