On the 3-dimensional Geroch Conjecture and the Positive Mass Theorem

Gonçalo Ruivo Lopes da Fonseca

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Supervisor: Prof. José Natário

Examination Committee

Chairperson: Prof. Miguel Abreu
Supervisor: Prof. José Natário
Member of the Committee: Prof. Pedro Girão
Member of the Committee: Prof. Rosa Sena-Dias

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Declaration

I declare that this document is an original work of my own authorship and that it fulfills all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

...para a minha avó Lurdes.

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Resumo

Inspirados em ideias apresentadas por Kazdan-Warner em [KW75b], e fazendo algumas correções à literatura, provamos nesta tese que um 3-toro, T^3, não possui uma métrica de curvatura escalar não-negativa – o caso tridimensional da Conjectura de Geroch, já generalizado para dimensões arbitrárias por Schoen-Yau [SY79] e Gromov-Lawson [GL80]. Para tal, partimos da suposição de que existe um 2-toro mínimo em T^3 e recorremos a argumentos geométricos para concluir que tal suposição impõe restrições nas métricas possíveis para a variedade, nomeadamente em relação à curvatura escalar.

Recorrendo à Teoria Geométrica da Medida, demonstramos que existe um 2-toro mínimo homologicamente não trivial, providenciando tanto os resultados de existência e regularidade, e concluímos que a suposição anteriormente mencionada é verificada.

Finalmente, relacionamos o teorema principal da tese com um resultado famoso da Teoria da Relatividade - o Teorema da Massa Positiva.

Palavras-chave: Conjetura de Geroch, Toro Mínimo, Curvatura Escalar, Corrente Rectificável, Teorema da Massa Positiva.

Abstract

Inspired by ideas presented by Kazdan-Warner in [KW75b], while making some corrections to the literature, we prove in this thesis that a 3-torus, T^3, does not admit a metric with non-negative scalar curvature - the three-dimensional case of the Geroch Conjecture, already generalized to arbitrary dimensions by Schoen-Yau [SY79] and Gromov-Lawson [GL80]. To do so, we start with the assumption that there is a minimal 2-torus in T^3 and we use geometric arguments to conclude that such an assumption imposes constraints on the possible metrics for the manifold, namely regarding the scalar curvature.

Using Geometric Measure Theory, we show that there is a minimal homologically non-trivial 2-torus, providing both the results of existence and regularity, and we conclude that the previously mentioned assumption indeed holds.

Finally, we relate the main theorem of this thesis with a famous result of Relativity - the Positive Mass Theorem.

Keywords: Geroch's Conjecture, minimal torus, scalar curvature, rectifiable current, Positive Mass Theorem.

Contents

Chapter 1

Background

This chapter serves as a brief presentation of general definitons and preliminary tools that will be used. If the reader is familiar with the basic ideas of Real Analysis (as presented in [RF10]) and Differential Geometry (mainly in [GN14]), this chapter can be skipped. To those unfamiliar with these statements, a warning is in order: these results will not be proven here, since they are outside the scope of the objective of this thesis, and this is not meant to be a thorough introduction.

1.1 Measure Theory

Definition 1.1.1 (Measure)**.** *A (outer) measure μ on $\mathbf{R}^n$ is a non-negative function on all subsets of $\mathbf{R}^n$ that is countably subadditive, i.e.*

$$A \subset \bigcup A_i \implies \mu(A) \leq \sum \mu(A_i).$$

A set $A \subset \mathbf{R}^n$ is said to be μ-measurable if, for all $E \subset \mathbf{R}^n$, we have $\mu(E) = \mu(A^C \cap E) + \mu(A \cap E)$. The class of measurable sets is a σ-algebra – closed under complementation, countable union and countable intersection. The smallest σ-algebra containing all open sets is the σ-algebra of Borel sets.

Definition 1.1.2 (Borel regular)**.** *A measure μ is said to be Borel regular if Borel sets are measurable and every subset of $\mathbf{R}^n$ is contained in a Borel set of the same measure.*

The Lebesgue measure, $\mathcal{L}^n$, is the unique Borel regular, translation invariant measure on $\mathbf{R}^n$, such that the unit cube $[0,1]^n$ has measure 1.

Definition 1.1.3 (Restriction of a measure to a set)**.** *Let $E \subset \mathbf{R}^n$ and μ be a measure on $\mathbf{R}^n$. The restriction of μ to E, $\mu \llcorner E$, is defined as*

$$(\mu \llcorner E)(A) = \mu(E \cap A).$$

Definition 1.1.4 (Densities)**.** *Let μ be a measure on $\mathbf{R}^n$. For $1 \leq m \leq n$ and $a \in \mathbf{R}^n$, we define the*

m-dimensional density of μ at a, $\Theta^m(\mu,\, a)$, by

$$\Theta^m(\mu,\, a) = \lim_{r\to 0} \frac{\mu\left(\mathbf{B}^n(a,r)\right)}{\alpha_m r^m},$$

where α_m is the Lebesgue measure of the closed unit ball $\mathbf{B}^m$ in $\mathbf{R}^m$.

Now, let $A \subset \mathbf{R}^n$. For $1 \le m \le n$ and $a \in \mathbf{R}^n$, we define the m-dimensional density of A at a with respect to the measure μ, $\Theta^m(A,\, a)$, by

$$\Theta^m(A,\, a) = \lim_{r\to 0} \frac{\mu\left(A \cap \mathbf{B}^n(a,r)\right)}{\alpha_m r^m}.$$

Definition 1.1.5 (Lipschitz function)**.** *A function $f : \mathbf{R}^m \to \mathbf{R}^n$ is Lipschitz if there exists $C > 0$ such that for $x,\, y \in \mathbf{R}^m$*

$$|f(x) \;-\; f(y)| \le C|x - y|.$$

The smallest constant C is called the Lipschitz constant and is denoted by $\operatorname{Lip} f$.

A useful result about Lipschitz functions is that they can be approximated in a strong sense by C^1 functions. By strong sense it is meant that they coincide except in a set of small measure. This will allow the substitution of Lipschitz functions for C^1 diffeomorphisms later on.

Proposition 1. *Let $A \subset \mathbf{R}^m$ and $f : A \to \mathbf{R}^n$ be a Lipschitz function. Given $\varepsilon > 0$, there is a C^1 function $g : \mathbf{R}^m \to \mathbf{R}^n$ such that*

$$\mathcal{L}^m \left\{x \in A \,:\, f(x) \ne g(x)\right\} \le \varepsilon.$$

We will also use this section to present some tools and objects from Topology that will provide the building blocks for Differential Geometry.

Definition 1.1.6 (Topology)**.** *Let X be a nonempty set. A topology $\mathcal{T}$ for X is a collection of open subsets of X such that*

1. *both X and $\emptyset$ are open;*
2. *the intersection of any finite collection of open sets is open;*
3. *the union of any collection of open sets is open.*

A space equipped with the previous structure is designated a topological space. However, we want our spaces to behave in an appropriate manner, hence we require further properties to be satisfied.

Definition 1.1.7 (Hausdorff Space)**.** *Let M be a topological space. Then, M is Hausdorff if for each pair of distinct points $p_1, p_2 \in M$, there exist open neighborhoods V_1,V_2 of p_1 and p_2, respectively, such that*

$$V_1 \cap V_2 = \emptyset.$$

Definition 1.1.8 (Topological Manifold)**.** *Consider a topological space M satisfying the following properties:*

1. M *is Hausdorff;*

2. *for each* $p \in M$*, there is a neighbourhood* V *of* p *homeomorphic to an open subset* $U \subset \mathbf{R}^n$*;*

3. M *satisfies the second countability axiom, i.e. it has a countable basis for its topology.*

Then, M *is called a* n*-dimensional topological manifold.*

1.2 Differential Geometry

Let us start by definying the basic object of study of Differential Geometry – a differentiable manifold.

Definition 1.2.1 (Smooth Manifold)**.** *A smooth manifold is a (*n*-dimensional) topological manifold* M *with a family of parameterizations* $\phi_\alpha : U_\alpha \to M$ *defined on open sets* $U_\alpha \subset \mathbf{R}^n$*, such that:*

1. *the coordinate neighborhoods cover* M*, that is,* $\bigcup \phi_\alpha(U_\alpha) = M$*;*

2. *for each pair* α, β *such that*
$$W := \phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$$
the transition maps
$$\phi_\alpha^{-1} \circ \phi_\beta : \phi_\beta^{-1}(W) \to \phi_\alpha^{-1}(W) \quad \textit{and} \quad \phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(W) \to \phi_\beta^{-1}(W)$$
are C^∞*;*

3. *the family* $\{(\phi_\alpha, U_\alpha)\}$ *is maximal regarding to* $1.$ *and* $2.$*.*

Let M be a smooth, n-dimensional manifold and recall that for $p \in M$ one has the set of vectors tangent to M at p , T_pM, the n-dimensional tangent space.

Definition 1.2.2 (Wedge Product)**.** *Given* $u_1,\ u_2,\ v_1,\ v_2$ *vectors in* T_pM*, define the wedge product as the operation satisfying:*

1. *(multilinear)* $(cu_1) \wedge v_1 = c(u_1 \wedge v_1) = u_1 \wedge (cv_1)$ *and* $(u_1 + v_1) \wedge (u_2 + v_2) = (u_1 \wedge u_2) + (u_1 \wedge v_2) + (v_1 \wedge u_2) + (v_1 \wedge v_2)$*;*

2. *(alternating)* $u_1 \wedge v_1 = -v_1 \wedge u_1$*.*

If $\{e_1, ..., e_n\}$ is a basis for T_pM, then the corresponding space of m-vectors, given by linear combinations of $\{e_{i_1} \wedge ... \wedge e_{i_m} : i_1 < ... < i_m\}$, is denoted by $\Lambda_m(T_pM)$ and has dimension $\binom{n}{m}$. A m-vector v is said to be *simple* if it can be written as a single wedge product.

Now, recall that we also have a dual space to T_pM called the cotangent space to M at p, and denoted by T_p^*M, which is the set of covectors. Analogously to $\Lambda_m(T_pM)$, one can define its dual, $\Lambda^m(T_pM)$, which is the space of linear combinations of $\left\{e_{i_1}^* \wedge ... \wedge e_{i_m}^* : i_1 < ... < i_m\right\}$, where $\{e_1^*, ..., e_n^*\}$ is a basis for T_p^*M.

Definition 1.2.3 (Differential Form)**.** *A differential m-form is a smooth map $\omega : M \to \Lambda^m(T^*M)$ that to each $p \in M$ assigns an element $\omega_p \in \Lambda^m(T_p^*M)$.*

One should also recall the uses these forms have, as they provide the natural objects of integration, as well as providing an orientation to the manifold, if orientable (equivalent to the existence of a volume form – non-vanishing n-dimensional form on M).

In particular, let $\mathbf{R}^n$ be the ambient space. We denote by $\mathcal{D}^m$ the set of differential forms with compact support, whose dual will be of use later on. Recall also that we define the pullback, $f^*\omega$, of a m-differential form $\omega \in \mathcal{D}^m$ by a C^∞ map f as

$$(f^*\omega)_p\,(X_1,\,\ldots,X_m) = \omega_{f(p)}(df_p(X_1),\ldots,df_p(X_m)),$$

where $X_1\,\ldots,X_m \in T_p\mathbf{R}^n$.

Definition 1.2.4 (Mass norm, Comass norm)**.** *Assuming that there exists an inner product on T_pM, naturally extended to $\Lambda_m(T_pM)$ and $\Lambda^m(T_p^*M)$, define the mass norm $\|v\|$ and comass norm $\|\omega\|^*$, respectively, as*

$$\|\omega\|^* = \sup\left\{|\,\langle v,\,\omega\rangle\,|\;:\;v\textit{ unit, simple }m\textit{-vector}\right\},$$
$$\|v\| = \sup\{|\,\langle v,\,\omega\rangle\,|\;:\;\|\omega\|^* = 1\},$$

such that they are dual to each other.

More generally speaking, one can define tensors on a vector space V. A (covariant) k-tensor on V is a multilinear function T defined on $V \times ... \times V$ (k times V). The set of all covariant k-tensors is itself a vector space and is denoted by $\mathcal{T}^k(V^*)$. Analogously, we can define a contravariant m-tensor on the dual vector space V^*, which then gives us the space $\mathcal{T}^m(V)$ of contravariant tensors.

In general, we have

Definition 1.2.5 (Tensor Field)**.** *A (k,m)-tensor field is a map that, to each point $p \in M$, assigns a tensor $T \in \mathcal{T}^{k,m}(T_p^*M, T_pM)$.*

Some examples of such tensor fields are:

1. A vector field X is a $(0,1)$-tensor field, in other words, a contravariant 1-tensor field.

2. A differential form is nothing more than an alternating covariant m-tensor field on M.

One particularly useful tensor field is the following.

