Derived categories of coherent sheaves and integral functors

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Abstract

We provide an introduction to the theory of derived categories and derived functors. To achieve this, we begin by studying the triangulated structure on the homotopy category of complexes over an abelian category $\mathscr{A}$, and define its derived category $D(\mathscr{A})$ by formally inverting quasi-isomophisms. In this way, the derived category, although not abelian, inherits a canonical structure of a triangulated category, and derived functors are defined as initial objects in the category of extensions that preserve the distinguished triangles. We apply these constructions to the abelian category Coh_X of coherent sheaves on a smooth projective variety X, with the help of tools such as spectral sequences and δ-functors. Finally, we introduce integral functors. Given two such varieties X and Y, these are geometrically motivated functors $D^b(\mathrm{Coh}_X) \to D^b(\mathrm{Coh}_Y)$ between the derived categories, which are extensively used in present day Algebraic Geometry and Mathematical Physics.

Keywords: Derived Categories, Derived Functors, Integral Functors, Fourier-Mukai Transforms, Homological Algebra.

1. Introduction

The field of Homological Algebra reached maturity in 1956 with the publication of [CE56]. There, Cartan and Eilenberg defined short exact sequences and injective/projective resolutions of modules. Given a commutative ring A, an A-module I is said to be ***injective*** if, for each monomorphism of A-modules $i\colon M \hookrightarrow N$ and morphism $f\colon M \to I$, there exists an arrow $g\colon N \to I$ such that $f = g \circ i$. Therefore, we can think of injective modules as ones giving extensions of morphisms to them. An ***injective resolution*** of $M \in \mathrm{Mod}_A$ is then a complex $I^\bullet$ of injective A-modules such that the sequence $0 \to M \to I^\bullet$ is exact. These resolutions always exist.

In the ***homotopy category*** $K(\mathrm{Mod}_A)$ of complexes of A-modules, any two injective resolutions of M are isomorphic, via a uniquely defined isomorphism. In fact, recalling the terminology of ***homotopy equivalence*** as the the notion of isomorphism in $K(\mathrm{Mod}_A)$, given an A-module homomorphism $M \to N$, and injective resolutions $0 \to M \to I^\bullet_M$ and $0 \to N \to I^\bullet_N$, there is a *unique* arrow $I^\bullet_M \to I^\bullet_N$ making the square

$$\begin{array}{ccc} M & \longrightarrow & I^\bullet_M \\ \downarrow & & \downarrow \exists\,! \\ N & \longrightarrow & I^\bullet_N \end{array} \tag{1}$$

commute in $K(\mathrm{Mod}_A)$. In particular, $I^\bullet_M \to I^\bullet_N$ is a ***quasi-isomorphism*** (or ***quis*** for short), that is, for each i, the induced map $H^i(I^\bullet_M) \to H^i(I^\bullet_N)$ on cohomology is an isomorphism of abelian groups.

In [CE56], we also find the definition of a right iterated satellite. Given a short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$$

of A-modules, and other $P \in \mathrm{Mod}_A$, the functor $\mathrm{Hom}(P, -)\colon \mathrm{Mod}_A \to \mathrm{Ab}$ yields a complex

$$0 \longrightarrow \mathrm{Hom}(P, M) \longrightarrow \mathrm{Hom}(P, N) \longrightarrow \mathrm{Hom}(P, K) \longrightarrow 0 \tag{2}$$

which may not be exact anymore at $\mathrm{Hom}(P, K)$. For Cartan and Eilenberg, the ***i-th iterated satellite*** ($i \geq 0$) of $\mathrm{Hom}(P, -)$ was a functor $\mathrm{Ext}^i(P, -)\colon \mathrm{Mod}_A \to \mathrm{Ab}$ defined as follows. Given $M \in \mathrm{Mod}_A$, we take a resolution $0 \to M \to I^\bullet$ by injective modules, and apply $\mathrm{Hom}(P, -)$ to $0 \to I^\bullet$, in order to get the complex

$$0 \longrightarrow \mathrm{Hom}(P, I^0) \longrightarrow \mathrm{Hom}(P, I^1) \longrightarrow \mathrm{Hom}(P, I^2) \longrightarrow \dots \ .$$

We set $\mathrm{Ext}^i(P, M)$ to be the cohomology of the complex above at $\mathrm{Hom}(P, I^i)$. This object is well-defined up to canonical isomorphism, by the fact that any two injective resolutions of M are homotopy equivalent, via a unique homotopy, as previously asserted. The collection $\{\mathrm{Ext}^i(P, -)\}_i$ has a nice property, in that it extends the Hom sequence (2) to a sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(P, M) \longrightarrow \mathrm{Hom}(P, N) \longrightarrow \mathrm{Hom}(P, K) \\ \hookrightarrow \mathrm{Ext}^1(P, M) \longrightarrow \mathrm{Ext}^1(P, N) \longrightarrow \mathrm{Ext}^1(P, K) \\ \hookrightarrow \mathrm{Ext}^2(P, M) \longrightarrow \cdots \end{array}$$

which is now exact at every object.

