On the 3-dimensional Geroch Conjecture and the Positive Mass Theorem

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1 Introduction

Inspired by ideas presented by Kazdan and Warner in [KW75b], while making some corrections to the literature, we prove in this thesis that a 3-torus, T^3, does not admit a metric with non-negative scalar curvature - the three-dimensional case of the Geroch Conjecture, already generalized to arbitrary dimensions by Schoen-Yau [SY79] and Gromov-Lawson [GL80]. To do so, we start with the assumption that there is a minimal 2-torus in T^3 and we use geometric arguments to conclude that such an assumption imposes constraints on the possible metrics for the manifold, namely regarding the scalar curvature.

Using Geometric Measure Theory, we show that there is a minimal homologically non-trivial 2-torus, providing both the results of existence and regularity, and we conclude that the previously mentioned assumption indeed holds.

Finally, we relate the main theorem of this thesis with a famous result of Relativity – the Positive Mass Theorem – as it implies a weaker version of its rigidity statement.

2 Non-Positive Scalar Curvature

A word fo caution is due, as Einstein's notation will be used and we will also denote the parameter t by the coordinate index 0, following [Nat21]. Furthermore, recall the Gauss lemma, from which we will obtain the coordinate system which we will use.

Lemma 1. *Let (M, g) be a Riemannian manifold and $S \subset M$ a hypersurface with unit normal vector field η. The hypersurfaces S_t, obtained from S by moving a distance t along the geodesics with initial condition η, orthogonal to S, remain orthogonal to the geodesics.*

Now, consider a 3-dimensional torus, T^3, while assuming that it admits a 2-dimensional stable minimizing torus T^2, and suppose there is a Riemannian metric in T^3 given in the Gauss lemma form, i.e.

$$g = dt^2 + h_{ij}(t, x)dx^i dx^j,$$

such that the level sets of t are Riemannian manifolds themselves, with an induced metric $h(t) = h_{ij}dx^i dx^j$ and a second fundamental form given by

$$K(t) = \frac{1}{2}\frac{\partial h_{ij}}{\partial t} dx^i dx^j.$$

Proposition 1. *The scalar curvatures of T^3 and its hypersurfaces are related by the equation:*

$$R = \bar{R} - 2\frac{\partial}{\partial t} K_i^i - (K_i^i)^2 - K_{ij}K^{ij}, \tag{1}$$

where R, $\bar{R}$ are the scalar curvatures of T^3 and its hypersurfaces, respectively.

Using (1.1) we can prove a weaker version of the 3-dimensional Geroch conjecture, thus concluding that a metric with positive scalar curvature is incompatible with the asssumption of a stable, minimal T^2.

Theorem 1. *Suppose T^3 admits a stable area minimizing 2-torus T^2. Then, T^3 does not admit a metric g of positive scalar curvature.*

Proof. We have a volume form given by $\sigma = \sqrt{\det(h)} dx^1 \wedge dx^2$, and recall that for any matrix-valued function M we have the identity $\partial_t \det(M) = \det(M) \operatorname{tr}(M^{-1}\partial_t M)$. Computing the formulas for the first and second variation of the volume form we eventually obtain:

$$\frac{\partial}{\partial t}\sigma = K_i^i \sigma, \quad \text{and} \quad \frac{\partial^2}{\partial t^2}\sigma = \left\{ \frac{\partial K_i^i}{\partial t} + (K_i^i)^2 \right\} \sigma.$$

By rewriting equation (1.1), we see that $\frac{\partial}{\partial t} K_i^i = \frac{1}{2}\left\{\bar{R} - R - (K_i^i)^2 - K_{ij}K^{ij}\right\}$, hence we arrive at the following formula for the second variation of the volume form:

$$\frac{\partial^2}{\partial t^2}\sigma = \frac{1}{2}\left\{\bar{R} - R + (K_i^i)^2 - K_{ij}K^{ij}\right\}\sigma.$$

Since the first variation has to vanish, we conclude from the first formula that $\int_{T^2} K_i^i \sigma = 0$. In fact, more is true, as it is well known that $K_i^i = 0$ pointwise for minimal surfaces. Now, assuming that the 2-torus is a stable minimizer, the second variation has to be non-negative, and given that $K_i^i = 0$, we conclude that

$$\frac{1}{2}\int_{T^2} \left\{\bar{R} - R - K_{ij}K^{ij}\right\}\sigma \geq 0.$$

