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Hierarchical optimization of fiber reinforced composites for natural frequencies

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ABSTRACT: This paper has the aim of designing laminated composite structures to maximize vibration frequencies using concepts of hierarchical topology optimization, a structural optimization branch whose purpose is to simultaneously design the material distribution layout of a structure in two distinct levels: One macro-mechanical (or structural, global) and the other micro-mechanical (or material, local). In the macro-level, design in terms of optimal material distribution on the general layout of the structure is taken into account. In the micro-level, the constitutive properties optimal design is searched, in terms of defining the distribution of material phases in unit cells of microstructure. This general approach is here extended to the laminated composites of interest: At the macro-level, optimal orientations and fiber volume fractions are defined for unidirectional composite material layers and, at the micro-level, it is designed the shape of the reinforcement fibers. The objective is to maximize the first and second natural vibration frequencies of laminated plates subjected to constraints on the limit of total fiber volume fraction employed. In this problem multiple eigenvalues are possible, leading to non-differentiability.

1 INTRODUCTION

The concepts of hierarchical optimization (Rodrigues, Guedes, & Bendsøe 2002) were recently applied (Ferreira, Rodrigues, Guedes, & Hernandes 2014) to compliance minimization (maximization of stiffness) of laminated fiber reinforced composites, considering a two-level optimization problem to design structure and its material. In the formulation employed, the structure level (also called macro-mechanical level) is the first problem level, where the goal is to find optimal unidirectional fiber orientations and fiber volume fractions over the composites' layers. The second is the material level (or micro-mechanical level), where the aim is to determine optimal cross-sectional shapes of reinforcement fibers parameterized as elliptical and with variable dimensions. The results obtained showed that the stiffness of the composite laminates can be increased by varying the material microstructure.

Here, the same hierarchical optimization concepts are used to investigate problems of maximization of natural vibration frequencies of laminated plates. As in the first work dealing with compliance minimization, the variation of the material stiffness with the microstructure is taken into account by asymptotic homogenization techniques (Guedes & Kikuchi 1990) and response surface methods (Lauridsen et al. 2001). The layers orientations of the laminated composites are chosen using discrete material optimization techniques (Stegmann & Lund 2005). Moreover, mass density variation of the material being designed is now necessary and trivially derived analytically.

2 PROBLEM FORMULATION

The following equation shows a standard finite element eigenvalue problem for natural vibration frequencies of a structure:

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\Phi = 0. \tag{1}$$

In the Problem (1) above, **K** and **M** are respectively stiffness and mass global matrices, both real, symmetric and of order *n*, the number of degrees of freedom of the discrete system, ω^2 is an eigenvalue and Φ its corresponding eigenvector. It is assumed that this problem renders the following set of eigenvalues:

$$\omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_j^2 \leq \ldots \leq \omega_n^2. \tag{2}$$

It also renders the corresponding set of eigenvectors:

$$\Phi_1, \Phi_2, \ldots, \Phi_j, \ldots, \Phi_n. \tag{3}$$

In this work, the optimization problem of interest is as follows:

$$\min_{\substack{\mathbf{x},\boldsymbol{\rho} \\ \sum_{i=1}^{\ell}\rho_i V_i \le V_f}} \; \min_{\substack{\mathbf{a},\mathbf{b} \\ \rho_1 = \pi a_1 b_1 \\ \vdots \\ \rho_\ell = \pi a_\ell b_\ell}} - \left(\min_{\substack{\Phi \in \mathbb{R}^n \\ \Phi \neq 0 \\ \Phi_1^T \mathbf{K}\Phi = 0 \\ \vdots \\ \Phi_{j-1}^T \mathbf{K}\Phi = 0}} \bar{\omega}^2(\Phi) \right). \tag{4}$$

In the problem above:

$$\bar{\omega}^2(\Phi) = \frac{\Phi^T \mathbf{K} \Phi}{\Phi^T \mathbf{M} \Phi}, \quad \mathbf{K}(\mathbf{x}, \mathbf{a}, \mathbf{b}), \quad \mathbf{M}(\mathbf{a}, \mathbf{b}). \tag{5}$$

Therefore, in Problem (4), it is minimized the negative of the j-*th* least vibration eigenvalue of the structure, represented by its Rayleigh's quotient $\bar{\omega}^2(\Phi)$ (Bathe 1996) in Eq. (5). The choice of this eigenvalue is assured by the equality constraints in the most inner minimization where the property $\Phi_p^T \mathbf{K} \Phi_q = 0$ if $p \neq q$ of Problem (1) is used. This minimization leads to the maximization of the target eigenvalue.

