Department of Mathematics, IST — Probability and Statistics Unit

Introduction to Stochastic Processes

1st. Test ("Recurso") 2nd. Semester — 2013/14
Duration: 1h30m **2014/07/01** — **8AM**, Room C9

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 0 — Introduction to Stochastic Processes **2.5 points**

A computer operates in discrete time units (slots). A priority task arises with probability p at the beginning of each slot, independently of other slots, and requires one full slot to be complete. With this in mind, let us call a slot *busy* (resp. *idle*) if within this slot the computer executes (resp. does not execute) a priority task.

(a) Identify the c.d.f. of the time index of the first *busy* slot. **(1.0)**

- **Stochastic process**
  $\{X_i : i \in \mathbb{N}\} \stackrel{i.i.d.}{\sim}$ Bernoulli(p)
  X_i = indicator of the state of i^{th} slot (*busy*=1, *idle*=0)
- **R.v.**
  T_1 = time index of the first *busy* slot
  $T_1 = U_1 \stackrel{form.}{\sim}$ Geometric(p)
- **P.f. of T_1**
  $P(T_1 = x) \stackrel{form.}{=} (1-p)^{x-1}\, p,\ x \in \mathbb{N}$
- **C.d.f. of T_1**
  For $t \in \mathbb{R}$,

$$P(T_1 \le t) = \begin{cases} 0, & t < 1 \\ \sum_{x \le t} P(T_1 = x) = \sum_{x=1}^{\lfloor t \rfloor} (1-p)^{x-1}\, p & \\ \qquad = p\, \frac{1-(1-p)^{\lfloor t \rfloor}}{1-(1-p)} = 1 - (1-p)^{\lfloor t \rfloor}, & t \ge 1. \end{cases}$$

(b) Calculate the probability that 5 out of the first 10 slots were *busy*, given that 15 out of the first 20 slots were *idle*. **(1.5)**

- **New r.v.**
  S_n = number of *busy* slots out of the first n slots
- **Conditional distribution**
  $S_m \mid S_n = k \stackrel{form.}{\sim}$ HyperG(n, m, k), $0 \le m \le n$, $0 \le k \le n$
- **P.f.**
  $P(S_m = x \mid S_n = k) = \frac{\binom{m}{x}\binom{n-m}{k-x}}{\binom{n}{k}}$, $x = \max\{0, k-n+m\}, \ldots, \min\{k, m\}$
- **Requested probability**
  Considering $n = 20$, $m = 10$, $k = 20 - 15$ and $x = 5$ yields

$$\begin{aligned} P(S_{10} = 5 \mid S_{20} = 20 - 15) &= \frac{\binom{10}{5}\binom{20-10}{(20-15)-5}}{\binom{20}{20-15}} \\ &= \frac{\frac{10!}{5!\,5!}}{\frac{20!}{5!\,15!}} \\ &= \frac{10!\,15!}{5!\,20!} \\ &= \frac{10 \times \cdots \times 6}{20 \times \cdots \times 16} \\ &= \frac{21}{1292} \\ &\simeq 0.016254 \end{aligned}$$

Group 1 — Poisson Processes **9.0 points**

1. You get email messages according to a Poisson process at a rate of $\lambda = 0.2$ messages per hour.

   (a) Suppose you have not checked your email for a whole day. What is the probability of finding 2 new messages in the first 12 hours and no new messages in the last 16 hours? **(1.5)**

