Department of Mathematics, IST — Probability and Statistics Unit

Introduction to Stochastic Processes

1st. Test ("Recurso") — 2nd. Semester — **2012/13**

Duration: **1h30m** — **2013/09/24** — **8AM**, Room V1.11

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Introduction to Stochastic Processes **2.5 points**

Let $\{N(t): t \geq 0\}$ be a Poisson process of intensity λ and T_0 be an independent r.v. such that $P(T_0 = -1) = P(T_0 = 1) = \frac{1}{2}$, and define $T(t) = T_0 \times (-1)^{N(t)}$.

$\{T(t): t \geq 0\}$ is an example of burst noise (also called popcorn or random telegraph signal noise), a type of electronic noise that occurs in semiconductors.

(a) *Derive the expected value and the variance of $T(t)$.* (1.0)

- **Stochastic processes**
  $\{N(t): t \geq 0\} \sim PP(\lambda)$
  $\{T(t) = T_0 \times (-1)^{N(t)}: t \geq 0\}$ burst noise process
- **R.v.**
  $N(t) \sim \text{Poisson}(\lambda t)$
  $P(T_0 = -1) = P(T_0 = 1) = \frac{1}{2}$
  $T_0 \perp\!\!\!\perp \{N(t): t \geq 0\}$
- **Requested expected value**

$$\begin{aligned} E[T(t)] &= E\left[T_0 \times (-1)^{N(t)}\right] \\ &\overset{T_0 \perp\!\!\!\perp N(t)}{=} E(T_0) \times E\left[(-1)^{N(t)}\right] \\ &= \left[(-1) \times \frac{1}{2} + (+1) \times \frac{1}{2}\right] \times E\left[(-1)^{N(t)}\right] \\ &= 0 \times E\left[(-1)^{N(t)}\right] \text{ (the 2nd. factor is finite; see (b))} \\ &= 0 \end{aligned}$$

- **Requested variance**

$$\begin{aligned} V[T(t)] &\overset{E[T(t)]=0}{=} E[T^2(t)] \\ &= E\left[T_0^2 \times (-1)^{2N(t)}\right] \\ &\overset{N(t) \in \mathbb{N}_0}{=} E(T_0^2 \times 1) \\ &= (-1)^2 \times \frac{1}{2} + (+1)^2 \times \frac{1}{2} \\ &= 1. \end{aligned}$$

(b) *Is $\{T(t): t \geq 0\}$ a (second order weakly) stationary process?* (1.5)

Note: $\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$; $\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$.

- **Investigating the 2nd. order weak stationarity**
  On one hand $E[T(t)] \overset{(a)}{=} 0$, hence, constant for all $t \geq 0$.
  On the other hand, if we recall that

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \\ \sinh x &= \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

we get, for $t, s \geq 0$,

$$\begin{aligned} cov(T(t), T(t+s)) &\overset{E[T(z)]=0,\, z \geq 0}{=} E\left[T(t) \times T(t+s)\right] \\ &= E\left[T_0^2 \times (-1)^{N(t)+N(t+s)}\right] \\ &= E\left[T_0^2 \times (-1)^{2N(t)+[N(t+s)-N(t)]}\right] \\ &\overset{indep.\, inc.}{=} E(T_0^2) \times E\left[(-1)^{2N(t)}\right] \times E\left[(-1)^{N(t+s)-N(t)}\right] \\ &\overset{(a),\, station.\, inc.}{=} 1 \times E(1) \times E\left[(-1)^{N(s)}\right] \\ [&= \sum_{n=0}^{+\infty} (-1)^n \times P[N(s) = n] \\ &= \sum_{n=0}^{+\infty} (-1)^{2n} \times P[N(s) = 2n] \\ &\quad + \sum_{n=0}^{+\infty} (-1)^{2n+1} \times P[N(s) = 2n+1] \\ &\overset{N(s) \sim Poi(\lambda s)}{=} \sum_{n=0}^{+\infty} e^{-\lambda s} \frac{(\lambda s)^s}{(2n)!} - \sum_{n=0}^{+\infty} e^{-\lambda s} \frac{(\lambda s)^s}{(2n+1)!} \\ &= e^{-\lambda s} \times [\cosh(\lambda s) - \sinh(\lambda s)] \\ &= e^{-\lambda s} \times \left(\frac{e^{\lambda s} + e^{-\lambda s}}{2} - \frac{e^{\lambda s} - e^{-\lambda s}}{2}\right) \\ &= e^{-2\lambda s},] \end{aligned}$$

which does not depend on t and only depends on the time lag s. Consequently, we are dealing with a second order weakly stationary process.

