Wave scattering by a compressible thermal boundary layer

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July 2019

Abstract

Sound propagation is considered in a thermal boundary layer with a linear temperature profile between the wall temperature and the temperature of an isothermal free stream; the flow velocity is constant with arbitrary magnitude. The sound field in the thermal boundary layer is specified by confluent hypergeometric functions: it is matched to plane waves in the free stream by conditions of continuity of the pressure and particle displacement at the edge of the thermal boundary layer. These specify for a wave incident from the free stream, the reflection factor back into the free stream and transmission factor into the thermal boundary layer. The reflection and transmission factor are generally complex, and their amplitude and phase are plotted as a function of the angle of incidence. For given angle of incidence, the amplitude and phase of the acoustic pressure are plotted against distance from the wall over width of the thermal boundary layer and into the free stream. The scattering factors and acoustic pressure are plotted for a wide range of free stream Mach numbers, temperature gradients in the thermal boundary layer and frequencies from the compactness to the ray approximations.

Keywords: Thermal Boundary Layer, Sound Scattering, Acoustic Wave Propagation

1. Introduction

1.1. Motivation

The acoustic wave equation in a homentropic unidirectional shear flow has been solved exactly, in the literature, only for three velocity profiles: linear, hyperbolic tangent or exponential. The acoustic wave equation in a shear flow in the isentropic case, when the entropy is constant only along streamlines, allows the entropy and temperature to vary from one stream line to the other. It is interesting to understand how temperature gradients together with shear flow affects sound propagation. Considering temperature gradients transverse to the shear flow together with a linear velocity profile, for a homenergetic shear flow [7, 8], or constant enthalpy, it is possible to relate the sound speed to the mean flow velocity and specify the temperature profile in the boundary layer. This dissertation considers a linear temperature profile with a uniform flow so that the effect of temperature gradient can be isolated and studied.

1.2. Topic Overview

The acoustic wave equation in an isentropic unidirectional shear flow allows both the velocity and the temperature to vary transversely to the flow direction. The flow velocity is assumed to be constant, in order to isolate the effects of the temperature gradient, for a linear temperature profile matched to an isothermal free stream.

The acoustic wave equation in the linear temperature profile has a singularity at the point of vanishing sound speed, outside the thermal boundary layer. The solution around this point has infinite radius of convergence, and covers the whole of the thermal boundary layer leading to a confluent hypergeometric equation [20].

The acoustic field in the thermal boundary layer may have a critical layer where the pressure vanishes. The acoustic field consists of a superposition of upward and downward propagating waves which can be combined so as to satisfy the impedance boundary conditions at the wall. An incident wave from the free stream gives rise to a reflected wave back into the free stream and a transmitted wave into the thermal boundary layer.

The matching of the acoustic pressure and displacement at the edge of the thermal boundary layer specify the reflection and transmission coefficients.

2. Exact solution in a thermal boundary layer

2.1. Acoustic wave equation in an anisothermal shear flow

The acoustic wave equation for a plane unidirectional isentropic shear flow is [12, 17, 15, 4]

$$
\begin{aligned}
&(\omega - kU)P'' + 2[kU' + (\omega - kU)c'/c]P' \\
&+(\omega - kU)[(\omega - kU)^2/c^2 - k^2]P = 0,
\end{aligned} \tag{1}
$$

where $P(y;k,\omega)$ is the spectrum of the acoustic pressure *p(x; y, t)* for a wave of frequency ω and horizontal wavenumber k at transverse station y.

$$
p(x,y,t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega \int_{-\infty}^{+\infty} \mathrm{d}k\, exp[i(kx - \omega t)]P(y;k,\omega); \tag{2}
$$

and prime denotes derivative with regard to y, e.g. $P' \equiv \mathrm{d}P/\mathrm{d}y$. The shear flow is unidirectional (3a)

$$
\vec{U} = \vec{e_x}U(y); \tag{3a}
$$

$$
[c(y)]^2 = \gamma RT(y), \tag{3b}
$$

the sound speed may vary across the stream lines, e.g. (3b) for a perfect gas of constant R, adiabatic exponent γ, and temperature $T(y)$. The unidirectional shear flow (3a) corresponds to a constant mean flow pressure; in the homentropic case, since the entropy is also constant, all other thermodynamic quantities are constant, including sound speed and temperature, so the flow is isothermal. In the non-homentropic isentropic case, the entropy is conserved along the streamlines, and can vary across the streamlines, leading to transverse temperature and sound speed gradients (3b).

In order to isolate temperatures effects, the mean flow velocity is assumed to be constant (4a):

$$
U(y) = const \equiv U: \tag{4a}
$$

$$
P'' + 2(c'/c)P' + [(\omega - kU)^2/c^2 - k^2]P = 0 \tag{4b}
$$

and the wave equation (1) reduces to (4b). Next is considered a linear temperature profile in a thermal boundary layer of width L, matched to an isothermal free stream outside:

$$
[c(y)]^2 = c_\infty^2 \quad \text{if} y \geq L \tag{5a}
$$

$$
[c(y)]^2 = c_0^2 + \frac{(c_\infty^2 - c_0^2)y}{L} \quad \text{if} 0 \leq y \leq L \tag{5b}
$$

where c_∞ is the sound speed in the free stream and c_0 the value at the wall. The simplest case is the propagation in the uniform (4a) and isothermal (6a) free stream, when the wave equation (4b) reduces to (6b):

$$
y \geq L: \tag{6a}
$$

$$
P''_\infty + [(\omega - kU)^2/c_\infty^2 - k^2]P_\infty = 0 \tag{6b}
$$

This corresponds to plane waves (7b) with transverse wavenumber (7a)

$$
\kappa \equiv |[(\omega - Uk)/c_\infty]^2 - k^2|^{1/2} \tag{7a}
$$

$$
P''_\infty + \kappa^2 P_\infty = 0 \tag{7b}
$$

whose solution is elementary.