Definition 1.2.6 (Riemannian Metric)**.** *A Riemmanian metric g on a smooth manifold M is a covariant 2-tensor field satisfying:*

1. *(symmetric) $g(u,\,v) = g(v,\,u)$ for any $u, v \in T_pM$;*

2. *(positive-definite) $g(u,\,u) > 0$ for all $u \in T_pM - \{0\}$.*

Therefore, a Riemannian metric is a smooth assignment of an inner product to each T_pM. A smooth manifold M equipped with a Riemannian metric g is called a Riemannian manifold, and the pairing is denoted by $(M,\, g)$. Such objects are the subject of Riemannian Geometry.

One of the most important tools of Riemannian Geometry is the Gauss-Bonnet theorem, which connects assertions about curvature (geometry of the surface) and the Euler characteristic (topology of the surface), and will be used later on. Recall also that, for two-dimensional manifolds, the scalar curvature is twice the Gaussian curvature.

Theorem 1 (Gauss-Bonnet)**.** *Let (M, g) be an oriented, compact 2-dimensional manifold with Gauss curvature K, and let X be a vector field in M with isolated singularities $p_1,\, p_2,\, ..., p_k$. Then*

$$\int_M K = 2\pi \sum_{i=1}^k I_{p_i} = 2\pi\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M, and I_{p_i} is the index of X at p_i.

Chapter 2

Positive Scalar Curvature

Given a 3-dimensional torus, T^3, and assuming that it admits a 2-dimensional stable minimizing torus T^2, we will prove in this chapter a weak version of the 3-dimensional case of the Geroch Conjecture, i.e. that the scalar curvature of T^3 cannot be positive. To do so, we deduce the relation between the scalar curvatures of the ambient manifold and of the miminal 2-torus. For subsequent computations, Einstein's notation will be used and we will denote the parameter t by the coordinate index 0, following [Nat21].

2.1 Gauss' Lemma

Lemma 1. *Let (M, g) be a Riemannian manifold and $S \subset M$ a hypersurface with unit normal vector field η. The hypersurfaces S_t, obtained from S by moving a distance t along the geodesics with initial condition η, orthogonal to S, remain orthogonal to the geodesics.*

Proof. Let us start by defining Gaussian normal coordinates (which will be used later on). For each $p \in S$, let γ be the geodesic starting at p with initial tangent vector η_p, and let $(x^1, x^2, ..., x^{n-1})$ be local coordinates in S parameterizing p. Moving along γ by a parameter t gives us, for a small neighbourhood of p, a coordinate chart $(t, x^1, x^2, ..., x^{n-1})$ called Gaussian normal coordinates.

We claim that the geodesics remain orthognal to the hypersurfaces S_t defined by the level sets of t. Clearly, for $t = 0$ we "remain in" S, therefore we have orthogonality. Now, let ∂_i be the coordinate basis fields for $i = 1, ..., n-1$, and recall that such fields commute. Then,

$$\begin{aligned}
\partial_t \langle \partial_t, \partial_i \rangle &= \langle \nabla_{\partial_t} \partial_t, \partial_i \rangle + \langle \partial_t, \nabla_{\partial_t} \partial_i \rangle \\
&= \langle \partial_t, \nabla_{\partial_t} \partial_i \rangle = \langle \partial_t, \nabla_{\partial_i} \partial_t \rangle \\
&= \frac{1}{2} \partial_i \langle \partial_t, \partial_t \rangle \\
&= 0,
\end{aligned}$$

since the normalization $g(\eta,\eta) = \langle \partial_t, \partial_t \rangle = 1$ is preserved by parallel transport and, by the geodesic equation, $\nabla_{\partial_t}\partial_t = 0$.

We then have that $\langle \partial_t, \partial_i \rangle$ is independent of t, and consequently, given that it vanishes on the hypersurface S, it remains zero. □

2.2 Ricci and Scalar Curvature in Gauss Coordinates

Suppose there is a Riemannian metric in T^3 given in the Gauss lemma form, i.e.

$$g = dt^2 + h_{ij}(t,x)dx^i dx^j,$$

such that the level sets of *t* are Riemannian manifolds themselves, with an induced metric $h(t) = h_{ij}dx^i dx^j$ and a second fundamental form given by

$$K(t) = \frac{1}{2}\frac{\partial h_{ij}}{\partial t}dx^i dx^j.$$

Proposition 2. *The scalar curvatures of T^3 and its hypersurfaces are related by the equation:*

$$R = \bar{R} - 2\frac{\partial}{\partial t}K^i_i - (K^i_i)^2 - K_{ij}K^{ij}, \tag{2.1}$$

where R, $\bar{R}$ are the scalar curvatures of T^3 and its hypersurfaces, respectively.

Proof. The result in this theorem follows from some simple computations. In this metric, the Christoffel symbols are:

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{l0}\left\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right\} \\
&= \frac{1}{2}\left\{\frac{\partial g_{j0}}{\partial x_i} + \frac{\partial g_{0i}}{\partial x_j} - \frac{\partial h_{ij}}{\partial t}\right\} \\
&= -\frac{1}{2}\frac{\partial h_{ij}}{\partial t} \\
&= -K_{ij}; \\
\Gamma^i_{jk} &= \frac{1}{2}g^{li}\left\{\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l}\right\} \\
&= \frac{1}{2}g^{0i}\left\{\frac{\partial g_{k0}}{\partial x_j} + \frac{\partial g_{0j}}{\partial x_k} - \frac{\partial g_{jk}}{\partial t}\right\} \\
&+ \frac{1}{2}h^{li}\left\{\frac{\partial h_{kl}}{\partial x_j} + \frac{\partial h_{lj}}{\partial x_k} - \frac{\partial h_{jk}}{\partial x_l}\right\} \\
&= 0 + \bar{\Gamma}^i_{jk} \\
&= \bar{\Gamma}^i_{jk};
\end{aligned}$$

where $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols of the induced metric h, and also:

$$
\begin{aligned}
\Gamma^i_{0j} &= \frac{1}{2} g^{li} \left\{ \frac{\partial g_{jl}}{\partial t} + \frac{\partial g_{l0}}{\partial x_k} - \frac{\partial g_{0j}}{\partial x_l} \right\} \\
&= 0 + \frac{1}{2} h^{li} \left\{ \frac{\partial g_{jl}}{\partial t} + 0 \right\} \\
&= h^{li} \frac{1}{2} \frac{\partial h_{jl}}{\partial t} \\
&= h^{li} K_{jl} \\
&= K^i_j; \\
\Gamma^i_{00} &= \frac{1}{2} g^{li} \left\{ \frac{\partial g_{0l}}{\partial t} + \frac{\partial g_{l0}}{\partial t} - \frac{\partial g_{00}}{\partial x_l} \right\} = 0; \\
\Gamma^0_{0j} &= \frac{1}{2} g^{l0} \left\{ \frac{\partial g_{jl}}{\partial t} + \frac{\partial g_{l0}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_l} \right\} = 0.
\end{aligned}
$$

Given the previous Christoffel symbols, it is now possible to compute the Riemannian curvature tensor coefficients and, consequently, the Ricci tensor coefficients. Firstly, for the Riemann tensor we have:

$$
\begin{aligned}
R^j_{0i0} &= \Gamma^l_{00}\Gamma^j_{il} - \Gamma^l_{i0}\Gamma^j_{0l} + \frac{\partial \Gamma^j_{00}}{\partial x_i} - \frac{\partial \Gamma^j_{i0}}{\partial t} \\
&= -K^l_i K^j_l - \frac{\partial}{\partial t} K^i_j \\
&= -\frac{\partial}{\partial t} K^i_j - K_{is} h^{sl} h_{lr} K^{rj} \\
&= -\frac{\partial}{\partial t} K^i_j - K_{il} K^{lj}; \\
R^0_{i0j} &= \Gamma^r_{ij}\Gamma^0_{0r} - \Gamma^r_{0j}\Gamma^0_{ir} + \frac{\partial \Gamma^0_{ij}}{\partial t} - \frac{\partial \Gamma^0_{0j}}{\partial x_i} \\
&= -\frac{\partial K_{ij}}{\partial t} - \Gamma^l_{0j}\Gamma^0_{il} - \Gamma^0_{0j}\Gamma^0_{i0} \\
&= -\frac{\partial K_{ij}}{\partial t} + K^l_j K_{il}; \\
R^s_{ij0} &= \Gamma^l_{i0}\Gamma^s_{jl} - \Gamma^l_{j0}\Gamma^s_{il} + \frac{\partial \Gamma^s_{i0}}{\partial x_j} - \frac{\partial \Gamma^s_{j0}}{\partial x_i} \\
&= K^l_i \bar{\Gamma}^s_{jl} - K^l_j \bar{\Gamma}^s_{il} + \frac{\partial K^s_i}{\partial x_j} - \frac{\partial K^s_j}{\partial x_i} - \frac{\partial K^s_j}{\partial x_i} \\
&- K^l_j \bar{\Gamma}^s_{il} + K^s_l \bar{\Gamma}^l_{ij} - K^s_l \bar{\Gamma}^l_{ij} + \frac{\partial K^s_i}{\partial x_j} + K^l_i \bar{\Gamma}^s_{jl} \\
&= -\bar{\nabla}_i K^s_j + \bar{\nabla}_j K^s_i; \\
R^s_{ijk} &= \Gamma^r_{ik}\Gamma^s_{jr} - \Gamma^r_{jk}\Gamma^s_{ir} + \frac{\partial \Gamma^s_{ik}}{\partial x_j} - \frac{\partial \Gamma^s_{jk}}{\partial x_i} \\
&= \frac{\partial \bar{\Gamma}^s_{ik}}{\partial x_j} - \frac{\partial \bar{\Gamma}^s_{jk}}{\partial x_i} + \bar{\Gamma}^l_{ik}\bar{\Gamma}^s_{jl} - \bar{\Gamma}^l_{jk}\bar{\Gamma}^s_{il} \\
&+ \Gamma^0_{ik}\Gamma^s_{j0} - \Gamma^0_{jk}\Gamma^s_{i0} \\
&= \bar{R}^s_{ijk} - K^s_j K_{ik} + K^s_i K_{jk},
\end{aligned}
$$

where $\bar{R}^s_{ijk}$ are the components for the Riemann curvature tensor of the induced metric h. Now, for the Ricci tensor components, we derive:

$$\begin{aligned}
R_{00} &= R^i_{0i0} = -\partial_t K^i_i - K_{ij}K^{ij};\\
R_{i0} &= R^j_{ij0} = -\bar{\nabla}_i K^j_j + \bar{\nabla}_j K^j_i;\\
R_{ij} &= R^0_{i0j} + R^l_{ilj} = -\frac{\partial K_{ij}}{\partial t} + K^l_j K_{il} + \bar{R}^l_{ilj} - K^l_l K_{ij} + K^l_i K_{lj}.
\end{aligned}$$

Noting that $K^l_i K_{lj} = K_{ir}h^{rl}h_{ls}K^s_j = K^l_j K_{il}$ we conclude that

$$R_{ij} = \bar{R}_{ij} - \frac{\partial K_{ij}}{\partial t} + 2K^l_j K_{il} - K^l_l K_{ij}.$$

Before computing the relationship between the scalar curvatures of the ambient manifold and of the level sets of t, we need to compute the time derivative of the inverse induced metric. Recalling that $K^j_i = \Gamma^j_{0i} = h^{jl}K_{li}$ and also that $K_{ij} = \frac{1}{2}\frac{\partial h_{ij}}{\partial t}$, we then have

$$\begin{aligned}
K^{ij} = h^{jl}K^i_l = h^{jl}h^{ik}K_{lk}&\\
&= \frac{1}{2}h^{jl}h^{ik}\frac{\partial h_{lk}}{\partial t} =\\
&= \frac{1}{2}h^{jl}\left\{\frac{\partial(h^{ik}h_{lk})}{\partial t} - h_{lk}\frac{\partial h^{ik}}{\partial t}\right\}\\
&= \frac{1}{2}h^{jl}\partial_t\delta^i_l - \frac{1}{2}h^{jl}h_{lk}\frac{\partial h^{ik}}{\partial t}\\
&= -\frac{1}{2}\delta^j_k\frac{\partial h^{ik}}{\partial t}\\
&= -\frac{1}{2}\frac{\partial h^{ij}}{\partial t},
\end{aligned}$$

which consequently gives us

$$\frac{\partial h^{ij}}{\partial t} = -2K^{ij}. \tag{2.2}$$

For the scalar curvature, it follows from definition and the previous computations that

$$\begin{aligned}
R = R_{00} + 2g^{0i}R_{0i} + h^{ij}R_{ij} =&\\
= -\partial_t K^i_i - K_{ij}K^{ij} + 0 + h^{ij}\left\{\bar{R}_{ij} - \frac{\partial K_{ij}}{\partial t} + 2K^l_j K_{il} - K^l_l K_{ij}\right\}&\\
= -\partial_t K^i_i - K_{ij}K^{ij} + \bar{R} - h^{ij}\frac{\partial K_{ij}}{\partial t} + 2h^{ij}K^l_j K_{il} - h^{ij}K^l_l K_{ij}&\\
= -\partial_t K^i_i - K_{ij}K^{ij} + \bar{R} - \frac{\partial(h^{ij}K_{ij})}{\partial t} + K_{ij}\frac{\partial h^{ij}}{\partial t} + 2K^{il}K_{il} - K^l_l K^i_i.&
\end{aligned}$$

However, due to (2.2), we then have that

$$R = -2\partial_t K^i_i - K_{ij}K^{ij} + \bar{R} - 2K_{ij}K^{ij} + 2K^{ij}K_{ij} - (K^i_i)^2,$$

and so

$$R = \bar{R} - 2\frac{\partial}{\partial_t} K_i^i - (K_i^i)^2 - K_{ij}K^{ij}.$$

□

2.3 Minimizing Condition

The objective of this section is to use (2.1) to prove a weaker version of the 3-dimensional Geroch conjecture, thus concluding the goal of the chapter. Essentiallly, we want to prove that a metric with positive scalar curvature is incompatible with the asssumption of a stable, minimal T^2.