Group 2 — Poisson Processes **9.5 points**

1. *Requests arrive to a web server according to a Poisson process having rate equal to λ requests per hour.*

   (a) *Suppose two requests arrived during the first hour. What is the probability that both requests arrived during the first 20 minutes?* (1.5)

   - **Stochastic process**
     $\{N(t): t \geq 0\} \sim PP(\lambda)$
     $N(t) =$ number of requests to a web server by time t (time in hours)
     $N(t) \sim \text{Poisson}(\lambda t)$

- **Requested probability**
  According to Exercise 1.47,

$$(N(s) \mid N(t) = n) \sim \text{Binomial}(n, s/t),\ 0 < s < t.$$

Thus,

$$\begin{aligned} P[N(1/3) = 2 \mid N(1) = 2] &= P_{Binomial(n=2,s/t=1/3)}(2) \\ &= \binom{2}{2} \times (1/3)^2 \times (1-1/3)^{2-2} \\ &= (1/3)^2 \\ &= \frac{1}{9}. \end{aligned}$$

- **Alternatively...**

$$\begin{aligned} P[N(1/3) = 2 \mid N(1) = 2] &= \frac{P[N(1/3) = 2,\ N(1) = 2]}{P[N(1) = 2]} \\ &= \frac{P[N(1/3) = 2,\ N(1) - N(1/3) = 2-2]}{P[N(1) = 2]} \\ &\overset{indep.\,incr.}{=} \frac{P[N(1/3) = 2] \times P[N(1) - N(1/3) = 2-2]}{P[N(1) = 2]} \\ &\overset{station.\,incr.}{=} \frac{P[N(1/3) = 2] \times P[N(1-1/3) = 0]}{P[N(1) = 2]} \\ &\overset{N(t)\sim Poi(\lambda t)}{=} \frac{e^{-\lambda/3}\frac{(\lambda/3)^2}{2!} \times e^{-2\lambda/3}\frac{(2\lambda/3)^0}{0!}}{e^{-\lambda}\frac{\lambda^2}{2!}} \\ &= (1/3)^2 \\ &= \frac{1}{9} \end{aligned}$$

(b) *Derive $P\left[S_1 < \frac{1}{3} \text{ or } S_2 < \frac{1}{3} \mid N(1) = 2\right]$, where $N(1)$ and S_i ($i = 1, 2$) represent the number of requests arrived during the first hour and the time the i^{th} request arrived to the web server.* (1.5)

- **Requested probability**
  Capitalizing once again on the fact $(N(s) \mid N(t) = n) \sim \text{Binomial}(n, s/t)$, $0 < s < t$, we conclude that

$$\begin{aligned} P\left[S_1 < 1/3 \text{ or } S_2 < 1/3 \mid N(1) = 2\right] &= P[N(1/3) \geq 1 \mid N(1) = 2] \\ &= \sum_{i=1}^{2} P[N(1/3) = i \mid N(1) = 2] \\ &\overset{(a)}{=} P[N(1/3) = 1 \mid N(1) = 2] + \frac{1}{9} \\ &= P_{Binomial(n=2,s/t=1/3)}(1) + \frac{1}{9} \\ &= \binom{2}{1} \times (1/3)^1 \times (1-1/3)^{2-1} + \frac{1}{9} \\ &= \frac{5}{9} \end{aligned}$$

- **Alternatively...**
  Given that

$$(S_1, \ldots, S_n \mid N(t) = n) \sim (Y_{(1)}, \ldots, Y_{(n)}),\ n \in \mathbb{N},$$

where $Y_i \overset{i.i.d.}{\sim} Y \sim \text{Uniform}(0, t)$, $i = 1, \ldots, n$. Furthermore, since $t = 1$, we can add that

$$\begin{aligned} F_Y(y) &= \begin{cases} 0, & y \leq 0 \\ y, & 0 < y < 1 \\ 1, & y \geq 1 \end{cases} \\ f_Y(y) &= \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \\ F_{Y_{(i)}}(y) &\overset{form.}{=} 1 - F_{Binomial(n,F_Y(y)=y)}(i-1) \\ &= 1 - F_{Binomial(n,y)}(i-1),\ 0 < y < 1 \\ f_{Y_{(1)},\ldots,Y_{(n)}}(y_{(1)}, \ldots, y_{(n)}) &= n! \times \prod_{i=1}^{n} f_Y(y_{(i)}) \\ &= n!, 0 < y_{(1)} \leq \cdots \leq y_{(n)} < 1. \end{aligned}$$

