Department of Mathematics, IST — Probability and Statistics Unit

Introduction to Stochastic Processes

2nd. Test ("Recurso") | 2nd. Semester — **2013/14**

Duration: **1h30m** | **2014/07/01** — **9:45 PM**, Room C9

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Renewal Processes **1.5 points**

Customers arrive at a bus depot according to a renewal process with i.i.d. inter-arrival times with mean $\mu < +\infty$. As soon as there are k ($k \in \mathbb{N}$) customers waiting at the depot, a shuttle is immediately dispatched to (instantly) clear all the k customers. Let $X(t)$ denote the number of customers in the depot at time t. **(1.5)**

After having identified the regenerative epochs of the stochastic process $\{X(t) : t \geq 0\}$, derive the long-run proportion of time the bus depot has j ($j \in \{0, 1, \ldots, k-1\}$) customers?

- **Renewal process**
  $\{N(t) : t \geq 0\}$
  $N(t) =$ number of customers that arrived at the bus depot up to time t
- **Inter-renewal times**
  $X_i \stackrel{i.i.d.}{\sim} X$, $i \in \mathbb{N}$
  $E(X) = \mu < +\infty$
- **Important**
  As soon as there are k ($k \in \mathbb{N}$) customers waiting at the depot, a shuttle is immediately dispatched to (instantly) clear all the k customers.
- **Regenerative process**
  $\{X(t) : t \geq 0\}$
  $X(t) =$ number of customers in the depot at time t
- **Regenerative epochs**
  $S_n =$ dispatch time of the n^{th} shuttle
  $S_1 = \sum_{i=1}^{k} X_i, \quad S_2 = \sum_{i=k+1}^{2k} X_i, \quad \ldots, \quad S_n = \sum_{i=(n-1)k+1}^{nk} X_i$
- **Requested proportion**
  The long-run proportion of time the bus depot has j ($j \in \{0, 1, \ldots, k-1\}$) customers is equal to

$$\begin{aligned} P_j &= \lim_{t \to +\infty} P[X(t) = j] \\ &= \frac{E(U_j)}{E(S_1)} \end{aligned}$$

where

$$\begin{aligned} E(S_1) &= \sum_{i=1}^{k} E(X_i) \\ &\stackrel{X_i \stackrel{i.i.d.}{\sim} X}{=} k\, E(X) \\ &= k\, \mu \\ E(U_j) &= E(\text{time between the arrivals of customers } j \text{ and } j+1) \\ &= E(X_{j+1}) \\ &= E(X) \\ &= \mu. \end{aligned}$$

Consequently,

$$\begin{aligned} P_j &= \frac{\mu}{k\, \mu} \\ &= \frac{1}{k}. \end{aligned}$$

Group 2 — Discrete time Markov chains **9.5 points**

1. Dental records included the classification of molar teeth essentially according to the number of dental caries on their 5 surfaces: *no caries* (state 1); *caries in one surface* (state 2); *caries in more than one surface* (state 3); and *filling* (state 4). These records led to the following TPM governing the *transitions* of a molar between those 4 states in consecutive semesterly visits to the dentist:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the DTMC $\{X_n : n \in \mathbb{N}\}$, where X_n represents the state of a molar at the beginning of the n^{th} semesterly visit to the dentist.

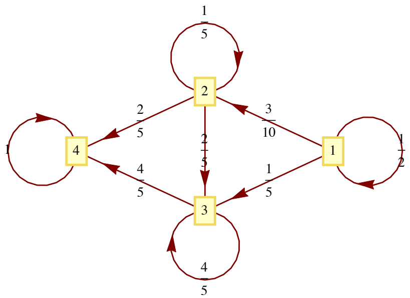
(a) Draw the associated transition diagram and classify the states of this DTMC. **(1.0)**

- **DTMC**
  $\{X_n : n \in \mathbb{N}\}$
  $X_n =$ state of a molar at the beginning of the n^{th} semesterly visit to the dentist
- **State space**
  $\mathcal{S} = \{1, 2, 3, 4\}$
  $1 =$ *no caries*
  $2 =$ *caries in one surface*
  $3 =$ *caries in more than one surface*
  $4 =$ *filling*

- **TPM**

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Transition diagram**



- **Classification of the states of the DTMC**
  - States 1, 2 and 3 are all transient because we may never return to any of these states.
    [$\{1\}, \{2\}, \{3\}$ are open and transient (communicating) classes.]
  - Since $P_{44} = 1$, state 4 is absorbing (hence recurrent) — once we reach this state we never leave it.
    [$\{4\}$ is a closed and recurrent (communicating) class.]