The general integral is a superposition of upward C_+ and downward C_- propagating waves;

$$
\kappa = \Re(\kappa): \tag{8a}
$$

$$
P_\infty(y;k,\omega) = C_+\, exp(\mathrm{i}\,\kappa\, y) + C_-\, exp(-\mathrm{i}\,\kappa\, y), \tag{8b}
$$

if the transverse wavenumber (7a) is real. If the transverse wavenumber (7a) is imaginary:

$$
\kappa = \mathrm{i}\,|\kappa|: \tag{9a}
$$

$$
P_\infty(y;k,\omega) = C_1\, exp(-|\kappa|\, y) + C_2\, exp(|\kappa|\, y), \tag{9b}
$$

then the acoustic field is a superposition of evanescent C_1 and divergent C_2 waves; the latter can be suppressed setting $C_2 = 0$. The condition for propagation in the free stream, e.g. that (7a) is real, is satisfied if either (10a) or (10b) are met:

$$
\omega - Uk > kc_\infty \quad or \tag{10a}
$$

$$
-(\omega - Uk) > kc_\infty. \tag{10b}
$$

There are two cases to consider: (i) for upstream propagation (11a) the (10a,10b) imply respectively (11b,11c):

$$
0 < k = |k|: \tag{11a}
$$

$$
\omega > |k|(U + c_\infty) \quad or \tag{11b}
$$

$$
\omega < |k|(U - c_\infty); \tag{11c}
$$

(ii) for downstream propagation (12a) then (10a,10b) imply respectively (12b,12c)

$$
0 > k = -|k|: \tag{12a}
$$

$$
\omega > -|k|(U + c_\infty) \quad or \tag{12b}
$$

$$
\omega < |k|(c_\infty - U). \tag{12c}
$$

In the thermal boundary layer the wave equation (4b)has variable coefficients (5), so there is no elementary solution (8b).

2.2. Thermal boundary layer with linear profile

In the thermal boundary layer (13a) the sound speed profile is given by (5b)≡(13b)

$$0 \le y \le L : \tag{13a}$$
$$[c(y)]^2 = c_0^2(1 + ay/L), \tag{13b}$$
$$a = (c_\infty/c_0)^2 - 1, \tag{13c}$$

where the dimensionless temperature parameter is a positive $a > 0$ if the free stream is hotter than the wall $T_\infty > T_0$, negative $a < 0$ otherwise $T_0 > T_\infty$ and zero $a = 0$ in the intermediate isothermal case $T_0 = T_\infty$. On substitution of (13b), the acoustic wave equation (4b) in the thermal boundary becomes:

$$\begin{aligned}&(1 + ay/L)P'' + (a/L)P' \\ &+\{[(\omega - Uk)/c_0]^2 - k^2(1 + ay/L)\}P = 0.\end{aligned} \tag{14}$$

The linear ordinary differential equation with variable coefficients has only one regular singularity at finite distance, at the point where the sound speed would vanish. When $c(y_1) = 0$:

$$y_1 = -L/a = \begin{cases} > 0 & \text{if } c_0 > c_\infty, \quad (15a) \\ < 0 & \text{if } c_0 < c_\infty, \quad (15b) \end{cases}$$

which: (i) for wall cooler than the free stream $0 < T_0 < T_\infty$ lies below the wall (15b) outside the physical region of interest $y_1 < 0 \le y < \infty$; (ii) for wall hotter than the free stream $T_0 > T_\infty > 0$ the singularity lies above the wall (15a) and is excluded by matching to the isothermal free stream before $0 < L < y_1$ which is always the case because $y_1/L > 1$ for $c_0 > c_\infty$.

The other two singularities of (14) are the points at infinity, of which: (i) $y = -\infty$ lies below the wall outside the physical region of interest; (ii) $y = +\infty$ lies outside the thermal boundary layer, where the solution of (14) is not needed, because it is replaced by the solutions (8a,8b) of (6a,6b). Thus the solution around the singularity at finite distance (15) has infinite radius of convergence, $-\infty < y \le +\infty$ covers the whole of the thermal boundary layer $0 \le y \le L$. The preceding remarks suggests using (16a) an independent variable (16b) proportional to the distance from the singularity made dimensionless multiplying by the horizontal wavenumber.

$$Q(z) \equiv P(y) : \tag{16a}$$
$$z = k(y - y_1) = k(y + L/a). \tag{16b}$$

The wall $y = 0$ and width $y = L$ of the boundary layer corresponds to the positions:

$$y = 0 : \quad z_1 = kL/a = kL/[(c_\infty/c_0)^2 - 1] \tag{17a}$$
$$y = L : \quad z_2 = kL(1 + 1/a) = kL/[1 - (c_0/c_\infty)^2], \tag{17b}$$

Note that (17a,17b) always have the same sign $z_1, z_2 > 0$ so the origin $z = 0$ never lies between them.