Hence,

$$\begin{aligned} P\left[S_1 < 1/3 \text{ or } S_2 < 1/3 \mid N(1) = 2\right] &= P\left[S_1 < 1/3 \mid N(1) = 2\right] \\ &\quad + P\left[S_2 < 1/3 \mid N(1) = 2\right] \\ &\quad - P\left[S_1 < 1/3,\ S_2 < 1/3 \mid N(1) = 2\right] \\ &\overset{n=2}{=} F_{Y_{(1)}}(1/3) + F_{Y_{(2)}}(1/3) - F_{Y_{(1)},Y_{(2)}}(1/3, 1/3), \end{aligned}$$

where

$$\begin{aligned} F_{Y_{(1)}}(1/3) &= 1 - F_{Binomial(n=2,F_Y(1/3)=1/3)}(1-1) \\ &= 1 - \binom{2}{0} \times (1/3)^0 \times (1-1/3)^{2-0} \\ &= \frac{5}{9} \\ F_{Y_{(2)}}(1/3) &= 1 - F_{Binomial(n=2,F_Y(1/3)=1/3)}(2-1) \\ &= 1 - \binom{2}{0} \times (1/3)^0 \times (1-1/3)^{2-0} - \binom{2}{1} \times (1/3)^1 \times (1-1/3)^{2-1} \\ &= \frac{1}{9} \\ F_{Y_{(1)},Y_{(2)}}(1/3) &= \int_0^{1/3} \int_0^{y_{(2)}} 2!\, dy_{(1)}\, dy_{(2)} \\ &= \int_0^{1/3} 2y_{(2)}\, dy_{(2)} \\ &= y_{(2)}^2 \Big|_0^{1/3} \\ &= \frac{1}{9}. \end{aligned}$$

We finally obatin

$$P[S_1 < 1/3 \text{ or } S_2 < 1/3 \mid N(1) = 2] = \frac{5}{9} + \frac{1}{9} - \frac{1}{9} = \frac{5}{9}.$$

2. *An insurance company feels that a randomly chosen policyholder will make claims according to a conditional Poisson process with rate uniformly distributed over (0,1) and time measured in years.*

(a) *Derive the mean value and variance of the number of claims made by that policyholder in t years.* (1.5)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim Conditional\,PP(\text{Uniform}(0,1))$
  $N(t) =$ number of claims by time t
- **Random arrival rate**
  $\Lambda \sim \text{Uniform}(0,1)$
  $E(\Lambda) \stackrel{form.}{=} \frac{0+1}{2} = \frac{1}{2}$
  $V(\Lambda) \stackrel{form.}{=} \frac{(1-0)^2}{12} = \frac{1}{12}$
- **Distribution of N(t) conditional to $\Lambda = \lambda$, etc.**
  $(N(t) \mid \Lambda = \lambda) \sim \text{Poisson}(\lambda t)$
  $E[N(t) \mid \Lambda = \lambda] = \lambda t$
  $V[N(t) \mid \Lambda = \lambda] = \lambda t$
- **Requested expected value**

$$\begin{aligned} E[N(t)] &= E\{E[N(t) \mid \Lambda]\} \\ &= E(\Lambda t) \\ &= \frac{1}{2} \times t \end{aligned}$$

- **Requested variance**

$$\begin{aligned} V[N(t)] &= V\{E[N(t) \mid \Lambda]\} + E\{V[N(t) \mid \Lambda]\} \\ &= V(\Lambda t) + E(\Lambda t) \\ &= \frac{1}{12} \times t^2 + \frac{1}{2} \times t \\ &= \frac{t(t+6)}{12}. \end{aligned}$$

(b) *Compute the probability that the policyholder makes exactly one claim in one year.* (2.5)

- **Random arrival rate**
  $\Lambda \sim \text{Uniform}(0,1)$
  $G(\lambda) = F_\Lambda(\lambda) = \lambda,\ 0 < \lambda < 1$
  $g(\lambda) = f_\Lambda(\lambda) = 1,\ 0 < \lambda < 1$
- **F.p. of $N(t)$**
  $P[N(t+s) - N(s) = n] \stackrel{form.}{=} \int_0^{+\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, dG(\lambda)$

- **Requested probability**
  Considering $t = 1$, $s = 0$ and $n = 1$ in the previous formula and using integration by parts,[1] we get

$$\begin{aligned} P[N(1) = 1] &= \int_0^{+\infty} e^{-\lambda}\lambda\, dG(\lambda) \\ &= \int_0^1 \lambda e^{-\lambda}\, d\lambda \\ &= -\lambda e^{-\lambda}\Big|_0^1 + \int_0^1 e^{-\lambda}\, d\lambda \\ &= -e^{-1} - e^{-\lambda}\Big|_0^1 \\ &= 1 - 2e^{-1} \\ &\simeq 0.2642. \end{aligned}$$

3. *Consider a maternity ward in a hospital. A delivery may result in one, two or three births with probabilities 0.9, 0.08, and 0.02, respectively.*

Admit the number of deliveries forms a Poisson process with rate 10 deliveries per day.