   - **Stochastic process**
     $\{N(t) : t \ge 0\} \sim PP(\lambda = 0.2)$
     $N(t)$ = number of new messages by time t (time in hours)
   - **Relevant distribution**
     $N(t) \sim$ Poisson(λt)
     $P[N(t) = x] = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$, $x \in \mathbb{N}_0$
   - **Requested probability**
     We want to obtain

$$P[N(12) = 2, N(24) - N(24-16) = 0] = \star,$$

which is equal to

$$\begin{aligned} \star &= P[N(12) = 2, N(24) - N(8) = 0] \\ &= P[N(8) = 2, N(12) - N(8) = 0, N(24) - N(12) = 0] \\ &\stackrel{indep.\ incr.}{=} P[N(8) = 2] \times P[N(12) - N(8) = 0] \times P[N(24) - N(12) = 0] \\ &\stackrel{station.\ incr.}{=} P[N(8) = 2] \times P[N(12-8) = 0] \times P[N(24-12) = 0] \\ &= P[N(8) = 2] \times P[N(4) = 0] \times P[N(12) = 0] \\ &\stackrel{N(t) \sim Poi(0.2t)}{=} \frac{e^{-0.2 \times 8} \times (0.2 \times 8)^2}{2!} \times e^{-0.2 \times 4} \times e^{-0.2 \times 12} \\ &= \frac{e^{-4.8} \times 1.6^2}{2} \\ &\simeq 0.010534. \end{aligned}$$

[Alternatively,

$$\begin{aligned}
\star &= P[N(8)=2, N(24)-N(8)=0] \\
&\overset{indep.\,incr.}{=} P[N(8)=2] \times P[N(24)-N(8)=0] \\
&\overset{station.\,incr.}{=} P[N(8)=2] \times P[N(24-8)=0] \\
&= P[N(8)=2] \times P[N(16)=0] \\
&\overset{N(t)\sim Poi(0.2t)}{=} \frac{e^{-0.2\times 8} \times (0.2\times 8)^2}{2!} \times e^{-0.2\times 16} \\
&= \frac{e^{-4.8}\times 1.6^2}{2} \\
&\simeq 0.010534.]
\end{aligned}$$

(b) Obtain the probability that you will have to wait for more than 6 hours to get (exactly) 3 new messages. **(1.0)**

- **R.v.**
  S_n = time of the arrival of the n^{th} new message
  $S_n \overset{form.}{\sim} \text{Erlang}(n,\lambda)$
- **Requested probability**

$$\begin{aligned}
P(S_n > t) &= 1 - F_{S_n}(t) \\
&\overset{form.}{=} 1 - [1 - F_{Poisson(\lambda t)}(n-1)] \\
&\overset{\lambda=0.2,t=6,n=3}{=} F_{Poisson(0.2\times 6)}(3-1) \\
&= F_{Poisson(1.2)}(2) \\
&\overset{tables}{=} 0.8795.
\end{aligned}$$

2. A bank undergoes inspections, which occur according to a homogeneous Poisson process with intensity λ. This bank fails an inspection at time s if its current amount of capital, $a \times s$ ($a > 0$), does not exceed a nonnegative random amount U, where the p.d.f. of U is equal to $f_U(u) = \frac{1}{(1+u)^2}$, $u \geq 0$. **(2.0)**

   Find the probability the bank will pass all the inspections done up to time t (write it in terms of λ, a and t).

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim PP(\lambda = 5)$
  $N(t)$ = number of inspections by time t
- **Non-homogenous Bernoulli splitting**
  An inspection, which occurred at time s ($0 < s < t$), will lead to a failure with probability

$$\begin{aligned}
p(s) &= P(as \leq U) \\
&= \int_{as}^{+\infty} f_U(u)\,du \\
&= \int_{as}^{+\infty} \frac{1}{(1+u)^2}\,du
\end{aligned}$$

$$\begin{aligned}
p(s) &= -\frac{1}{1+u}\Big|_{as}^{+\infty} \\
&= \frac{1}{1+as},
\end{aligned}$$

for $as \geq 0$. Then the number of inspections this bank fails until time t, $N_f(t)$, results from a non-homogenous Bernoulli splitting of $\{N(t) : t \geq 0\}$. As a consequence,

$$N_f(t) \overset{form.}{\sim} \text{Poisson}\left(\lambda \int_0^t p(s)\,ds\right),$$

where

$$\begin{aligned}
\int_0^t p(s)\,ds &= \int_0^t \frac{1}{1+as}\,ds \\
&= \frac{1}{a}\ln(1+as)\Big|_0^t \\
&= \frac{1}{a}\ln(1+at).
\end{aligned}$$