(b) Admit the initial state X_1 is uniformly distributed in the state space and obtain: **(1.5)**

(i) the probability that a molar has caries in more than one surface at the beginning of the third visit to the dentist;

(ii) $P(X_3 = 3 \mid X_1 = 3)$.

- **Initial state**
  $X_1 \sim \text{Uniform}(\{1, 2, 3, 4\})$

- **1st. requested probability**
  Since the initial state of this DTMC is X_1 (instead of X_0) we have to adapt the results in the list of formulae:

$$\begin{aligned} \underline{\alpha} &= [P(X_1 = i)]_{i \in \mathcal{S}} \\ &= [1/4 \quad 1/4 \quad 1/4 \quad 1/4] \\ \underline{\alpha}^n &= [P(X_{n+1} = i)]_{i \in \mathcal{S}} \\ &\stackrel{form.}{=} \underline{\alpha} \times \mathbf{P}^n. \end{aligned}$$

Thus,

$$\begin{aligned} \underline{\alpha}^2 &= [P(X_{2+1} = i)]_{i \in \mathcal{S}} \\ &= \underline{\alpha} \times \mathbf{P}^2 \\ P(X_{2+1} = 3) &= \underline{\alpha} \times \mathbf{P} \times \text{3rd. column of } \mathbf{P} \\ &= \underline{\alpha} \times \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix} \\ &= [1/4 \quad 1/4 \quad 1/4 \quad 1/4] \times \begin{bmatrix} \frac{13}{50} \\ \frac{4}{25} \\ \frac{1}{25} \\ 0 \end{bmatrix} \\ &= \frac{23}{200} \end{aligned}$$

- **2nd. requested probability**

$$\begin{aligned} P(X_3 = 3 \mid X_1 = 3) &= P^2_{33} \\ [&= \text{3rd. row of } \mathbf{P} \times \text{3rd. column of } \mathbf{P} \\ &= [0 \quad 0 \quad 1/5 \quad 4/5] \times \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix}] \\ &= \frac{1}{25} \text{ (from the previous calculations).} \end{aligned}$$

(c) Given that $X_1 = 1$, calculate the expected number of months until the molar is classified in state 4.[1] **(2.0)**

- **Important**
  Let

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

be the substochastic matrix governing the transitions between the transient states ($T = \{1, 2, 3\}$), and

$$\tau = \inf\{n \in \mathbb{N} : X_n \notin T\}$$

be the number of semesterly visits until the molar is classified in state 4.
Then [(see Prop. 3.116)] the result in the footnote yields

[1]The following result may come handy in this line: $\begin{bmatrix} \frac{1}{2} & -\frac{3}{10} & -\frac{1}{5} \\ 0 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 0 & \frac{4}{5} \end{bmatrix}^{-1} = \begin{bmatrix} 2 & \frac{3}{4} & \frac{7}{8} \\ 0 & \frac{5}{4} & \frac{5}{8} \\ 0 & 0 & \frac{5}{4} \end{bmatrix}$.

$$
\begin{aligned}
[E(\tau \mid X_1 = i)]_{i\in T} &\overset{form.}{=} (\mathbf{I}-\mathbf{Q})^{-1}\times \underline{1}\\
&= \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}\frac{1}{2}&\frac{3}{10}&\frac{1}{5}\\0&\frac{1}{5}&\frac{2}{5}\\0&0&\frac{1}{5}\end{bmatrix}\right)^{-1}\times \underline{1}\\
&= \begin{bmatrix}\frac{1}{2}&-\frac{3}{10}&-\frac{1}{5}\\0&\frac{4}{5}&-\frac{2}{5}\\0&0&\frac{4}{5}\end{bmatrix}^{-1}\times \underline{1}\\
&= \begin{bmatrix}2&\frac{3}{4}&\frac{7}{8}\\0&\frac{5}{4}&\frac{5}{8}\\0&0&\frac{5}{4}\end{bmatrix}\times\begin{bmatrix}1\\1\\1\end{bmatrix}\\
&= \begin{bmatrix}\frac{29}{8}\\\frac{15}{8}\\\frac{5}{4}\end{bmatrix}.
\end{aligned}
$$