Substituting (16a,16b) in (14) leads to the linear differential equation with variable coefficients:

$$zQ'' + Q' + (b - z)Q = 0, \tag{18}$$

which involves a single dimensionless parameter:

$$b \equiv [(\omega - Uk)/c_0]^2 L/ka. \tag{19}$$

The differential equation (18) has one regular singularity at the origin and other singularities at infinity. In the particular case (20a) it would reduce to (20b) which differs only on the last sign ($-zQ$ instead of zQ) from a Bessel equation of order zero.

$$b = 0 : \tag{20a}$$
$$zQ'' + Q' - zQ = 0. \tag{20b}$$

The Bessel equation is a particular case of the confluent hypergeometric equation:

$$\zeta F'' + (\beta - \zeta)F' - \alpha F = 0, \tag{21}$$

which involves two parameters (α, β). It is shown next that the differential equation (14) can be reduced to the confluent hypergeometric type (21) via changes of dependent $Q \to F$ and independent $z \to \zeta$ variables; this specifies the constants (α, β) in terms of b, for any value of the latter.

2.3. Transformation to a confluent hypergeometric equation

The change of dependent variable:

$$Q(z) = exp(\nu z)J(z), \tag{22}$$

leads from (18) to the differential equation:

$$zJ'' + (1 + 2\nu z)J' + [(\nu^2 - 1)z + \nu + b]J = 0, \tag{23}$$

where the parameters ν can be chosen at will. The differential equation (23) is close to the confluent hypergeometric type (21) by suppression the coefficient of z in the coefficient in square brackets, leading to the choice (24a, 24b) of parameters and to the change of dependent variable (24c)

$$\nu^2 = 1 : \tag{24a}$$
$$\nu_\pm = \pm 1, \tag{24b}$$
$$Q(z) = exp(\pm z)J_\pm(z). \tag{24c}$$

Substituting (24a) in (23) specifies:

$$zJ''_\pm + (1 \pm 2z)J'_\pm + (b \pm 1)J_\pm = 0, \qquad (25a, b)$$

the two differential equations satisfied one by J_+ and other by J_-.

Both differential equations (25) reduce via slightly different changes of independent variable (26a)

$$\zeta = \mp 2z : \qquad (26a)$$
$$J_\pm(z) = F(\zeta), \qquad (26b)$$

to a confluent hypergeometric equation for (26b):

$$\zeta F'' + (1 - \zeta)F' - [(1 \pm b)/2]F = 0. \qquad (27)$$

The latter is a [10, 19, 18] confluent hypergeometric equation (21)≡(27) with parameters:

$$\beta = 1, \qquad (28a)$$
$$\alpha = (1 \pm b)/2. \qquad (28b)$$

The confluent hypergeometric equation has two linearly independent solutions, one which is always [2, 5] a function of the fist kind:

$$\begin{aligned} &F(\alpha; 1; \zeta) \\ &= 1 + \sum_{n=0}^{\infty} \zeta^n (n!)^{-2} \alpha(\alpha + 1) \cdots (\alpha + n - 1); \end{aligned} \qquad (29)$$

the Pochhammer symbol (30a) was used in (29)

$$\begin{aligned} (\alpha)_n &\equiv \alpha(\alpha + 1) \cdots (\alpha + n - 1) \\ &= \Gamma(\alpha + n)/\Gamma(\alpha), \end{aligned} \qquad (30a)$$
$$(1)_n = n!, \qquad (30b)$$

which is related to the Gamma function [21, 6] and implies (30b).

In the case of parameter $\beta = 1$ a solution of the confluent hypergeometric equation linearly independent from (29) is specified by [11, 16, 13] the function of the second kind:

$$\begin{aligned} &G(\alpha; 1; \zeta) \\ &= \log \zeta \lim_{\sigma \to 0} \sum_{n=0}^{\infty} \zeta^{n+\sigma} (\alpha + \sigma)_n [(1 + \sigma)_n]^{-2} \\ &+ \lim_{\sigma \to 0} \sum_{n=0}^{\infty} \zeta^{n+\sigma} (\alpha + \sigma)_n [(1 + \sigma)_n]^{-2} \\ &\{\Psi(\alpha + \sigma + n) - \Psi(\alpha + \sigma) \\ &+ 2\Psi(1 + \sigma) - 2\Psi(1 + \sigma + n)\}; \end{aligned} \qquad (31)$$

The confluent hypergeometric function of the second kind (31) consists of two terms: (i) a logarithm of the independent variable ζ multiplied by the function of the first kind (29):

$$G(\alpha; 1; \zeta) = \log \zeta F(\alpha; 1; \zeta) + N(\alpha; 1; \zeta); \qquad (32)$$

(ii) plus a complementary function

$$\begin{aligned} &N(\alpha; 1; \zeta) = \\ &\sum_{n=0}^{\infty} \zeta^n (n!)^{-2} (\alpha)_n [\Psi(\alpha + n) - \Psi(\alpha) - 2\Psi(1 + n)]. \end{aligned} \qquad (33)$$

The last factor $+2\Psi(1)$ in (31) was omitted from the square brackets in (33) because it is a multiple of the function of the first kind (29), and can be included in the latter.