Theorem 2. *Suppose T^3 admits a stable area minimizing 2-torus T^2. Then, T^3 does not admit a metric g of positive scalar curvature.*

Proof. Assume that there exists a area minimizing 2-torus T^2, in the 3-torus T^3, with an induced metric given by Gauss coordinates and volume form given by $\sigma = \sqrt{\det(h)}dx^1 \wedge dx^2$, and recall that for any matrix-valued function M we have the identity $\partial_t \det(M) = \det(M)\operatorname{tr}(M^{-1}\partial_t M)$. Computing the formulas for the first and second variation of the volume form we obtain:

$$\begin{aligned}
\frac{\partial}{\partial t}\sigma &= \frac{\partial\sqrt{\det(h_{ij})}}{\partial t}dx^1 \wedge dx^2 = \\
&= \frac{1}{2}\det(h_{ij})\frac{1}{\sqrt{\det(h_{ij})}}(h^{ij}\frac{\partial h_{ij}}{\partial t})dx^1 \wedge dx^2 \\
&= \sqrt{\det(h_{ij})}h^{ij}K_{ij}dx^1 \wedge dx^2 \\
&= K_i^i\sigma. \\
\frac{\partial^2}{\partial t^2}\sigma &= \frac{\partial}{\partial t}(K_i^i\sigma) \\
&= \frac{\partial K_i^i}{\partial t}\sigma + K_i^i\frac{\partial}{\partial t}\sigma \\
&= \left\{\frac{\partial K_i^i}{\partial t} + (K_i^i)^2\right\}\sigma.
\end{aligned}$$

By rewriting equation (2.1), we see that

$$\frac{\partial}{\partial t}K_i^i = \frac{1}{2}\left\{\bar{R} - R - (K_i^i)^2 - K_{ij}K^{ij}\right\},$$

hence we arrive at the following formula for the second variation of the volume form:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\sigma &= \frac{1}{2}\left\{\bar{R} - R - (K_i^i)^2 - K_{ij}K^{ij}\right\}\sigma + (K_i^i)^2\sigma \\
&= \frac{1}{2}\left\{\bar{R} - R + (K_i^i)^2 - K_{ij}K^{ij}\right\}\sigma.
\end{aligned}$$

By the previous area minimizing assumption, since the first variation has to vanish, we conclude from the first formula that $\int_{T^2} K^i_i \sigma = 0$. In fact, more is true, as it is well known that $K^i_i = 0$ pointwise for minimal surfaces. Now, assuming that the 2-torus is a stable minimizer, the second variation has to be non-negative, which in turn gives us

$$\frac{1}{2}\int_{T^2} \left\{\bar{R} - R + (K^i_i)^2 - K_{ij}K^{ij}\right\} \sigma \geq 0,$$

and given that $K^i_i = 0$, we conclude that

$$\frac{1}{2}\int_{T^2} \left\{\bar{R} - R - K_{ij}K^{ij}\right\} \sigma \geq 0.$$

Furthermore, by the Gauss-Bonnet theorem, since the torus T^2 is a 2-dimensional, oriented and compact Riemannian manifold such that its Euler Characteristic is zero, $\chi(T^2) = 0$, we have that

$$\int_{T^2} \bar{R} = 0.$$

Consequently,

$$-\frac{1}{2}\int_{T^2} R + K_{ij}K^{ij} \geq 0 \Leftrightarrow \int_{T^2} R + K_{ij}K^{ij} \leq 0.$$

Note that $K_{ij}K^{ij}$ is a sum of squares, hence positive. Therefore, if the 3-torus T^3 admits an area minimizing 2-torus, T^2, there is no metric on the 3-torus with positive scalar curvature.

$\square$

Chapter 3

Non-Negative Scalar Curvature

Following results and ideas from [KW75b], through pointwise conformal deformations of a given metric, we will show that the only metric with non-negative scalar curvature on T^3 is the flat metric, which is the 3-dimensional case of the Geroch Conjecture.

3.1 Scalar Curvature of Conformal Deformation

First, in order to provide some backgroung to the operator used in [KW75a], some computations are mandatory. Given a smooth, positive function $u : T^3 \to \mathbf{R}$, we will consider the following conformal deformation of (T^3, g):

$$\tilde{g} = u^4 g.$$

Now, for the result of this section.

Proposition 3. *The scalar curvature $\tilde{R}$ of $\tilde{g}$ is related to the curvature R of g by the equation:*

$$u^5 \tilde{R} = -8\Delta u + Ru. \tag{3.1}$$

Proof. Again, computing the Christoffel symbols for this new metric, we have

$$
\begin{aligned}
\tilde{\Gamma}^k_{ij} &= \frac{1}{2}\tilde{g}^{lk}\left\{\frac{\partial \tilde{g}_{jl}}{\partial x_i} + \frac{\partial \tilde{g}_{li}}{\partial x_j} - \frac{\partial \tilde{g}_{ij}}{\partial x_l}\right\} \\
&= \frac{1}{2u^4}g^{lk}\left\{\frac{\partial (u^4 g_{jl})}{\partial x_i} + \frac{\partial (u^4 g_{li})}{\partial x_j} - \frac{\partial (u^4 g_{ij})}{\partial x_l}\right\} \\
&= \frac{1}{2}g^{lk}\left\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right\} \\
&+ \frac{1}{2u^4}g^{lk}\left\{\frac{\partial u^4}{\partial x_i}g_{jl} + \frac{\partial u^4}{\partial x_j}g_{li} - \frac{\partial u^4}{\partial x_l}g_{ij}\right\} = \\
&= \Gamma^k_{ij} + \frac{4}{2u^4}g^{lk}u^3\left\{\frac{\partial u}{\partial x_i}g_{jl} + \frac{\partial u}{\partial x_j}g_{li} - \frac{\partial u}{\partial x_l}g_{ij}\right\} \\
&= \Gamma^k_{ij} + 2g^{lk}\left\{\frac{\partial_i u}{u}g_{jl} + \frac{\partial_j u}{u}g_{li} - \frac{\partial_l u}{u}g_{ij}\right\} = \\
&= \Gamma^k_{ij} + 2\left\{\partial_i(\ln(u))g^k_j + \partial_j(\ln(u))g^k_i - \partial^k(\ln(u))g_{ij}\right\}.
\end{aligned}
$$

For notation purposes, define $D^k_{ij} = \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}$. Then, the coefficients of the Riemann curvature for this new conformal metric are

$$
\begin{aligned}
\tilde{R}^s_{ijk} &= \tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{jr} - \tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{ir} + \frac{\partial \tilde{\Gamma}^s_{ik}}{\partial x_j} - \frac{\partial \tilde{\Gamma}^s_{jk}}{\partial x_i} \\
&= (\Gamma^r_{ik} + D^r_{ik})(\Gamma^s_{jr} + D^s_{jr}) - (\Gamma^r_{jk} + D^r_{jk})(\Gamma^s_{ir} + D^s_{ir}) + \frac{\partial(\Gamma^s_{ik} + D^s_{ik})}{\partial x_j} - \frac{\partial(\Gamma^s_{jk} + D^s_{jk})}{\partial x_i} = \\
&= \Gamma^r_{ik}\Gamma^s_{jr} - \Gamma^r_{jk}\Gamma^s_{ir} + \frac{\partial \Gamma^s_{ik}}{\partial x_j} - \frac{\partial \Gamma^s_{jk}}{\partial x_i} + \frac{\partial D^s_{ik}}{\partial x_j} - \frac{\partial D^s_{jk}}{\partial x_i} \\
&+ \Gamma^r_{ik}D^s_{jr} + \Gamma^s_{jr}D^r_{ik} - \Gamma^r_{jk}D^s_{ir} - \Gamma^s_{ir}D^r_{jk} + D^r_{ik}D^s_{jr} - D^r_{jk}D^s_{ir} = \\
&= R^s_{ijk} + \frac{\partial D^s_{ik}}{\partial x_j} - \frac{\partial D^s_{jk}}{\partial x_i} + \Gamma^r_{ik}D^s_{jr} + \Gamma^s_{jr}D^r_{ik} - \Gamma^r_{jk}D^s_{ir} - \Gamma^s_{ir}D^r_{jk} + D^r_{ik}D^s_{jr} - D^r_{jk}D^s_{ir}.
\end{aligned}
$$

Now, we compute the components of the Ricci curvature tensor. First, note that

$$
\begin{aligned}
\nabla_i D^l_{jk} &= \nabla_i \tilde{\Gamma}^l_{jk} - \nabla_i \Gamma^l_{jk} = \\
&= \partial_i \tilde{\Gamma}^l_{jk} + \tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^m_{ij}\tilde{\Gamma}^l_{mk} - \tilde{\Gamma}^l_{jk}\tilde{\Gamma}^m_{ik} \\
&- \partial_i \Gamma^l_{jk} - \Gamma^l_{im}\Gamma^m_{jk} + \Gamma^m_{ij}\Gamma^l_{mk} + \Gamma^l_{jk}\Gamma^m_{ik} = \\
&= \partial_i D^l_{jk} + (\Gamma^l_{im} + D^l_{im})(\Gamma^m_{jk} + D^m_{jk}) - (\Gamma^m_{ij} + D^m_{ij})(\Gamma^l_{mk} + D^l_{mk}) \\
&- (\Gamma^l_{jk} + D^l_{jk})(\Gamma^m_{ik} + D^m_{ik}) - \Gamma^l_{im}\Gamma^m_{jk} + \Gamma^m_{ij}\Gamma^l_{mk} + \Gamma^l_{jk}\Gamma^m_{ik} = \\
&= \partial_i D^l_{jk} + \Gamma^l_{im}D^m_{jk} + D^l_{im}\Gamma^m_{jk} - \Gamma^m_{ij}D^l_{mk} - \Gamma^l_{mk}D^m_{ij} + \Gamma^l_{jk}D^m_{ik} + \Gamma^m_{ik}D^l_{jk} \\
&+ D^l_{im}D^m_{jk} - D^m_{ij}D^l_{mk} - D^l_{jk}D^m_{ik},
\end{aligned}
$$

hence, while being careful to both add and subtract the necessary components, we see that

$$\begin{aligned}
\tilde{R}_{ij} &= R^k_{ikj} + \partial_k D^k_{ij} - \partial_i D^k_{kj} + \Gamma^r_{ij} D^k_{kr} + \Gamma^k_{kr} D^r_{ij} - \Gamma^r_{kj} D^k_{ir} - \Gamma^k_{ir} D^r_{kj} + D^r_{ij} D^k_{kr} - D^r_{kj} D^k_{ir} = \\
&= \left\{ (\partial_k D^k_{ij} + \Gamma^k_{kr} D^r_{ij} + \Gamma^r_{ij} D^k_{kr}) - \Gamma^r_{ki} D^k_{rj} - \Gamma^k_{rj} D^r_{ki} - \Gamma^k_{ij} D^r_{kj} - \Gamma^r_{kj} D^k_{ij} + D^r_{ij} D^k_{kj} - D^r_{ki} D^k_{rj} - D^k_{ij} D^r_{kj} \right\} \\
&- \left\{ (\partial_i D^k_{kj} + \Gamma^r_{kj} D^k_{ir} + \Gamma^k_{ir} D^r_{kj}) - \Gamma^r_{ki} D^k_{rj} - \Gamma^k_{rj} D^r_{ki} - \Gamma^k_{ij} D^r_{kj} - \Gamma^r_{kj} D^k_{ij} + D^r_{kj} D^k_{ij} - D^r_{ki} D^k_{rj} - D^k_{kj} D^r_{ij} \right\} \\
&+ D^r_{ij} D^k_{kr} - D^r_{kj} D^k_{ir} = \\
&= R_{ij} + \nabla_k D^k_{ik} - \nabla_i D^k_{kj} + D^r_{ij} D^k_{kr} - D^r_{kj} D^k_{ir}.
\end{aligned}$$

