Department of Mathematics, IST — Probability and Statistics Unit

# Introduction to Stochastic Processes

**2nd. Test** 2nd. Semester — **2012/13**

Duration: **1h30m** **2013/06/11** — **8AM**, Room P8

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

## Group 1 — Renewal Processes **4.0 points**

*The duration $Z$ (in years) of a component is a non negative r.v. with c.d.f. $G(z) = 1 - e^{-z}$, for $z \geq 0$. However, Clotilde replaces the component by a new one as soon as the old one either breaks down or reaches the age of $A = 1$ year. Suppose that the time spent replacing a component is fixed and equal to $\lambda = 1$ month.*

(a) *Obtain the expected time between two consecutive replacements (of components).* (2.5)

- **R.v.**
  $Z$ = duration (in years) of a component
  $G(z) = P(Z \leq z) = 1 - e^{-z}$, $z \geq 0$
  $g(z) = \frac{dG(z)}{dz} = e^{-z}$, $z \geq 0$, i.e., $Z \sim$ Exponential(1)
- **Up time**
  $U$ = time a component is used (system is UP)
  $U = \min\{Z, A\} = \begin{cases} Z, & Z < A \\ A, & Z \geq A \end{cases}$
- **Down time**
  $D$ = time spent replacing a component = $\lambda$ (system is DOWN)
- **Duration of the up-down cycle**
  $X = U + D$
- **Expected duration of the up-down cycle**
  By integration by parts,[1]

$$\begin{aligned} E(U) &= \int_0^A z \times g(z)\, dz + \int_A^{+\infty} A \times g(z)\, dz \\ &= \int_0^A z \times e^{-z}\, dz + A \times P(Z > A) \\ &= -z \times e^{-z}\Big|_0^A + \int_0^A e^{-z}\, dz + A \times [1 - G(A)] \\ &= -A \times e^{-A} - e^{-z}\Big|_0^A + A \times e^{-A} \\ &= 1 - e^{-A} \\ &= 1 - e^{-1}. \end{aligned}$$

[1]In case you forgot: $\begin{cases} u = z \\ v' = e^{-z} \end{cases}$ $\begin{cases} u' = 1 \\ v = -e^{-z} \end{cases}$ $\int uv' = uv - \int u'v$



Moreover,

$$\begin{aligned} E(D) &= \lambda \\ &= \frac{1}{12}. \end{aligned}$$

Consequently,

$$\begin{aligned} E(X) &= E(U) + E(D) \\ &= (1 - e^{-1}) + \frac{1}{12} \\ &\simeq 0.715454. \end{aligned}$$

(b) *Determine the long-run proportion of time that is spent replacing components.* (1.5)

- **State variable**
  $X(t) = \begin{cases} 1, & \text{if the component is being used (system is UP) at time } t \\ 0, & \text{if the component is being replaced (system is DOWN) at time } t \end{cases}$
- **Alternating renewal process**
  $\{X(t) : t \geq 0\}$
- **Long-run proportion of time spent replacing components**
  Let

$$\begin{aligned} Q(t) &= P[X(t) = 0] \\ &= P(\text{system is DOWN at time } t). \end{aligned}$$

Then the long-run proportion of time spent replacing components is given by:

$$\begin{aligned} \lim_{t \to +\infty} Q(t) &\stackrel{[Prop.\, 2.106]}{=} \frac{E(D)}{E(U) + E(D)} \\ &= \frac{\frac{1}{12}}{(1 - e^{-1}) + \frac{1}{12}} \\ &\simeq 0.116476. \end{aligned}$$

## Group 2 — Discrete time Markov chains **9.0 points**

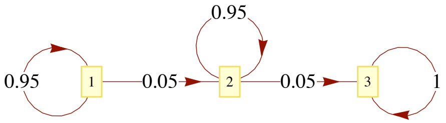
1. *A drilling machine can be in any of 3 different states of alignment, labeled 1 for the best, 2 for the next, down to the worst state 3. From one week to the next, it either stays in its current state with probability 0.95, or moves to the next lower state with probability 0.05. Furthermore, if state 3 is reached the machine is certain to remain in this state indefinitely.*