One year after the release of [CE56], Grothendieck axiomatized the notion of ***abelian category*** in the celebrated *Tôhoku* article, [Gro57]. From the archetypical example of the category Mod_A of modules over a commutative ring A, one could generalize the notion of injective object to any abelian category. However, the French mathematician noted that there were several cases of abelian categories where one would have to deal with resolutions that were not given by injective/projective objects. As an example, in the category Ab_X of sheaves of abelian groups over a topological space X, any such sheaf $\mathscr{F}$ can be embedded in its Godement sheaf, $\mathrm{Gode}(\mathscr{F})$, [God58]. This is just the sheaf defined over each open subset $U \subseteq X$ as the product of the stalks of $\mathscr{F}$ at points in U,

$$\mathrm{Gode}(\mathscr{F})(U) \coloneqq \prod_{p \in U} \mathscr{F}_p \ ,$$

with the projection maps giving the restrictions. With this type of embeddings, one can build an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathrm{Gode}^0(\mathscr{F}) \longrightarrow \mathrm{Gode}^1(\mathscr{F}) \longrightarrow \mathrm{Gode}^2(\mathscr{F}) \longrightarrow \dots \ ,$$

that is, a resolution of $\mathscr{F}$ by Godement sheaves. The sheaves $\mathrm{Gode}^i(\mathscr{F})$ have the property of being ***flasque*** (or flabby), *i.e.* for each pair of open subsets $U \subseteq V \subseteq X$, the restriction map $\mathrm{Gode}(\mathscr{F})(V) \to \mathrm{Gode}(\mathscr{F})(U)$ is surjective. Given a morphism of sheaves $\mathscr{F} \to \mathscr{G}$ on X, and resolutions $\mathscr{F} \to \mathscr{R}_1^\bullet$ and $\mathscr{G} \to \mathscr{R}_2^\bullet$ by flasque sheaves, there may be two distinct dashed arrows making the diagram

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{R}_1^\bullet \\ \downarrow & & \vdots\!\!\downarrow\!\downarrow \\ \mathscr{G} & \longrightarrow & \mathscr{R}_2^\bullet \end{array}$$

commute in $K(\mathrm{Ab}_X)$, unlike in the case of (1). Therefore, a resolution of a sheaf by flasque sheaves need not be unique up to homotopy, but only up to quasi-isomorphism. This fact led Grothendieck to invent a new category $D(\mathrm{Ab}_X)$, together with a functor $K(\mathrm{Ab}_X) \to D(\mathrm{Ab}_X)$, such that quasi-isorphisms in $K(\mathrm{Ab}_X)$ became canonically isomorphic in $D(\mathrm{Ab}_X)$ – the ***derived category*** of Ab_X.

2. Derived categories

A proper definition of the derived category $D(\mathscr{A})$ of an abelian category $\mathscr{A}$ first appeared in Verdier's 1967 PhD thesis, titled appropriately *Des catégories dérivées des catégories abéliennes*, which he conducted under the supervision of Grothendieck. This text was only published later in the nineties in *Société Mathématique de France*'s journal *Astérisque*, [Ver96].

Verdier's construction of $D(\mathscr{A})$ relies on a structure of distinguished triangles in the homotopy category $K(\mathscr{A})$, which we now explain briefly.

2.1. Triangulated categories

Definition. Let $\mathscr{D}$ be an additive category and $T \colon \mathscr{D} \to \mathscr{D}$ an automorphism. A ***triangle*** in $(\mathscr{D}, T)$ is a collection of three objects and three arrows of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A).$$

A ***morphism of triangles*** is the data of the three vertical downward morphisms in the following commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & T(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ D & \xrightarrow{x} & E & \xrightarrow{y} & F & \xrightarrow{z} & T(D) \end{array} .$$

The morphism above is an isomorphism if α, β, γ are isomorphisms. The following axioms specify a class of triangles which we call ***distinguished***:

A1) i) Any triangle of the form $A \xrightarrow{\mathrm{id}_A} A \to 0 \to T(A)$ is distinguished.
   ii) Any triangle isomorphic to a distinguished triangle is itself distinguished.
   iii) For each morphism $f\colon A \to B$, there exists a distinguished triangle of the form

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow T(A) .$$

A2) The triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is distinguished if and only if the triangle $B \xrightarrow{g} C \xrightarrow{h} T(A) \xrightarrow{-T(f)} T(B)$ is distinguished.

A3) Every solid diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & T(D) \end{array}$$

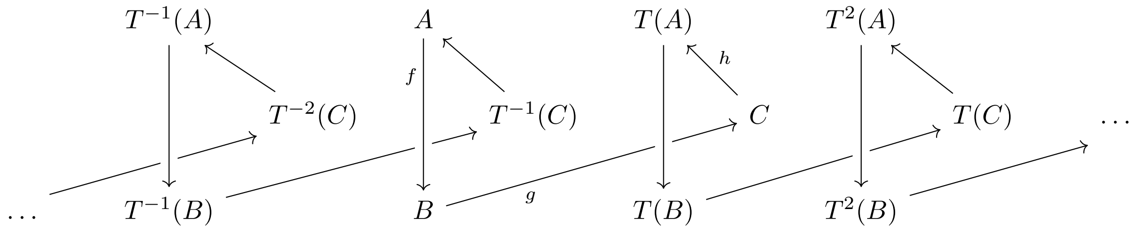
whose rows are distinguished triangles can be completed *(not necessarily uniquely)* by γ to a morphism of triangles.

The pair $(\mathscr{D}, T)$ endowed with a class of distinguished triangles is said to be a ***triangulated category***.

Remark. We have purposefully left out Verdier's fourth axiom in the definition above. This axiom is usually called the octahedron axiom – see ($\star$) Remark 2.2.3 in the main text for more information[1].

The axioms of the definition above provide a great deal of information. For example,

- The composition of any two consecutive morphisms in a distinguished triangle is zero, ($\star$, Proposition 2.2.4).
- Any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$, can be included in an helix of morphisms



  and, according to the statement above, this helix is a complex in $\mathscr{D}$.

- Given a morphism of distinguished triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & T(A') \end{array} ,$$

  if two of the morphisms f, g and h are isomorphisms, then so is the third, ($\star$, Proposition 2.2.8).

[1] From now on, we use the symbol ($\star$) to refer to the main text.