Considering each term on the right-hand side, and recalling that $\nabla_k g = 0$ and $\nabla_j \ln(u) = \partial_j \ln(u)$, we have:

$$\begin{aligned}
\nabla_i D^k_{kj} &= 2\nabla_i \left[\partial_k(\ln(u)) g^k_j + \partial_j(\ln(u)) g^k_k - \partial^k(\ln(u)) g_{jk} \right] = \\
&= 2 \left\{ \nabla_i(\partial_j \ln u) + \nabla_i(\partial_j \ln(u)) g^k_k - \nabla_i(\partial_j \ln(u)) \right\} \\
&= 6\nabla_i(\partial_j \ln(u)). \\
\nabla_k D^k_{ij} &= 2\nabla_k \left[\partial_i(\ln(u)) g^k_j + \partial_j(\ln(u)) g^k_i - \partial^k(\ln(u)) g_{ij} \right] = \\
&= 2 \left[g^k_j \nabla_k(\partial_i \ln(u)) + g^k_i \nabla_k(\partial_j \ln(u)) - \nabla_k(\partial^k \ln(u)) g_{ij} \right] = \\
&= 2 \left[\nabla_j(\partial_i \ln(u)) + \nabla_i(\partial_j \ln(u)) - \nabla_k(\partial^k ln(u)) g_{ij} \right] \\
&= 4\nabla_i(\partial_j \ln(u)) - 2\Delta \ln(u) g_{ij}.
\end{aligned}$$

$$\begin{aligned}
D^r_{ij} D^k_{kr} &= 4 \left[\partial_i(\ln(u)) g^r_j + \partial_j(\ln(u)) g^r_i - \partial^r(\ln(u)) g_{ij} \right] \left[\partial_k(\ln(u)) g^k_r + \partial_r(\ln(u)) g^k_k - \partial^k(\ln(u)) g_{kr} \right] = \\
&= 12 \left[\partial_i(\ln(u)) g^r_j + \partial_j(\ln(u)) g^r_i - \partial^r(\ln(u)) g_{ij} \right] \partial_r(\ln(u)) = \\
&= 12 \left[\partial_i(\ln(u)) \partial_j(\ln(u)) + \partial_j(\ln(u)) \partial_i(\ln(u)) - |\operatorname{grad}(\ln(u))|^2 g_{ij} \right] = \\
&= 24 \partial_i(\ln(u)) \partial_j(\ln(u)) - 12 |\operatorname{grad}(\ln(u))|^2 g_{ij}.
\end{aligned}$$

$$\begin{aligned}
D^r_{kj} D^k_{ir} &= 4 \left[\partial_k(\ln(u)) g^r_j + \partial_j(\ln(u)) g^r_k - \partial^r(\ln(u)) g_{kj} \right] \left[\partial_i(\ln(u)) g^k_r + \partial_r(\ln(u)) g^k_i - \partial^k(\ln(u)) g_{ir} \right] = \\
&= 4\partial_k(\ln(u)) \left\{ g^r_j g^k_r \partial_i(\ln(u)) + g^r_j g^k_i \partial_r(\ln(u)) - g^r_j g_{ir} \partial^k(\ln(u)) \right\} \\
&+ 4\partial_j(\ln(u)) \left\{ g^r_k g^k_r \partial_i(\ln(u)) + g^r_k g^k_i \partial_r(\ln(u)) - g^r_k g_{ir} \partial^k(\ln(u)) \right\} \\
&- 4\partial^r(\ln(u)) \left\{ g_{kj} g^k_r \partial_i(\ln(u)) + g_{kj} g^k_i \partial_r(\ln(u)) - g_{kj} g_{ir} \partial^k(\ln(u)) \right\} = \\
&= 4 \left\{ \partial_j(\ln(u)) \partial_i(\ln(u)) + \partial_j(\ln(u)) \partial_i(\ln(u)) + 3\partial_j(\ln(u)) \partial_i(\ln(u)) - 2\partial_k(\ln(u)) \partial^k(\ln(u)) g_{ij} \right\} = \\
&= 20 \partial_i(\ln(u)) \partial_j(\ln(u)) - 8 |\operatorname{grad}(\ln(u))|^2 g_{ij}.
\end{aligned}$$

Therefore, the Ricci coefficients are

$$
\begin{aligned}
\tilde{R}_{ij} &= R^k_{ikj} - \nabla_i D^k_{kj} + \nabla_k D^r_{ik} + D^r_{ij} D^k_{kr} - D^r_{kj} D^k_{ir} \\
&= R_{ij} + 4\nabla_i(\partial_j \ln(u)) - 2\Delta \ln(u) g_{ij} - 6\nabla_i(\partial_j \ln(u)) + 24\partial_i(\ln(u))\partial_j(\ln(u)) \\
&- 12|\operatorname{grad}(\ln(u))|^2 g_{ij} - 20\partial_i(\ln(u))\partial_j(\ln(u)) + 8|\operatorname{grad}(\ln(u))|^2 g_{ij} = \\
&= R_{ij} - 2\nabla_i(\partial_j \ln(u)) - 2\Delta \ln(u) g_{ij} + 4\partial_i(\ln(u))\partial_j(\ln(u)) - 4|\operatorname{grad}(\ln(u))|^2 g_{ij}.
\end{aligned}
$$

Given the previous Ricci coefficients, we can now compute the scalar curvature of the conformally related metric. However, as the equation (3.1) suggests, we want to work with the function u and not with $\ln(u)$, hence

$$
|\operatorname{grad}(\ln(u))|^2 = \partial_k \ln(u) \partial^k \ln(u) = \frac{\partial_k u}{u}\frac{\partial^k u}{u} = \frac{|\operatorname{grad}(u)|^2}{u^2},
$$

and also

$$
\Delta(\ln(u)) = \nabla^k(\frac{\partial_k u}{u}) = \partial^k(\frac{1}{u})\partial_j u + \frac{1}{u}\nabla^k(\partial_k u) = -\frac{|\operatorname{grad}(u)|^2}{u^2} + \frac{\Delta u}{u}.
$$

Therefore, the scalar curvature is

$$
\begin{aligned}
\tilde{R} &= \frac{1}{u^4} g^{ij}\left\{R_{ij} - 2\nabla_i(\partial_j \ln(u)) - 2\Delta \ln(u) g_{ij} + 4\partial_i(\ln(u))\partial_j(\ln(u)) - 4|\operatorname{grad}(\ln(u))|^2 g_{ij}\right\} = \\
&= \frac{1}{u^4}R - \frac{2}{u^4}\nabla^j\nabla_j(\ln(u)) - \frac{6}{u^4}\Delta \ln(u) + \frac{4}{u^4}\partial^j(\ln(u))\partial_j(\ln(u)) - \frac{12}{u^4}|\operatorname{grad}(\ln(u))|^2 = \\
&= \frac{1}{u^4}R - \frac{8}{u^4}\Delta \ln(u) - \frac{8}{u^4}|\operatorname{grad}(\ln(u))|^2 = \\
&= \frac{1}{u^4}R - \frac{8}{u^5}\Delta u + \frac{8}{u^6}|\operatorname{grad}(u)|^2 - \frac{8}{u^6}|\operatorname{grad}(u)|^2 = \\
&= \frac{1}{u^4}R - \frac{8}{u^5}\Delta u.
\end{aligned}
$$

Finally, multiplying both sides by u^5, we arrive at (3.1). $\square$

3.2 Zero Scalar Curvature

Using the same notation as in [KW75b], consider now the elliptic differential operator given by $L_g u \equiv -8\Delta u + Ru = \tilde{R}u^5$. Since T^3 is a compact manifold, we can take $\lambda_1(g)$ as the lowest eigenvalue of the operator with a corresponding positive eigenfunction ψ. Consider also the following two lemmas.

Lemma 2. *Let M be a compact and connected manifold, with $\dim(M) \geq 3$. Then, M admits a metric pointwise conformal to g with positive (zero, or negative) scalar curvature if and only if $\lambda_1(g) > 0$ ($\lambda_1(g) = 0$, or $\lambda_1(g) < 0$, respectively).*

Proof. Let $\bar{R}$ be the scalar curvature of the metric $\bar{g}$ pointwise conformal to g with conformal factor the eigenfunction ψ corresponding to the first eigenvector, and let

$$
\langle u, v\rangle_{L^2} = \int_M uv
$$

be the L^2 the inner product on $C^\infty(M)$. Then,

$$\begin{aligned}
&L_g\psi = \bar{R}\psi^5 \implies \langle L_g\psi,\, \psi\rangle_{L^2} = \left\langle \bar{R}\psi^5,\, \psi\right\rangle_{L^2} \implies \\
&\langle \lambda_1(g)\psi,\, \psi\rangle_{L^2} = \left\langle \bar{R}\psi^5,\, \psi\right\rangle_{L^2} \implies \\
&\lambda_1(g)\,\langle \psi,\, \psi\rangle_{L^2} = \left\langle \bar{R}\psi^5,\, \psi\right\rangle_{L^2},
\end{aligned}$$

which gives us the "only if" part.

For the other implication, let $\lambda_1(g)$ be the first eigenvalue as previously stated. Rewriting the equation, we see that

$$L_g\psi = \lambda_1(g)\psi \iff L_g\psi = \lambda_1(g)\psi^{-4}\psi^5$$

which gives a conformal metric $\bar{g} = \psi^4 g$ with scalar curvature $\bar{R} = \lambda_1(g)\psi^{-4}$, whose sign depends only on $\lambda_1(g)$. □

Lemma 3. *Let M be a compact manifold that does not admit a metric with positive scalar curvature. Then, any metric with zero scalar curvature must have zero Ricci curvature.*

Proof. Let g be a metric with zero scalar curvature, $R_g = 0$. Then, by the previous lemma, $\lambda_1(g) = 0$. Suppose also that the associated Ricci curvature, S, is not zero and, for notation purposes, write $g_t = g(t) = g - tS$, $L_t = L_{g(t)}$, and $\lambda_1(t) = \lambda_1(g(t))$ for t sufficiently small.

Considering the normalized eigenfunction, we have the equation $L_t\psi(t) = \lambda_1(t)\psi(t)$ and, differentiating both sides with respect to t, we get

$$\begin{aligned}
&\partial_t(L_t\psi(t)) = \partial_t(\lambda_1(t)\psi(t)) \iff \\
&L_t'\psi(t) + L_t\psi'(t) = \lambda_1'(t)\psi(t) + \lambda_1(t)\psi'(t).
\end{aligned}$$

Now, taking the inner product (defined in the previous Lemma) with $\psi(t)$, since the eigenfunction is normalized, we see that

$$\begin{aligned}
&\langle L_t'\psi(t),\, \psi(t)\rangle_{L^2} + \langle L_t\psi'(t),\, \psi(t)\rangle_{L^2} = \langle \lambda_1'(t)\psi(t),\, \psi(t)\rangle_{L^2} + \langle \lambda_1(t)\psi'(t),\, \psi(t)\rangle_{L^2} \iff \\
&\langle L_t'\psi(t),\, \psi(t)\rangle_{L^2} + \langle L_t\psi'(t) - \lambda_1(t)\psi'(t),\, \psi(t)\rangle_{L^2} = \lambda_1'(t)\,\langle \psi(t),\, \psi(t)\rangle_{L^2} \iff \\
&\langle L_t'\psi(t),\, \psi(t)\rangle_{L^2} + \langle (L_t - \lambda_1(t))\,\psi'(t),\, \psi(t)\rangle_{L^2} = \lambda_1'(t),
\end{aligned}$$

which, at $t = 0$ is:

$$\langle L_0'\psi(0),\, \psi(0)\rangle_{L^2} + \langle (L_0 - \lambda_1(0))\,\psi'(0),\, \psi(0)\rangle_{L^2} = \lambda_1'(0).$$

We claim that the operator $L_0 - \lambda_1(0)$ is self-adjoint and, since $(L_0 - \lambda_1(0))\,\psi(0) = 0$, we arrive at

$$\langle L_0'\psi(0),\, \psi(0)\rangle_{L^2} = \lambda_1'(0). \tag{3.2}$$

Let $\tilde{\nabla}$, ∇ be the Levi-Civita connections of $g(t)$ and g, respectively, and denote by $C(t)$ the tensor which gives us the difference between the two connections, i.e.:

$$C(t, X, Y) = (\tilde{\nabla}_X - \nabla_X)Y \iff \nabla_i Y^j = \tilde{\nabla}_i Y^j - C^j_{ik} Y^k.$$