The problem design variables are in the vectors $\mathbf{x}$ and $\boldsymbol{\rho}$ at the macro-level and $\mathbf{a}$ and $\mathbf{b}$ at the micro-level. The $\mathbf{x}$ is devoted to material orientation selection by the discrete material optimization method (Stegmann & Lund 2005). The $\boldsymbol{\rho}$ contains the individual layers' fiber volume fractions and the $\mathbf{a}, \mathbf{b}$ define the semi-axis of the elliptical cross sections of the layers' reinforcement fibers, part of the material unit cell considered for the layers. The components of these vectors are respectively treated by ρ_i, a_i and b_i, referred to an i-*th* layer of material whose total number of layers is ℓ. Ferreira (2013) shows details on the definition of these design variables.

The equality constraints $\rho_i = \pi a_i b_i$, in number ℓ and imposed in the micro-level, establish the dependency between the $\mathbf{a}, \mathbf{b}$ and the $\boldsymbol{\rho}$ variables and guarantee that the layers' fiber volume fractions found in both levels is the same. The inequality constraint $\sum_{i=1}^{\ell} \rho_i V_i \le V_f$, imposed in the macro-level, defines the limit V_f of fiber volume fraction utilization in the whole composite.

In terms of the layer-to-layer variable material properties taken into account, the influence of the design variables in Problem (4) is seen in Eqs. (6) and (7). In the first equation, the stiffness parameters $\bar{\mathbf{Q}}_i$ of an i-*th* layer of a laminate vary with a_i, b_i microstructural parameters and $w_k(\mathbf{x})$ material selection weighting interpolation functions according to the discrete material optimization method. The candidate orientations are $0°/\pm 45°/90°$ and $\bar{\mathbf{Q}}_i^{\theta}$ means material oriented according to the angle θ:

$$\bar{\mathbf{Q}}_i(\mathbf{x}, a_i, b_i) = w_1(\mathbf{x})\bar{\mathbf{Q}}_i^{-45}(a_i, b_i) + w_2(\mathbf{x})\bar{\mathbf{Q}}_i^{0}(a_i, b_i) + \\ + w_3(\mathbf{x})\bar{\mathbf{Q}}_i^{45}(a_i, b_i) + w_4(\mathbf{x})\bar{\mathbf{Q}}_i^{90}(a_i, b_i). \tag{6}$$

In Eq. (7) the mass density d_i of the i-*th* layer varies with the ρ_i fiber volume fraction and mass densities of fiber and matrix, respectively d_f and d_m:

$$d_i = \rho_i d_f + (1 - \rho_i) d_m. \tag{7}$$

The properties in Eqs. (6) and (7) are used to calculate, respectively, $\mathbf{K}$ and $\mathbf{M}$ matrices in Eq. (5).

3 OPTIMALITY NECESSARY CONDITIONS

The Lagrangian function of Problem (4) is written as:

$$\mathcal{L} = -\left(\min_{\substack{\Phi \in \mathbb{R}^n \\ \Phi \neq 0}} \bar{\omega}^2(\Phi) \right) + \lambda_V \left(\sum_{i=1}^{\ell} \rho_i V_i - V_f \right) + \\ + \sum_{i=1}^{\ell} \lambda_i \left(\rho_i - \pi a_i b_i \right) + \sum_{p=1}^{j-1} \lambda_p \left(\Phi_p K \Phi \right). \tag{8}$$