Furthermore, by the Gauss-Bonnet theorem, as the Euler Characteristic is zero, $\chi(T^2) = 0$, we have that $\int_{T^2} \bar{R} = 0$. Consequently,

$$-\frac{1}{2}\int_{T^2} R + K_{ij}K^{ij} \geq 0 \Leftrightarrow \int_{T^2} R + K_{ij}K^{ij} \leq 0.$$

Note that $K_{ij}K^{ij}$ is a sum of squares, hence positive. Therefore, if the 3-torus T^3 admits an area minimizing 2-torus, T^2, there is no metric on the 3-torus with positive scalar curvature. □

3 Non-Negative Scalar Curvature

Following results and ideas from [KW75b], we will show that the only metric with non-negative scalar curvature on T^3 is the flat metric, which is the 3-dimensional case of the Geroch Conjecture. Given a smooth, positive function $u : T^3 \to \mathbf{R}$, consider the conformal deformation $\tilde{g} = u^4 g$ of (T^3, g). Then, we have:

Proposition 2. *The scalar curvature $\tilde{R}$ of $\tilde{g}$ is related to the curvature R of g by the equation*

$$u^5 \tilde{R} = -8\Delta u + Ru. \tag{2}$$

Consider now the induced elliptic differential operator given by $L_g u \equiv -8\Delta u + Ru = \tilde{R}u^5$, as in [KW75a]. Since T^3 is a compact manifold, we can take $\lambda_1(g)$ as the lowest eigenvalue of the operator with a corresponding positive eigenfunction ψ, from which some remarkable assessments about the existence of metrics, with certain curvatures, can be made, as given by the two lemmas.

Lemma 2. *Let M be a compact and connected manifold, with $\dim(M) \geq 3$. Then, M admits a metric pointwise conformal to g with positive (zero, or negative) scalar curvature if and only if $\lambda_1(g) > 0$ ($\lambda_1(g) = 0$, or $\lambda_1(g) < 0$, respectively).*

Lemma 3. *Let M be a compact manifold that does not admit a metric with positive scalar curvature. Then, any metric with zero scalar curvature must have zero Ricci curvature.*

Given the two previous lemmas, we now have an interesting restriction to both curvature and metric of a 3-torus allowing a stable area-minimizing T^2. From them, we get the following result.

Theorem 2. *Suppose we have a 3-torus (T^3, g) that admits a stable minimal T^2, with g such that the scalar curvature satisfies $R \geq 0$. Then, g is the flat metric (and $R = 0$).*

Proof. By Theorem 1, if T^3 admits a stable minimal T^2, there is no metric with positive scalar curvature for T^3, so (as to have $R \geq 0$) either it is identically zero in all of T^3 or it is positive somewhere.

By Lemma 3, if the metric has zero scalar curvature everywhere, then we must have zero Ricci curvature as well, hence (in a 3-manifold) the Riemann curvature is also identically zero and, consequently, g is flat.

Consider the case of R being positive somewhere. Take $\bar{\psi}$ to be the normalized eigenfunction of L_g with the eigenvalue $\lambda_1(g)$. Now, multiplying both sides by $\bar{\psi}$ and taking the integral over T^3, noting it is a compact manifold, we eventually get

$$L_g \bar{\psi} = \lambda_1(g)\bar{\psi} \iff 8 \int_{T^3} |\operatorname{grad}(\bar{\psi})|^2 + \int_{T^3} R\bar{\psi}^2 = \lambda_1(g).$$

Both integrals on the LHS cannot be simultaneously zero. Suppose the first integral vanishes; then $\operatorname{grad}(\bar{\psi})$ is identically zero and, therefore, the eigenfunction $\bar{\psi}$ is a positive constant and the second integral has to be strictly positive. On the other hand, if the second integral is zero, then one has that the eigenfunction vanishes where $R > 0$. However, by definition, $\bar{\psi}$ can't be zero everywhere and, consequentely, its gradient has to be non-zero.