- **Requested probability**

$$\begin{aligned}
P[N_f(t)=0] &= e^{-\lambda\times\frac{1}{a}\ln(1+at)} \\
&= (1+at)^{-\frac{\lambda}{a}}.
\end{aligned}$$

3. Suppose that particle emissions due to radioactive decay by an unstable substance are governed by a non-homogeneous Poisson process with intensity function $\lambda(t) = \frac{1}{1+t}$, $t \geq 0$.

   (a) Compute the expected number of particle emissions in the interval $(0,t]$. **(1.0)**

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim NHPP$
  $N(t)$ = number of particle emissions by time t
- **Intensity function**
  $\lambda(t) = \frac{1}{1+t}$, $t \geq 0$
- **Mean value function**

$$\begin{aligned}
m(t) &= E[N(t)] \\
&= \int_0^t \lambda(s)\,ds \\
&= \int_0^t \frac{1}{1+s}\,ds \\
&= \ln(1+s)|_0^t \\
&= \ln(1+t),\ t \geq 0
\end{aligned}$$

   (b) Find the probability that the first particle emission occurred within the first t time units. **(1.5)**

- **R.v.**
  S_n = time of the n^{th} particle emission

- **P.d.f.**
  $f_{S_n}(t) \overset{form.}{=} \lambda(t)\, e^{-m(t)}\, \frac{[m(t)]^{n-1}}{(n-1)!},\ t \geq 0$
- **Requested probability**

$$\begin{aligned} P(S_1 \leq t) &= \int_{-\infty}^{t} f_{S_1}(s)\, ds \\ &= \int_0^t \lambda(s)\, e^{-m(s)} ds \\ &= \int_0^t \frac{1}{1+s}\, e^{-\ln(1+s)} ds \\ &= \int_0^t \frac{1}{(1+s)^2} ds \\ &= -\frac{1}{1+s}\bigg|_0^t \\ &= 1 - \frac{1}{1+t} \end{aligned}$$

[Alternatively,

$$\begin{aligned} P(S_1 \leq t) &= P[N(t) \geq 1] \\ &= 1 - P[N(t) = 0] \\ &\overset{N(t)\sim Poisson(m(t))}{=} 1 - e^{-m(t)} \\ &= 1 - e^{-\ln(1+t)} \\ &= 1 - \frac{1}{1+t}.] \end{aligned}$$

4. Suppose the number of claims generated by a small portfolio of insurance policies is governed by a Poisson process with rate $\lambda = 2$ (claims per month). Individual claim amounts will be 1 or 2 with probabilities 0.6 and 0.4, respectively.

   Obtain the p.g.f. of the monthly claims amount and use it to derive the probability that this r.v. is equal to 1. **(2.0)**

   - **Relevant stochastic processes**
     $\{N(t) : t \geq 0\} \sim PP(\lambda = 2)$
     $N(t) =$ number of claims up to month t
     $N(t) \sim$ Poisson(λt)
   - **Another stochastic process**
     $\left\{X(t) = \sum_{i=1}^{N(t)} Y_i : t \geq 0\right\} \sim$ Compound $PP(\lambda, Y)$
     $X(t) =$ total claims amount up to time t
   - **R.v. et al.**
     $Y_i =$ amount of the i^{th} claim
     $Y_i \overset{i.i.d.}{\sim} Y$
     $P(Y = y) = \begin{cases} 0.6, & y = 1 \\ 0.4, & y = 2 \end{cases}$
     $\{Y_i : i \in \mathbb{N}\}$ indep. of $\{N(t) : t \geq 0\}$