(a) *Obtain the probability of at least one twin being born on a given day.* (1.0)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim PP(\lambda = 10 \text{ deliveries per day})$
  $N(t) =$ number of deliveries by time t
  $N(t) \sim \text{Poisson}(\lambda t)$
- **Split process**
  $\{N_{twins}(t) : t \geq 0\} \sim PP(\lambda p)$
  $N_{twins}(t) =$ number of deliveries leading to twins by time t
  $p = P(\text{delivery leading to twins}) = 0.08$
  $N_{twins}(t) \sim \text{Poisson}(10 \times 0.08 \times t = 0.8t)$
- **Requested probability**

$$\begin{aligned} P[N_{twins}(t) \geq 1] &= 1 - P[N_{twins}(t) = 0] \\ &\stackrel{t=1}{=} 1 - e^{-0.8}\frac{0.8^0}{0!} \\ &= 1 - e^{-0.8} \\ &\simeq 0.550671. \end{aligned}$$

(b) *Calculate an approximate value to the probability that there will be more than 700 births in a 8 week period.* (1.5)

- **Relevant stochastic process**
  $\left\{X(t) = \sum_{i=1}^{N(t)} Y_i : t \geq 0\right\} \sim \text{Compound } PP(\lambda, Y)$
  $X(t) =$ total number of births by time t

[1] In case you forgot: $\begin{cases} u = \lambda \\ v' = e^{-\lambda} \end{cases}$ $\begin{cases} u' = 1 \\ v = -e^{-\lambda} \end{cases}$ $\int uv' = uv - \int u'v$

- **R.v. et al.**
$Y_i =$ number of births/babies in the i^{th} delivery
$Y_i \overset{i.i.d.}{\sim} Y$
$P(Y=y) = \begin{cases} 0.9, & y=1 \\ 0.08, & y=2 \\ 0.02, & y=3 \end{cases}$
$\{Y_i : i \in \mathbb{N}\}$ indep. of $\{N(t) : t \geq 0\} \sim PP(\lambda = 10)$

- **Requested probability** (approximate value)
According to the formulae, $E[X(t)] = \lambda t \times E(Y)$ and $V[X(t)] = \lambda t E(Y^2)$. Hence, for $t = 8 \times 7 = 56$ days,

$$\begin{aligned} E[X(56)] &= 10 \times 56 \times (1 \times 0.9 + 2 \times 0.08 + 3 \times 0.02) \\ &= 560 \times 1.12 \\ &= 627.2 \\ V[X(56)] &= 10 \times 56 \times (1^2 \times 0.9 + 2^2 \times 0.08 + 3^2 \times 0.02) \\ &= 560 \times \times 1.4 \\ &= 784 \end{aligned}$$

Thus,

$$\begin{aligned} P[X(56) > 700] &\simeq 1 - \Phi\left(\frac{700 - E[X(56)]}{\sqrt{V[X(56)]}}\right) \\ &= 1 - \Phi\left(\frac{700 - 627.2}{\sqrt{784}}\right) \\ &= 1 - \Phi\left(\frac{72.8}{28}\right) \\ &= 1 - \Phi(2.6) \\ &\overset{table}{=} 1 - 0.9953 \\ &= 0.0047. \end{aligned}$$

Group 3 — Renewal Processes **8.0 points**

1. *Planes land at Heathrow airport at the times of a renewal process with inter-renewal distribution* $\chi^2_{(4)}$.

(a) *Compute and interpret* $\lim_{t\to+\infty} \frac{m(t)}{t}$. (1.0)

- **Renewal process**
$\{N(t) : t \geq 0\}$
$N(t) =$ number of airplanes that landed by time t

- **Inter-renewal times**
$X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$
$X \sim \chi^2_{(4)}$
$\mu = E(X) \overset{form.}{=} 4$

- **Requested limit**
According to the elementary renewal theorem (ERT) (see formulae!),

$$\begin{aligned} \lim_{t\to+\infty} \frac{m(t)}{t} &= \frac{1}{\mu} \\ &= \frac{1}{4}. \end{aligned}$$

- **Interpretation**
In the long-run expect that one airplane lands every 4 time units.