3. Multiple internal reflections in the thermal boundary layer

3.1. Vanishing of the acoustic pressure at the critical layer

The general solution of the differential equation (25) is a linear combination of confluent hypergeometric functions of the first (29) and second (32, 33) kind with parameters (28a, 28b); and variable (26a):

$$\begin{aligned} J_\pm(z) = &A_\pm F(1/2 \pm b/2; 1; \mp 2z) \\ &+ B_\pm G(1/2 \pm b/2; 1; \mp 2z). \end{aligned} \qquad (34)$$

Substitution of (34) in (24c) shows that:

$$\begin{aligned} Q_\pm(z, b) = exp(\pm z)\{&A_\pm F(1/2 \pm b/2; 1; \mp 2z) \\ &+ B_\pm G(1/2 \pm b/2; 1; \mp 2z)\}, \end{aligned} \qquad (35)$$

the leading term corresponds to a divergent or evanescent wave in the direction of increasing or decreasing z, respectively for Q_+ and Q_-. The variable sound speed in the thermal boundary layer causes multiple internal reflections of all orders, represented by the series in curly brackets. The change of dependent (16a) and independent (16b) variable:

$$0 \le y \le L : \qquad (36a)$$

$$\begin{aligned} &P(y) = exp[\pm k(y + L/a)] \\ &\{A_\pm F(1/2 \pm b/2; 1; \mp 2k(y + L/a)) \\ &+ B_\pm G(1/2 \pm b/2; 1; \mp 2k(y + L/a))\}, \end{aligned} \qquad (36b)$$

specifies the acoustic pressure in the thermal boundary layer.

The confluent hypergeometric function of the second kind (32) has a logarithmic singularity, specified by (26a, 16b)

$$log\zeta = log(\mp 2z) = log[\mp 2k(y + L/a)], \qquad (37)$$

at the point (15) where the sound speed vanishes. Although this lies outside the flow region of interest, the physical reasons for the singularity deserve consideration. The point where the sound speed vanishes corresponds to a singularity of the differential equations (14), (18), (23) and (25) because the accumulation of acoustic energy at a point can lead to an 'infinite' amplitude, in linear non-dissipative theory. In reality non-linear effects will limit the amplitude, and it is dissipation which may cause partial absorption of the sound wave. As for a shock wave, which is a non-linear process involving dissipation, the change of amplitude across a critical layer can be calculated in the limit of zero "thickness'. The method to be used for a critical layer [1, 14, 3] in the case of frequency ω and time dependence can be adapted to the wavenumber k and spatial dependence. The passage across the singularity changes the sign of the argument of the logarithm in (37) and thus adds a term in:

$$log|2k(y+L/a)| + \begin{cases} 0 & \text{if } y > -L/a, \quad (38a) \\ \mathrm{i}\pi & \text{if } y < -L/a, \quad (38b) \end{cases}$$

This specifies the discontinuity across the critical layer.

The two expressions (35) for the acoustic pressure field in the thermal boundary layer differ only in the signs of z and b:

$$Q_+(z;b) = Q_-(-z;-b). \tag{39}$$

There are four particular integrals in (35), viz. the functions of the first F and second G kinds with upper or lower signs, corresponding to $(F, G(\pm z; \pm b))$; only two can be linearly independent because the differential equation is of order two. Thus the functions of the second kind can be avoided, since $F(\pm z; \pm b)$ are linearly independent. The general integral can be expressed as their linear combination.

$$0 \le y \le L: \tag{40a}$$

$$\begin{aligned} P(y;k;\omega) = T\{&exp[k(y+L/a)] \\ &F(1/2+b/2;1;-2k(y+L/a)] \\ &-Dexp[-k(y+L/a)] \\ &F[1/2-b/2;1;+2k(y+L/a)]\}. \end{aligned} \tag{40b}$$

The amplitude of the downward propagating wave in the thermal boundary layer is the transmission factor T, if a wave of unit amplitude is incident outside the boundary layer (7a, 7b):

$$y \ge L: P_\infty(y) = exp(-\mathrm{i}\kappa y) + R\, exp(\mathrm{i}\kappa y), \tag{41}$$

giving rise to a reflected wave whose amplitude is the reflection factor R. The acoustic pressure field inside (40) and outside (41) the thermal boundary layer involves two scattering parameters, namely the reflection R and transmission T factors, which are determined subsequently by applying two matching conditions. The constant D in (40) is determined next from the boundary condition at the wall.

3.2. Rigid or impedance wall boundary conditions

The impedance boundary condition at the wall:

$$P(0;k;\omega) = -Z(k,\omega)V(0;k;\omega), \tag{42}$$

relates the acoustic pressure perturbation spectrum (2) to the spectrum of the transverse acoustic velocity:

$$\begin{aligned} &v_y(x,y,t) = \\ &\int_{-\infty}^{+\infty} \mathrm{d}\omega \int_{-\infty}^{+\infty} \mathrm{d}k\, exp[\mathrm{i}(kx-\omega t)]\, V(y,k,\omega). \end{aligned} \tag{43}$$

The transverse component of the linearized inviscid momentum equation:

$$\rho_0[\partial v_y/\partial t + U(y)\,\partial v_y/\partial x] + \partial p/\partial y = 0, \tag{44}$$

relates the two spectra by:

$$\rho_0 \mathrm{i}[kU(y)-\omega]\, V(y,k,\omega) + \mathrm{d}P(y,k,\omega)/\mathrm{d}y = 0. \tag{45}$$

For a uniform flow (4a) at the wall this simplifies to:

$$\begin{aligned} \lim_{y\to 0} \mathrm{d}P(y,k,\omega)/\mathrm{d}y &= -\mathrm{i}\rho_0[kU-\omega]\, V(0,k,\omega) \\ &= \mathrm{i}\rho_0[kU-\omega]\, P(0,k,\omega)/Z(k,\omega), \end{aligned} \tag{46}$$

where the impedance boundary condition (42) was used.