Note that at $t = 0$ we have $\nabla_i = \tilde{\nabla}_i$ (or, put in another way, $C(0) = 0$). Furthermore, since the connections are symmetric, our tensor $C(t)$ is symmetric as well, and given that

$$\tilde{\nabla}_k g_{ij} = \nabla_k g_{ij} - C^s_{ki} g_{sj} - C^s_{kj} g_{is} \iff 0 = \nabla_k g_{ij} - C_{jki} - C_{ikj},$$

$$\tilde{\nabla}_i g_{jk} = \nabla_i g_{jk} - C^s_{ij} g_{sk} - C^s_{ik} g_{js} \iff 0 = \nabla_i g_{jk} - C_{kij} - C_{jik},$$

$$\tilde{\nabla}_j g_{ki} = \nabla_j g_{ki} - C^s_{jk} g_{si} - C^s_{ji} g_{ks} \iff 0 = \nabla_k g_{ij} - C_{ijk} - C_{kji},$$

through algebraic manipulation of the previous equations, we arrive at the expression:

$$C^k_{ij} = \frac{1}{2} g^{ks}(t) \left\{ \nabla_i g_{js}(t) + \nabla_j g_{is}(t) - \nabla_s g_{ij}(t) \right\}. \tag{3.3}$$

To compute the relation between both metric's curvature tensors, note first that:

$$\begin{aligned}
\tilde{\nabla}_i \tilde{\nabla}_j X^k &= \tilde{\nabla}_i (\nabla_j X^k + C^k_{js} X^s) \\
&= \nabla_i \nabla_j X^k + C^k_{is} \nabla_j X^s - C^s_{ij} \nabla_s X^k \\
&\quad + \nabla_i C^k_{js} X^s + C^k_{js} \nabla_i X^s + C^k_{il} C^l_{js} X^s - C^l_{ij} C^k_{ls} X^s.
\end{aligned}$$

Then, for the curvature tensor we have

$$\begin{aligned}
\tilde{R}^k_{ijs} X^s = \left(\tilde{\nabla}_i \tilde{\nabla}_j - \tilde{\nabla}_i \tilde{\nabla}_j \right) X^k &= \nabla_i \nabla_j X^k + C^k_{is} \nabla_j X^s - C^s_{ij} \nabla_s X^k + \nabla_i C^k_{js} X^s \\
&+ C^k_{js} \nabla_i X^s + C^k_{il} C^l_{js} X^s - C^l_{ij} C^k_{ls} X^s - \nabla_j \nabla_i X^k - C^k_{js} \nabla_i X^s + C^s_{ij} \nabla_s X^k \\
&- \nabla_j C^k_{is} X^s - C^k_{is} \nabla_j X^s - C^k_{jl} C^l_{is} X^s + C^l_{ij} C^k_{ls} X^s = \\
&= (\nabla_i \nabla_j - \nabla_j \nabla_i) X^k + (\nabla_j C^k_{is} - \nabla_i C^k_{js} + C^k_{il} C^l_{js} - C^k_{jl} C^l_{is}) X^s \\
&= (R^k_{ijs} - \nabla_j C^k_{is} + \nabla_i C^k_{js} + C^k_{il} C^l_{js} - C^k_{jl} C^l_{is}) X^s,
\end{aligned}$$

whence we see that

$$\tilde{R}^k_{ijs} = R^k_{ijs} - \nabla_j C^k_{is} + \nabla_i C^k_{js} + C^k_{il} C^l_{js} - C^k_{jl} C^l_{is}. \tag{3.4}$$

Now, since we are linearizing the scalar curvature, we want the derivatives, at $t = 0$, of the previous formulas (3.3) and (3.4). First, recal the formula for the derivative of the inverse of a matrix, A, dependent on a parameter t, that is $\frac{d}{dt} A^{-1} = -A^{-1} \left(\frac{d}{dt} A \right) A^{-1}$. Then, we have

$$\begin{aligned}
&\frac{d}{dt}_{|t=0} g_{ij}(t) = -S_{ij} \\
&\frac{d}{dt}_{|t=0} g^{ij}(t) = -g^{ij}(0) \left(\frac{d}{dt}_{|t=0} g_{ij}(t) \right) g^{ij}(0) = S^{ij}.
\end{aligned}$$

Furthermore, we easily get from the previous computations

$$\begin{aligned}
\frac{d}{dt}_{|t=0} C^k_{ij}(t) &= \frac{1}{2}\left(\frac{d}{dt}_{|t=0} g^{ks}(t)\right)\{\nabla_i g_{js}(0) + \nabla_j g_{is}(0) - \nabla_s g_{ij}(0)\} \\
&\quad + \frac{1}{2} g^{ks}(0)\frac{d}{dt}_{|t=0}\left(\nabla_i g_{js}(t) + \nabla_j g_{is}(t) - \nabla_s g_{ij}(t)\right) = \\
&= 0 + \frac{1}{2} g^{ks}\left(\nabla_i\left(\frac{d}{dt}_{|t=0} g_{js}(t)\right) + \nabla_j\left(\frac{d}{dt}_{|t=0} g_{is}(t)\right) - \nabla_s\left(\frac{d}{dt}_{|t=0} g_{ij}(t)\right)\right) \\
&= -\frac{1}{2} g^{ks}\left(\nabla_i S_{js} + \nabla_j S_{is} - \nabla_s S_{ij}\right) \\
&= -\frac{1}{2}\left(\nabla_i S^k_j + \nabla_j S^k_i - \nabla^k S_{ij}\right).
\end{aligned}$$

From $C(0) = 0$, we see that the derivative at $t = 0$ of a product of components of C, such as $C^k_{il}C^l_{js}$, will vanish. Consequently, we obtain the following derivative for (3.4):

$$\frac{d}{dt}_{|t=0} R^k_{ijs} = \nabla_j\left(\frac{d}{dt}_{|t=0} C^k_{is}\right) - \nabla_i\left(\frac{d}{dt}_{|t=0} C^k_{js}\right) \implies$$

$$\begin{aligned}
\implies \frac{d}{dt}_{|t=0} S_{ij} &= \frac{d}{dt}_{|t=0} R^k_{ikj} = \nabla_k\left(\frac{d}{dt}_{|t=0} C^k_{ij}\right) - \nabla_i\left(\frac{d}{dt}_{|t=0} C^k_{kj}\right) = \\
&= -\frac{1}{2}\nabla_k\left(\nabla_i S^k_j + \nabla_j S^k_i - \nabla^k S_{ij}\right) + \frac{1}{2}\nabla_i\left(\nabla_k S^k_j + \nabla_j S^k_k - \nabla^k S_{kj}\right) \\
&= -\frac{1}{2}\left(\nabla_k\nabla_i S^k_j + \nabla_k\nabla_j S^k_i - \nabla_k\nabla^k S_{ij} - \nabla_k\nabla_i S^k_j - \nabla_i\nabla_j S^k_k + \nabla_i\nabla^k S_{kj}\right) \\
&= -\frac{1}{2}\left(\nabla_i\nabla^k S_{kj} + \nabla_k\nabla_j S^k_i - \nabla_k\nabla^k S_{ij} - \nabla_i\nabla_j S^k_k\right),
\end{aligned}$$

which is the differential for the components of the Ricci curvature tensor.

Computing now $L'_0 = \frac{d}{dt}_{|t=0}(-8\Delta + R)$ for an arbitrary function $f \in C^\infty$, we see that

$$\begin{aligned}
\frac{d}{dt}_{|t=0}\Delta_{g(t)} f &= \frac{d}{dt}_{|t=0}(g^{ij}(t)\tilde{\nabla}_i(\partial_j f)) = \\
&= \left(\frac{d}{dt}_{|t=0} g^{ij}(t)\right)\nabla_i\partial_j f + g^{ij}\frac{d}{dt}_{|t=0}\left(\nabla_i - C^k_{ij}\right)\partial_j f \\
&= S^{ij}\left(\nabla_i\partial_j f\right) - g^{ij}\frac{d}{dt}_{|t=0}(C^k_{ij})\partial_k f \\
&= S^{ij}\left(\nabla_i\partial_j f\right) + \frac{1}{2} g^{ij}\left(\nabla_i S^k_j + \nabla_j S^k_i - \nabla^k S_{ij}\right)\partial_k f \\
&= S^{ij}\left(\nabla_i\partial_j f\right) + \nabla_i S^{ik}\partial_k f - \frac{1}{2}\nabla^k S^i_i\partial_k f, \\
\frac{d}{dt}_{|t=0} R_{g(t)} &= \frac{d}{dt}_{|t=0}(g^{ij}(t)S_{ij}) = \\
&= \left(\frac{d}{dt}_{|t=0} g^{ij}(t)\right) S_{ij} + g^{ij}\left(\frac{d}{dt}_{|t=0} S_{ij}\right) \\
&= S^{ij}S_{ij} - \frac{1}{2} g^{ij}\left\{\nabla_i\nabla^k S_{kj} + \nabla_k\nabla_j S^k_i - \nabla_k\nabla^k S_{ij} - \nabla_i\nabla_j S^k_k\right\} \\
&= S^{ij}S_{ij} + \nabla_i\nabla^i S^k_k - \nabla_i\nabla_j S^{ij} = S^{ij}S_{ij} + \Delta S^j_j - \nabla_i\nabla_j S^{ij}.
\end{aligned}$$

Therefore, from the previous computations, we have

$$L_0'\psi = -8S^{ij}\left(\nabla_i\partial_j\psi\right) - 8\nabla_i S^{ij}\partial_j\psi + 4\nabla^j S_i^i\partial_j\psi + \left(S^{ij}S_{ij} + \Delta S_j^j - \nabla_i\nabla_j S^{ij}\right)\psi,$$

and so, writing $\psi = \psi(0)$, (3.2) becomes:

$$\langle -8S^{ij}\left(\nabla_i\partial_j\psi\right) - 8\nabla_i S^{ij}\partial_j\psi + 4\nabla^j S_i^i\partial_j\psi + \left(S^{ij}S_{ij} + \Delta S_j^j - \nabla_i\nabla_j S^{ij}\right)\psi,\, \psi\rangle_{L^2} = \lambda_1'(0).$$

Furthermore, noting that by the assumption of the lemma, we have both $R = S_i^i = 0$ and $\lambda_1(0) = 0$, then $L_0\psi(0) = -8\Delta\psi(0) = \lambda_1(0)\psi(0) = 0 \implies \psi(0) = \psi$ is constant and, proceeding through integration by parts to take out all the derivatives of S, we have

$$\begin{aligned}
\int_M -8S^{ij}\left(\nabla_i\partial_j\psi\right)\psi - 8\nabla_i S^{ij}(\partial_j\psi)\psi + 4\nabla^j S_i^i(\partial_j\psi)\psi + \left(S^{ij}S_{ij} + \Delta S_j^j - \nabla_i\nabla_j S^{ij}\right)\psi^2 \\
&= \int_M S^{ij}S_{ij}\psi^2 + 2\int_M \nabla_j S^{ij}(\nabla_i\psi)\psi \\
&= \int_M S^{ij}S_{ij}\psi^2 \\
&= \langle S,\, \psi^2 S\rangle_{L^2} \\
&= \psi(0)^2\langle S, S\rangle_{L^2}.
\end{aligned}$$

We then conclude that

$$\lambda_1'(0) = c\langle S,\, S\rangle_{L^2} > 0.$$

Hence, for t sufficiently small, our manifold admits a metric, $\bar{g} = g - tS$, such that $\lambda_1(\bar{g}) > 0$. Then, by the previous Lemma, this metric has positive scalar curvature, which contradits the very assumption of Lemma 3. Therefore, the Ricci tensor must vanish, $S = 0$.

To conclude this proof, note that for any functions $u,\, v \in C^\infty(M)$ we have

$$\begin{aligned}
\langle\left(L_0 - \lambda_1(0)\right)u,\, v\rangle_{L^2} &= \int_M \left(-8\Delta u + R_g u - \lambda_1(0)u\right)\, v \\
&= -8\int_M \Delta u\, v = 8\int_M \partial^i u\,\partial_i v \\
&= -8\int u\Delta v = \langle u,\, \left(L_0 - \lambda_1(0)\right)v\rangle_{L^2}\,,
\end{aligned}$$

hence our claim that $L_0 - \lambda_1(0)$ is self-adjoint was correct. □

Given the two previous lemmas, we now have an interesting restriction to both curvature and metric of a 3-torus allowing a stable area-minimizing T^2. From them, we get the following result.

Theorem 3. *Suppose we have a 3-torus (T^3, g) that admits a stable minimal T^2, with g such that the scalar curvature satisfies $R \geq 0$. Then, g is the flat metric (and $R = 0$).*

Proof. We concluded in the previous chapter (Theorem 2) that, if T^3 admits a stable minimal T^2, there is no metric with positive scalar curvature for T^3. Furthermore, note that $M = T^3$ is a compact 3-dimensional manifold.

Now, suppose we have $R \geq 0$. Hence, the scalar curvature vanishes on T^2, and it is either identically zero in all of T^3 or it is positive somewhere. By Lemma 3, if the metric has zero scalar curvature everywhere, then we must have zero Ricci curvature as well. However, given that we are working with a 3-manifold, having zero Ricci curvature implies that the Riemann curvature is also identically zero and, consequently, g is flat.