(b) *Derive the renewal function* $m(t)$ *of this renewal process, by using the Laplace-Stieltjes transform method and capitalizing on the table of important Laplace transforms in the formulae.* (2.5)

- **Deriving the renewal function**
Since the inter-renewal times are continuous r.v. the LST of the inter-renewal distribution is given by

$$\begin{aligned} \tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &= E(e^{-sX}) \\ &= M_X(-s) \\ &\overset{form.}{=} \left(\frac{1/2}{1/2 + s}\right)^{4/2} \\ &\overset{form.}{=} \frac{1}{(2s+1)^2}. \end{aligned}$$

Moreover, the LST of the renewal function can be obtained in terms of the one of F:

$$\begin{aligned} \tilde{m}(s) &\overset{form.}{=} \frac{\tilde{F}(s)}{1 - \tilde{F}(s)} \\ &= \frac{1}{(2s+1)^2} \times \frac{1}{1 - \frac{1}{(2s+1)^2}} \\ &= \frac{1}{4} \times \frac{1}{s(s+1)}. \end{aligned}$$

Taking advantage of the LT in the formulae, we successively get:

$$\begin{aligned} \frac{dm(t)}{dt} &= LT^{-1}[\tilde{m}(s), t] \\ &= LT^{-1}\left[\frac{1}{4} \times \frac{1}{s(s+1)}, t\right] \\ &= \frac{1}{4} \times LT^{-1}\left[\frac{1}{s(s+1)}, t\right] \\ &= \frac{1}{4} \times \frac{e^{-0\times t} - e^{-1\times t}}{1 - 0} \\ &= \frac{1 - e^{-t}}{4} \end{aligned}$$

$$\begin{aligned} m(t) &= \int_0^t \frac{1-e^{-x}}{4}\,dx \\ &= \frac{t}{4} + \frac{e^{-t}}{4} - \frac{1}{4},\ t \geq 0. \end{aligned}$$

2. *Admit that at time 0 we started to install a component of a mechanical system. The duration Z of this component is a r.v. with c.d.f. G. When the component breaks down it is replaced by a new/similar one and this replacement takes a fixed time equal to λ.*

   Consider the stochastic process $\{N(t) : t \geq 0\}$, where $N(t)$ represents the number of completed replacements by time t.

   (a) *Derive a renewal-type equation for $E[Y(t)]$, the expected value of the residual life at time t of the stochastic process. (Do not try to solve it!)* (3.0)

   - **Renewal process**
     $\{N(t) : t \geq 0\}$
     $N(t) =$ number of completed replacements by time t
   - **Inter-renewal times**
     $X_i \overset{i.i.d.}{\sim} X \overset{st}{=} Z + \lambda$, $i \in \mathbb{N}_0$, where
     $Z =$ duration of a component
     $\lambda =$ time spent replacing a component
   - **Important r.v.**
     $Y(t) =$ residual life at time t
   - **Renewal-type equation**
     Applying the renewal argument, that is, conditioning on the time of the first renewal, $X_1 = x$ (which coincides with the time the first component broke down), we have
     – for $0 < x \leq t$,
     $$E[Y(t) \mid X_1 = x] = E[Y(t-x)]$$
     – for $x > t$,
     $$E[Y(t) \mid X_1 = x] = x - t.$$
     Consequently,
     $$\begin{aligned} E[Y(t)] &= \int_0^{+\infty} E[Y(t) \mid X_1 = x]\,dF(x) \\ &= \int_0^t E[Y(t-x)]\,dF(x) + \int_t^{+\infty} (x-t)\,dF(x), \end{aligned}$$
     where $F(x) = P(X \leq x) = P(X_1 \leq x)$.

   (b) *Determine the limiting value of $E[Y(t)]$ when $Z \sim$ Exponential(ξ^{-1}).* (1.5)

   - **R.v.**
     $Z \sim$ Exponential(ξ^{-1})

   - **Requested limit**
     $$\begin{aligned} \lim_{t\to+\infty} E[Y(t)] &\overset{form}{=} \frac{E(X^2)}{2E(X)} \\ &= \frac{E[(Z+\lambda)^2]}{2E(Z+\lambda)} \\ &= \frac{E(Z^2) + 2\lambda E(Z) + \lambda^2}{2[E(Z)+\lambda]} \\ &= \frac{V(Z) + E^2(Z) + 2\lambda E(Z) + \lambda^2}{2[E(Z)+\lambda]} \\ &= \frac{\xi^2 + \xi^2 + 2\lambda\xi + \lambda^2}{2(\xi+\lambda)} \\ &= \xi + \frac{\lambda^2}{2(\xi+\lambda)}. \end{aligned}$$