The specific impedance is defined (47a) by the ratio to the impedance of a plane wave:

$$z(k,\omega) = Z(k,\omega)/(\rho_0 c): \tag{47a}$$

$$0 = c\, z(k,\omega)\, P'(0,k,\omega) + \mathrm{i}[\omega - kU]P(0,k,\omega), \tag{47b}$$

simplifying the boundary condition to (47b). From (40) follows the second term of (47b):

$$P(0;k,\omega) = T(E_+ - E_- D), \tag{48a}$$

$$E_\pm \equiv exp(\pm kL/a)\, F(1/2 \pm b/2; 1; \mp 2kL/a). \tag{48b}$$

The derivative of the confluent hypergeometric function:

$$\mathrm{d}[F(\alpha;\beta;\xi)]/\mathrm{d}\xi = (\alpha/\beta)\, F(1+\alpha; 1+\beta; \xi), \quad (49)$$

appears in:

$$\begin{gathered}\mathrm{d}P(y;k;\omega)/\mathrm{d}y = \\ kT\{exp[k(y+L/a)]\, F(1/2+b/2; 1; -2k(y+L/a)) \\ +Dexp[-k(y+L/a)]\, F(1/2-b/2; 1; 2k(y+L/a))\} \\ -kT\{exp[k(y+L/a)](1+b) \\ F(3/2+b/2; 2; -2k(y+L/a))\} \\ +D\{exp[-k(y+L/a)](1-b) \\ F(3/2-b/2; 2; 2k(y+L/a))\}\end{gathered} \quad (50)$$

This is evaluated at the wall:

$$P'(0;k,\omega) = kT(E_+ + E_- D) - kT(E^+ + E^- D), \quad (51a)$$

$$E^\pm \equiv (1\pm b)\, exp(\pm kL/a)\, F(3/2 \pm b/2; 2; \mp 2kL/a), \quad (51b)$$

for substitution in (47b).

The impedance boundary condition at the wall (47b) is rewritten using (48a), (51a):

$$\begin{aligned}&\mathrm{i}\{(U-\omega/k)/[c\, z(k,\omega)]\}\,(E_+ - E_- D) \\ &\quad = E_+ + E_- D - (E^+ + E^- D);\end{aligned} \quad (52)$$

solving for D specifies:

$$\begin{aligned}D = {} & [\mathrm{i}(U-\omega/k)E_+ - c(E_+ - E^+)z(k,\omega)] \\ & /[\mathrm{i}(U-\omega/k)E_- + c(E_- - E^-)z(k,\omega)];\end{aligned} \quad (53)$$

this appears in the combination of upward and downward propagating waves in the thermal boundary layer (40) which complies with the impedance boundary condition (42). In the particular case of a rigid wall (54a), then (53) simplifies to (54b):

$$z(k,\omega) = \infty : \quad (54a)$$

$$D = -(E_+ - E^+)/(E_- - E^-). \quad (54b)$$

Having determined D in (40), it remains to determine the transmission factor T together with the reflection factor R in (41), from the matching conditions of the acoustic pressure and displacement at the edge of the thermal boundary layer.

3.3. Matching of the acoustic pressure and particle displacement

The matching conditions at the edge of the thermal boundary layer concern the continuity acoustic pressure p and normal particle displacement $\vec{\xi}$ perturbations.

It is shown that continuity of particle displacement, leads to the same condition as the continuity of acoustic velocity or acceleration, i.e. P' is continuous.

Thus the matching conditions to be applied:

$$P_\infty(L) = P(L), \quad (55a)$$

$$P'_\infty(L) = P'(L), \quad (55b)$$

concern the continuity of pressure (55a) and its transverse derivative (55b) at the junction $y = L$ of the thermal boundary layer (40) and isothermal free stream (41). The continuity of the pressure (55a) yields:

$$exp(-\mathrm{i}\kappa L) + R\, exp(\mathrm{i}\kappa L) = T(H_+ - H_- D), \quad (56)$$

where:

$$H_\pm = exp(\pm\vartheta kL)\, F(1/2 \pm b/2; 1; \mp 2\vartheta kL), \quad (57)$$

$$\begin{aligned}\vartheta \equiv 1 + 1/a &= 1 + 1/[(c_\infty/c_0)^2 - 1] \\ &= T_\infty/(T_\infty - T_0).\end{aligned} \quad (58)$$

The continuity of the transverse component of the displacement leads to:

$$\begin{aligned}&\mathrm{i}\kappa[R\, exp(\mathrm{i}\kappa L) - exp(-\mathrm{i}\kappa L)] \\ &\quad = kT(H_+ + H_- D) - kT(H^+ + H^- D),\end{aligned} \quad (59)$$

where

$$H^\pm \equiv (1\pm b) exp(\pm\vartheta kL)\, F(3/2 \pm b/2; 2; \mp 2\vartheta kL), \quad (60)$$

The differentiation formula for the confluent hypergeometric function (49) was used in (50) when substituting (40) in (55b) to obtain (59). The system of equations (56) and (59) specifies the reflection and transmission factors.