- **P.g.f. of $X(t)$**
  It is given by

$$\begin{aligned} P_{X(t)}(s) &= E[s^{X(t)}] \\ &= E\left\{E\left[s^{X(t)} \mid N(t)\right]\right\}, \end{aligned}$$

where $E\left[s^{X(t)} \mid N(t)\right]$ is a r.v., which takes value

$$\begin{aligned} E\left[s^{X(t)} \mid N(t) = n\right] &= E\left[s^{\sum_{i=1}^{N(t)} Y_i} \mid N(t) = n\right] \\ &= E\left[\prod_{i=1}^{N(t)} s^{Y_i} \mid N(t) = n\right] \\ &\overset{Y_i \overset{i.i.d.}{\sim} Y,\, Y_i \perp\!\!\!\perp N(t)}{=} E\left\{\left[E\left(s^Y\right)\right]^n\right\} \end{aligned}$$

with probability $P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n \in \mathbb{N}_0$. Moreover, since the p.g.f. of Y and $N(t)$ are equal to

$$\begin{aligned} P_Y(s) &= E(s^Y) \\ &= \sum_y s^y \times P(Y = y) \\ &= s^1 \times 0.6 + s^2 \times 0.4 \\ &= 0.6s + 0.4s^2 \end{aligned}$$

$$\begin{aligned} P_{N(t)}(s) &= E[s^{N(t)}] \\ &\overset{form.}{=} e^{-\lambda t\,(1-s)} \end{aligned}$$

(respectively), we get

$$\begin{aligned} P_{X(t)}(s) &= E\left\{\left[E\left(s^Y\right)\right]^{N(t)}\right\} \\ &= P_{N(t)}[P_Y(s)] \\ &= e^{-\lambda t\,[1-P_Y(s)]} \\ &= e^{-\lambda t\,(1-0.6s-0.4s^2)} \\ &\overset{\lambda=2, t=1}{=} e^{-2\,(1-0.6s-0.4s^2)}. \end{aligned}$$

- **Requested probability**

$$\begin{aligned} P[X(t) = 1] &\overset{form.}{=} \frac{dP_{X(t)}(s)}{ds}\bigg|_{s=0} \\ &= -\lambda t\,(-0.6 - 0.8s) \times e^{-\lambda t\,(1-0.6s-0.4s^2)}\Big|_{s=0} \\ &= \lambda t \times 0.6 \times e^{-\lambda t} \\ &\overset{\lambda=2, t=1}{=} 1.2\, e^{-2} \\ &\simeq 0.162402. \end{aligned}$$

- **Obs**. — Since $N(t)$ and the Y_i are independent r.v., we could also obtain

$$P[X(t) = 1] = P[N(t) = 1,\, Y_1 = 1] = \frac{e^{-\lambda t}\,(\lambda t)^1}{1!} \times 0.6 = 1.2\, e^{-2} \simeq 0.162402.$$

Group 2 — Renewal Processes **8.5 points**

1. Replacements of an electronic part take a negligible time and occur according to a renewal process whose inter-renewal time:

   - is equal to zero with probability p (the replacing part is defective and leads to another replacement);
   - follows an exponential distribution with parameter λ with probability $1-p$.

(a) What is the long-run rate at which replacements occur? Interpret it. **(1.5)**

Note: Every inter-renewal time is a mixture of two r.v.

- **Renewal process**
  $\{N(t): t \geq 0\}$
  $N(t) =$ number of replacements until time t
- **Inter-renewal times**
  $X_i \overset{i.i.d.}{\sim} X$, $i \in \mathbb{N}$
- **Expected inter-renewal time**
  Since X is a mixture of two r.v. — one taking value 0 (with *weight* p) and another one exponentially distributed with parameter λ (with *weight* $1-p$) —, its expected value is given by a convex linear combination of expected values:

$$\begin{aligned}\mu &= E(X) \\ &= p \times 0 + (1-p) \times \frac{1}{\lambda} \\ &= \frac{1-p}{\lambda}.\end{aligned}$$

- **Requested long-run rate**
  According to the SLLN for renewal processes (see formulae!),

$$\begin{aligned}\lim_{t\to+\infty} \frac{N(t)}{t} &\overset{w.p.1}{=} \frac{1}{\mu} \\ &= \frac{\lambda}{1-p}.\end{aligned}$$

- **Interpretation**
  In the long-run a replacement takes place every $\mu = \frac{1-p}{\lambda}$ time units.