   *Let $\{X_n : n \in \mathbb{N}_0\}$ be a discrete time Markov chain (DTMC), where $X_0$ denotes the initial state and $X_n$ represents the state of alignment of the machine at the end of week $n$.*

   (a) *Draw the associated transition diagram and determine the transition probability matrix (TPM).* (0.5)

   - **DTMC**
     $\{X_n : n \in \mathbb{N}_0\}$
     $X_0$ = initial state of alignment
     $X_n$ = state of alignment at the end of week $n$

- **State space**
  $\mathcal{S} = \{1, 2, 3\}$
  $1 =$ best alignment
  $2 =$ average alignment
  $2 =$ worst alignment
- **Transition diagram**
  According to the description in the test, we are dealing with the following transition diagram:




- **TPM**
  Follows from the transition diagram above:

$$\begin{aligned}
\mathbf{P} &= [P_{ij}]_{i,j\in\mathcal{S}} \\
&= [P(X_{n+1} = j \mid X_n = i)]_{i,j\in\mathcal{S}},\ n \in \mathbb{N}_0 \\
&= [P(X_1 = j \mid X_0 = i)]_{i,j\in\mathcal{S}} \\
&= \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.95 & 0.05 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

(b) *Admit the initial distribution is* $\underline{\alpha} = [0.5 \quad 0.5 \quad 0]$ *and obtain not only* $P(X_1 = 1)$ *but also* $P(X_3 = 2, X_1 = 1)$. (1.5)

- **1st. requested probability**
  Since

$$\begin{aligned}
\underline{\alpha} &= [P(X_0 = i)]_{i\in\mathcal{S}} \\
&= [0.5 \quad 0.5 \quad 0] \\
\underline{\alpha}^n &= [P(X_n = i)]_{i\in\mathcal{S}} \\
&\overset{[(3.8)]}{=} \underline{\alpha} \times \mathbf{P}^n,
\end{aligned}$$

  we get

$$\begin{aligned}
\underline{\alpha}^1 &= [P(X_1 = i)]_{i\in\mathcal{S}} \\
&= \underline{\alpha} \times \mathbf{P} \\
&= [0.5 \quad 0.5 \quad 0] \times \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.95 & 0.05 \\ 0 & 0 & 1 \end{bmatrix} \\
&= [0.475 \quad 0.5 \quad 0.025]
\end{aligned}$$

  and conclude that $P(X_1 = 1) = 0.475$.
- **1st. requested probability (alternative solution)**
  $P(X_1 = 1) = \sum_{i\in\mathcal{S}} P(X_0 = i) \times P(X_1 = 1 \mid X_0 = i) = \underline{\alpha} \times$ 1st. column of $\mathbf{P}$ $= [0.5 \quad 0.5 \quad 0] \times [0.95 \quad 0 \quad 0]^\top = 0.475$



- **2nd. requested probability**

$$\begin{aligned}
P(X_3 = 2, X_1 = 1) &= P(X_1 = 1) \times P(X_3 = 2 \mid X_1 = 1) \\
&= P(X_1 = 1) \times P(X_2 = 2 \mid X_0 = 1) \\
&= P(X_1 = 1) \times P^2_{12} \\
&= P(X_1 = 1) \times \text{1st. row of } \mathbf{P} \times \text{2nd. column of } \mathbf{P} \\
&= 0.475 \times [0.95 \quad 0.05 \quad 0] \times \begin{bmatrix} 0.05 \\ 0.95 \\ 0 \end{bmatrix} \\
&= 0.475 \times 0.095 \\
&= 0.045125.
\end{aligned}$$

2. *Evaristo owns a restaurant in a region where the daily weather is governed by a four-state DTMC, with states 1 (sunny), 2 (very humid), 3 (cloudy) and 4 (rainy), and TPM:*

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.6 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{bmatrix}$$

   (a) *What is the long-run proportion of days which are cloudy?* (2.0)

   **Note**: *Check the footnote!*[2]

   - **DTMC**
     $\{X_n : n \in \mathbb{N}_0\}$
     $X_0 =$ initial weather
     $X_n =$ weather on day $n$
   - **State space**
     $\mathcal{S} = \{1, 2, 3, 4\}$
     $1 =$ sunny
     $2 =$ very humid
     $3 =$ cloudy
     $4 =$ rainy
   - **TPM**