So $\lambda_1(g) > 0$, and by Lemma 2, T^3 admits a pointwise conformal metric g_1 with positive scalar curvature, which contradicts Theorem 1. Therefore, we must have $R = 0$ and g must be the flat metric. $\square$

4 Existence of Minimal 2-torus

In this section we will show that, in fact, there exists a stable minimal 2-torus in T^3 and, consequently, all the previous results follow. In other words, we show that we can drop the assumption of T^3 admiting a stable minimal torus from previous theorems.

However, to do so we require some heavier hardware – Geometric Measure Theory (GMT), mainly following [Mor16]. The idea is to take a set of generalized surfaces (called rectifiable currents) and, in this set, we consider a sequence of surfaces with area decreasing to an infimum.

4.1 Measures and Currents

Before taking a convergent subsequence and showing that, in fact, this limit exists and is the surface of least area, we need some other tools. First, the useful measure for our work is:

Definition 4.1 (Hausdorff measure)**.** *Let α_m be the Lebesgue measure of the closed unit ball $\mathbf{B}^m(0,1) \subset \mathbf{R}^n$. For any $A \subset \mathbf{R}^n$, the m-dimensional Hausdorff measure is defined by*

$$\mathcal{H}^m(A) = \lim_{\delta \to 0} \inf_{\substack{A \subset \bigcup S_j \\ \mathrm{diam}(S_j) \leq \delta}} \sum \alpha_m \left(\frac{\mathrm{diam}(S_j)}{2} \right)^2,$$

where the infimum is taken over all the countable coverings $\{S_j\}$ of A with $\mathrm{diam}(S_j) \leq \delta$.

As δ decreases, the infimum itself is non-decreasing, and therefore the limit exists (allowing $0 \leq \mathcal{H}^m(A) \leq \infty$). We present now the sets that will be the generalized surfaces of GMT, for which recall the definition of Lipschitz functions introduced earlier.

Definition 4.2 (Rectifiable Set)**.** *Let $E \subset \mathbf{R}^n$. We say E is $(\mathcal{H}^m,\, m)$ rectifiable if:*

1. *$\mathcal{H}^m(E) < \infty$;*
2. *$\mathcal{H}^m$-almost all of E is contained in $\bigcup \mathrm{im}(f_i)$, where f_i are countably many Lipschitz functions from $\mathbf{R}^m$ to $\mathbf{R}^n$.*

Proposition 3. *On the previous definition, one can substitute the Lipschitz functions by C^1-diffeomorphisms f_j on compact domains with disjoint images. Moreover, the Lipschitz constants of f_j and f_j^{-1} can be taken near 1.*

Definition 4.3 (Tangent Cones)**.** *Let $E \subset \mathbf{R}^n$ and $a \in \mathbf{R}^n$. Considering the m-dimensional density $\Theta^m(E,\, a)$ with respect to the Hausdorff measure, $\mathcal{H}^m$, the tangent cone of E at a is defined by*

$$\mathrm{Tan}(E,\, a) = \mathbf{R}_0^+ \left[\bigcap_{\varepsilon > 0} \mathrm{clos} \left\{ \frac{x-a}{|x-a|} : x \in E,\, 0 < |x-a| < \varepsilon \right\} \right],$$

and the cone of approximate tangent vectors of E at a is given by

$$\mathrm{Tan}^m(E,\, a) = \bigcap_{\Theta^m(E-S,\, a)=0} \mathrm{Tan}(S,\, a).$$

For almost all points a in a rectifiable set E, the tangent cone $\mathrm{Tan}^m(E,\, a)$ is in fact a tangent plane [Mor16]. Moreover, an orientation of a m-dimensional rectifiable, positive measure set of $E \subset \mathbf{R}^n$ is a choice out of uncountably many possible orientation for each $\mathrm{Tan}^m(E,\, a)$.

Recall the definition of $\mathcal{D}^m$ from the introduction. Its dual space, $\mathcal{D}_m$, is the space of m-dimensional currents, as given a differential form ω and a oriented rectifiable set S, we have the induced linear functional:

$$\omega \mapsto S(\omega) = \int_S \left\langle \vec{S}(x),\, \omega \right\rangle \, d\mathcal{H}^m,$$

where $\vec{S}(x)$ is the unit m-vector associated with the oriented tangent plane to S at x.