Consider the case of R being positive somewhere, in other words, the case where T^3 admits a non-flat metric with non-negative scalar curvature. Take $\bar{\psi}$ to be the normalized eigenfunction of L_g with the eigenvalue $\lambda_1(g)$. Now, multiplying both sides by $\bar{\psi}$ and taking the integral over T^3, we get

$$\begin{aligned}
&L_g\bar{\psi} = \lambda_1(g)\bar{\psi} \\
&\iff -8\Delta\bar{\psi} + R\bar{\psi} = \lambda_1(g)\bar{\psi} \\
&\implies -8\bar{\psi}\Delta\bar{\psi} + R\bar{\psi}^2 = \lambda_1(g)\bar{\psi}^2 \\
&\implies -8\int_{T^3}\bar{\psi}\Delta\bar{\psi} + \int_{T^3}R\bar{\psi}^2 = \lambda_1(g)\int_{T^3}\bar{\psi}^2 \\
&\iff -8\int_{T^3}\operatorname{div}\left[\operatorname{grad}(\bar{\psi})\right]\bar{\psi} + \int_{T^3}R\bar{\psi}^2 = \lambda_1(g).
\end{aligned}$$

Recalling that T^3 is a compact manifold, we now have

$$\begin{aligned}
&-8\int_{T^3}\operatorname{div}\left[\operatorname{grad}(\bar{\psi})\right]\bar{\psi} + \int_{T^3}R\bar{\psi}^2 = \lambda_1(g) \\
&\iff -8\int_{T^3}\operatorname{div}\left[\bar{\psi}\operatorname{grad}(\bar{\psi})\right] + 8\int_{T^3}|\operatorname{grad}(\bar{\psi})|^2 + \int_{T^3}R\bar{\psi}^2 = \lambda_1(g) \\
&\iff 8\int_{T^3}|\operatorname{grad}(\bar{\psi})|^2 + \int_{T^3}R\bar{\psi}^2 = \lambda_1(g).
\end{aligned}$$

We argue that $\lambda_1(g)$ is positive since, on the left side, both integrals cannot be simultaneously zero. Suppose the first integral vanishes; then $\operatorname{grad}(\bar{\psi})$ is identically zero. Therefore, the eigenfunction $\bar{\psi}$ is a positive constant and the second integral, by our assumption that R is positive somewhere, has to be strictly positive. On the other hand, if the second integral is zero, then one has that the eigenfunction vanishes where $R > 0$. However, by definition, $\bar{\psi}$ can't be zero everywhere and, consequentely, its gradient has to be non-zero. Ergo, the first integral is stricly positive.

Given that $\lambda_1(g) > 0$, by Lemma 2, T^3 admits a pointwise conformal metric g_1 with positive scalar curvature, yet such result contradicts the previous chapter and our assumption of a stable minimal 2-torus. Therefore, we must have $R = 0$ and g must be the flat metric. $\square$

Chapter 4

Existence of a Minimal 2-torus

In this chapter we will show that, in fact, there exists a stable minimal 2-torus in T^3 and, consequently, all the previous results follow. In other words, we show that we can drop the assumption of T^3 admiting a stable minimal torus from previous theorems.

However, to do so we require some heavier hardware – Geometric Measure Theory (GMT), mainly following [Mor16] – whose basic notions will be presented in section 4.1. The intuiton behind the result follows from Real Analysis: we take a set of generalized surfaces (called rectifiable currents) and, in this set, we consider a sequence of surfaces with area decreasing to an infimum. By taking a convergent subsequence and showing that, in fact, this limit exists and is the surface of least area, we conclude the proof.

Now, to proceed as was described, there are some issues one must adress, in section 4.2. For example, we must have compactness of the set of surfaces for the existence of the area-minimizing limit. Furthermore, there is the concern about the regularity of such area-minimizing surfaces, i.e. are these limits some undesired generalized objects or are they smooth manifolds?

By adressing these issues, in section 4.3, we derive the existence of a smooth stable minimal 2-torus in T^3, and thus conclude the proof of the 3-dimensional case of the Geroch Conjecture.

4.1 Measures and Currents

In this section, the basic tools and notions required to prove the desired results will be introduced, in a brief and concise manner. Let us start by defining the measure, for all subsets of $\mathbf{R}^n$, that we will use when constructing the alternative to surfaces as classical submanifolds.

Recall that the definition of the diameter of a subset S of $\mathbf{R}^n$ is

$$\operatorname{diam}(S) = \sup\{|x-y| : x,\, y \in S\}.$$

Definition 4.1.1 (Hausdorff measure)**.** *Let α_m be the Lebesgue measure of the closed unit ball $\mathbf{B}^m(0,1) \subset \mathbf{R}^n$. For any $A \subset \mathbf{R}^n$, the m-dimensional Hausdorff measure is defined by*

$$\mathcal{H}^m(A) = \lim_{\delta\to 0} \inf_{\substack{A\subset\bigcup S_j \\ \operatorname{diam}(S_j)\le\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^2,$$

where the infimum is taken over all the countable coverings $\{S_j\}$ of A with $\operatorname{diam}(S_j) \le \delta$.

As δ decreases, the infimum itself is non-decreasing, and therefore the limit exists (allowing $0 \le \mathcal{H}^m(A) \le \infty$). Furthermore, this measure is Borel regular, and the n-dimensional case agrees with the Lebesgue measure in $\mathbf{R}^n$, i.e. $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbf{R}^n$.

We present now the sets that will be the generalized surfaces of GMT. Recall the definition of Lipschitz functions introduced earlier.

Definition 4.1.2 (Rectifiable Set)**.** *Let $E \subset \mathbf{R}^n$. We say E is $(\mathcal{H}^m, m)$ rectifiable if:*

1. *$\mathcal{H}^m(E) < \infty$;*
2. *$\mathcal{H}^m$-almost all of E is contained in $\bigcup \operatorname{im}(f_i)$, where f_i are countably many Lipschitz functions from $\mathbf{R}^m$ to $\mathbf{R}^n$.*

Proposition 4. *On the previous definition, one can substitute the Lipschitz functions by C^1-diffeomorphisms f_j on compact domains with disjoint images. Moreover, the Lipschitz constants of f_j and f_j^{-1} can be taken near 1.*

Sketch of Proof. The idea is to procceed by repetition until exhaustion. Divide the domains of the Lipschitz functions as to assume they have at most diameter 1 and, due to Proposition 1 – Chapter 1, replace the first Lipschitz function g by a C^1 approximation f.

By taking a small portion of the domain of f so that $\operatorname{im}(f) \subset E$ and Df is approximately constant (and small), we get injectivity of f. Through a linear transformation we get $\operatorname{Lip} g \approx \operatorname{Lip} g^{-1} \approx 1$, and replacing the domain by a compact set we obtain 1% of $\operatorname{im}(f)$. Repeating the process for all the remaining Lipschtiz functions gives us 1% of the set E, and the rest can be exhausted by repetition. □

Definition 4.1.3 (Tangent Cones)**.** *Let $E \subset \mathbf{R}^n$ and $a \in \mathbf{R}^n$. Considering the m-dimensional density $\Theta^m(E, a)$ with respect to the Hausdorff measure, $\mathcal{H}^m$, the tangent cone of E at a is defined by*

$$\operatorname{Tan}(E, a) = \mathbf{R}_0^+ \left[\bigcap_{\varepsilon>0} \operatorname{clos}\left\{\frac{x-a}{|x-a|} : x \in E,\, 0 < |x-a| < \varepsilon\right\}\right],$$

and the cone of approximate tangent vectors of E at a is given by

$$\operatorname{Tan}^m(E, a) = \bigcap_{\Theta^m(E-S,\, a)=0} \operatorname{Tan}(S, a).$$

For almost all points a in a rectifiable set E, the tangent cone $\operatorname{Tan}^m(E, a)$ is in fact a tangent plane [Mor16]–3.12. Moreover, an orientation of a m-dimensional rectifiable subset of $A \subset \mathbf{R}^n$ is a choice of orientation for each $\operatorname{Tan}^m(A, a)$. Note that every rectifiable, positive measure set has uncountably many possible orientations.

Recall the definition of $\mathcal{D}^m$ from the introduction. Its dual space, $\mathcal{D}_m$, is the space of m-dimensional currents, intuitively viewed as such by analogy with electrical currents. That is, given a differential form ω and a oriented rectifiable set S, the action of S on ω (given by integrating a form ω over S) induces a linear functional on smooth differential forms:

$$\omega \mapsto S(\omega) = \int_S \left\langle \vec{S}(x), \omega \right\rangle \, d\mathcal{H}^m,$$

where $\vec{S}(x)$ is the unit m-vector associated with the oriented tangent plane to S at x.

Generally, given an m-vector-field $\vec{v}$, we would get, by duality, a current T (just not a very adequate or interesting one) if we substituted $\vec{S}$ by $\vec{v}$. However, this would not give us a surface in the sense we need. It is precisely the integration over S with the orientation given by $\vec{S}$ that provide us with a rectifiable current/surface.

Definition 4.1.4 (Boundary/Support of Currents)**.** *Let $S \in \mathcal{D}_m$ be a m-dimensional current.*

The boundary of S is the $(m-1)$-dimensional current defined by

$$\partial S(\omega) = S(d\omega).$$

The support of S is the smallest closed set C such that

$$\operatorname{supp}(\omega) \cap C = \emptyset \Rightarrow S(\omega) = 0.$$

Definition 4.1.5 (Spaces of Currents)**.**

1. $\mathcal{D}_m$ *is the space of m-dimensional currents in $\mathbf{R}^n$;*

2. $\mathcal{E}_m = \{T \in \mathcal{D}_m : \operatorname{supp}(T)$ *is compact*$\}$*;*

3. $\mathcal{R}_m = \{T \in \mathcal{E}_m : T$ *is an oriented rectifiable set with integer multiplicities and finite measure*$\}$ *is the space of rectifiable currents;*

4. $\mathcal{P}_m = \{$*integral polyhedral chains*$\}$ *is the additive subgroup of $\mathcal{E}_m$ generated by classicaly oriented simplices;*

5. $\mathbf{I}_m = \{T \in \mathcal{R}_m : \partial T \in \mathcal{R}_{m-1}\}$ *is the set of rectifiable currents T whose boundary is a rectifiable current, the so-called integral currents.*

6. $\mathcal{F}_m = \{T + \partial S : T \in \mathcal{R}_m, S \in \mathcal{R}_{m+1}\}$ *is the set of integral flat chains.*

The last two spaces allow us to see how well-behaved the boundary operator ∂ is.

Proposition 5. *The boundary operator ∂ maps $\mathbf{I}_m$ to $\mathbf{I}_{m-1}$ and $\mathcal{F}_m$ to $\mathcal{F}_{m-1}$. Moreover,* $\operatorname{supp}(\partial T) \subset \operatorname{supp}(T)$.

Proof. Let $T \in \mathbf{I}_m$, then (by definition) $\partial T \in \mathcal{R}_{m-1}$. Furthermore, $\partial(\partial T) = 0 \in \mathcal{R}_{m-2}$, hence $\partial T \in \mathbf{I}_{m-1}$.

Now, let $J \in \mathcal{F}_m$. By definiton, $J = T + \partial S$ with $T \in \mathcal{R}_m$ and $S \in \mathcal{R}_{m+1}$. Then, $\partial J = \partial T$ with $T \in \mathcal{R}_m$ which gives us $\partial J \in \mathcal{F}_{m-1}$.

Let $\omega \in \mathcal{D}^{m-1}$ be a differential form such that $\mathrm{supp}(\omega) \cap \mathrm{supp}(T) = \emptyset$. Consequently, $\mathrm{supp}(d\omega) \cap \mathrm{supp}(T) = \emptyset$ and we have

$$\partial T(\omega) = T(d\omega) = 0.$$

Therefore, $\mathrm{supp}(\partial T) \subset \mathrm{supp}(T)$. $\square$

Definition 4.1.6 (Mass; Flat norm). *We define on the space of currents $\mathcal{D}_m$ the seminorms*

$$\mathbf{M}(T) = \sup\left\{T(\omega) \,:\, \sup \|\omega(x)\|^* \leq 1\right\}$$

and

$$\mathfrak{F}(T) = \inf\left\{\mathbf{M}(A) + \mathbf{M}(B) : \, T = A + \partial B,\, A \in \mathcal{R}_m,\, B \in \mathcal{R}_{m+1}\right\},$$

called the mass and flat norm, respectively.

From the definitions, it is clear that the flat norm topology is weaker than the mass norm topology. Moreover, it will be shown, in the next section, that the flat norm topology is the natural topology to get compactness. Before proceeding, let us prove some results that will be required in the next section.