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.6 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{bmatrix}$$

[2] *The following results may come handy in this and the next lines:*

$$\begin{bmatrix} 1.6 & 0.8 & 0.9 & 0.7 \\ 0.6 & 1.7 & 0.8 & 0.9 \\ 0.4 & 0.9 & 1.9 & 0.8 \\ 0.8 & 0.6 & 0.7 & 1.9 \end{bmatrix}^{-1} \simeq \begin{bmatrix} 0.815 & -0.213 & -0.264 & -0.088 \\ -0.135 & 0.850 & -0.194 & -0.271 \\ 0.022 & -0.334 & 0.712 & -0.149 \\ -0.309 & -0.056 & -0.090 & 0.704 \end{bmatrix} \text{ and } \begin{bmatrix} 0.6 & -0.2 & -0.1 \\ -0.4 & 0.7 & -0.2 \\ -0.6 & -0.1 & 0.9 \end{bmatrix}^{-1} \simeq \begin{bmatrix} 2.723 & 0.848 & 0.491 \\ 2.143 & 2.143 & 0.714 \\ 2.054 & 0.804 & 1.518 \end{bmatrix}$$

- **Obs.**
We are dealing with an irreducible DTMC with finite state space. Hence, all states are positive recurrent[, by Prop. 3.55]. Furthermore, the DTMC looks aperiodic.

- **Stationary distribution**
Since the DTMC is irreducible positive recurrent and aperiodic we can add that

$$\lim_{n\to+\infty} P_{ij}^n = \pi_j > 0,\ i,j \in \mathcal{S},$$

where $\{\pi_j : j \in \mathcal{S}\}$ is the unique stationary distribution and satisfies the following system of equations:

$$\begin{cases} \pi_j = \sum_{i\in\mathcal{S}} \pi_i P_{ij},\ j \in \mathcal{S} \\ \sum_{j\in\mathcal{S}} \pi_j = 1. \end{cases}$$

Equivalently [(see Prop. 3.68)], the row vector denoting the stationary distribution, $\underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$, is given by

$$\underline{\pi} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1},$$

where:
$\underline{1} = [1 \quad \cdots \quad 1]$ a row vector with $\#\mathcal{S}$ ones;
$\mathbf{I}$ = identity matrix with rank $\#\mathcal{S}$;
$\mathbf{P} = [P_{ij}]_{i,j\in\mathcal{S}}$ is the TPM;
**ONE** is the $\#\mathcal{S} \times \#\mathcal{S}$ matrix all of whose entries are equal to 1.
By capitalizing on the first inverse in the footnote, we obtain

$$\begin{aligned}
\underline{\pi} &= \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1} \\
&= \underline{1} \times \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.6 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right)^{-1} \\
&= \underline{1} \times \begin{bmatrix} 1.6 & 0.8 & 0.9 & 0.7 \\ 0.6 & 1.7 & 0.8 & 0.9 \\ 0.4 & 0.9 & 1.9 & 0.8 \\ 0.8 & 0.6 & 0.7 & 1.9 \end{bmatrix}^{-1} \\
&\simeq [1 \quad 1 \quad 1 \quad 1] \times \begin{bmatrix} 0.815 & -0.213 & -0.264 & -0.088 \\ -0.135 & 0.850 & -0.194 & -0.271 \\ 0.022 & -0.334 & 0.712 & -0.149 \\ -0.309 & -0.056 & -0.090 & 0.704 \end{bmatrix} \\
&= [0.393 \quad 0.247 \quad 0.164 \quad 0.196].
\end{aligned}$$

(With the *previous and incorrect data* the result is slightly different [0.400 0.249 0.166 0.185].)
Thus, the long-run proportion of days which are cloudy is equal to [the sum of the entries of the 3rd. column of $(\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1}$]:

$$\pi_3 \simeq 0.164.$$



(b) *The daily profit of Evaristo's restaurant is weather dependent:* $c(1) = 200$, $c(2) = 150$, $c(3) = 100$, $c(4) = 50$. (1.0)
*What is Evaristo's long-run profit per day?*