We can also define the boundary of a current and its support. Let $S \in \mathcal{D}_m$ be a m-dimensional current. Then, the boundary of S is the $(m-1)$-dimensional current defined by $\partial S(\omega) = S(d\omega)$, and the support of S is the smallest closed set C, such that $\mathrm{supp}(\omega) \cap C = \emptyset \Rightarrow S(\omega) = 0$.

Definition 4.4 (Spaces of Currents)**.**

1. *$\mathcal{D}_m$ is the space of m-dimensional currents in $\mathbf{R}^n$;*

2. *$\mathcal{E}_m = \{T \in \mathcal{D}_m \,:\, \mathrm{supp}(T)$ is compact$\}$;*

3. *$\mathcal{R}_m = \{T \in \mathcal{E}_m : T$ is an oriented rectifiable set with integer multiplicities and finite measure$\}$ is the space of rectifiable currents;*

4. *$\mathcal{P}_m = \{$integral polyhedral chains$\}$ is the additive subgroup of $\mathcal{E}_m$ generated by classicaly oriented simplices;*

5. *$\mathbf{I}_m = \{T \in \mathcal{R}_m \,:\, \partial T \in \mathcal{R}_{m-1}\}$ is the set of rectifiable currents T whose boundary is a rectifiable current, the so-called integral currents.*

6. *$\mathcal{F}_m = \{T + \partial S \,:\, T \in \mathcal{R}_m,\, S \in \mathcal{R}_{m+1}\}$ is the set of integral flat chains.*

The last two spaces allow us to see how well-behaved the boundary operator ∂ is.

Proposition 4. *The boundary operator ∂ maps $\mathbf{I}_m$ to $\mathbf{I}_{m-1}$ and $\mathcal{F}_m$ to $\mathcal{F}_{m-1}$. Moreover, $\mathrm{supp}(\partial T) \subset \mathrm{supp}(T)$.*

Definition 4.5 (Mass; Flat norm)**.** *We define on the space of currents $\mathcal{D}_m$ the seminorms*

$$\mathbf{M}(T) = \sup \left\{ T(\omega) \,:\, \sup \|\omega(x)\|^* \leq 1 \right\}$$

and

$$\mathfrak{F}(T) = \inf \left\{ \mathbf{M}(A) + \mathbf{M}(B) :\, T = A + \partial B,\, A \in \mathcal{R}_m,\, B \in \mathcal{R}_{m+1} \right\},$$

called the mass and flat norm, respectively.

It will be shown that the flat norm topology is the natural topology to get compactness.

Definition 4.6 (Push-forward of a Current). *Let $T \in \mathcal{E}_m(\mathbf{R}^n)$ be a current with compact support, $\omega \in \mathcal{D}^m(\mathbf{R}^p)$ an arbitrary m-differential form and $f : \mathbf{R}^n \to \mathbf{R}^p$ a C^∞-map. Then, the push-forward $f_*T \in \mathcal{D}_m(\mathbf{R}^p)$ is defined by*

$$(f_*T)(\omega) = T(f^*\omega),$$

where $f^*\omega$ is the pullback of ω by f.

Theorem 3 ([Fed96] - 4.1.28). *$T \in \mathcal{E}_m$ is a rectifiable current iff given $\varepsilon > 0$, there exists an integral polyhedral chain $P \in \mathcal{P}_m(\mathbf{R}^v)$ and a Lipschitz function $f : \mathbf{R}^v \to \mathbf{R}^n$ such that*

$$\mathbf{M}(T - f_*P) < \varepsilon,$$

where f_* is the push-forward by f.

Corollary 1.