Finally, since we are dealing with semesterly visits (i.e., visits every six months), the expected number of months until the molar is classified in state 4, given that the molar has initially no caries, equals

$$
\begin{aligned}
E(\tau \mid X_1 = 1)\times 6 &= \frac{29}{8}\times 6\\
&= \frac{87}{4}\\
&= 21.75.
\end{aligned}
$$

(d) What is the probability that the molar reaches state 4 before state 3, given $X_1 = 2$? **(2.0)**

Note: You may have to consider state 3 absorbing, identify substochastic matrices $\mathbf{Q}$ and $\mathbf{R}$ and calculate $(\mathbf{I}-\mathbf{Q})^{-1}\times\mathbf{R}$.

- **Important**

  To calculate the requested probability, we have to consider another DTMC, whose states 3 (*caries in more than one surface*) and 4 (*filling*) are absorbing and whose associated TPM is

$$
\mathbf{P}^\star = \begin{bmatrix}\frac{1}{2}&\frac{3}{10}&\frac{1}{5}&0\\0&\frac{1}{5}&\frac{2}{5}&\frac{2}{5}\\0&0&1&0\\0&0&0&1\end{bmatrix}.
$$

  The substochastic matrices governing the transitions between the transient states ($T=\{1,2\}$) of this DTMC and the transitions from the transient to the absorbing states ($\overline{T}=\{3,4\}$) are

$$
\begin{aligned}
\mathbf{Q} &= \begin{bmatrix}\frac{1}{2}&\frac{3}{10}\\0&\frac{1}{5}\end{bmatrix}\\
\mathbf{R} &= \begin{bmatrix}\frac{1}{5}&0\\\frac{2}{5}&\frac{2}{5}\end{bmatrix},
\end{aligned}
$$

  respectively.

- **Requested probability**

  Keeping in mind that

$$
\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix},
$$

  we get

$$
\begin{aligned}
\mathbf{U} &= [u_{ik}]_{i\in T,\,k\in\overline{T}}\\
&= [P(\text{reach absorbing state } k \mid X_1 = i)]_{i\in T,\,k\in\overline{T}}\\
&\overset{form}{=} (\mathbf{I}-\mathbf{Q})^{-1}\times\mathbf{R}\\
&= \left(\begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}\frac{1}{2}&\frac{3}{10}\\0&\frac{1}{5}\end{bmatrix}\right)^{-1}\times\mathbf{R}\\
&= \begin{bmatrix}\frac{1}{2}&-\frac{3}{10}\\0&\frac{4}{5}\end{bmatrix}^{-1}\times\mathbf{R}\\
&= \frac{1}{\frac{1}{2}\times\frac{4}{5}}\begin{bmatrix}\frac{4}{5}&\frac{3}{10}\\0&\frac{1}{2}\end{bmatrix}\times\mathbf{R}\\
&= \begin{bmatrix}2&\frac{3}{4}\\0&\frac{5}{4}\end{bmatrix}\times\begin{bmatrix}\frac{1}{5}&0\\\frac{2}{5}&\frac{2}{5}\end{bmatrix}\\
&= \begin{bmatrix}\frac{7}{10}&\frac{3}{10}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}.
\end{aligned}
$$

  Therefore the probability that the molar reaches state 4 before state 3, given that $X_1 = 2$, is equal to

$$
\begin{aligned}
u_{24} &= P(X_\tau = 4 \mid X_1 = 2)\\
&= \frac{1}{2}.
\end{aligned}
$$

2. Let $\{X_n : n\in\mathbb{N}_0\}$ be a branching process such that $X_0 = 1$ and admit the number of offspring per individual has p.g.f. given by $\frac{p}{1-(1-p)s}$, where $0<p<1$.

   Prove that:

   (a) if $0<p<\frac{1}{2}$ then the extinction probability is equal to $\frac{p}{1-p}$. **(2.0)**

   - **Branching process**

     $\{X_n : n\in\mathbb{N}_0\}$

     $X_n =$ size of generation n

   - **Initial state**

     $X_0 = 1$ (single initial individual)

   - **State space**

     $\mathcal{S} = \mathbb{N}_0$

   - **Number of offspring per individual**

     $Z_l \equiv Z_{l,n} =$ number of offspring of the l^{th} individual of generation n