Definition. Let $(\mathscr{D}, T)$ and $(\mathscr{N}, S)$ be triangulated categories. An additive functor $F\colon \mathscr{D} \to \mathscr{N}$ is called ***exact*** if:

i) There is a functor isomorphism $F \circ T \overset{\cong}{\Longrightarrow} S \circ F$.

ii) Any distinguished triangle $A \to B \to C \to T(A)$ in $\mathscr{D}$ is mapped via F to a distinguished triangle $F(A) \to F(B) \to F(C) \to S(F(A))$ in $\mathscr{N}$, where $F(T(A))$ is identified with $S(F(A))$ via the isomorphism in i).

Definition. Let $(\mathscr{D}, T)$ be a triangulated category. If $\mathscr{D}'$ is a full additive subcategory of $\mathscr{D}$, we say that $(\mathscr{D}', T)$ is a ***full triangulated subcategory*** if $\mathscr{D}'$ is invariant under shift and, for any distinguished triangle $A \to B \to C \to T(A)$ in $\mathscr{D}$ with $A, B \in \mathrm{Obj}(\mathscr{D}')$, C is isomorphic to an object in $\mathscr{D}'$.

2.2. The homotopy category is triangulated

Throughout this section let $\mathscr{A}$ stand for an abelian category, $\mathrm{Com}_{\mathscr{A}}$ for its category of complexes and $K(\mathscr{A})$ for its homotopy category of complexes.

Definition ($\star$, 2.3.12)**.** The left shift by 1 is the automorphism $[1]\colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$ that sends $A^\bullet \in \mathrm{Com}_{\mathscr{A}}$ to $(A^\bullet[1], d_{A^\bullet[1]})$, with $d_{A^\bullet[1]} := -d_{A^\bullet}$ and the i-th term of $A^\bullet[1]$ being A^{i+1}. Similarly, one defines $[1]\colon K(\mathscr{A}) \to K(\mathscr{A})$.

Definition ($\star$, 2.3.17)**.** Given a chain map $f\colon A^\bullet \to B^\bullet$ in $\mathrm{Com}_{\mathscr{A}}$, the ***cone*** of f is the chain complex $(\mathrm{cone}\,(f)\,, d_{\mathrm{cone}(f)})$, where $(\mathrm{cone}\,(f))^i := A^{i+1} \oplus B^i$ and the differential is

$$d^i_{\mathrm{cone}(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}$$

for every i.

Proposition ($\star$, Definition/Proposition 2.3.19 and Proposition 2.3.20)**.** Given any chain map $f\colon A^\bullet \to B^\bullet$, there is a short exact sequence of chain complexes

$$0 \longrightarrow B^\bullet \xrightarrow{\ \tau_f\ } \mathrm{cone}\,(f) \xrightarrow{\ \pi_f\ } A^\bullet[1] \longrightarrow 0, \tag{3}$$

where τ_f and π_f are natural maps given, for each degree i, by the canonical injection $B^i \hookrightarrow A^{i+1} \oplus B^i$, and the canonical projection $A^{i+1} \oplus B^i \twoheadrightarrow A^{i+1}$, respectively. Consequently, we have a long exact sequence

$$\cdots \longrightarrow H^i(A^\bullet) \xrightarrow{\ \partial^{i-1}\ } H^i(B^\bullet) \longrightarrow H^i(\mathrm{cone}\,(f)) \longrightarrow H^{i+1}(A^\bullet) \xrightarrow{\ \partial^{i}\ } H^{i+1}(B^\bullet) \longrightarrow \cdots$$

and the connecting homomorphism is actually $\partial^i = H^{i+1}(f)$.

Corollary ($\star$, Corollary 2.3.21)**.** A chain map $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if $\mathrm{cone}\,(f)$ is acyclic.

Proposition ($\star$, Proposition 2.3.24)**.** $K(\mathscr{A})$ has a structure of a triangulated category by choosing the automorphism $[1]\colon K(\mathscr{A}) \to K(\mathscr{A})$, and by specifying the distinguished triangles to triangles isomorphic to ones of the form (3).

2.3. The derived category is triangulated

Definition ($\star$, Definition 2.4.1)**.** Let $\mathscr{A}$ be an abelian category. The ***derived category*** of $\mathscr{A}$ is a category $D(\mathscr{A})$, together with a functor $Q\colon K(\mathscr{A}) \to D(\mathscr{A})$ that sends quasi-isomorphisms in $K(\mathscr{A})$ to isomorphisms in $D(\mathscr{A})$. Moreover, Q should be initial with respect to this property, *i.e.* given any category $\mathscr{C}$ with a functor $F\colon K(\mathscr{A}) \to \mathscr{C}$ that sends quasi-isomorphisms in $K(\mathscr{A})$ to isomorphisms in $\mathscr{C}$, there exists a unique arrow $G\colon D(\mathscr{A}) \to \mathscr{C}$ making the diagram

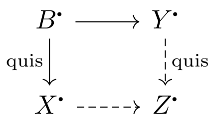
$$\begin{array}{ccc} K(\mathscr{A}) & \xrightarrow{\ F\ } & \mathscr{C} \\ {\scriptstyle Q}\big\downarrow & \nearrow_{\exists\,!\,G} & \\ D(\mathscr{A}) & & \end{array} \tag{4}$$

commute.

Proposition ($\star$, Propositions 2.1.5 and 2.4.2)**.** In the notation above, the derived category $D(\mathscr{A})$ exists. We say that $D(\mathscr{A})$ is obtained by localization of $K(\mathscr{A})$ with respect to quasi-isomorphisms.