1. *$\{T \in \mathcal{R}_m : \operatorname{supp}(T) \subset \mathbf{B}^n(0,r)\}$ is $\mathbf{M}$ complete.*
2. *$\{T \in \mathcal{F}_m : \operatorname{supp}(T) \subset \mathbf{B}^n(0,r)\}$ is $\mathfrak{F}$ complete.*

Definition 4.7 (General Flat norm). *For any $T \in \mathcal{D}_m$, define the more general flat norm as*

$$\mathbf{F}(T) = \sup\{T(\omega) : \omega \in \mathcal{D}^m, \ \|\omega(x)\|^* \leq 1, \text{ and } \ \|d\omega(x)\|^* \leq 1 \text{ for all } x\}$$

$$= \min\{\mathbf{M}(A) + \mathbf{M}(B) : T = A + \partial B, \, A \in \mathcal{E}_m, \, B \in \mathcal{E}_{m+1}\}.$$

Proposition 5 (Lower Semicontinuity). *Given $T_i, T \in \mathcal{D}_m$ such that $T_i \xrightarrow{\mathbf{F}} T$, then*

$$\mathbf{M}(T) \leq \liminf \mathbf{M}(T_i)$$

4.2 Compactness Theorem

As previously stated, we want to work with a set $\mathcal{S}$ of surfaces which is compact under a natural topology. To do so we require two theorems – the Deformation Theorem and the Closure Theorem – from which we get the compactness of a suitable set, under the flat norm topology.

Theorem 4 (Deformation Theorem). *Let $T \in \mathbf{I}_m(\mathbf{R}^n)$ and $\varepsilon > 0$. Then, there are $P \in \mathcal{P}_m(\mathbf{R}^n)$, $\mathcal{Q} \in \mathbf{I}_m(\mathbf{R}^n)$ and $S \in \mathbf{I}_{m+1}(\mathbf{R}^n)$ such that the following conditions hold, for $\gamma = 2n^{2m+2}$:*

1. *$T = P + \mathcal{Q} + \partial S$;*
2. *$\operatorname{supp}(P) \cup \operatorname{supp}(\mathcal{Q}) \cup \operatorname{supp}(S) \subset \{x : \operatorname{dist}(x, \operatorname{supp}(T)) \leq 2n\varepsilon\}$;*
3. *$\mathbf{M}(P) \leq \gamma[\mathbf{M}(T) + \varepsilon\mathbf{M}(\partial T)]$,*

   $\mathbf{M}(\partial P) \leq \gamma\mathbf{M}(\partial T)$,

   $\mathbf{M}(\mathcal{Q}) \leq \gamma\varepsilon\mathbf{M}(\partial T)$

   $\mathbf{M}(S) \leq \gamma\varepsilon\mathbf{M}(T)$.

Corollary 2. *The set* $\mathcal{S} = \{T \in \mathbf{I}_m : \mathrm{supp}(T) \subset \mathbf{B}^n(0, c_1),\, \mathbf{M}(T) \le c_2,\, \mathbf{M}(\partial T) \le c_3\}$ *is totally bounded under* $\mathfrak{F}$.

Proof. By the Deformation Theorem 4 , each $T \in \mathcal{S}$ can be approximated by a polyhedral chain P, such that $\mathbf{M}(P) \le \gamma\,(c_2 + \varepsilon c_3)$ and $\mathrm{supp}(P) \subset \mathbf{B}^n(0, c_1 + 2n\varepsilon)$, in a ε-grid. However, there are only finitely many chains P, therefore $\mathcal{S}$ is totally bounded. ☐

We then have totally boundedness of our set, all we are missing is completeness. Recall the definition of $\mathcal{F}_m$ and that $\{T \in \mathcal{F}_m : \mathrm{supp}(T) \subset \mathbf{B}^n(0, r)\}$ is $\mathfrak{F}$ complete.

Theorem 5 (Closure Theorem)**.**

1. *$\mathbf{I}_m$ is $\mathbf{F}$-closed in $\mathbf{N}_m$;*
2. *$\mathbf{I}_{m+1} = \{T \in \mathcal{R}_{m+1} \,:\, \mathbf{M}(\partial T) < \infty\}$;*
3. *$\mathcal{R}_m = \{T \in \mathcal{F}_m \,:\, \mathbf{M}(T) < \infty\}$, consequently,*
4. *$\mathcal{S} = \{T \in \mathbf{I}_m : \mathrm{supp}(T) \subset \mathbf{B}^n(0, r),\, \mathbf{M}(T) \le c,\, \mathbf{M}(\partial T) \le c\}$ is complete under $\mathfrak{F}$.*

Proof. Proofs of assertions 1. to 3. will be omitted, as we want to prove and use assertion 4..