In Eq. (8), λ_V, λ_i and λ_p are Lagrange multipliers: the first positive and the others unrestricted in sign. The stationarity conditions for the Lagrangian function above, with respect to the problem design variables, assume the form:

$$\nabla_{\mathbf{x}} \mathcal{L} = -\nabla_{\mathbf{x}} \left(\min_{\substack{\Phi \in \mathbb{R}^n \\ \Phi \neq 0}} \bar{\omega}^2(\Phi) \right) + \\ + \nabla_{\mathbf{x}} \left[\sum_{p=1}^{j-1} \lambda_p (\Phi_p^T \mathbf{K} \Phi) \right] = \mathbf{0}, \tag{9}$$

$$\nabla_{\boldsymbol{\rho}} \mathcal{L} = \nabla_{\boldsymbol{\rho}} \left[\lambda_V \left(\sum_{i=1}^{\ell} \rho_i V_i \right) + \left(\sum_{i=1}^{\ell} \lambda_i \rho_i \right) \right] = \mathbf{0}, \tag{10}$$

$$\nabla_{\mathbf{a}} \mathcal{L} = -\nabla_{\mathbf{a}} \left(\min_{\substack{\Phi \in \mathbb{R}^n \\ \Phi \neq 0}} \bar{\omega}^2(\Phi) \right) + \\ + \nabla_{\mathbf{a}} \left\{ -\left(\sum_{i=1}^{\ell} \lambda_i \pi b_i \right) + \left[\sum_{p=1}^{j-1} \lambda_p (\Phi_p^T \mathbf{K} \Phi) \right] \right\} = \mathbf{0}, \tag{11}$$

$$\nabla_{\mathbf{b}} \mathcal{L} = -\nabla_{\mathbf{b}} \left(\min_{\substack{\Phi \in \mathbb{R}^n \\ \Phi \neq 0}} \bar{\omega}^2(\Phi) \right) + \\ + \nabla_{\mathbf{b}} \left\{ -\left(\sum_{i=1}^{\ell} \lambda_i \pi a_i \right) + \left[\sum_{p=1}^{j-1} \lambda_p (\Phi_p^T \mathbf{K} \Phi) \right] \right\} = \mathbf{0}, \tag{12}$$

Eqs. (9) to (12) are first order optimality conditions of Problem (4), which hold for stationary points not at the design variables bounds. In such equations, the gradient of a function with respect to a set of design variables $\mathbf{z}$ is indicated by the operator $\nabla_{\mathbf{z}}(\,\cdot\,)$. Moreover, the derivative of an eigenvalue in Problem (4) with respect to a variable $z_i \in \mathbf{z}$ is given by:

$$\frac{\partial \bar{\omega}^2(\Phi)}{\partial z_i} = \Phi^T \left[\frac{\partial \mathbf{K}}{\partial z_i} - \bar{\omega}^2(\Phi) \frac{\partial \mathbf{M}}{\partial z_i} \right] \Phi \tag{13}$$

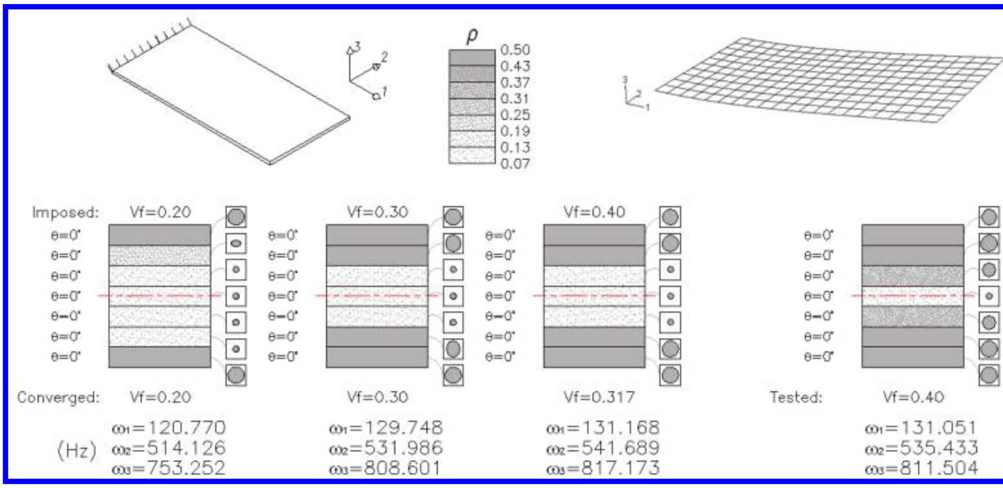

Figure 1. Results for maximizing the first natural frequency (flexural mode) of a 7-layer rectangular laminated plate.