Definition 4.1.7 (Push-forward of a Current). *Let $T \in \mathcal{E}_m(\mathbf{R}^n)$ be a current with compact support, $\omega \in \mathcal{D}^m(\mathbf{R}^p)$ an arbitrary m-differential form and $f : \mathbf{R}^n \to \mathbf{R}^p$ a C^∞-map. Then, the push-forward $f_*T \in \mathcal{D}_m(\mathbf{R}^p)$ is defined by*

$$(f_*T)(\omega) = T(f^*\omega),$$

where $f^*\omega$ is the pullback of ω by f.

Theorem 4 ([Fed96] - 4.1.28). *$T \in \mathcal{E}_m$ is a rectifiable current iff given $\varepsilon > 0$, there exists an integral polyhedral chain $P \in \mathcal{P}_m(\mathbf{R}^v)$ and a Lipschitz function $f : \mathbf{R}^v \to \mathbf{R}^n$ such that*

$$\mathbf{M}(T - f_*P) < \varepsilon,$$

where f_* is the push-forward by f.

Corollary 1.

1. *$\{T \in \mathcal{R}_m : \mathrm{supp}(T) \subset \mathbf{B}^n(0,r)\}$ is $\mathbf{M}$ complete.*
2. *$\{T \in \mathcal{F}_m : \mathrm{supp}(T) \subset \mathbf{B}^n(0,r)\}$ is $\mathfrak{F}$ complete.*

Proof. By the previous theorem, we see that the first set indeed is $\mathbf{M}$ complete.

For the second set, let F_i be a Cauchy sequence in $\{T \in \mathcal{F}_m : \mathrm{supp}(T) \subset \mathbf{B}^n(0,r)\}$. We can assume, taking a subsequence if necessary, that $\mathfrak{F}(F_{i+1} - F_i) < 2^{-i}$, and let us rewrite $F_{i+1} - F_i = T_i + \partial S_i$ such that $\mathbf{M}(T_i) + \mathbf{M}(S_i) < 2^{-i}$, with $T_i \in \mathcal{R}_m$ and $S_i \in \mathcal{R}_{m+1}$. Then

$$\sum_{i=2}^{\infty} \mathfrak{F}(F_{i+1} - F_i) < \infty \quad \text{and} \quad \sum_{i=2}^{\infty} \mathbf{M}(T_i) + \mathbf{M}(S_i) < \infty,$$

and therefore, by $\mathbf{M}$ completeness of the first set, we get that $\sum T_i$ and $\sum S_i$ converge to two rectifiable currents, $T \in \mathcal{R}_m$ and $S \in \mathcal{R}_{m+1}$, respectively. Consequently, we conclude that $F = F_1 + \sum T_i + \partial(\sum S_i)$ and

$$\mathfrak{F}(F - F_j) \leq \sum_{i=j+1}^{\infty} \mathbf{M}(T_i) + \mathbf{M}(S_i) \to 0 \text{ as } j \to \infty,$$

that is, we get $F_i \to F_1 + T + \partial S \in \mathcal{F}_m$. □

Definition 4.1.8 (General Flat norm). *For any $T \in \mathcal{D}_m$, define the more general flat norm as*

$$\mathbf{F}(T) = \sup \{T(\omega) \, : \omega \in \mathcal{D}^m, \, \|\omega(x)\|^* \leq 1, \textit{ and } \quad \|d\omega(x)\|^* \leq 1 \textit{ for all x}\}$$

$$= \min \{\mathbf{M}(A) + \mathbf{M}(B) : \, T = A + \partial B, \, A \in \mathcal{E}_m, \, B \in \mathcal{E}_{m+1}\}.$$

The second equality is proved using the Hahn-Banach Theorem. It is possible to define more general spaces of currents with this norm, and introduce the notion of normal current.

Before proceeding to the next section, let us end with a demonstration of a property of the mass $\mathbf{M}$ which will be usefull later on.

Proposition 6 (Lower Semicontinuity). *Given $T_i, T \in \mathcal{D}_m$ such that $T_i \xrightarrow{\mathbf{F}} T$, then*

$$\mathbf{M}(T) \leq \liminf \mathbf{M}(T_i)$$

Proof. For the case where $\mathbf{M}(T)$ is finite, take $\varepsilon > 0$ and choose a differential form $\omega \in \mathcal{D}^m$, with $\|\omega(x)\|^* \leq 1$, such that $\mathbf{M}(T) \leq T(\omega) + \varepsilon$. Then, taking the limit with respect with $\mathbf{F}$

$$\mathbf{M}(T) \leq T(\omega) + \varepsilon = \lim T_i(\omega) + \varepsilon \leq \liminf \mathbf{M}(T_i) + \varepsilon.$$

For the case where $\mathbf{M}(T) = \infty$, take $\varepsilon > 0$ and choose a differential form $\omega \in \mathcal{D}^m$, with $\|\omega(x)\|^* \leq 1$, such that $T(\omega) > \frac{1}{\varepsilon}$. Then,

$$\liminf \mathbf{M}(T_i) \geq \lim T_i(\omega) > \frac{1}{\varepsilon}.$$

Therefore, $\liminf \mathbf{M}(T_i) = \infty$. □

4.2 The Compactness Theorem

As previously stated, we want to work with a set $\mathcal{S}$ of surfaces which is compact under a natural topology. To do so we require two theorems – the Deformation Theorem and the Closure Theorem – from which we get the compactness of a suitable set.

Theorem 5 (Deformation Theorem). *Let $T \in \mathbf{I}_m(\mathbf{R}^n)$ and $\varepsilon > 0$. Then, there are $P \in \mathcal{P}_m(\mathbf{R}^n)$, $\mathcal{Q} \in \mathbf{I}_m(\mathbf{R}^n)$ and $S \in \mathbf{I}_{m+1}(\mathbf{R}^n)$ such that the following conditions hold, for $\gamma = 2n^{2m+2}$:*

1. $T = P + \mathcal{Q} + \partial S$;

2. $\operatorname{supp}(P) \cup \operatorname{supp}(\mathcal{Q}) \cup \operatorname{supp}(S) \subset \{x : \operatorname{dist}(x, \operatorname{supp}(T)) \leq 2n\varepsilon\}$;

3. $\mathbf{M}(P) \leq \gamma\,[\mathbf{M}(T) + \varepsilon\mathbf{M}(\partial T)]$,

   $\mathbf{M}(\partial P) \leq \gamma\mathbf{M}(\partial T)$,

   $\mathbf{M}(\mathcal{Q}) \leq \gamma\varepsilon\mathbf{M}(\partial T)$

   $\mathbf{M}(S) \leq \gamma\varepsilon\mathbf{M}(T)$.

Corollary 2. *The set* $\mathcal{S} = \{T \in \mathbf{I}_m : \operatorname{supp}(T) \subset \mathbf{B}^n(0, c_1),\, \mathbf{M}(T) \leq c_2,\, \mathbf{M}(\partial T) \leq c_3\}$ *is totally bounded under* $\mathfrak{F}$.

Proof. By the Deformation Theorem 5 , each $T \in \mathcal{S}$ can be approximated by a polyhedral chain P, such that $\mathbf{M}(P) \leq \gamma\,(c_2 + \varepsilon c_3)$ and $\operatorname{supp}(P) \subset \mathbf{B}^n(0, c_1 + 2n\varepsilon)$, in a ε-grid. However, there are only finitely many chains P, therefore $\mathcal{S}$ is totally bounded. ☐

We then have totally boundedness of our set, all we are missing is completeness. Recall the definition of $\mathcal{F}_m$ and that $\{T \in \mathcal{F}_m : \operatorname{supp}(T) \subset \mathbf{B}^n(0, r)\}$ is $\mathfrak{F}$ complete.

Theorem 6 (Closure Theorem)**.**

1. $\mathbf{I}_m$ *is* $\mathbf{F}$*-closed in* $\mathbf{N}_m$;

2. $\mathbf{I}_{m+1} = \{T \in \mathcal{R}_{m+1} \,:\, \mathbf{M}(\partial T) < \infty\}$;

3. $\mathcal{R}_m = \{T \in \mathcal{F}_m \,:\, \mathbf{M}(T) < \infty\}$*, consequently,*

4. $\mathcal{S} = \{T \in \mathbf{I}_m : \operatorname{supp}(T) \subset \mathbf{B}^n(0, r),\, \mathbf{M}(T) \leq c,\, \mathbf{M}(\partial T) \leq c\}$ *is complete under* $\mathfrak{F}$.

Proof. Proofs of assertions 1. to 3. will be omitted, as we want to prove and use assertion 4..

Let T_i be a Cauchy sequence in $\mathcal{S}$. By completeness of $\{T \in \mathcal{F}_m : \operatorname{supp}(T) \subset \mathbf{B}^n(0, r)\}$, there is a limit T in $\mathcal{F}_m$. By lower semicontinuity of mass $\mathbf{M}(T) \leq c$ and $\mathbf{M}(\partial T) \leq c$, hence we have, by 3., $T \in \mathcal{R}_m$ and, consequently, by 2., $T \in \mathbf{I}_m$. Therefore, the limit of a Cauchy sequence in $\mathcal{S}$ exists in $\mathcal{S}$, which gives us completeness. ☐

Corollary 3 (Compactness Theorem)**.** *For a closed ball* K *in* $\mathbf{R}^n$ *and* $0 \leq c < \infty$*, the set*

$\mathcal{S} = \{T \in \mathbf{I}_m : \operatorname{supp}(T) \subset K,\, \mathbf{M}(T) \leq c,\, \mathbf{M}(\partial T) \leq c\}$ *is* $\mathfrak{F}$ *compact.*

Proof. By Corollary 2 and the Closure Theorem (Theorem 6), the set is both totally bounded and complete, hence compact. ☐

The range of this theorem is further extended when we substitute K for a C^1 compact Riemannian submanifold of $\mathbf{R}^n$. Via C^1-embeddings into Euclidean space, it can then be generalized to any compact C^1 Riemannian manifold $(M,\, g)$.

4.3 Existence and Regularity of Minimal Surfaces

We can now extract a convergent subsequence from any sequence of rectifiable currents. By defining the homology class of a rectifiable current T as the set of rectifiable currents S such that $S - T = \partial X$ for some rectifiable current X, we now possess the necessary framework to prove the existence of homologically non-trivial minimizing surfaces in arbitray C^1 manifolds.

Theorem 7. *Let M be a compact C^1-Riemannian manifold and T be a rectifiable current in M. Then, among the currents S such that $S - T = \partial X$ in M, there is one that minimizes area.*

Proof. Take S_i to be a sequence of rectifiable currents in M with decreasing areas to

$$\inf \{\mathbf{M}(S) \,:\, S - T = \partial X, \text{ for some rectifiable current X}\} .$$

Since $\operatorname{supp}(S_i) \subset M$, by the Compactness Theorem, we have a subsequence that converges to a rectifiable current $\bar{S}$ such that, by continuity of ∂ and lower semicontinuity of $\mathbf{M}$, $\bar{S} - T = \partial X$, and

$$\mathbf{M}(\bar{S}) = \inf \{\mathbf{M}(S) \,:\, S - T = \partial X, \text{ for some rectifiable current} X\} .$$

Now, if $\mathfrak{F}(S_i - S)$ is small then, by definition of the flat norm, $S_i - S = A + \partial B$ with both the masses $\mathbf{M}(A)$, $\mathbf{M}(B)$ small as well. Assume M is isometrically embedded in $\mathbf{R}^n$, and take the minimal surface Y_1 such that its border coincides with A, i.e. $\partial Y_1 = A$ (whose existence can be deduced by similar arguments). Since $\mathbf{M}(Y_1) = \mathbf{M}(A)$ is small, Y_1 can be retracted onto Y in M, thus we have $\partial Y = A$. Now, we have $S_i - S = \partial Y + \partial B$ which means that for each i, S and S_i differ only by a boundary. Therefore, they are on the same homology class. $\square$

As the previous result states, we now have a stable rectifiable current of least area in the homology class of T. However, we do not know how geometrically ”well-behaved” this current is, i.e. we lack knowledge of its regularity. To address this, we make use of a theorem by Wendell Fleming [Fle62], whose proof is omitted, that guarantees the interior regularity for 2-dimensional currents.

Theorem 8 (Regularity for the 2-dimensional hypersurface)**.** *Any 2-dimensional, area-minimizing rectifiable current T in a 3-dimensional manifold M is a smooth, embedded submanifold.*

Remark. It is worth mentioning that the regularity theorem holds true for $(n-1)$-dimensional, volume minimizing rectifiable currents in n-dimensional Riemannian manifolds (i.e. maintaining codimension 1) up to $n \leq 7$. For higher dimensions, singularities of geometrical nature start to occur.

Chapter 5

Further Results

In this chapter, by combining the previous theorems we obtain our main result, the proof of the Geroch Conjecture for the three dimensional torus. Furthermore, we also estabilish a relation between this conjecture and a particular case of the Positive Mass Theorem [Nat21].