The system of equations (56, 59) can be put into the matrix form:

$$\begin{bmatrix} 1 & S_+ \\ 1 & S_- \end{bmatrix} \begin{bmatrix} R \\ T \end{bmatrix} = \begin{bmatrix} -1 \\ +1 \end{bmatrix} exp(-2\mathrm{i}\kappa L), \quad (61)$$

where the scattering factors are given by:

$$S_+ = exp(-\mathrm{i}\kappa L)[H_- D - H_+], \quad (62a)$$

$$S_- = exp(-\mathrm{i}\kappa L)[\mathrm{i}(k/\kappa)[(H_+ - H^+) + D(H_- - H^-)]], \quad (62b)$$

The determinant of the system (61) is given by:

$$
\begin{aligned}
S &= S_- - S_+ \\
&= exp(-\mathrm{i}\kappa L)\{[-\mathrm{i}(k/\kappa)H^+ + (1+\mathrm{i}k/\kappa)H_+] \\
&\quad +D[-\mathrm{i}(k/\kappa)H^- + (\mathrm{i}k/\kappa - 1)H_-]\}.
\end{aligned} \tag{63}
$$

The vanishing of (63) would correspond to resonance; since (63) is complex it would be fortuitous that both the real and imaginary parts vanish simultaneous for real k. Resonance would be possible in general for complex k only. Also $\kappa = 0$ is excluded, since the incident wave (41) must have a vertically propagating component. Bearing in mind that the determinant (63) is finite and non-zero (64), then (61) can be inverted (64b)

$$S \neq 0, \infty : \tag{64a}$$

$$S \begin{bmatrix} R \\ T \end{bmatrix} = \begin{bmatrix} S_- & -S_+ \\ -1 & +1 \end{bmatrix} \begin{bmatrix} -1 \\ +1 \end{bmatrix} exp(-2\mathrm{i}\kappa L); \tag{64b}$$

this specifies explicitly the reflection and transmission coefficients:

$$\{R, T\} = S^{-1}\, exp(-2\mathrm{i}\kappa L)\{-S_- - S_+, 2\}, \tag{65a,b}$$

in terms of (63, 57, 60). The boundary condition at the wall is met (48b, 51b) by (53) for an impedance wall and by (54b) for a rigid wall. Thus all conditions on the acoustic field have been met, allowing the plotting of the scattering coefficients and the acoustic pressure.

4. Implementation

4.1. Scattering coefficients and waveforms of acoustic pressure

Several combinations of the parameters of the problem are used for plotting the amplitudes and phase of the reflection and transmission factors and acoustic pressure.

4.2. Parameters of the problem

The reflection and transmission coefficients are generally complex, and can be plotted in terms of amplitude and phase:

$$R(\theta) = |R|\, exp[\mathrm{i}\arg(R)], \tag{66a}$$

$$T(\theta) = |T|\, exp[\mathrm{i}\arg(T)], \tag{66b}$$

and depend on the reference angle of propagation θ, which is implied in one of the six parameters of the problem:

$$T_0, \quad T_\infty, \quad L, \quad U, \quad \omega, \quad k; \tag{67}$$

the six parameters in the problem are the temperature at the wall T_0 and in the free stream T_∞, the thickness L of the thermal boundary layer, the free stream velocity U, the wave frequency ω and horizontal wavenumber k. These six parameters can be combined into four dimensionless parameters. An obvious choice of the first parameter is the free stream Mach number, which is given values ranging from zero, to low speed, subsonic, sonic and supersonic:

$$M_\infty \equiv U/c_\infty = 0, \quad 0.3, \quad 0.8, \quad 1, \quad 2.5, \tag{68}$$

since a unidirectional shear flow (3a) is incompressible and there is no restriction on Mach number.

A second dimensionless parameter is the ratio of temperatures in the free stream and at the wall, including lower and higher, with small or large deviation from the isothermal case.

$$\mu \equiv T_\infty/T_0 = (c_\infty/c_0)^2 = 0.1, \quad 0.5, \quad 2, \quad 5, \quad 10. \tag{69}$$

This specifies two other dimensionless parameters (13c) and (58):

$$a = \mu - 1 = -0.9, \quad -0.5, \quad 1, \quad 4, \quad 9; \tag{70}$$

$$\vartheta = \mu/(\mu-1) = -1/9, \quad -1, \quad 2, \quad 5/4, \quad 10/9. \tag{71}$$

The third parameter is the frequency made dimensionless using the sound speed in the free stream c_∞ and thickness L of the thermal boundary layer:

$$
\begin{aligned}
\Omega &= \omega L/c_\infty = 2\pi L/c_\infty\tau = 2\pi L/\lambda_\infty \\
&= 0.1, \quad 0.3, \quad 1, \quad 3, \quad 10;
\end{aligned} \tag{72}
$$

it compares the thickness of the thermal boundary layer with the wavelength at infinity, i.e. distance in a period covered at the sound speed in the free stream: it is given a range of values from large corresponding to ray theory to small corresponding to the compactness approximation, including intermediate values of order unity; for later neither the ray nor the compactness approximations would do, and an exact theory is necessary.