- **Vector of profits/rewards**

$$\underline{c} = [c(j)]_{j\in\mathcal{S}} = \begin{bmatrix} 200 \\ 150 \\ 100 \\ 50 \end{bmatrix}$$

- **Long-run expected profit per time unit**
[According to Prop. 3.81,]

$$\begin{aligned}
\underline{\pi} \times \underline{c} &= \sum_{j\in\mathcal{S}} \pi_j \times c(j) \\
&\simeq 0.393 \times 200 + 0.247 \times 150 + 0.164 \times 100 + 0.196 \times 50 \\
&= 141.85.
\end{aligned}$$

(With the *previous and incorrect data* the result is slightly different: $0.400 \times 200 + 0.249 \times 150 + 0.166 \times 100 + 0.185 \times 50 = 143.2$ instead of 141.85.)

(c) *Determine the expected number of days until it rains, given that the weather is now cloudy.* (2.0)
**Note**: *Check the footnote!*

- **Initial/present state**
$X_0 = i$

- **Important**
To obtain the expected number of days until it rains, given $X_0 = i$, we have to consider another CTMC where state 4 (rainy) is absorbing. The associated TPM is

$$\mathbf{P}' = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.6 & 0.1 & 0.1 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- **Requested expected value**
Let

$$\mathbf{Q} = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.4 & 0.3 & 0.2 \\ 0.6 & 0.1 & 0.1 \end{bmatrix}$$

be the substochastic matrix governing the transitions between the states in $T = \{1, 2, 3\}$, the class of transient states of this new DTMC, and

$$\tau = \inf\{n \in \mathbb{N}_0 : X_n \notin T\}$$

be the number of transitions/days until it rains. Then [(see Prop. 3.116)] the 2nd. inverse in the footnote yields

$$\begin{aligned}
[E(\tau \mid X_0 = i)]_{i=1,2,3} &= (\mathbf{I}-\mathbf{Q})^{-1} \times \underline{1} \\
&= \ldots \\
&= \begin{bmatrix} 0.6 & -0.2 & -0.1 \\ -0.4 & 0.7 & -0.2 \\ -0.6 & -0.1 & 0.9 \end{bmatrix}^{-1} \times \underline{1} \\
&\simeq \begin{bmatrix} 2.723 & 0.848 & 0.491 \\ 2.143 & 2.143 & 0.714 \\ 2.054 & 0.804 & 1.518 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 4.062 \\ 5.000 \\ 4.376 \end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
E(\text{days until it rains} \mid \text{the weather is now cloudy}) &= E(\tau \mid X_0 = 3) \\
&\simeq 4.376.
\end{aligned}$$

(d) *Compute the probability that a sunny day will occur before it rains, given that the weather today is very humid.* (2.0)

**Note:** *You may have to consider states 1 and 4 absorbing, eventually relabel the states, identify substochastic matrices* $\mathbf{Q}$ *and* $\mathbf{R}$ *and calculate* $(\mathbf{I}-\mathbf{Q})^{-1} \times \mathbf{R}$.

- **Important**

  To calculate the requested probability, we have to consider once again another CTMC. In this case states 1 (sunny) and 4 (rainy) are absorbing and the resulting TPM equals

$$\mathbf{P}^\star = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.6 & 0.1 & 0.1 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

  The substochastic matrix governing the transitions between the states in $T = \{2,3\}$, the class of transient states of this new DTMC is

$$\mathbf{Q} = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}.$$

  As for the transitions from the transient to the absorbing states, they are governed by the substochastic matrix

$$\mathbf{R} = \begin{bmatrix} 0.4 & 0.1 \\ 0.6 & 0.2 \end{bmatrix}.$$



- **Requested probability**

  Keeping in mind that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

  we get

$$\begin{aligned}
\mathbf{U} &= [P(\text{reach absorbing state } k \mid X_0 = i)]_{i \in T,\, k \notin T} \\
&= [u_{ik}]_{i \in T,\, k \notin T} \\
&= (\mathbf{I}-\mathbf{Q})^{-1} \times \mathbf{R} \\
&= \begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.9 \end{bmatrix}^{-1} \times \begin{bmatrix} 0.4 & 0.1 \\ 0.6 & 0.2 \end{bmatrix} \\
&= \frac{1}{0.7 \times 0.9 - (-0.2) \times (-0.1)} \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.1 \\ 0.6 & 0.2 \end{bmatrix} \\
&= \frac{1}{0.61} \begin{bmatrix} 0.48 & 0.13 \\ 0.46 & 0.15 \end{bmatrix} \\
&\simeq \begin{bmatrix} 0.787 & 0.213 \\ 0.754 & 0.246 \end{bmatrix}.
\end{aligned}$$