Let T_i be a Cauchy sequence in $\mathcal{S}$. By completeness of $\{T \in \mathcal{F}_m : \mathrm{supp}(T) \subset \mathbf{B}^n(0, r)\}$, there is a limit T in $\mathcal{F}_m$. By lower semicontinuity of mass $\mathbf{M}(T) \le c$ and $\mathbf{M}(\partial T) \le c$, hence we have, by 3., $T \in \mathcal{R}_m$ and, consequently, by 2., $T \in \mathbf{I}_m$. Therefore, the limit of a Cauchy sequence in $\mathcal{S}$ exists in $\mathcal{S}$, which gives us completeness. ☐

Corollary 3 (Compactness Theorem)**.** *For a closed ball K in $\mathbf{R}^n$ and $0 \le c < \infty$, the set*
$\mathcal{S} = \{T \in \mathbf{I}_m : \mathrm{supp}(T) \subset K,\, \mathbf{M}(T) \le c,\, \mathbf{M}(\partial T) \le c\}$ *is $\mathfrak{F}$ compact.*

Proof. By Corollary 2 and the Closure Theorem (Theorem 5), the set is both totally bounded and complete, hence compact. ☐

Remark. The range of this theorem is further extended when we substitute K for a C^1 compact Riemannian submanifold of $\mathbf{R}^n$. Via C^1-embeddings into Euclidean space, it can then be generalized to any compact C^1 Riemannian manifold $(M,\, g)$.

We can now extract a convergent subsequence from any sequence of rectifiable currents. By defining the homology class of a rectifiable current T as the set of rectifiable currents S such that $S - T = \partial X$ for some rectifiable current X, we have:

Theorem 6. *Let M be a compact C^1-Riemannian manifold and T be a rectifiable current in M. Then, among the currents S such that $S - T = \partial X$ in M, there is one that minimizes area.*

We do not know how geometrically ”well-behaved” this current is, i.e. we lack knowledge of its regularity, however, using a theorem by Wendell Fleming [Fle62] we guarantee the interior regularity for 2-dimensional currents.

Theorem 7 (Regularity for the 2-dimensional hypersurface)**.** *Any 2-dimensional, area-minimizing rectifiable current T in a 3-dimensional manifold M is a smooth, embedded submanifold.*

4.3 Main Conjecture and Generalizations

We finally arrive at our main result, as we have derived the existence of a smooth stable minimal 2-torus in T^3, and thus conclude the proof of the 3-dimensional case of the Geroch Conjecture.

Theorem 8 (Geroch Conjecture)**.** *There is no metric g with positive scalar curvature, R, on the 3-torus T^3. Furthermore, if $R \geq 0$, then g is flat and $R = 0$.*

Proof. We have already concluded that there exists a homologically non-trivial minimizing 2-torus in T^3, which means we can relax the existence constraint of the previous theorems. Therefore, the first part of the result follows trivially from Theorem 1, and the second part from Theorem 2. □

Note that, essentially, we only used the compactness of T^3 and the existence of a homologically non-trivial 2-torus. Hence, this result can be generalized to other 3-dimensional manifolds, satisfying the previous properties, for example, the connected sum $T^3 \# M$ with M a smooth, compact and connected 3-dimensional manifold, for instance T^3 again. For other examples, let $f : T^2 \to T^2$ be a orientation-preserving diffeomorphism. Consider the infinitely many inequivalent torus bundles (see [Hat80] for more information) constructed by taking the Cartesian product of T^2 and the unit interval $I = [0, 1]$, and gluing the two components of the boundary via f, that is:

$$M = \frac{T^2 \times I}{(0, x) \sim (1, f(x))}.$$

Clearly, we can apply the previous theorem to these constructions. Notice, moreover, that if f is the identity, the resulting bundle is just the 3-torus T^3.

To end this section, we will show that the assumptions of the theorem are in fact necessary, through examples where the conjecture fails. Consider the manifold $(\mathbf{R}^+ \times T^2,\, g)$, where $g = dt^2 + (f(t))^2 \left(d\theta^2 + d\varphi^2\right)$ is the metric and $f(t)$ is a positive function.