An explicit description of $D(\mathscr{A})$ can be given as follows:

- The objects of $D(\mathscr{A})$ are the same as those of $\mathrm{Com}_{\mathscr{A}}$ or $K(\mathscr{A})$, that is, chain complexes $A^\bullet$ whose terms A^i are objects of $\mathscr{A}$.
- ($\star$, Proposition 2.4.2) shows that that class of quis in $K(\mathscr{A})$ has the following properties:
  i) It is multiplicatively closed (*i.e.*, for every $A^\bullet \in K(\mathscr{A})$, $\mathrm{id}_{A^\bullet}$ is a quis, and the composition of any two composable quis is again a quis).
  ii) Given a solid diagram



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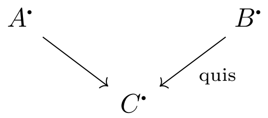
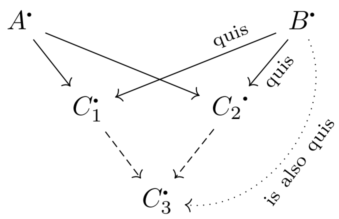
  there exist dashed arrows completing the diagram to a commutative square.
  iii) Given a morphism $f\colon A^\bullet \to B^\bullet$, if there exists a quis $C^\bullet \xrightarrow{s} A^\bullet$ such that $f \circ s$ is nulhomotopic, then there exists a quis $B^\bullet \xrightarrow{s'} D^\bullet$ such that $s' \circ f$ is nulhomotopic.
- A ***right roof*** from $A^\bullet$ to $B^\bullet$ in $K(\mathscr{A})$ is a diagram of the form



(5)

  where $C^\bullet$ is some complex. We can define an equivalence class on the set of right roofs by declaring two right roofs from $A^\bullet$ to $B^\bullet$ to be equivalent if there exists a third right roof from $A^\bullet$ to $B^\bullet$ dominating the other two:



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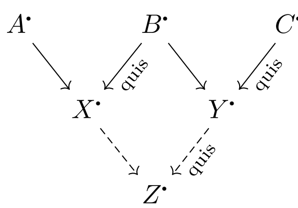
  By this we mean that there exist dashed arrows $C_1^\bullet \dashrightarrow C_3^\bullet \dashleftarrow C_2^\bullet$ such that the composition $B^\bullet \to C_2^\bullet \dashrightarrow C_3^\bullet$ is a quis and, in addition, the two inscribed triangles commute.

- The composition of the equivalence class of the right roof $A^\bullet \to X^\bullet \xleftarrow{\text{quis}} B^\bullet$ with the equivalence class of the right roof $B^\bullet \to Y^\bullet \xleftarrow{\text{quis}} C^\bullet$ is given in the following way: we fill in the solid diagram



  by completing $X^\bullet \xleftarrow{\text{quis}} B \to Y^\bullet$ to the shown commutative square (which is always possible, as claimed). The composition of the equivalence classes is defined as the equivalence class of the "big" right roof $A^\bullet \to Z^\bullet \leftarrow C^\bullet$. One can check that this well-defined, ($\star$, Definition/Proposition 2.1.2).
- Finally, morphisms $A^\bullet \to B^\bullet$ in $D(\mathscr{A})$ are given by equivalence classes of right roofs of the form (5).

- One can also show that $D(\mathscr{A})$ inherits a canonical structure of an additive category from $K(\mathscr{A})$. Indeed, one can do addition with right roofs by "reducing to common denominators", ($\star$, Proposition 2.1.6).

- The functor $Q\colon K(\mathscr{A}) \to D(\mathscr{A})$ is defined as $Q(A^\bullet) = A^\bullet$ for every $A^\bullet \in K(\mathscr{A})$, and sends $f \in \mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet)$ to the equivalence class of the right roof

$$\begin{array}{ccccc} A^\bullet & & & & B^\bullet \\ & \overset{f}{\searrow} & & \overset{\mathrm{id}_{B^\bullet}}{\swarrow} & \\ & & B^\bullet & & \end{array}$$

  In addition, Q is an exact functor of triangulated categories.

- There are well defined cohomology functors on the derived category, arising from the universal property of $Q\colon K(\mathscr{A}) \to D(\mathscr{A})$:

$$\begin{array}{ccc} K(\mathscr{A}) & \xrightarrow{H^i} & \mathscr{A} \\ {\scriptstyle Q}\big\downarrow & \nearrow & \\ D(\mathscr{A}) & & \end{array}.$$

- In particular, any two quasi-isomorphic complexes become canonically isomorphic in $D(\mathscr{A})$. ($\star$, Corollary 2.4.4) shows that a complex $A^\bullet$ is the zero object in $D(\mathscr{A})$ if and only if $A^\bullet$ is ***acyclic***, that is, $H^i(A^\bullet) = 0$ for every i.

With this description $D(\mathscr{A})$ inherits a canonical structure of a triangulated category from the one in $K(\mathscr{A})$, by choosing the automorphism $[1]\colon D(\mathscr{A}) \to D(\mathscr{A})$, and by declaring the distinguished triangles to be triangles that are isomorphic to the image under $Q\colon K(\mathscr{A}) \to D(\mathscr{A})$ of canonical distinguished triangles in $K(\mathscr{A})$, that is, triangles of the form (3), ($\star$, Proposition 2.4.7).

One advantage of working with the derived category $D(\mathscr{A})$ is that there is a one-to-one correspondence between short exact sequences in $\mathrm{Com}_{\mathscr{A}}$ and distinguished triangles in $D(\mathscr{A})$, unlike in the homotopy category $K(\mathscr{A})$. Indeed, consider the following statement.