  Thus, the probability that a sunny day (state 1) will occur before it rains (state 4), given that the weather today is very humid (state 2) is equal to

$$\begin{aligned}
u_{21} &= P(X_\tau = 1 \mid X_0 = 2) \\
&\simeq 0.787.
\end{aligned}$$

## Group 3 — Continuous time Markov chains **7.0 points**

1. *Admit that the number of customers in a drive-in banking system at time* $t$, $X(t)$, *is governed by a birth and death process* $\{X(t) : t \geq 0\}$ *with rates equal to:* $\lambda_j = \lambda$, *for* $j \in \mathbb{N}_0$; *and* $\mu_j = \min\{j, 2\} \times \mu$, *for* $j \in \mathbb{N}$.

   (a) *Write the Kolmogorov's forward differential equations in terms of* $P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$, *for* $j \in \mathbb{N}_0$. *(Do not try to solve them!)* (1.5)

   - **Birth and death process**

     $\{X(t) : t \geq 0\}$

     $X(t) =$ number of customers in the drive-in banking system at time $t$

   - **Birth and death rates**

     $\lambda_j = \lambda$, $j \in \mathbb{N}_0$

     $$\mu_j = \begin{cases} \mu, & j = 1 \\ 2\mu, & j = 2, 3, \ldots \end{cases}$$

   - **Kolmogorov's forward differential equations**

     Note that

$$\begin{aligned} P_j(t) &\equiv P_{0j}(t) = P[X(t)=j \mid X(0)=0],\ j \in \mathbb{N}_0 \\ P_{-1}(t) &= \lambda_{-1} = \mu_0 = 0 \end{aligned}$$

therefore the Kolmogorov's forward differential equations

$$\frac{d\,P_j(t)}{dt} = P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j+\mu_j),\ j \in \mathbb{N}_0$$

reads as follows:

$$\begin{aligned} \frac{d\,P_0(t)}{dt} &= P_1(t)\,\mu - P_0(t)\,\lambda; \\ \frac{d\,P_1(t)}{dt} &= P_0(t)\,\lambda + P_2(t)\,2\mu - P_1(t)(\lambda+\mu); \\ \frac{d\,P_j(t)}{dt} &= P_{j-1}(t)\,\lambda + P_{j+1}(t)\,2\mu - P_j(t)\,(\lambda+2\mu),\ j=2,3,\ldots \end{aligned}$$

(b) *After having admitted that* $\rho = \frac{\lambda}{2\mu} < 1$, *prove that the equilibrium probabilities* $P_j = \lim_{t\to+\infty} P_j(t)$ *are given by:* $P_0 = \frac{1-\rho}{1+\rho}$; *and* $P_j = 2\,\frac{1-\rho}{1+\rho}\,\rho^j$, *for* $j \in \mathbb{N}$. (2.5)

- **Ergodicity condition**
  $\rho = \frac{\lambda}{2\mu} < 1$
- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P_j(t)$

$$\begin{aligned} P_0 &= \left(1 + \sum_{n=1}^{+\infty} \frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{n-1}}{\mu_1\,\mu_2\,\ldots\,\mu_n}\right)^{-1} \\ &= \left[1 + \frac{\lambda}{\mu}\sum_{n=1}^{+\infty}\left(\frac{\lambda}{2\mu}\right)^{n-1}\right]^{-1} \\ &= \left[1 + 2\rho\sum_{n=1}^{+\infty}\rho^{n-1}\right]^{-1} \\ &= \left(1+\frac{2\rho}{1-\rho}\right)^{-1} \\ &= \frac{1-\rho}{1+\rho} \\ P_j &= P_0 \times \frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{j-1}}{\mu_1\,\mu_2\,\ldots\,\mu_j} \\ &= P_0 \times \frac{\lambda}{\mu}\left(\frac{\lambda}{2\mu}\right)^{j-1} \\ &= \frac{1-\rho}{1+\rho} \times 2\rho\,\rho^{j-1} \\ &= 2\,\frac{1-\rho}{1+\rho}\,\rho^j,\ j \in \mathbb{N}. \end{aligned}$$