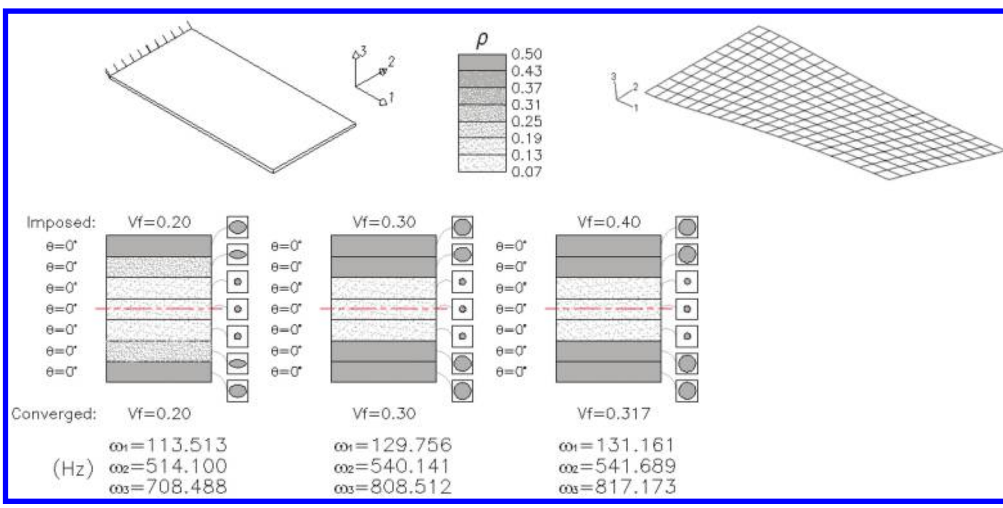

Figure 2. Results for maximizing the second natural frequency (torsional mode) of a 7-layer rectangular laminated plate.

Derivatives like in Eq. (13), where mass normalization was considered to the eigenvectors, are involved in the gradients in Eqs. (9), (11) and (12). However, when the target eigenvalue is multiple (Seyranian, Lund, & Olhoff 1994), non-differentiability happens for the objective function and the terms which constrain the target eigenvalue to the *j-th* one. In this case, the optimality conditions need to take into account the concept of generalized gradients from non-smooth functions (Rodrigues, Guedes, & Bendsøe 1995). In this paper, this multiplicity is only monitored in the results to come.

4 SOLUTION STRATEGY

Problem (4) is here solved using a hierarchical optimization approach, where optimum points are searched iteratively and simultaneously in the macro and micro levels. This approach is directly adapted from the one used for compliance minimization in Ferreira (2013), where full details are given, and is the same employed in Ferreira, Rodrigues, & Guedes (2014). Its main characteristics are that it uses gradients, is based on approximating eigenvalues by the Canfield's Rayleigh's quotient approximation (Canfield 1990) (moving asymptotes approximation (Svanberg 1987) terms are also used) and the optimization solver employed is the conjugated gradient (Vanderplaats 2005). Moreover, multiplicity of eigenvalues is monitored during the optimization iterations as mentioned before.

5 RESULTS

This section presents results for the optimization Problem (4) for two cases of laminated plates, for

