Department of Mathematics, IST — Probability and Statistics Unit

Introduction to Stochastic Processes

2nd. Test ("RECURSO") 2nd. Semester — **2012/13**

Duration: **1h30m** **2013/06/24** — **9:45AM**, Room V1.11

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Renewal Processes **4.0 points**

Satellites are launched according to a Poisson process with rate λ. Each satellite will, independently, orbit the earth for a random time with c.d.f. G and expected value μ^{-1}. Moreover, if at least one satellite is orbiting, then messages can be transmitted and we say that the system is functional.

(a) *Let $X(t)$ denote the number of satellites orbiting at time t.* (1.5)

After having related this system to the $M/G/\infty$ queueing system, determine $\lim_{t\to+\infty} P[X(t) = j \mid X(0) = 0]$.

- **Analogy with the $M/G/\infty$**

  We can view this communication system as an $M/G/\infty$ queueing system where:
  - a satellite launching corresponds to an arrival, and these arrivals occur according to a Poisson process with rate λ;
  - G is the self-service distribution with expected value μ^{-1}.

- **Stochastic process**

  $\{X(t) : t \geq 0\}$

  $X(t)$ = number of satellites orbiting at time t

  = number of customers in the $M/G/\infty$ queueing system at time t

- **Limiting probabilities**

  Since $\lim_{t\to+\infty}(X(t) = j \mid X(0) = 0) \overset{form.}{\sim} \text{Poisson}(\lambda/\mu)$, we have

$$\lim_{t\to+\infty} P[X(t) = j \mid X(0) = 0] = e^{-\lambda/\mu}\frac{(\lambda/\mu)^j}{j!},\ j \in \mathbb{N}_0.$$

(b) *By making use of (a) and viewing the system as an alternating renewal process that is* ON *at time t if $X(t) > 0$ and* OFF *at time t if $X(t) = 0$, determine the expected time that the system remains functional.* (2.5)

- **State variable**

$$Z(t) = \begin{cases} 0, & \text{if there is no satellite orbiting at time } t \\ 1, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } X(t) = 0 \text{ (communication system is OFF)} \\ 1, & \text{if } X(t) > 0 \text{ (communication system is ON)} \end{cases}$$

- **Alternating renewal process**

  $\{Z(t) : t \geq 0\}$

- **Down/Off time**

  D = time comm. system is DOWN/OFF

  $E(D) = \frac{1}{\lambda}$ because the time between consecutive launches are i.i.d. and Exponential(λ) so are the time until the 1st. launch and the period from the time the last operating satellite breaks down to the time of the next launch.

- **Up/On time**

  U = time comm. system is UP/ON

  $E(U)$ is to be determined!

- **Long-run proportion of time system is DOWN/OFF**

  On one hand,

$$\lim_{t\to+\infty} P[Z(t) = 0] \overset{[Prop. 2.106]}{=} \frac{E(D)}{E(D)+E(U)} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+E(U)}.$$

  On the other hand the long-run proportion of time the comm. system is DOWN/OFF is given by:

$$\lim_{t\to+\infty} P[Z(t) = 0] = \lim_{t\to+\infty} P[X(t) = 0 \mid X(0) = 0] \overset{(a)}{=} e^{-\lambda/\mu}.$$

  Equating these two results we can add that:

$$\frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+E(U)} = e^{-\lambda/\mu}$$
$$E(U) = \frac{e^{\lambda/\mu}-1}{\lambda}.$$

Group 2 — Discrete time Markov chains **9.0 points**

1. *A particle moves among 4 vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $1-p$. Let $\{X_n : n \in \mathbb{N}_0\}$ be a discrete time Markov chain (DTMC), where X_0 denotes the initial state and X_n represents the position of the particle at step n.*

   (a) *Draw the associated transition diagram and determine the transition probability matrix (TPM).* (1.0)

   - **DTMC**

     $\{X_n : n \in \mathbb{N}_0\}$

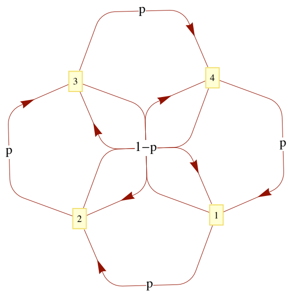
     X_0 = initial position of the particle