Proposition ($\star$, Proposition 2.4.9)**.** Let $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ be a short exact sequence in $\mathrm{Com}_{\mathscr{A}}$. Then, there is a natural quasi-isomorphism $\alpha\colon \mathrm{cone}\,(f) \to C^\bullet$, and so the triangle

$$A^\bullet \xrightarrow{\quad f \quad} B^\bullet \xrightarrow{\quad g \quad} C^\bullet \xrightarrow{\pi_f \circ \alpha^{-1}} A^\bullet[1]$$

is distinguished in $D(\mathscr{A})$, where α^{-1} is the inverse of α in $D(\mathscr{A})$, and $\pi_f\colon \mathrm{cone}\,(f) \to A^\bullet[1]$ is the natural projection. Moreover, any distinguished triangle in $D(\mathscr{A})$ is isomorphic to one obtained in this way.

3. Derived functors

Consider the following proposition.

Proposition 3.1. Let $\mathscr{A}, \mathscr{B}$ be abelian categories and $F\colon \mathscr{A} \to \mathscr{B}$ an additive functor. Denote by $Q_A\colon K(\mathscr{A}) \to D(\mathscr{A})$ the natural functor of the localization (and similarly for Q_B), and by $K(F)\colon K(\mathscr{A}) \to K(\mathscr{B})$ the functor one obtains by applying F term-wise to complexes in $K(\mathscr{A})$. Then, there exists an exact functor of triangulated categories $D(\mathscr{A}) \dashrightarrow D(\mathscr{B})$ making the diagram

$$\begin{array}{ccc} D(\mathscr{A}) & \dashrightarrow & D(\mathscr{B}) \\ {\scriptstyle Q_A}\big\uparrow & & \big\uparrow{\scriptstyle Q_B} \\ K(\mathscr{A}) & \xrightarrow{K(F)} & K(\mathscr{B}) \end{array}$$

commute if and only if F is exact.

In particular, if F is only left-exact, strict commutativity of the diagram above is impossible. The idea of the right derived functor RF of such a F is to extend the morphism $K(F)\colon K(\mathscr{A}) \to K(\mathscr{B})$ to the derived categories, but weakening the condition of strict commutativity. However, it will not be possible to define this extension over all $D(\mathscr{A})$, but only to a full triangulated subcategory.

Definition/Proposition ($\star$, Definition/Proposition 3.1.2)**.** The category $K^+(\mathscr{A})$ is the full triangulated subcategory of $K(\mathscr{A})$ whose objects are complexes $A^\bullet$ that are ***bounded below***, *i.e.* $\exists\ N$ such that $A^i = 0$ for all $i \leq N$.

The localization of $K^+(\mathscr{A})$ with respect to quasi-isomorphisms is a full triangulated subcategory of $D(\mathscr{A})$, which we denote by $D^+(\mathscr{A})$. The canonical functor $D^+(\mathscr{A}) \to D(\mathscr{A})$ is exact.

Definition 3.2. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories. The ***right derived functor*** of F is a pair (RF, η) consisting of an exact functor of triangulated categories $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$, together with a natural transformation $\eta\colon Q_B \circ K^+(F) \Rightarrow RF \circ Q_A$, which we represent diagrammatically by

$$\begin{array}{ccccc} K^+(\mathscr{A}) & & \xrightarrow{\quad Q_B\circ K^+(F)\quad} & & D^+(\mathscr{B}) \\ & \searrow^{Q_A} & \Downarrow \eta & \nearrow_{RF} & \\ & & D^+(\mathscr{A}) & & \end{array} .$$

The pair (RF, η) is required to satisfy the following universal property: for any other pair $(G\colon D^+(\mathscr{A}) \to D^+(\mathscr{B}), \gamma\colon Q_B \circ K^+(F) \Rightarrow G \circ Q_A)$, there exists a unique natural transformation $\varepsilon\colon RF \Rightarrow G$ such that the diagram

$$\begin{array}{ccccc} Q_B \circ K^+(F) & & \xRightarrow{\qquad\gamma\qquad} & & G \circ Q_A \\ & \overset{\eta}{\searrow} & & \overset{\varepsilon\circ Q_A}{\nearrow} & \\ & & RF \circ Q_A & & \end{array}$$

commutes.

The theorem below is central to our discussion. Its proof is in ($\star$, Section 3.2).

Theorem ($\star$, Definition 3.1.4 and Theorem 3.1.10)**.** In the notation above, RF exists if there exists a class of objects $\mathcal{R} \subseteq \mathrm{Obj}(\mathscr{A})$ verifying three axioms:

A1) $\mathcal{R}$ is closed under direct sums;

A2) $K^+(F)$ maps any bounded below acyclic complex with terms in $\mathcal{R}$ into an acyclic complex with terms in $\mathscr{B}$;

A3) every object in $\mathscr{A}$ is a subobject of an object in $\mathcal{R}$.

In this case, $\mathcal{R}$ is said to be ***adapted*** to F.

We now summarize the construction of RF.

- First of all, given $A^\bullet \in K^+(\mathscr{A})$, axiom A3 implies the existence of $R^\bullet \in K^+(\mathcal{R})$ and a quis $q\colon A^\bullet \to R^\bullet$. We say that such a q is a ***quasi-resolution*** of $A^\bullet$ by a bounded below complex of F-adapted objects. It can be shown that giving quasi-resolutions is a functorial procedure, *i.e.* given $f \in \mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet)$, we can find quasi-resolutions of $A^\bullet$ and $B^\bullet$ and a dashed arrow,

$$\begin{array}{ccc} A^\bullet & \xrightarrow{\text{quis}} & R^\bullet_A \\ {\scriptstyle f}\downarrow & & \downarrow \\ B^\bullet & \xrightarrow{\text{quis}} & R^\bullet_B \end{array} ,$$

such that the diagram above commutes.