(b) Derive the renewal function $m(t)$ of this stochastic process. **(3.0)**

Notes: The following LST may come handy in the auxiliary calculations: $p + (1-p) \times \frac{\lambda}{\lambda+s}$. Recall that $LT^{-1}[1,\, t] = \delta(t)$, where $\delta(t)$ denotes the Dirac delta function.

- **C.d.f. of X**

$$\begin{aligned}F(x) &= P(X \leq 0) \\ &= \begin{cases} 0, & x < 0 \\ p, & x = 0 \\ p + (1-p) \times \int_0^x \lambda\, e^{-\lambda s}\, ds = 1 - (1-p)e^{-\lambda x}, & x > 0 \end{cases}\end{aligned}$$

- **Deriving the renewal function**
  Since the X is the mixed r.v. we have just described, its LST is given by

$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &= p \times e^{-s\times 0} + (1-p) \int_0^{+\infty} e^{-sx} \lambda e^{-\lambda x} dx \\ &= p + (1-p) \times M_{Exp(\lambda)}(-s) \\ &\overset{form.}{=} p + (1-p) \times \frac{\lambda}{\lambda+s}.\end{aligned}$$

Moreover, the LST of the renewal function can be obtained in terms of $\tilde{F}$:

$$\begin{aligned}\tilde{m}(s) &\overset{form.}{=} \frac{\tilde{F}(s)}{1-\tilde{F}(s)} \\ &= \frac{p + (1-p) \times \frac{\lambda}{\lambda+s}}{1 - p - (1-p) \times \frac{\lambda}{\lambda+s}} \\ &= \frac{\frac{p\lambda + ps + \lambda - p\lambda}{\lambda+s}}{\frac{(1-p)\,(\lambda+s-\lambda)}{\lambda+s}} \\ &= \frac{p}{1-p} + \frac{\lambda}{(1-p)s}.\end{aligned}$$

Taking advantage of the LT in the formulae and on the fact that $LT^{-1}[1,\, t] = \delta(t)$, where $\delta(t)$ denotes the Dirac delta function, we successively get:

$$\begin{aligned}\frac{dm(t)}{dt} &= LT^{-1}[\tilde{m}(s),\, t] \\ &= LT^{-1}\left[\frac{p}{1-p} + \frac{\lambda}{(1-p)s},\, t\right] \\ &= \frac{p}{1-p} \times LT^{-1}[1,\, t] + \frac{\lambda}{1-p} \times LT^{-1}\left[\frac{1}{s}, t\right] \\ &= \frac{p}{1-p} \times \delta(t) + \frac{\lambda}{1-p} \\ m(t) &= \int_0^t \left[\frac{p}{1-p} \times \delta(x) + \frac{\lambda}{1-p}\right] dx \\ &= \frac{p}{1-p} + \frac{\lambda\, t}{1-p},\ t \geq 0.\end{aligned}$$

(c) Show that the renewal function obtained in (b) verifies the elementary renewal theorem. **(1.0)**

- Verification of the elementary renewal theorem (ERT)

$$\begin{aligned}\lim_{t\to+\infty} \frac{m(t)}{t} &= \lim_{t\to+\infty} \frac{\frac{p}{1-p} + \frac{\lambda\, t}{1-p}}{t} \\ &= \frac{\lambda}{1-p} \\ &= \frac{1}{\mu},\end{aligned}$$

thus, verifying the ERT.