     X_n = position of the particle at step n

   - **State space**

     $\mathcal{S} = \{1, 2, 3, 4\}$

- **Transition diagram**
According to the description in the test, we are dealing with the following transition diagram:



- **TPM**
Follows from the transition diagram above:

$$
\begin{aligned}
\mathbf{P} &= [P_{ij}]_{i,j\in\mathcal{S}} \\
&= [P(X_{n+1} = j \mid X_n = i)]_{i,j\in\mathcal{S}},\ n \in \mathbb{N}_0 \\
&= [P(X_1 = j \mid X_0 = i)]_{i,j\in\mathcal{S}} \\
&= \begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix}
\end{aligned}
$$

(b) *Classify the states of this DTMC. Is this MC periodic? If that is the case, identify its period.* (1.5)

- **Classification of the states**
Since
  - $\mathcal{S} = \{1,2,3,4\}$ is **finite**,
  - all the states communicate with each other, i.e., $\mathcal{S}$ constitutes the **sole closed communication class**, hence **irreducible**,

  the DTMC is **positive recurrent**[, by Prop. 3.55].

- **Periodicity of the states**
A close inspection of the transition diagram leads to the conclusion that we can return to state 1 in $2,4,6,\ldots$ steps but not in an odd number of steps. In fact, if we obtain the entry $(1,1)$ of a few powers of $\mathbf{P}$, for $0 < p < 1$:

$$
\begin{aligned}
(\mathbf{P}^2)_{11} &= \text{1st. row of } \mathbf{P} \times \text{1st. column of } \mathbf{P} \\
&= 2p(1-p) \\
&\neq 0
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{P}^3)_{11} &= \text{1st. row of } \mathbf{P} \times \text{1st. column of } \mathbf{P}^2 \\
&= [0 \quad p \quad 0 \quad 1-p] \times \begin{bmatrix} 2p(1-p) \\ 0 \\ (1-p)^2 + p^2 \\ 0 \end{bmatrix} \\
&= 0 \\
(\mathbf{P}^4)_{11} &= \text{1st. row of } \mathbf{P}^2 \times \text{1st. column of } \mathbf{P}^2 \\
&= [2p(1-p) \quad 0 \quad (1-p)^2 + p^2 \quad 0] \times \begin{bmatrix} 2p(1-p) \\ 0 \\ (1-p)^2 + p^2 \\ 0 \end{bmatrix} \\
&\neq 0 \\
&\vdots
\end{aligned}
$$

Thus, state 1 had period $d = 2$.
Moreover, since periodicity is a class property and the DTMC is irreducible, all its states have the same period $d = 2$ and the MC is periodic.

2. *Clotilde travels economy (state 1) or first class (state 2) once a month, according to a DTMC governed by the following TPM:*

$$
\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0.6 & 0.4 \end{bmatrix}
$$

(a) *What is the long-run proportion of months in which Clotilde travels economy?* (2.0)

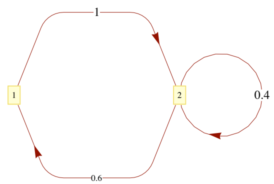
- **DTMC**
$\{X_n : n \in \mathbb{N}\}$
X_n = class in which Clotilde travelled on month n

- **State space**
$\mathcal{S} = \{1,2\}$

- **TPM**

$$
\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0.6 & 0.4 \end{bmatrix}
$$

- **Important**
We are dealing with an irreducible DTMC with finite state space $\mathcal{S} = \{1,2\}$. Hence, all states are positive recurrent[, by Prop. 3.55]. Furthermore, a close look at the corresponding transition diagram



leads to the conclusion that the DTMC is aperiodic — after all we can return to state 1 (resp. state 2) after $2, 3, 4, \ldots$ steps (resp. $1, 2, 3, 4, \ldots$).