2. *A corporate computing center has two computers of the same capacity. The jobs arriving at the center are of two types, internal and external jobs. These jobs arrive according to two independent Poisson processes with rates 18 internal jobs per hour and 15 external jobs per hour. The service times are i.i.d. r.v. exponentially distributed with mean 3 minutes.*



(a) *Find the average waiting time per arriving job, when the two computers handle both types of jobs.* (2.0)

- **Birth and death queueing system**
  $M/M/m$
- **Arrival process/rate**
  We ought to note that mergi two independent PP with rates $\lambda_{int}$ and $\lambda_{ext}$ leads to a PP having rate

$$\begin{aligned} \lambda &= \lambda_{int} + \lambda_{ext} \\ &= (18+15) \\ &= 33 \text{ jobs per hour} \end{aligned}$$

- **Service times/rate**
  $S_i \overset{i.i.d.}{\sim} \text{Exponential}(\mu^{-1}=3)$
  $\mu = 1$ job per 3 minutes $\equiv 20$ jobs per hour
- **Servers**
  $m=2$ because the two computers handle both types of jobs
- **Traffic intensity/ergodicity condition**

$$\begin{aligned} \rho &= \frac{\lambda}{m\,\mu} \\ &= \frac{33}{2\times 20} \\ &= 0.825 \\ &< 1 \end{aligned}$$

- **Performance measure (in the long-run)**
  $W_q$ = time (in hours) an arriving job waits until it starts being "served"
- **Requested expected value**

$$\begin{aligned} E(W_q) &\overset{form}{=} \frac{C(m,m\rho)}{m\,\mu(1-\rho)} \\ &= \frac{C(2,2\rho)}{2\times\mu(1-\rho)} \\ &\overset{form}{=} \frac{\frac{2\rho^2}{1+\rho}}{2\times\mu(1-\rho)} \\ &= \frac{\rho^2}{\mu(1-\rho^2)} \\ &= \frac{0.825^2}{20\times(1-0.825^2)} \\ &\simeq 0.106556 \text{ hours} \\ &\simeq 6.393346 \text{ minutes} \end{aligned}$$

(b) *Obtain the average waiting time per arriving internal job, when one computer is used exclusively for internal jobs and the other for external jobs. Comment the result in light of (a).* (1.0)

- **Another birth and death queueing system (internal jobs)**
  $M_{int}/M/m_{int}$
- **Arrival rate**
  $\lambda_{int} = 18$ INTERNAL jobs per hour
- **Service rate**
  $\mu_{int} = \mu = 20$ INTERNAL jobs per hour
- **Servers**
  $m_{int} = 1$ because only one computer is handling INTERNAL jobs
- **Traffic intensity/ergodicity condition**

$$\begin{aligned}\rho_{int} &= \frac{\lambda_{int}}{m_{int}\,\mu_{int}} \\ &= \frac{18}{1 \times 20} \\ &= 0.9 \\ &< 1\end{aligned}$$

- **Performance measure (in the long-run)**
  $W_q^{int}$ = time (in hours) an arriving INTERNAL job waits until it starts being "served"
- **Requested expected value**

$$\begin{aligned}E(W_q^{int}) &\stackrel{form}{=} \frac{\rho_{int}}{\mu_{int}(1-\rho_{int})} \\ &= \frac{0.9}{20 \times (1-0.9)} \\ &\simeq 0.45 \text{ hours} \\ &\simeq 27 \text{ minutes}\end{aligned}$$

- **Comment**
  Unsurprisingly, $E(W_q^{int})$ is much larger than $E(W_q)$, i.e., it is more efficient to have both computers handling both types of jobs.