- Moreover, by axioms A1 and A2, we can show that, given any $A^\bullet \in K^+(\mathscr{A})$, there exists an isomorphism in $D^+(\mathscr{A})$ from $A^\bullet$ to some $R^\bullet \in K^+(\mathcal{R})$, which is represented by a right roof of the form

$$\begin{array}{ccccc} A^\bullet & & & & R^\bullet \\ & \overset{\text{quis}}{\searrow} & & \overset{\text{quis}}{\swarrow} & \\ & & T^\bullet & & \end{array}$$

where $T^\bullet$ is also in $K^+(\mathcal{R})$. By applying $K^+(F)\colon K(\mathscr{A}) \to K(\mathscr{B})$, we get a right roof

$$\begin{array}{ccccc} K^+(F)(A^\bullet) & & & & K^+(F)(R^\bullet) \\ & \searrow & & \swarrow_{\text{quis}} & \\ & & K^+(F)(T^\bullet) & & \end{array} \tag{6}$$

in $K^+(\mathscr{B})$.

- In the notation above, we define $RF(A^\bullet) := K^+(F)(R^\bullet)$, and $\eta(A^\bullet)$ to be the morphism in $D^+(\mathscr{B})$ determined by the roof (6). We refer the reader to ($\star$, Section 3.2) for details.
- In this way, we can think of $\eta(A^\bullet)$ as measuring the difference, in $D^+(\mathscr{B})$, between the complex $K^+(F)(A^\bullet)$ that one obtains from $A^\bullet$ by term-wise application of F, and its image $RF(A^\bullet)$ under the construction of the right derived functor. Indeed, by the universal property of (RF, η), when F is exact, $RF(A^\bullet) = K^+(F)(A^\bullet)$, and η is just the identity.

Under certain conditions on the abelian category $\mathscr{A}$, there is a class of objects in $\mathscr{A}$ which is adapted to any left exact functor $F\colon \mathscr{A} \to \mathscr{B}$.

Theorem 3.3 ($\star$, Theorem 3.4.10)**.** Let $\mathscr{A}$ be an abelian category with enough injectives, and $F\colon \mathscr{A} \to \mathscr{B}$ a left exact functor. Then the class $\mathcal{I}$ of injective objects in $\mathscr{A}$ is F-adapted.

The next definition asserts how we can recover the classical derived functors of Cartan-Eilenberg[2] from Verdier's "total" derived functor.

Definition ($\star$, Definitions 3.3.1 and 3.3.8)**.** Let $\mathscr{A}, \mathscr{B}$ be abelian categories and $F\colon \mathscr{A} \to \mathscr{B}$ a left exact functor. Suppose that $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is everywhere defined. The ***i-th (higher) right derived functor*** of F is the functor $R^iF\colon \mathscr{A} \to \mathscr{B}$ defined as $A \mapsto H^i(RF(A))$, where A is seen as a complex concentrated in degree 0.

We say that $A \in \mathrm{Obj}(\mathscr{A})$ is ***F-acyclic*** if $R^iF(A) = 0$ for $i \geq 1$.

It is easy to see that any object in F-adapted class is F-acyclic ($\star$, Lemma 3.3.9). Something less obvious to show is that, if the right derived functor RF exists, one can compute it using F-acyclic resolutions ($\star$, Corollary 3.4.3).

The collection of higher right derived functors $\{R^iF\colon \mathscr{A} \to \mathscr{B}\}_{i\geq 0}$ is an example of a more general construction of homological algebra, that of a δ-functor. This concept is introduced in ($\star$, Subsection 3.3.1).

Composition of derived functors is straightforward under the formalism we describe. Indeed, we have the following result.

Proposition 3.4 ($\star$, Proposition 3.6.1)**.** Let $F_1\colon \mathscr{A} \to \mathscr{B}$ and $F_2\colon \mathscr{B} \to \mathscr{C}$ be two left exact functors. Suppose there exist subclasses $\mathcal{R}_A \subseteq \mathrm{Obj}(\mathscr{A})$ and $\mathcal{R}_B \subseteq \mathrm{Obj}(\mathscr{B})$ which are adapted to F_1 and F_2, respectively. If $F_1(\mathcal{R}_A) \subseteq \mathcal{R}_B$, then there is a natural isomorphism of functors $R(F_2 \circ F_1) \xRightarrow{\cong} R(F_2) \circ R(F_1)$.

Verdier's construction of derived functors simplified the composition of (higher) right derived functors immensely. Indeed, these compositions had been previously studied by Grothendieck using spectral sequences, [Gro57]. By Hartshorne's wording, the proposition above "*shows the convenience of derived functors in the context of derived categories. What used to be a spectral sequence becomes now simply a composition of functors. (And of course one can recover the old spectral sequence from this proposition by taking cohomology and using the spectral sequence of a double complex)*", [Har66, Pag. 60]. Nonetheless, spectral sequences remain a very useful gadget in homological algebra. They are introduced in ($\star$, Appendix C).

3.1. A few remarks

The construction of right derived functors $RF\colon D^+(\mathscr{A}) \to D(\mathscr{B})$ can be generalized in order not to require F to be defined over abelian categories. We give a precise definition of what we mean.

Definition ($\star$, Definition 3.2.3)**.** Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and $V\colon K^+(\mathscr{A}) \to K(\mathscr{B})$ an exact functor of triangulated categories. The ***right derived functor*** of V is a pair (RV, η) consisting of an exact functor of triangulated categories $RV\colon D^+(\mathscr{A}) \to D(\mathscr{B})$, and a natural transformation $\eta\colon Q_B \circ V \Rightarrow RV \circ Q_A$. This pair is required to be initial with respect to any other such pair.