- **Stationary distribution**
Since the DTMC is irreducible positive recurrent and aperiodic we can add that

$$\lim_{n\to+\infty} P_{ij}^n = \pi_j > 0,\ i, j \in \mathcal{S},$$

where $\{\pi_j : j \in \mathcal{S}\}$ is the unique stationary distribution and satisfies the following system of equations:

$$\begin{cases} \pi_j = \sum_{i\in\mathcal{S}} \pi_i P_{ij},\ j \in \mathcal{S} \\ \sum_{j\in\mathcal{S}} \pi_j = 1. \end{cases}$$

$$\begin{cases} \pi_1 = \pi_2 \times 0.6 \\ (\pi_2 = \pi_1 \times 1 + \pi_2 \times 0.4) \\ \pi_1 + \pi_2 = 1 \end{cases}$$

$$\begin{cases} \pi_1 = (1-\pi_1) \times 0.6 \\ \pi_2 = 1 - \pi_1 \end{cases}$$

$$\begin{cases} \pi_1 = \frac{0.6}{1+0.6} = \frac{3}{8} \\ \pi_2 = \frac{5}{8}. \end{cases}$$

Hence, the long-run proportion of months in which Clotilde travels economy is equal to:

$$\pi_1 = 0.375.$$

- **Alternatively...**
Equivalently [(see Prop. 3.68)], the row vector denoting the stationary distribution, $\underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$, is given by

$$\underline{\pi} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1},$$

where:

$\underline{1} = [1 \quad \cdots \quad 1]$ a row vector with $\#\mathcal{S}$ ones;

$\mathbf{I}$ = identity matrix with rank $\#\mathcal{S}$;

$\mathbf{P} = [P_{ij}]_{i,j\in\mathcal{S}}$ is the TPM;

$\mathbf{ONE}$ is the $\#\mathcal{S} \times \#\mathcal{S}$ matrix all of whose entries are equal to 1.

By capitalizing on the fact that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

we obtain

$$\begin{aligned} \underline{\pi} &= \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1} \\ &= \underline{1} \times \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0.6 & 0.4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \\ &= \underline{1} \times \begin{bmatrix} 2 & 0 \\ 0.4 & 1.6 \end{bmatrix}^{-1} \\ &= \underline{1} \times \frac{1}{2 \times 1.6 - 0 \times 0.4} \begin{bmatrix} 1.6 & -0 \\ -0.4 & 2 \end{bmatrix} \\ &= [0.375 \quad 0.625]. \end{aligned}$$

Thus, the long-run proportion of months in which Clotilde travels economy is equal to [the sum of the entries of the 1st. column of $(\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1}$]:

$$\pi_1 = 0.375.$$

(b) *The monthly amount spent by Clotilde is equal to 200 (resp. 400) if she travels economy (resp. first class).* (1.0)
What is Clotilde's long-run expense per month?

- **Vector of expenses**

$$\begin{aligned} \underline{c} &= [c(j)]_{j\in\mathcal{S}} \\ &= \begin{bmatrix} 200 \\ 400 \end{bmatrix} \end{aligned}$$

- **Long-run expected expense per month**
[According to Prop. 3.81,]

$$\begin{aligned} \underline{\pi} \times \underline{c} &= \sum_{j\in\mathcal{S}} \pi_j \times c(j) \\ &= 0.375 \times 200 + 0.625 \times 400 \\ &= 325. \end{aligned}$$

(c) *Given that her first trip was in first class, compute the probability that Clotilde travels economy at least once in the next 5 months.* (2.0)

- **Requested probability**
Since the event *Clotilde travels economy at least once in the next 5 months* is the complement of *Clotilde only travels first class in the next 5 months*, we get, namely by using the Markov property:

$$\begin{aligned} 1 - P(X_j = 2, j = 2, \ldots, 6 \mid X_1 = 2) &= 1 - \prod_{j=1}^{5} P(X_{j+1} = 2 \mid X_j = 2) \\ &= 1 - (P_{22})^5 \\ &= 1 - 0.4^5 \\ &= 0.98976. \end{aligned}$$

3. *Admit a single cell is put in a Petri dish and it either dies with probability* $p = \frac{1}{4}$ *or divides itself in two identical cells with probability* $1 - p = \frac{3}{4}$. (1.5)

Identify a stochastic process capable of describing the dynamics of the (live) cell population in the Petri dish and obtain the probability of extinction of this population.