2. Clotilde is selling a certain article by phone. The duration of the phonecalls are independent mixed r.v. with common c.d.f. given by **(3.0)**

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 3x(1-x), & 0 < x < \tau \\ 1, & x \geq \tau, \end{cases}$$

where τ is a constant in $(0, \frac{1}{2}]$. An article is considered sold if Clotilde manages to persuade the customer to buy the article before time τ.

Find the value of τ that maximizes the number of sold articles per time unit in the long-run.

Notes: Recall that for any nonnegative r.v. X, $E(X) = \int_0^{+\infty} [1 - F_X(x)]\, dx$. It is convenient to deal with the objective function $\frac{6(1-\tau)}{2-3\tau+2\tau^2}$.

- **Renewal process**
  $\{N(t) : t \geq 0\}$
  $N(t)$ = number of phonecalls by time t

- **Inter-renewal times**
  $X_n \overset{i.i.d.}{\sim} X$, $n \in \mathbb{N}$

  $$F(x) = F_X(x) = \begin{cases} 0, & x \leq 0 \\ 3x(1-x), & 0 < x < \tau \\ 1, & x \geq \tau, \end{cases}$$

- **Reward renewal process**
  $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \geq 0\}$
  $R(t)$ = number of articles sold until time t

  $$R_n = \begin{cases} 1, & \text{if } X_n < \tau \text{ (i.e., if the } n^{th} \text{ phonecall led to the sale of an article)} \\ 0, & \text{otherwise} \end{cases}$$

  $(X_n, R_n) \overset{i.i.d.}{\sim} (X, R)$, $n \in \mathbb{N}$

- **Expected inter-renewal time**

$$\begin{aligned} E(X) &\overset{X \geq 0}{=} \int_0^{+\infty} [1 - F_X(x)]\, dx \\ &= \int_0^{\tau} [1 - 3x(1-x)]\, dx \\ &= \int_0^{\tau} (1 - 3x + 3x^2)\, dx \\ &= \left(x - \frac{3x^2}{2} + x^3 \right) \Big|_0^{\tau} \\ &= \tau - \frac{3\tau^2}{2} + \tau^3 \end{aligned}$$

- **Expected number of articles sold per phonecall**

$$\begin{aligned} E(R) &= 1 \times P(X < \tau) + 0 \times P(X \geq \tau) \\ &= 3\tau(1-\tau) \end{aligned}$$

- **Number of articles sold per time unit in the long-run**
  Since $E(X), E(R) < +\infty$, we can add that

$$\frac{R(t)}{t} \overset{w.p.1}{\rightarrow} \frac{E(R)}{E(X)},$$

where $\frac{E(R)}{E(X)}$ represents the number of articles sold per time unit in the long-run. Moreover,

$$\begin{aligned} \frac{E(R)}{E(X)} &= h(\tau) \\ &= \frac{3\tau(1-\tau)}{\tau - \frac{3\tau}{2} + \tau^3} \\ &= \frac{6(1-\tau)}{2 - 3\tau^2 + 2\tau^2}. \end{aligned}$$

- **Maximizing the number of articles sold per time unit in the long-run**

$$\tau \in (0, 1/2] \; : \; \frac{dh(\tau)}{d\tau} = 0 \quad \left(\text{and } \frac{d^2h(\tau)}{d\tau^2} < 0 \right)$$

$$-6(2 - 3\tau + 2\tau^2) - 6(1-\tau)(-3 + 4\tau) = 0$$

$$-12 + 18\tau - 12\tau^2 + 18 - 24\tau - 18\tau + 24\tau^2 = 0$$

$$12\tau^2 - 24\tau + 6 = 0$$

$$2\tau^2 - 4\tau + 1 = 0$$

$$\tau = \frac{4 \pm \sqrt{16 - 8}}{4}$$

$$\tau = 1 - \frac{\sqrt{2}}{2}$$

$$\tau \simeq 0.292893.$$