The fourth parameter is the horizontal wavenumber made dimensionless multiplying by thickness of the thermal boundary layer:

$$\varepsilon \equiv kL = 0.1, \quad 0.3, \quad 1, \quad 3, \quad 10; \tag{73}$$

it is given the same range of values (73) the dimensionless frequency (72). For a sound wave propagating in a medium at rest at an angle θ to the horizontal, the horizontal wavenumber is given by (74a) and is related to the dimensionless frequency

(72) by (74b). Thus when combining (72) and (73) then $\varepsilon < \Omega$ would represent propagating and $\varepsilon > \Omega$ evanescent waves in a medium at rest at the free stream temperature. Taking into account the free stream velocity, the conditions of propagation and evanescence are respectively real and imaginary transverse wavenumber (7a); e.g. in dimensionless form:

$$k = (\omega/c_\infty)\cos\theta, \tag{74a}$$

$$\varepsilon \equiv kL = (\omega L/c_\infty)\cos\theta = \Omega\cos\theta, \tag{74b}$$

$$\delta = \kappa L = |(\Omega - \varepsilon M_\infty)^2 - \varepsilon^2|^{1/2}. \tag{74c}$$

In the thermal boundary layer the sound field is not sinusoidal, and there is no transverse wavenumber.

In addition to the free stream Mach number (68), free stream to wall temperature ratio (69), dimensionless frequency (72) and horizontal compactness (73) the fifth parameter of this problem is the specific wall impedance (47a) that is given four values:

$$z = \infty, \quad 1, \quad \mathrm{i}, \quad 1+\mathrm{i}, \tag{75}$$

corresponding respectively to rigid, resistive, inductive and mixed wall.

4.3. Amplitude and phase of the reflection and transmission coefficient

The amplitude and phase of the reflection and transmission coefficients (66) are plotted as functions of the reference angle:

$$0 \le \theta \le \pi : \tag{76a}$$

$$-1 \le \varepsilon/\Omega = \cos\theta \le +1 \tag{76b}$$

The baseline case is a subsonic free stream (77a), at twice the wall temperature (77b) for a unit dimensionless frequency (77c) for a rigid wall (77d):

$$M_\infty \equiv U/c_\infty = 0.8, \tag{77a}$$

$$\mu = T_\infty/T_0 = 2, \tag{77b}$$

$$\Omega = \omega L/c_\infty = 1, \tag{77c}$$

$$z = \infty. \tag{77d}$$

While keeping the same baseline case are varied in turn the free stream Mach number (68), the ratio of free stream to wall temperature (69), the dimensionless frequency (72) and the wall impedance (75).

The five parameters (68, 69, 72, 73, 75) are sufficient to calculate the reflection and transmission coefficients (65).

$$\{R, T\} = (S_- - S_+)^{-1}\, exp(-2\mathrm{i}\delta)\{-S_- - S_+, 2\}, \tag{78}$$

where: $S_\pm$ are given by (62) involving (57, 60); D is given by (53) for an impedance wall and by (54b) for a rigid wall, using (48b, 51b); the confluent hypergeometric function of the first kind appears in (48b, 51b, 57, 60) and it is given by (29); in the latter expressions also appears the dimensionless parameter (19), which is dependent on those introduced before:

$$b = (\omega L/c_0 - kLU/c_0)^2 \frac{1}{kLa} = (\Omega - \varepsilon M_\infty)^2 \frac{\mu}{a\varepsilon}. \tag{79}$$

The range of angles θ is restricted (74b) to propagation in the free stream for real transverse wavenumber (7a) or (74c) or:

$$\kappa L = \Omega |(1 - M_\infty\cos\theta)^2 - \cos^2\theta|^{1/2}, \tag{80}$$

corresponding to

$$\theta \ge \theta_- \equiv \arccos[1/(1 + M_\infty)], \tag{81a}$$

$$\theta \le \theta_+ = \pi - \arccos[1/(M_\infty - 1)]. \tag{81b}$$

The lower limit θ_- applies to all Mach numbers and the upper limit θ_+ only above Mach 2. The values corresponding to (68) are:

$$\theta_- = \{0.0^\circ, 39.71^\circ, 56.25^\circ, 60^\circ, 73.40^\circ\}, \tag{82a}$$

$$\theta_+ = \{180^\circ, 180^\circ, 180^\circ, 180^\circ, 131.81^\circ\}. \tag{82b}$$

These angles apply to the reflection of acoustic waves back into the free stream. In the thermal boundary layer $cos(\theta)$ is replaced by ϵ/Ω according to the equation (74b). These two representations coincide at the interface between the free stream and the thermal boundary layer, and thus apply to both the reflection and transmission coefficients.

4.4. Acoustic pressure as a function of distance from the wall

The reflection and transmission coefficient appear in the acoustic pressure which is plotted as amplitude and phase (83b) versus distance from the wall made dimensionless dividing by the thickness of the thermal boundary layer (83a):

$$0 \le Y \equiv y/L \le 2 : \tag{83a}$$

$$P(Y) = |P| exp[\mathrm{i}\arg(P)], \tag{83b}$$

The range of distances (83a) includes: (i) the thermal boundary layer (40):

$$0 \le Y \le 1 : \tag{84a}$$

$$\begin{gathered} P(Y) = T\,[exp(\varepsilon(Y + 1/a)) \\ F(1/2 + b/2; 1; -2\varepsilon(Y + 1/a))] \\ -TD\,[exp(-\varepsilon(Y + 1/a)) \\ F(1/2 - b/2; 1; 2\varepsilon(Y + 1/a))], \end{gathered} \tag{84b}$$

(ii) an equal distance into the free stream:

$$1 \le Y \le 2 : \tag{85a}$$

$$P(Y) = exp(-\mathrm{i}Y\delta) + R\,exp(\mathrm{i}Y\delta). \tag{85b}$$

The two solutions (84) and (85) match in value (55a) and slope (55b) at the edge of the thermal boundary layer.