- **Branching process**

  $\{X_n : n \in \mathbb{N}_0\}$

  $X_0 = 1$ (one initial cell)

  $Z_{l,n-1}$ = number of "offspring" of the l^{th} individual of the $(n-1)^{th}$ generation, $n \in \mathbb{N}$

  $X_n = \sum_{l=1}^{X_{n-1}} Z_{l,n-1}$ = number of cells in the n^{th} generation

- **Assumptions**

  $Z_{l,n-1} \stackrel{i.i.d.}{\sim} Z_l$

  $$P_j = P(Z_l = j) = \begin{cases} \frac{1}{4}, & j = 0 \\ \frac{3}{4}, & j = 2 \\ 0 & \text{otherwise} \end{cases}$$

- **Extinction probability**

  Since

  $$\begin{aligned} \mu &= E(Z_l) \\ &= 0 \times \frac{1}{4} + 2 \times \frac{3}{4} \\ &= \frac{3}{2} \\ &> 1, \end{aligned}$$

  the extinction probability is[, according to Prop. 3.107,] the **smallest positive number** satisfying

  $$\begin{aligned} \pi_0 &= \sum_{j=0}^{+\infty} (\pi_0)^j \times P_j \\ \pi_0 &= (\pi_0)^0 \times \frac{1}{4} + (\pi_0)^2 \times \frac{3}{4} \\ 3\pi_0^2 - 4\pi_0 + 1 &= 0. \end{aligned}$$

  Hence,

  $$\begin{aligned} \pi_0 &= \frac{4 - \sqrt{4^2 - 4 \times 3 \times 1}}{2 \times 3} \\ &= \frac{4-2}{6} \\ &= \frac{1}{3}. \end{aligned}$$

Group 3 — Continuous time Markov chains **7.0 points**

1. *Consider a gas station with 2 pumps. Admit that the number of customers in this system at time t,* $X(t)$*, is governed by a birth and death process* $\{X(t) : t \geq 0\}$ *with rates equal to:* $\lambda_j = \lambda$*, for* $j = 0, 1$*; and* $\mu_j = j\mu$*, for* $j = 1, 2$.

   (a) *Write the Kolmogorov's forward differential equations in terms of* $P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$*, for* $j = 0, 1, 2$. *(Do not try to solve them!)* (1.5)

   - **Birth and death process**

     $\{X(t) : t \geq 0\}$

     $X(t)$ = number of customers in the gas station at time t

   - **Birth and death rates**

     $$\lambda_j = \begin{cases} \lambda, & j = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

     $$\mu_j = \begin{cases} j\mu, & j = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

   - **State space**

     $\mathcal{S} = \{0, 1, 2\}$

   - **Kolmogorov's forward differential equations**

     Note that

     $$P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0], \ j \in \mathcal{S}$$

     and since $\mathcal{S} = \{0, 1, 2\}$, we are dealing with

     $$\begin{aligned} P_{-1}(t) &= P_3(t) = 0 \\ \lambda_{-1} &= \lambda_2 = 0 \\ \mu_0 &= \mu_3 = 0. \end{aligned}$$

     Consequently, Kolmogorov's forward differential equations

     $$\frac{d\,P_j(t)}{dt} = P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j + \mu_j), \ j \in \mathcal{S}$$

     read as follows:

     $$\begin{aligned} \frac{d\,P_0(t)}{dt} &= P_1(t)\,\mu - P_0(t)\,\lambda; \\ \frac{d\,P_1(t)}{dt} &= P_0(t)\,\lambda + P_2(t)\,(2\mu) - P_1(t)(\lambda + \mu); \\ \frac{d\,P_2(t)}{dt} &= P_1(t)\,\lambda - P_2(t)\,(2\mu). \end{aligned}$$

   (b) *Consider* $\rho = \frac{\lambda}{2\mu}$ *and prove that the equilibrium probabilities* $P_j = \lim_{t \to +\infty} P_j(t)$ *are given by* $P_j = \frac{(2\rho)^j}{j!} / \sum_{k=0}^{2} \frac{(2\rho)^k}{k!}$*, for* $j = 0, 1, 2$. (2.5)

   - **Ergodicity condition**

     $\rho = \frac{\lambda}{2\mu} < +\infty.$[1]

   - **Equilibrium probabilities** $P_j = \lim_{t \to +\infty} P_j(t)$

[1] The traffic intensity ρ does not need to be smaller than 1 because the state space is finite.