$Z_l \overset{i.i.d.}{\sim} Z,\ l \in \mathbb{N}$

- **Distribution of the number of offspring per individual**
  $P_Z(s) = E(s^Z) = \sum_j s^j \times P(Z=j) = \sum_j s^j \times P_j = \frac{p}{1-(1-p)s},\ s \in [0,1]$
  $Z \overset{form.}{\sim} \text{Geometric}^\star(p)$
- **Obs.**
  $X_n = \sum_{l=1}^{X_{n-1}} Z_l,\ n \in \mathbb{N}$
- **Probability of extinction**
  Since

$$
\begin{aligned}
E(Z) \quad [&\overset{form}{=} \left.\frac{dP_Z(s)}{ds}\right|_{s=1} \\
&= \left.\frac{(1-p)p}{[1-(1-p)s]^2}\right|_{s=1}] \\
&\overset{form.}{=} \frac{1-p}{p}
\end{aligned}
$$

and

$$
\begin{aligned}
E(Z) &> 1 \\
\frac{1-p}{p} &> 1 \\
1-p &> p \\
p &< \frac{1}{2},
\end{aligned}
$$

the probability of extinction,

$$\pi \overset{form.}{=} \lim_{n\to+\infty} P(X_n = 0 \mid X_0 = 1),$$

is the smallest positive number satisfying

$$
\begin{aligned}
\pi &\overset{form.}{=} \sum_{j=0}^{+\infty} \pi^j \times P_j \\
&= P_Z(\pi) \\
&= \frac{p}{1-(1-p)\,\pi} \\
(1-p)\pi^2 - \pi + p &= 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\pi &= \frac{1-\sqrt{1-4(1-p)p}}{2(1-p)} \\
&= \frac{1-\sqrt{1-4p+4p^2}}{2(1-p)} \\
&= \frac{1-\sqrt{(1-2p)^2}}{2(1-p)} \\
&\overset{p<1/2}{=} \frac{1-(1-2p)}{2(1-p)} \\
&= \frac{p}{1-p}.
\end{aligned}
$$

(b) the probability that the process is extinct in the second generation equals $\frac{p}{1-(1-p)p}$. **(1.0)**

- **Requested probability**

$$
\begin{aligned}
\pi_2 &= P(X_2 = 0 \mid X_0 = 1) \\
&= P_2(0) \\
&\overset{form.}{=} P_Z[P_Z(0)] \\
&= P_Z\left[\frac{p}{1-(1-p)\times 0}\right] \\
&= P_Z(p) \\
&= \frac{p}{1-(1-p)\times p}.
\end{aligned}
$$

Group 3 — Continuous time Markov chains **9.0 points**

1. Consider a job shop that consists of 3 identical machines and 2 technicians. Suppose that the amount of time:

   - each machine operates before breaking down is exponentially distributed with mean 0.1^{-1};
   - a technician takes to fix a machine is exponentially distributed with parameter 0.4.

   Admit that all the times to breakdown and times to repair are independent r.v. and let $X(t)$ be the number of operating machines at time t.

   (a) After having identified the birth and death rates, write the Kolmogorov's forward differential equations in terms of $P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$, for $j \in \{0,1,2,3\}$. **(2.5)**
   Note: Consider that a conclusion of a repair (resp. a machine breakdown) corresponds to a *birth* (resp. *death*); do not try to solve the differential equations.

   - **CTMC**
     $\{X(t) : t \geq 0\}$
     $X(t) =$ number of operating machines at time t
   - **State space**
     $\mathcal{S} = \{0,1,2,3\}$
   - **Birth and death rates**
     Having in mind that we are dealing with a job shop with 3 machines and 2 technicians and considering that a conclusion of a repair (resp. a machine breakdown) corresponds to a *birth* (resp. *death*), we have:
     - birth rates (conclusion of repairs)
       (i) $j = 0 \to 0$ machines operating, 2 technicians busy repairing 2 machines, the conclusion of the shortest of the two repair occurs in an exponentially distributed time with rate
       $\lambda_0 = 2 \times 0.4 = 0.8$

(ii) $j = 1 \to 1$ machine operating, 2 technicians busy repairing 2 machines, the conclusion of the shortest of the two repair occurs in an exponentially distributed time with rate
$\lambda_1 = 2 \times 0.4 = 0.8$

(iii) $j = 2 \to 2$ machines operating, 1 technicians busy repairing 1 machine, the conclusion of the repair occurs in an exponentially distributed time with rate
$\lambda_2 = 0.4$