Theorem ($\star$, Theorem 3.2.4)**.** In the notation of the definition above, RV exists if $K^+(\mathscr{A})$ admits a triangulated subcategory $\mathcal{K}_V$ satisfying two conditions:

[2] That is, the right iterated satellites we mentioned in the introduction.

a1) if $A^\bullet \in \mathcal{K}_V$ is acyclic, $V(A^\bullet)$ is acyclic;

a2) for any $A^\bullet \in K^+(\mathscr{A})$, there exists $R^\bullet \in \mathcal{K}_V$ and a quis $A^\bullet \to R^\bullet$.

We also note that mostly everything we defined so far about derived functors admits a dual statement. Indeed, given a right exact functor $F\colon \mathscr{A} \to \mathscr{B}$, we have a dual definition of its left derived functor $LF\colon D^-(\mathscr{A}) \to D^-(\mathscr{B})$ between the ***bounded above*** derived categories of $\mathscr{A}$ and $\mathscr{B}$. There are some subtleties with the duality between these constructions, namely when it comes to the definition of morphisms in the derived categories. These subtleties are explained in ($\star$, Remark 2.1.3) and ($\star$, Subsection 3.2.5).

4. Application to coherent sheaves

We now give a *very* brief summary of ($\star$, Chapters 4 and 5). Until otherwise mentioned, let X and Y stand for Noetherian schemes over a field k.

We define the following notation for certain categories: Sch_k the category of Noetherian schemes over k, Mod_X the abelian category of $\mathscr{O}_X$-modules, and Vec_k the abelian category of k-vector spaces. The next table recalls some important functors of sheaf theory.

Name	*Required data*	*Symbol*	*Exactness*
Stalk	$p \in X$	$(-)_p\colon \mathrm{Mod}_X \to \mathrm{Vec}_k$	Exact
Sections	U open subset of X	$\Gamma(U,-)\colon \mathrm{Mod}_X \to \mathrm{Vec}_k$	Left exact
Pushforward	$f \in \mathrm{Mor}_{\mathrm{Sch}_k}(X,Y)$	$f_*\colon \mathrm{Mod}_X \to \mathrm{Mod}_Y$	Left exact
Tensor product	$\mathscr{F} \in \mathrm{Mod}_X$	$\mathscr{F} \otimes_{\mathscr{O}_X} (-)\colon \mathrm{Mod}_X \to \mathrm{Mod}_X$	Right exact
Pullback	$f \in \mathrm{Mor}_{\mathrm{Sch}_k}(X,Y)$	$f^*\colon \mathrm{Mod}_Y \to \mathrm{Mod}_X$	Right exact

Table 1: Important functors over categories of sheaves.

A ***coherent sheaf*** $\mathscr{F}$ on X is a sheaf of $\mathscr{O}_X$-modules with particularly nice algebrogeometric properties. For starters, the category Coh_X of coherent sheaves on X is abelian. In addition, if A is a Noetherian ring, there is an exact functor $(-)^\sim\colon \mathrm{Vec}_k^f \to \mathrm{Coh}_{\mathrm{Spec}\,A}$ taking any finitely generated k-vector space V to a coherent sheaf $\widetilde{V}$ over the affine scheme $\mathrm{Spec}\,A$. In fact, this functor is an equivalence, whose inverse is the global sections functor $\Gamma(\mathrm{Spec}\,A,-)\colon \mathrm{Coh}_{\mathrm{Spec}\,A} \to \mathrm{Vec}_k^f$, ($\star$, Proposition 4.1.8).

Under certain conditions, we can restrict the functors in the table above to the coherent setting. Indeed, if f is proper ($\star$, Definition 4.3.18), then we can restrict the pushforward to a functor $f_*\colon \mathrm{Coh}_X \to \mathrm{Coh}_Y$ ($\star$, Theorem 4.3.22). If f is flat ($\star$, Definition 4.3.31) then $f^*\colon \mathrm{Coh}_Y \to \mathrm{Coh}_X$ is actually exact ($\star$, Proposition 4.3.33). In particular, its left derived functor is just usual term-wise application of f^* to complexes, *i.e.* $Lf^* = K^-(f^*)$, in the notation we used in the last section. To simplify notation, we still denote this functor by $f^*\colon D^-(\mathrm{Coh}_Y) \to D^-(\mathrm{Coh}_X)$.

In ($\star$, Chapter 4), we study the construction of the derived functors of the half-exact functors of Table 1. It is relatively straightforward to construct $Rf_*\colon D^+(\mathrm{Mod}_X) \to D^+(\mathrm{Mod}_Y)$ since Mod_X has enough injective objects ($\star$, Proposition 4.1.2), and similarly for $R\Gamma(X,-)\colon D^+(\mathrm{Mod}_X) \to D^+(\mathrm{Vec}_k)$. The restriction of this functor to a right derived functor $Rf_*\colon D^+(\mathrm{Coh}_X) \to D^+(\mathrm{Coh}_Y)$ in the case of f being proper is much harder. Indeed, we do this by studying a bigger full abelian subcategory of Mod_X, that of ***quasicoherent sheaves***, QCoh_X. We then use the chain of inclusions $\mathrm{Coh}_X \subsetneq \mathrm{QCoh}_X \subsetneq \mathrm{Mod}_X$ to indirectly define the derived functor of the pushforward on $D^+(\mathrm{Coh}_X)$. This process is described in ($\star$, Subsection 4.3.4), and relies on the help of spectral sequences and δ-functors.

The tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} (-)\colon \mathrm{Mod}_X \to \mathrm{Mod}_X$ is not immediate to left derive as well, due to the lack of projective objects in Mod_X ($\star$, Example 4.1.4). In fact, the non-existence of projective objects holds in the smaller full abelian subcategories $\mathrm{Coh}_X \subsetneq \mathrm{QCoh}_X \subsetneq \mathrm{Mod}_X$ as well ($\star$, Example 4.1.32). The solution is to find a class of adapted objects other than the projective ones. For $\mathscr{F} \in \mathrm{Coh}_X$, this is possible for the class of locally free coherent sheaves ($\star$, Proposition 4.3.24), if we require X to be a projective scheme. With this adapted class, $L(\mathscr{F} \otimes_{\mathscr{O}_X} (-))\colon D^-(\mathrm{Coh}_X) \to D^-(\mathrm{Coh}_X)$ exists.

Having defined the derived functors

$$\begin{aligned} R\Gamma(X,-)\colon D^+(\mathrm{Coh}_X) &\to D^+(\mathrm{Vec}_k^f), \\ Rf_*\colon D^+(\mathrm{Coh}_X) &\to D^+(\mathrm{Coh}_Y), \\ L(\mathscr{F} \otimes_{\mathscr{O}_X} (-))\colon D^-(\mathrm{Coh}_X) &\to D^-(\mathrm{Coh}_X), \end{aligned}$$

and in order to find a common category to work over, we restrict these functors to the ***bounded derived category*** $D^b(\mathrm{Coh}_X)$. This is, as the name suggests, the full triangulated subcategory of $D(\mathrm{Coh}_X)$ whose objects are chain complexes of coherent sheaves that are simultaneously bounded above and bounded below. In addition, we also extend the definition of the derived tensor product to a bifunctor ($\star$, Proposition 4.3.29). At the end of ($\star$, Chapter 4), we are left with the following derived functors

$$\begin{aligned} R\Gamma(X,-)\colon D^b(\mathrm{Coh}_X) &\to D^b(\mathrm{Vec}_k^f), \\ Rf_*\colon D^b(\mathrm{Coh}_X) &\to D^b(\mathrm{Coh}_Y) && \text{for } f \text{ proper}, \\ (-)\otimes^L(-)\colon D^b(\mathrm{Coh}_X)\times D^b(\mathrm{Coh}_X) &\to D^b(\mathrm{Coh}_X), \\ f^*\colon D^b(\mathrm{Coh}_Y) &\to D^b(\mathrm{Coh}_X) && \text{for } f \text{ flat}, \end{aligned}$$

under the additional requirement that X is smooth. The three last functors are the main components of an integral functor. Indeed, consider the following definition.

Definition ($\star$, Definition 5.2.4)**.** Let X and Y be smooth projective varieties over k, and

$$X \xleftarrow{\;q\;} X\times Y \xrightarrow{\;p\;} Y$$

be the projections. The ***integral functor*** $\Phi_{\mathscr{P}^\bullet}^{X\to Y}$ with ***kernel*** $\mathscr{P}^\bullet \in D^b(\mathrm{Coh}_{X\times Y})$ is the functor

$$\begin{aligned} \Phi_{\mathscr{P}^\bullet}^{X\to Y}\colon D^b(\mathrm{Coh}_X) &\to D^b(\mathrm{Coh}_Y) \\ \mathscr{F}^\bullet &\mapsto Rp_*(q^*\mathscr{F}^\bullet \otimes^L \mathscr{P}^\bullet). \end{aligned}$$

We say that $\Phi_{\mathscr{P}^\bullet}^{X\to Y}$ is a ***Fourier-Mukai transform*** if it is an equivalence of categories. In this case, we say that X and Y are ***Fourier-Mukai partners***.

We prove that the composition of integral functors is again an integral functor in ($\star$, Proposition 5.2.7). As a last remark, let us state a celebrated result by Orlov, originally published in [Orl97].

Theorem ($\star$, Theorem 5.2.8)**.** Let $F\colon D^b(\mathrm{Coh}_X)\to D^b(\mathrm{Coh}_Y)$ be a fully faithful exact functor. Then there exists an object $\mathscr{P}^\bullet \in D^b(\mathrm{Coh}_{X\times Y})$, unique up to unique isomorphism, such that F is naturally isomorphic to the integral transform $\Phi_{\mathscr{P}^\bullet}^{X\to Y}$ with kernel $\mathscr{P}^\bullet$.

Corollary ($\star$, Corollary 5.2.9)**.** Any exact equivalence of categories $F\colon D^b(\mathrm{Coh}_X)\to D^b(\mathrm{Coh}_Y)$ is given by a Fourier-Mukai transform $\Phi_{\mathscr{P}^\bullet}^{X\to Y}$, with uniquely defined kernel $\mathscr{P}^\bullet$.

References

[CE56] Henry Cartan and Samuel Eilenberg. *Homological Algebra.* Princeton University Press, 1956.

[God58] Roger Godement. *Topologie algébrique et théorie des faisceaux.* 1958. ISBN: 2-7056-1252-1.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique, I. *Tohoku Mathematical Journal*, 2(9):119–221, 1957. English translation available at: `https://www.math.mcgill.ca/barr/papers/gk.pdf`.

[Har66] Robin Hartshorne. *Residues and Duality: Lecture Notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64.* Springer, Berlin, Heidelberg, 1st edition, 1966. ISBN: 978-3-540-03603-6.

[Orl97] D. O. Orlov. On equivalences of derived categories and K3 surfaces. *J. Math. Sci.*, 84:1361–1381, 1997.

[Ver96] Jean-Louis Verdier. *Des Catégories Dérivées des Catégories Abéliennes.* PhD thesis, Société Mathématique de France, 1996. Available at: `http://www.numdam.org/issue/AST_1996__239__R1_0.pdf`.