Firstly,

$$\begin{aligned} P_0 &= \left(1+\sum_{n=1}^{+\infty}\frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{n-1}}{\mu_1\,\mu_2\,\ldots\,\mu_n}\right)^{-1} \\ &= \left(1+\sum_{n=1}^{2}\frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{n-1}}{\mu_1\,\mu_2\,\ldots\,\mu_n}\right)^{-1} \\ &= \left(1+\frac{\lambda}{\mu}+\frac{\lambda^2}{2\mu^2}\right)^{-1} \\ &= \left[\sum_{k=0}^{2}\frac{(2\rho)^k}{k!}\right]^{-1}. \end{aligned}$$

Secondly,

$$\begin{aligned} P_j &= P_0 \times \frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{j-1}}{\mu_1\,\mu_2\,\ldots\,\mu_j} \\ &= P_0 \times \frac{\lambda}{\mu}\left(\frac{\lambda}{2\mu}\right)^{j-1} \\ &= \frac{\frac{(2\rho)^j}{j!}}{\sum_{k=0}^{2}\frac{(2\rho)^k}{k!}},\ j=1,2. \end{aligned}$$

2. *An air freight terminal has 2 loading docks. An aircraft which arrives when all docks are full is diverted to another terminal. The aircrafts arrive to the air freight terminal according to a Poisson process with rate 3 aircrafts per hour. The service times are i.i.d. r.v. exponentially distributed with mean equal to 2 hours.*

   (a) *Find the proportion of arriving aircrafts that are diverted to another terminal.* (2.0)

   - **Birth and death queueing system**
     $M/M/m/m$
   - **Arrival process/rate**
     $\lambda = 3$ aircrafts per hour
   - **Service times/rate**
     $S_i \stackrel{i.i.d.}{\sim}$ Exponential$(\mu^{-1}=2)$
     $\mu = \frac{1}{2}$ (1 aircraft every 2 hours)
   - **Servers; waiting area**
     $m/m = 2/2$ because the air freight terminal has 2 loading docks and an aircraft, which arrives when all docks are full, is diverted to another terminal (i.e., there is no waiting area)
   - **Traffic intensity/ergodicity condition**

$$\begin{aligned} \rho &= \frac{\lambda}{m\,\mu} \\ &= \frac{3}{2\times 1/2} = 3 \\ &< +\infty \end{aligned}$$

(see the previous footnote).

   - **Performance measure (in the long-run)**
     L_s = number of aircratfs in the air freight terminal
   - **Limiting probabilities**

$$P(L_s=j) \stackrel{form.}{=} \begin{cases} \frac{\frac{(m\rho)^j}{j!}}{\sum_{k=0}^{m}\frac{(m\rho)^k}{k!}}, & j=0,1,\ldots,m \\ 0, & j=m+1,m+2,\ldots \end{cases}$$

$$P(L_s=j) = \begin{cases} \frac{\frac{6^j}{j!}}{1+6+6^2/2} = \frac{\frac{6^j}{j!}}{25}, & k=0,1,2 \\ 0, & k=3,4,\ldots \end{cases}$$

   - **Requested probability**
     Since an aircraft is diverted to another terminal if upon its arrival all $m=2$ docks are full, we want to calculate

$$\begin{aligned} P(L_s=m) &\stackrel{form.}{=} B(m,m\rho) \\ &\stackrel{m=2}{=} \frac{\frac{6^2}{2!}}{25} \\ &= \frac{18}{25} \\ &= 0.72. \end{aligned}$$

   (b) *Obtain the average number of aircrafts in the air freight terminal.* (1.0)

   - **Requested expected value**

$$\begin{aligned} E(L_s) &\stackrel{form.}{=} m\rho\times[1-B(m,m\rho)] \\ &= 2\times 3\times\left(1-\frac{18}{25}\right) \\ &= \frac{42}{25} \\ &= 1.68. \end{aligned}$$