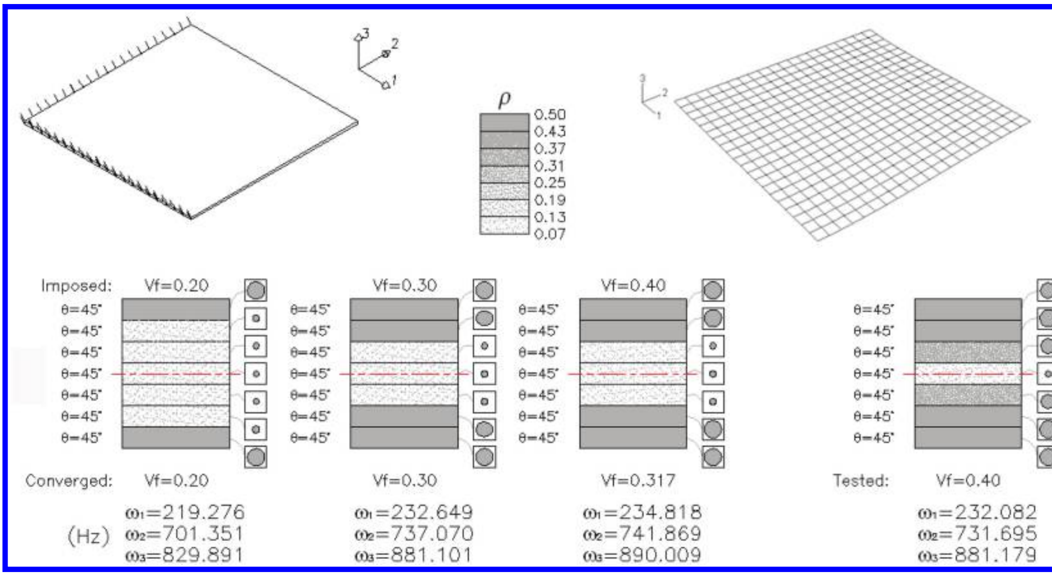

Figure 3. Results for maximizing the first natural frequency (flexural mode) of a 7-layer square laminated plate.

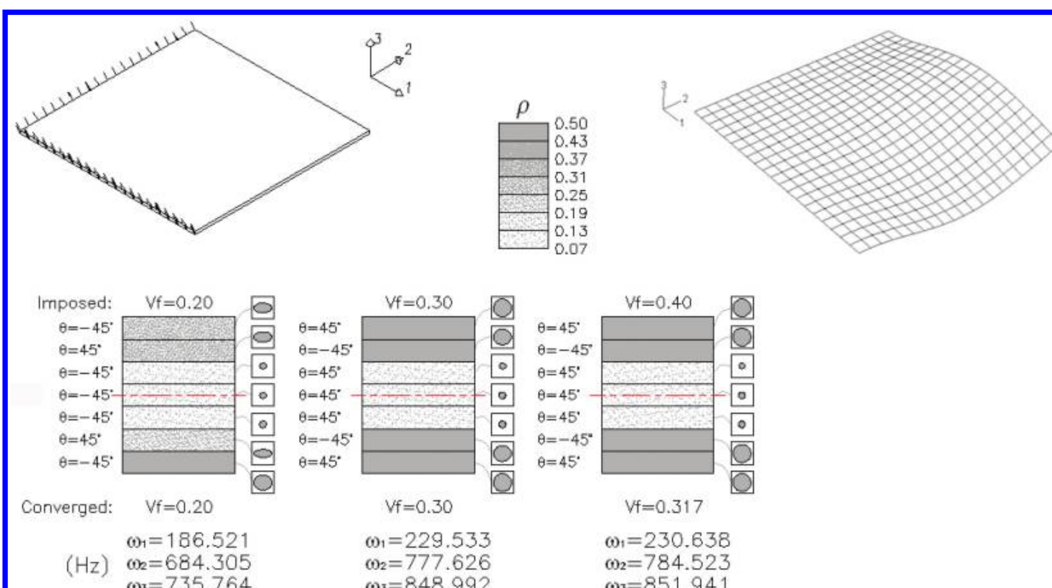

Figure 4. Results for maximizing the second natural frequency of a 7-layer laminated square plate. The mode depicted is torsional, the final for $V_f = 0.3, 0.4$.

maximization of first and second natural frequencies. The choice for these target eigenvalues is made by imposing, respectively, $j = 1$ and $j = 2$ in Problem (4). In all the cases, the laminates have seven layers and are designed in terms of layer-to-layer material orientation and microstructure as depicted in Section 2. Total fiber volume fractions admitted are $V_f = 0.2$, 0.3 and 0.4.

The first case considers a rectangular plate clamped in one of the shortest edges and free in the others. This structure and the results found for maximization of its first and second natural frequencies are illustrated in Fig. 1 and 2, respectively. The second case considers a square plate clamped in two perpendicular edges and free in the others. This plate and the results found for maximization of its first and second natural frequencies are illustrated in Fig. 3 and 4, respectively.

In Fig. 1 to 4, it can be seen that the results show strong similarities with the designs found for maximization of stiffness in Ferreira, Rodrigues, Guedes, & Hernandes (2014). These similarities are consonant with the mode shapes of the frequencies being maximized and the analogous elastic responses for displacements in the stiffness maximization cases. For example, results obtained for maximization of an eigenvalue correspondent to a flexural mode are similar to the results for stiffness maximization in an analogous bending case.