Then, after some computations following Cartan's structure equations we get the Ricci curvature coefficients and, subsequently, the scalar curvature is:

$$R = -\frac{2}{f^2}\left(2ff'' + (f')^2\right).$$

Notice that this metric admits positive scalar curvature if

$$2ff'' + (f')^2 < 0 \iff f'' < -\frac{(f'^2)}{2f},$$

which is satisfied, for instance, by $f(t) = \sqrt{t}$. Hence, if we drop the assumption of compactness, the theorem fails.

Next, let us consider the 3-sphere, S^3. It is a compact manifold and it clearly admits a metric with positive scalar curvature (the round metric). The theorem fails because S^3 does not have a homologically non-trivial 2-torus, something that can be easily seen given its homology groups.

5 Relation with the Positive Mass Theorem

Regarding the Positive Mass Theorem, let us first characterize the behaviour of the manifold at infinity, as the appropriate definition of mass is asymtoptic.

Definition 5.1 (Asymptotically Flat). *Let $(S,\, g)$ be a 3-dimensional Riemannian manifold. We say that $(S,\, g)$ is asymptotically flat if there exists:*

1. *a compact subset $K \subset S$ such that $S \setminus K$ is diffeomorphic to $\mathbf{R}^3 \setminus \overline{B_1(0)}$;*
2. *a chart at infinity $(x^1, .x^2, x^3)$ on $S \setminus K$ such that*

$$|g_{ij} - \delta_{ij}| + r|\partial_k g_{ij}| + r^2|\partial_k \partial_l g_{ij}| = O(r^{-p}), \text{ and } R = O(r^{-p}),$$

for some $p > \frac{1}{2}$ and $q > 3$, where δ is the Euclidean metric, $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and R is the scalar curvature of g.

Definition 5.2 (ADM Mass). *The ADM mass of an asymptotically flat Riemannian manifold $(S,\, g)$ is*

$$M = \lim_{r \to +\infty} \frac{1}{16\pi} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) \frac{x^i}{r},$$

where S_r is the a sphere of radius r in the chart at infinity (x^1, x^2, x^3).

This mass comes from varying the Einstein-Hilbert action, as to have an asymptotically defined Hamiltonian, and it does not depend on the choice of chart at infinity [Bar86].

Theorem 9 (Positive Mass Theorem). *Let (S, g) be a complete, aymptotically flat Riemmanian 3-manifold with non-negative scalar curvature, i.e. $R \geq 0$. Then:*

1. *Its ADM mass is non-negative, $M \geq 0$;*
2. *If $M = 0$ then (S, g) is isometric to $\mathbf{R}^3$ with the Euclidean metric.*

Given a torus T^3, we know from Theorem 8 that there is no metric g on T^3 with positive scalar curvature R. Furthermore, we know that if $R \geq 0$ then we have $R = 0$ and g is flat. A simple consequence, as seen in [Kaz], is:

Proposition 6. *Consider the manifold $(\mathbf{R}^3, g)$ such that*

1. *g is the standard Euclidean metric δ outside a compact set K;*
2. *$R_g \geq 0$, i.e. g has non-negative scalar curvature.*

Then, $g = \delta$ everywhere.

Proof. Let $d = \mathrm{diam}(K)$ and take $\varepsilon > 0$ so that we can include the compact set K inside a cube of edge $d+\varepsilon$. Identifying opposite faces results in a 3-torus that contains K and a "bit" of the outside, maintaining the assumptions of the proposition. Therefore, by Theorem 8, we know that g is flat and $R_g = 0$ in T^3, in particular inside the compact set K. Hence, g is the standard Euclidean metric everywhere. $\square$

The previous proposition can be seen as a corollary of the Conjecture we proved. However, how does it relate to the Positive Mass Theorem?

Assumption *1.* is a stronger version of the asymptotically flat requirement as, in fact, the manifold itself is already the flat Euclidean space outisde of K. Consequently, this assumption implies that the ADM mass vanishes, $M = 0$. Hence, this corollary is a special case of the Positive Mass Theorem, giving a weaker version of its rigidity statement.

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