(iv) $\lambda_3 = \lambda_4 = \cdots = 0$;

– death rates (machine breakdowns)

(i) $j = 1 \to 1$ machine operating, a breakdown will occur in an exponentially distributed time with rate
$\mu_1 = 0.1$

(ii) $j = 2 \to 2$ machines operating, the shortest of two possible breakdowns occurs in an exponentially distributed time with rate
$\mu_2 = 2 \times 0.1 = 0.2$

(iii) $j = 3 \to 3$ machines operating, the shortest of three possible breakdowns occurs in an exponentially distributed time with rate
$\mu_3 = 3 \times 0.1 = 0.3$

(iv) $\mu_4 = \mu_5 = \cdots = 0$.

- **Kolmogorov's forward differential equations**

Note that

$$\begin{aligned} P_j(t) &\equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0],\ j \in \mathbb{N}_0 \\ P_{-1}(t) &= P_4(t) = P_5(t) = \cdots = 0 \\ \lambda_{-1} &= \lambda_3 = \lambda_4 = \cdots = 0 \\ \mu_0 &= \mu_4 = \mu_5 = \cdots = 0, \end{aligned}$$

therefore the Kolmogorov's forward differential equations

$$\frac{d\,P_j(t)}{dt} \overset{form.}{=} P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j + \mu_j),\ j \in \mathcal{S} = \{0,1,2,3\},$$

read as follows:

$$\begin{aligned} \frac{d\,P_0(t)}{dt} &= P_1(t)\,\mu_1 - P_0(t)\,\lambda_0 \\ &= P_1(t) \times 0.1 - P_0(t) \times 0.8 \\ \frac{d\,P_1(t)}{dt} &= P_0(t) \times 0.8 + P_2(t) \times 0.2 - P_1(t) \times (0.8 + 0.1) \\ \frac{d\,P_2(t)}{dt} &= P_1(t) \times 0.8 + P_3(t) \times 0.3 - P_2(t) \times (0.4 + 0.2) \\ \frac{d\,P_3(t)}{dt} &= P_2(t) \times 0.4 - P_3(t) \times 0.3. \end{aligned}$$

(b) What are the equilibrium probabilities $P_j = \lim_{t\to+\infty} P_j(t)$? **(2.0)**

- **[Ergodicity condition**
Guaranteed because the finiteness of the state space (and the birth and death rates).]

- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P_j(t)$
It is well known that

$$\begin{aligned} P_0 &\overset{form.}{=} \left(1 + \sum_{n=1}^{+\infty} \frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{n-1}}{\mu_1\,\mu_2\,\ldots\,\mu_n}\right)^{-1} \\ P_j &\overset{form.}{=} \frac{\lambda_{j-1}}{\mu_j} P_{j-1},\ j \in \mathbb{N}. \end{aligned}$$

But since $\mathcal{S} = \{0,1,2,3\}$ we get

$$\begin{aligned} P_0 &= \left(1 + \sum_{n=1}^{3} \frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{n-1}}{\mu_1\,\mu_2\,\ldots\,\mu_n}\right)^{-1} \\ &= \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\,\lambda_1}{\mu_1\,\mu_2} + \frac{\lambda_0\,\lambda_1\,\lambda_2}{\mu_1\,\mu_2\,\mu_3}\right)^{-1} \\ &= \left(1 + \frac{0.8}{0.1} + \frac{0.8 \times 0.8}{0.1 \times 0.2} + \frac{0.8 \times 0.8 \times 0.4}{0.1 \times 0.2 \times 0.3}\right)^{-1} \\ &= \frac{3}{251} \\ &\simeq 0.011952 \\ P_1 &= \frac{\lambda_0}{\mu_1} \times P_0 \\ &= \frac{0.8}{0.1} \times \frac{3}{251} \\ &= \frac{24}{251} \\ &\simeq 0.0956178 \\ P_2 &= \frac{\lambda_1}{\mu_2} \times P_1 \\ &= \frac{0.8}{0.2} \times \frac{24}{251} \\ &= \frac{96}{251} \\ &\simeq 0.382470 \\ P_3 &= \frac{\lambda_2}{\mu_3} \times P_2 \\ &= \frac{0.4}{0.3} \times \frac{96}{251} \\ &= \frac{128}{251} \\ &\simeq 0.509960. \end{aligned}$$