The confluent hypergeometric functions of the first kind (29) appearing in (84, 48b, 51b, 57, 60) may be computed efficiently (86a) using the recurrence formula (86b) for the coefficients:

$$F(\alpha;\beta;\zeta) = 1 + \sum_{n=1}^{\infty} f_n (n!)^{-1} \zeta^n, \tag{86a}$$

$$f_{n+1} = [(\alpha+n)/(\beta+n)]\, f_n \tag{86b}$$

The baseline case for plotting pressure perturbation is the four parameters (77) plus the reference angle (87a) corresponding to (87b):

$$\theta = \pi/3 : \tag{87a}$$

$$\varepsilon = \Omega \cos\theta = \Omega/2 = 0.5. \tag{87b}$$

Keeping the baseline case (77, 87b) are varied in turn the Mach number (68), the ratio of free stream to wall temperature (69), the dimensionless frequency (72), the compactness parameter (73) and the wall impedance (75).

5. Results and Discussion

5.1. Amplitude and phase of reflection and transmission coefficients

Plots of amplitude and phase of reflection and transmissions factors were obtained and for all the variations of the problem, the modulus of the reflection factor was close to unity and the modulus of the transmission factor was much larger.

5.2. Pressure perturbation as a function of the distance from the wall

Pressure amplitude for acoustic-entropy waves increased away from the wall in the thermal boundary layer and for acoustic waves in the free stream decreased with distance from the wall resulting in a peak at the interface. The phase of the pressure perturbation almost does not vary except for phase jumps at the nodes with zero amplitude.

5.3. Conversion of acoustic to acoustic-entropy waves

The most strinking results is the large magnitude of the modulus of the transmission factor relatively to the reflection factor. The reflection and transmission factors are related by (61), that can be rewritten

$$R + S_{\pm} T = \mp \exp 2\mathrm{i}\kappa L, \tag{88}$$

so that for real propagating transverse wavenumber, the modulus is unity:

$$\kappa \text{ real}: \quad |R + S_{\pm} T| = 1. \tag{89}$$

For all values of the transverse wavenumber, either real or imaginary, corresponding respectively to propagating or evanescent acoustic waves in the free stream, the ratio is minus unity:

$$\frac{R + S_+ T}{R + S_- T} = -1, \tag{90}$$

implying that the transmission factor T for acoustic-entropy waves is related to the reflection factor R for acoustic waves by:

$$T = -\frac{2R}{S_+ + S_-}. \tag{91}$$

If the waves are all acoustic on both sides of the interface (92a) would be expected to hold leading to (92b,92c)

$$R + T = 1, \tag{92a}$$

$$R = \frac{S_+ + S_-}{S_+ + S_- - 2}, \tag{92b}$$

$$T = \frac{2}{2 - S_+ - S_-}. \tag{92c}$$

However there is no reason for (92) to hold since the distinct waves on opposite sides of the interface do not couple leading to total reflection of acoustic waves into the free stream:

$$|R| = 1. \tag{93}$$

In this case the modulus of the transmission factor for acoustic-entropy waves in the thermal boundary layer is:

$$|T| = \frac{2}{|S_+ + S_-|}. \tag{94}$$

If the coupling is weak (95a)

$$|S_+ + S_-| << 1, \tag{95a}$$

$$|T| >> 1, \tag{95b}$$

the modulus of the transmission factor is large (95b).

6. Conclusions

By analysing the obtained graphs, it is possible to conclude that in the free stream: the acoustic waves incident from the free stream have almost total reflection back since 'mode conversion' across the interface is mostly ineffective, pressure perturbation is nearly standing nodes, with almost constant phase, except for phase jumps at the nodes of zero amplitude and the nodes are more closely spaced at higher frequencies. In the thermal boundary layer: acoustic waves incident from the free stream cause moderate pressure and displacement perturbations of the interface, the same moderate pressure and displacement perturbations of the interface corresponds to larger amplitudes of the acoustic-entropy waves so the interface acts as an amplifier of acoustic waves incident from the free stream triggering instabilities of acoustic-entropy waves.

The distinction between acoustic waves in a free stream and acoustic-entropy waves in a thermal boundary layer, allows the interface to acts as a 'valve': in one direction it amplifies acoustic waves in the free stream into acoustic-entropy waves in the thermal boundary layer and in the opposite direction it attenuates the acoustic entropy waves in the thermal boundary layer into weaker acoustic waves in the free stream. Since the acoustic-entropy waves in the thermal boundary layer are insensitive to wall impedance, they 'shield' the acoustic waves in the free stream from wall effects. In conclusion the interface between acoustic waves in a free stream and acoustic-entropy waves in a thermal boundary layer has a 'valve' effect similar to critical layers for a variety of atmospheric waves [21, 9, 2].

There is some tendency to interpret all waves in fluids as acoustic waves, with less emphasis on vortical and entropy interactions this work demonstrates significant differences between acoustic waves in a free stream and acoustic-entropy waves in a thermal boundary layer. Instead of having a uniform stream a velocity boundary layer could be considered leading to vortical waves. The combinations of velocity and thermal boundary layers would lead to acoustic-vortical-entropy waves.

Acknowledgements

The author would like to thank Professor Luiz Manuel Braga da Costa Campos for his guidance and time spent during the development of this work.

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