In terms of microstructure, these similarities render a trend for high layer fiber volume fractions to be found in the outer laminate layers (in accordance with the V_f limit imposed), fibers with circular cross-sections to be found in problems of flexural modes and fibers with elliptical cross-sections to be found in problems of torsional modes. In terms of material orientations, similar comments could be made.

Based on these observations, it could be concluded that the mass variation in the problem has a negligible influence, since the results are stiffness dominated. In fact, the variation of layers' material densities d_i with ρ_i in Eq. (7) can be shown to be not big for these cases. However, the mass variation included in Problem (4) led to a limit of stiff material utilization with respect to eigenvalues maximization, as can be noticed in the cases where the total fiber volume fraction $V_f = 0.4$ was imposed as upper bound for presence of fiber material. In these cases, the converged total fiber volume fraction value obtained was $V_f = 0.317$. After this value, no increase in frequencies could be obtained with increase of fiber volume fraction used.

This behavior can be explained by analyzing the variation of ratios of stiffnesses per mass density of a laminate with the total fiber volume fraction V_f. For the rectangular plate, such analysis rendered the plots in Fig. 5. In this figure, D_{11} is the stiffness related to laminate bending, D_{66} is the stiffness related to torsion of the laminate (Jones 1999) and d is the mass density of the whole laminate. Both the D_{11}/d and D_{66}/d variations present peaks with $V_f = 0.317$, confirming that this is a limit for stiff material usage to maximize an eigenvalue in flexural and torsional modes, respectively. For the square plate, the same behavior can be expected.

Specific comments can be made for some results. Regarding the result for maximization of the second natural frequency of the rectangular plate with $V_f = 0.2$ in Fig. 2, the optimal design found with elliptical fibers is 7.4% better than a design with the same layer-to-layer fiber volume fraction distribution and circular fibers, in terms of second eigenvalue maximization. Regarding the result for maximization of the second natural frequency of the square plate with $V_f = 0.2$ in Fig. 4, the design found seems to be a local minima since in Fig. 3 a higher value of the same second natural frequency was obtained when maximizing the square plate first natural frequency for $V_f = 0.2$.

Finally, in none of the cases multiplicity of eigenvalues happened. Therefore, problems of non-differentiability of eigenvalues are not included in the optimal solutions found.

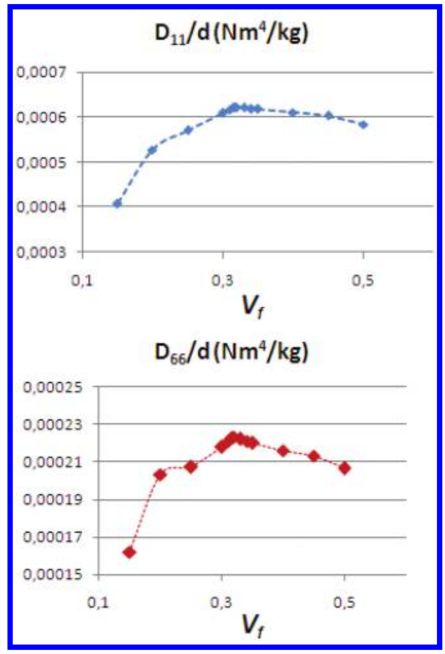

Figure 5. D_{11}/d and D_{66}/d variations with volume fraction V_f for the rectangular plate, obtained with circular fibers).

6 CONCLUSIONS

This paper presented results on the hierarchical optimization of laminated fiber reinforced composites for maximization of natural frequencies of plates. Structure and its material are simultaneously designed in the macro-mechanical and micro-mechanical levels, respectively.

The optimal designs here found are similar to previously encountered ones for stiffness maximization (Ferreira et al. 2014), both in terms of layers' material orientation and microstructure. This can be expected since a not big variation of layers' densities d_i with the design variables can be shown to exist in Problem (4) here dealt which, therefore, is stiffness dominated.

However the stiffness-to-mass ratio obtained for the laminates being optimized influenced some optimal designs by imposing a limit on stiff material usage when maximizing eigenvalues, as explained with the aid of Fig. 5.

A next step of this work is to consider constraints in the gap between two consecutive eigenvalues and optimize structures for target natural frequencies.

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