5.1 Generalizations and Counter-examples

Theorem 9 (Geroch Conjecture)**.** *There is no metric g with positive scalar curvature, R, on the 3-torus T^3. Furthermore, if $R \geq 0$, then g is flat and $R = 0$.*

Proof. We have already concluded that there exists a homologically non-trivial minimizing 2-torus in T^3, which means we can relax the existence constraint of the previous theorems. Therefore, the first part of the result follows trivially from Theorem 2 (section 2.3 - Chapter 2), and the second part from Theorem 3 (section 3.2 - Chapter 3). $\square$

Despite of working with the 3-torus, the main result of this thesis applies to other manifolds. What was essentially used was the compactness of T^3 and the existence of a homologically non-trivial 2-torus. Hence, this result can be generalized to other 3-dimensional manifolds, satisfying the previous properties, for example, the connected sum $T^3 \# M$ with M a smooth, compact and connected 3-dimensional manifold, for instance T^3 again. For other examples, let $f : T^2 \to T^2$ be a orientation-preserving diffeomorphism. Consider the infinitely many inequivalent torus bundles (see [Hat80] for more) constructed by taking the Cartesian product of T^2 and the unit interval $I = [0, 1]$, and gluing the two components of the boundary via f, that is:

$$M = \frac{T^2 \times I}{(0, x) \sim (1, f(x))}.$$

Clearly, we can apply the previous theorem to these constructions. Notice, moreover, that if f is the identity, the resulting bundle is just the 3-torus T^3.

To end this section, we will show that the assumptions of the theorem are in fact necessary, through examples where the conjecture fails. Consider the manifold $(\mathbf{R}^+ \times T^2,\, g)$, where $g = dt^2 + (f(t))^2 \left(d\theta^2 + d\varphi^2\right)$ is the metric and $f(t)$ is a positive function. Hence, we have the orthonormal frame

$$\left\{ \frac{\partial}{\partial t},\, \frac{1}{f(t)}\frac{\partial}{\partial \theta},\, \frac{1}{f(t)}\frac{\partial}{\partial \varphi} \right\},$$

and the dual co-frame

$$\{dt,\, f(t)d\theta,\, f(t)d\varphi\}\,.$$

Denoting by ω^i the respective elements of the coframe, one readily sees that

$$d\omega^t = 0, \quad d\omega^\theta = f'(t)dt \wedge d\theta, \quad \text{and} \quad d\omega^\varphi = f'(t)dt \wedge d\varphi.$$

Now, let

$$\begin{aligned} \omega_t^\theta &= a dt + b d\theta + c d\varphi; \\ \omega_t^\varphi &= \alpha dt + \beta d\theta + \gamma d\varphi; \\ \omega_\theta^\varphi &= h dt + k d\theta + l d\varphi. \end{aligned}$$

Then, by Cartan's first structure equation, we conclude that

$$\begin{aligned} & f'(t)dt \wedge d\theta = \omega^t \wedge \omega_t^\theta + d\omega^\varphi \wedge \omega_\varphi^\theta \iff \\ & f'(t)dt \wedge d\theta = b\, dt \wedge d\theta + c\, dt \wedge d\varphi - hf(t)\, dt \wedge d\varphi - kf(t)\, d\theta \wedge d\varphi \implies \\ & \qquad \implies b = f'(t) \wedge c = h = k = 0; \\ & f'(t)dt \wedge d\varphi = \omega^t \wedge \omega_t^\varphi + d\omega^\theta \wedge \omega_\theta^\varphi \iff \\ & f'(t)dt \wedge d\varphi = \beta\, dt \wedge d\theta + \gamma\, dt \wedge d\varphi + lf(t)\, d\theta \wedge d\varphi - \implies \\ & \qquad \implies \gamma = f'(t) \wedge \beta = l = 0; \\ & 0 = \omega^\theta \wedge \omega_\theta^t + \omega^\varphi \wedge \omega_\varphi^t \iff \\ & \qquad\quad 0 = -f(t)\, \{a\, d\theta \wedge dt + \alpha dt \wedge d\varphi\} \implies \\ & \qquad \implies a = \alpha = 0. \end{aligned}$$

Therefore, we have the following connection forms:

$$\begin{aligned} \omega_t^\theta &= f'(t)d\theta \implies d\omega_t^\theta = f''(t)dt \wedge d\theta; \\ \omega_t^\varphi &= f'(t)d\varphi \implies d\omega_t^\varphi = f''(t)dt \wedge d\varphi; \\ \omega_\theta^\varphi &= 0 \implies d\omega_\theta^\varphi = 0. \end{aligned}$$

Now, be Cartan's second structure equation we can see

$$\Omega_t^\theta = d\omega_t^\theta - \omega_t^\varphi \wedge \omega_\varphi^\theta = f''(t)dt \wedge d\theta = \frac{f''}{f}\omega^t \wedge \omega^\theta;$$
$$\Omega_t^\varphi = d\omega_t^\varphi - \omega_t^\theta \wedge \omega_\theta^\varphi = f''(t)dt \wedge d\varphi = \frac{f''}{f}\omega^t \wedge \omega^\varphi;$$
$$\Omega_\theta^\varphi = d\omega_\theta^\varphi - \omega_\theta^t \wedge \omega_t^\varphi = (f'(t)d\theta) \wedge (f'(t)d\varphi) = \left(\frac{f'}{f}\right)^2 \omega^\theta \wedge \omega^\varphi,$$

from which we get the non-zero curvature tensor coefficients:

$$R_{t\theta t}^\theta = \frac{f''}{f},$$
$$R_{t\varphi t}^\varphi = \frac{f''}{f},$$
$$R_{\theta\varphi\theta}^\varphi = \left(\frac{f'}{f}\right)^2.$$

Consequently, the Ricci curvature coefficients and, subsequently, the scalar curvature are:

$$R_{tt} = R_{\theta tt}^\theta + R_{\varphi tt}^\varphi = -2\frac{f''}{f},$$
$$R_{\theta\theta} = R_{t\theta\theta}^t + R_{\varphi\theta\theta}^\varphi = -\frac{f''}{f} - \left(\frac{f'}{f}\right)^2,$$
$$R_{\varphi\varphi} = R_{t\varphi\varphi}^t + R_{\theta\varphi\varphi}^\theta = -\frac{f''}{f} - \left(\frac{f'}{f}\right)^2,$$
$$R = R_{tt} + R_{\theta\theta} + R_{\varphi\varphi} =$$
$$= -2\frac{f''}{f} - 2\left(\frac{f''}{f} + \left(\frac{f'}{f}\right)^2\right) =$$
$$= -\frac{2}{f}\left(2f'' + \frac{(f')^2}{f}\right) =$$
$$= -\frac{2}{f^2}\left(2ff'' + (f')^2\right).$$

Notice that this metric admits positive scalar curvature if

$$2ff'' + (f')^2 < 0 \iff f'' < -\frac{(f'^2)}{2f},$$

which is satisfied, for instance, by $f(t) = \sqrt{t}$. Hence, if we drop the assumption of compactness, the theorem fails.

Next, let us consider the 3-sphere, S^3. It is a compact manifold and it clearly admits a metric with positive scalar curvature (the round metric). The theorem fails because S^3 does not have a homologically non-trivial 2-torus, something that can be easily seen given its homology groups.

5.2 Relation with the Positive Mass Theorem

Let's start by giving the necessary background for the Positive Mass Theorem. Obvsiously, one must first define what mass is. To do so, we need to characterize the behaviour of the manifold at infinity, as the appropriate definition of mass is asymtoptic.

Definition 5.2.1 (Asymptotically Flat)**.** *Let $(S,\, g)$ be a 3-dimensional Riemannian manifold. We say that $(S,\, g)$ is asymptotically flat if there exists:*

1. *a compact subset $K \subset S$ such that $S \setminus K$ is diffeomorphic to $\mathbf{R}^3 \setminus \overline{B_1(0)}$;*
2. *a chart at infinity $(x^1, .x^2, x^3)$ on $S \setminus K$ such that*

$$|g_{ij} - \delta_{ij}| + r|\partial_k g_{ij}| + r^2|\partial_k \partial_l g_{ij}| = O(r^{-p}), \text{ and } R = O(r^{-p}),$$

for some $p > \frac{1}{2}$ and $q > 3$, where δ is the Euclidean metric, $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and R is the scalar curvature of g.

Definition 5.2.2 (ADM Mass)**.** *The ADM mass of an asymptotically flat Riemannian manifold $(S,\, g)$ is*

$$M = \lim_{r \to +\infty} \frac{1}{16\pi} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) \frac{x^i}{r},$$

where S_r is the a sphere of radius r in the chart at infinity (x^1, x^2, x^3).

To give further context to the previous definition, the ADM mass comes from varying the Einstein-Hilbert action, in order to have an asymptotically defined Hamiltonian (hence we need asymtotically flat manifolds) that gives the total energy of the gravitational field. Moreover, while it is not a trivial conclusion, the ADM mass is well-defined, i.e. it does not depend on the choice of chart at infinity [Bar86].

Theorem 10 (Positive Mass Theorem)**.** *Let (S, g) be a complete, aymptotically flat Riemmanian 3-manifold with non-negative scalar curvature, i.e. $R \geq 0$. Then:*

1. *Its ADM mass is non-negative, $M \geq 0$;*
2. *If $M = 0$ then (S, g) is isometric to $\mathbf{R}^3$ with the Euclidean metric.*

Given a torus T^3, we know from Theorem 9 that there is no metric g on T^3 with positive scalar curvature R. Furthermore, we know that if $R \geq 0$ then we have $R = 0$ and g is flat. A simple consequence, as seen in [Kaz], is:

Proposition 7. *Consider the manifold $(\mathbf{R}^3, g)$ such that*

1. *g is the standard Euclidean metric δ outside a compact set K;*
2. *$R_g \geq 0$, i.e. g has non-negative scalar curvature.*

Then, $g = \delta$ everywhere.

Proof. Let $d = \mathrm{diam}(K)$ and take $\varepsilon > 0$ so that we can include the compact set K inside a cube of edge $d+\varepsilon$. Identifying opposite faces results in a 3-torus that contains K and a "bit" of the outside, maintaining the assumptions of the proposition. We then have a 3-torus (T^3, g) whose scalar curvature is positive, $R_g \geq 0$.

However, because of Theorem 9, we know that g is flat and $R_g = 0$ in T^3, in particular inside the compact set K. Hence, g is the standard Euclidean metric everywhere. □

The previous proposition can be seen as a corollary of the Conjecture we proved. However, how does it relate to the Positive Mass Theorem?

Assumption *1.* is a stronger version of the asymptotically flat requirement as, in fact, the manifold itself is already the flat Euclidean space outisde of K. Consequently, this assumption implies that the ADM mass vanishes, $M = 0$. Hence, this corollary is a special case of the Positive Mass Theorem, giving a weaker version of its rigidity statement.

Bibliography

[Bar86] Robert Bartnik, *The mass of an asymptotically flat manifold*, Communications on Pure and Applied Mathematics **39** (1986), no. 5, 661–693.

[Fed96] Herbert Federer, *Geometric measure theory*, 1^{st} ed., Springer International Publishing, 1996, ISBN: 978-3-642-62010-2.

[Fle62] W.H. Fleming, *On the oriented plateau problem*, Rendiconti del Circolo Matematico di Palermo **90** (1962), no. 11, 69–90, doi:10.1007/BF02849427.

[GL80] Mikhael Gromov and H. Blaine Lawson, *Spin and scalar curvature in the presence of a fundamental group. i*, Annals of Mathematics **111** (1980), no. 2, 209–230, doi.org/10.2307/1971198.

[GN14] Leonor Godinho and Jose Natario, *An introduction to riemannian geometry – with applications to mechanics and relativity*, 1^{st} ed., Springer International Publishing, 2014, ISBN 978-3-319-08666-8.

[Hat80] Allen Hatcher, *Notes on basic 3-manifold topology*, 1980.

[Kaz] Jerry L. Kazdan, *Positive energy in general relativity*, Seminaire Bourbaki.

[KW75a] Jerry L. Kazdan and F. W. Warner, *Existence and conformal deformation of metrics with prescribed gaussian and scalar curvatures*, The Annals of Mathematics **101** (1975), no. 2, 317–331, doi:10.2307/1970993.

[KW75b] ———, *Prescribing curvatures*, Proceedings of Symposia in Pure Mathematics **27** (1975), doi:10.1080/10618560701678647.

[Mor16] Frank Morgan, *Geometric measure theory - a beginner's guide*, 5^{th} ed., Academic Press, 2016, ISBN:978-0-12-804489-6.

[Nat21] Jose Natario, *An introduction to mathematical relativity*, 1^{st} ed., Springer International Publishing, 2021, ISBN: 978-3-030-65682-9.

[RF10] H.L. Royden and P.M. Fitzpatrick, *Real analysis*, 4^{th} ed., Pearson Mordern Classics, Pearson, 2010, ISBN 978-0-13-468949-4.

[SY79] R. Schoen and Shing-Tung Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, The Annals of Mathematics **110** (1979), no. 1, 127–142, doi:10.2307/1971247.