(c) Obtain the fraction of busy technicians in the long-run. **(1.5)**

- **R.v.**

$$\begin{aligned} Y(t) &= \text{number of busy technicians at time } t \\ &= \begin{cases} 2, & X(t) = 0, 1 \\ 1, & X(t) = 2 \\ 0, & X(t) = 3 \end{cases} \end{aligned}$$

- **Fraction of busy technicians in the long-run**

$$\lim_{t\to+\infty} E\left[\frac{Y(t)}{2}\right] = \frac{1}{2}\sum_{i=0}^{2} i \times \lim_{t\to+\infty} P[Y(t)=i]$$

$$\begin{aligned}\lim_{t\to+\infty} E\left[\frac{Y(t)}{2}\right] &= \frac{1}{2}\times[0\times P_3 + 1\times P_2 + 2\times(P_0+P_1)] \\ &= \frac{1}{2}\times\left[\frac{96}{251} + 2\times\left(\frac{2}{251}+\frac{24}{251}\right)\right] \\ &= \frac{75}{251} \\ &\simeq 0.298805.\end{aligned}$$

2. A small car rental company has 6 cars available. The costs (depreciation, insurance, maintenance, etc.) are of 60 monetary units per car per day. Admit:

   - customers arrive according to a Poisson process with a rate of 5 customers per day;
   - a customer rents a car for an exponential time with a mean of 1.5 days;
   - renting a car costs 110 monetary units per day;
   - arriving customers for which no car is available are lost.

   Consider this system in equilibrium and determine:

   (a) the fraction of arriving customers for which no car is available; **(1.5)**

   - **Birth and death queueing system**
     $M/M/m/m$
   - **Arrival process/rate**
     $\lambda = 5$ customers per day
   - **Service times/rate**
     $S_i \overset{i.i.d.}{\sim}$ Exponential$(\mu = 1.5^{-1})$
     $\mu = \frac{2}{3}$
   - **Servers; waiting area**
     $m/m = 6/6$ small car rental company has 6 cars available and arriving customers for which no car is available are lost.
   - **Traffic intensity/ergodicity condition**

$$\begin{aligned}\rho &= \frac{\lambda}{m\,\mu} \\ &= \frac{5}{6\times\frac{2}{3}} \\ &= \frac{5}{4} \\ &< +\infty\end{aligned}$$

   - **Performance measure (in the long-run)**
     $L_s =$ no. of customers with rented cars from this company (an arriving customer sees)

   - **Limiting probabilities**

$$P(L_s = j) \overset{form.}{=} \begin{cases} \frac{\frac{(m\rho)^j}{j!}}{\sum_{k=0}^{m}\frac{(m\rho)^k}{k!}} = \frac{m!}{k!\,(m\rho)^{m-k}} \times B(m, m\rho), & j = 0, 1, \ldots, m \\ 0, & j = m+1, m+2, \ldots, \end{cases}$$

     where $B(m, m\rho) = P(L_s = m) = \frac{\frac{(m\rho)^m}{m!}}{\sum_{j=0}^{m}\frac{(m\rho)^j}{j!}}$.

   - **Requested probability**
     Since a customer is lost if upon her/his arrival all $m = 6$ cars have been already rented and m, we want to calculate

$$\begin{aligned}P(L_s = m) &= B(m, m\rho) \\ &\overset{m=6,\rho=5/4}{=} \frac{\frac{(6\times5/4)^6}{6!}}{\sum_{j=0}^{6}\frac{(6\times5/4)^j}{j!}} \\ &= \frac{\frac{7.5^6}{6!}}{\sum_{j=0}^{6}\frac{7.5^j}{j!}} \\ &\simeq 0.361541.\end{aligned}$$

   (b) the mean profit per day. **(1.5)**

   - **R.v.**

$$\begin{aligned}Y &= \text{profit per day in equilibrium (rentals - costs)} \\ &= 110\times L_s - 60\times 6\end{aligned}$$

   - **Mean profit per day (in the long-run)**

$$\begin{aligned}E(Y) &= E(110\times L_s - 60\times 6) \\ &= 110\times E(L_s) - 60\times 6 \\ &\overset{form.}{=} 110\times m\,\rho\,[1 - B(m, m\rho)] - 60\times 6 \\ &\overset{(a)}{\simeq} 110\times 6\times\frac{5}{4}\times(1 - 0.361541) - 60\times 6 \\ &\simeq 166.729.\end{aligned}$$