Model Order Reduction for Aircraft Structural Analysis

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Abstract

The order, or dimension, of the structural dynamic models applied to airframe structures is considerably high. Consequently, the computation time involving these models can become unsustainable when it comes to MultiDisciplinary Optimization, like in the case of the NOVEMOR platform. This thesis studies the possibility of reducing the mentioned airframe models, thus resulting in a precise solution, but with less computational time spent in the solving process. Firstly, a research on the reduction methods was made, with focus on the ones which had applications to structural dynamics. This research revealed the prevalence of methods which used the concept of generalized coordinates. The vector basis which defines the vector space of these coordinates are formulated using Ritz vectors, free vibration eigenmodes or proper orthogonal modes. After defining this basis, the reduced models can be formulated using techniques like the Galerkin Projection or the Least Mean Square Reduction. After this research, some reference models were chosen and the most adequate reduction methods were applied to them. As a result of this implementation, a better understanding of the behaviour of these methods was obtained and an adequate selection of these reductions could be made in order to achieve the goal of this thesis: reducing an airframe structural model. A wing structure model from a commercial aircraft was formulated as a case study. The reduction methods applied in this model used the two techniques already mentioned above, exploiting the proper orthogonal modes. The reduction of this model resulted in a considerable decreasing of the computation time necessary for its solving process, maintaining, however, the precision of the solution.
Keywords: Model Order Reduction, Galerkin Projection, Least Mean Square Reduction, Structural Analysis, Proper Orthogonal Decomposition, Finite Element Method

1. Introduction

1.1. Motivation

The thesis in which this work is based on is made within the scope of a Multidisciplinary Design Optimization (MDO) framework that has been developed at IST for aircraft conceptual design. In order to have a better Optimization Framework, one of the key objectives is to have not too costly computations, so that a frequent model computation could be obtained without a large cost on simulation time, thereby easing the optimization process. To reach that goal in this framework, the aircraft structure, at a conceptual design phase, is modeled via beams and discretized by the finite element method (FEM). Nonetheless these structural simplifications of the high definition structural model are not the most accurate, since they neglect some of the physical and geometric properties of the model. In what concerns aerodynamics, despite the possibility of the utilization of a surrogate model applied to derivation of the aerodynamic forces of the entire aircraft, it still neglects the physical sense of the problem, because surrogate models are limited to fit a parametric function that best describes the behaviour of a certain sample of data, thus not bearing in mind the physical model of the system.[9]

Therefore the ideal solution would be a model that could keep the "physics" of the problem and at the same time reduce the order of the high fidelity model or high definition model (HDM). Since the mentioned framework has already explored the physical idealization and the surrogate model technique, there is one method left to be explored, which is called Model Order Reduction (MOR). This technique considers the partial differential equations or ordinary differential equations (ODEs) of the HDM and derives a much smaller model (less degrees of freedom) called reduced order model (ROM). Consequently this new model is supposed to take less time to reproduce the response of the system and the fact that it takes the model differential equations into consideration, will imply that the physics of the model are not neglected, hence improving the accuracy of the results when compared to the ones provided by the idealized structural models.

It is mandatory to state that this work is an extended abstract based on this author's master thesis, so not all information about the background research and about the implementation results are going to be presented here.

1.2. Objectives

The main goal for this work is to explore the main MOR techniques, which can be of interest for the structural analysis of aircraft structures. So to reach this goal, the following objectives are of importance for this article's thesis:

- Search for MOR methods applied in engineering problems and their latest scientific developments.
- Implementation of these techniques to structural models.
- Comparing the results of each technique on each structural model (benchamrk problems), with the purpose of establishing their pros and cons.
- Application of MOR to a structural model of an aircraft component.

Thus the structural analysis of several models will be presented, with focus on dynamic structural analysis, since the effects of the model reduction (decrease in simulation time) will be more noticed in this case comparing to the static analysis reduction.

2. Background

In this section some of the the MOR methods found in the research for the state of the art of this subject will be covered. All the methods found in this research are present in chapter 2 of the thesis. The focus of this research was done for the reduction methods with more relevance in the structural dynamics field of study. The methods presented in here are the ones used in the case study of the thesis.

If one wants to categorize the MOR methods, considering the type of variables employed in the reduced model, three types of reduction can be obtained, the ones using: physical coordinates; generalized coordinates; and hybrid coordinates.

The generalized coordinate methods require the computation of certain eigenproblems and generally they are developed through a coordinate transformation matrix, which makes the connection between the physical coordinates space of the structural HDM and the retained coordinates of the ROM [11]. This transformation matrix is called the reduced basis. The connection between the HDM space and the reduce one is usually made via orthogonal projections, which are commonly referred to as Galerkin projections [10], but there are also other alternatives to the Galerkin Projections, like the Least Mean Squares method (LMSQ) which is going to be mentioned in the next section of this article. The hybrid coordinates are a mix of these two types of coordinates.

Another issue involving all the MOR techniques is the assumption of non-linearities in a certain model. This issue will not be addressed here, but it can be found in several references, such as [26, 14, 13].

2.1. Proper Orthogonal Decomposition

This technique has the main purpose of reducing a large number of interdependent variables to a much smaller number of uncorrelated ones. The main idea is to find a basis, which contains several basis functions, usually referred as proper orthogonal modes (POM). This basis will be the reference for the projection between the physical coordinates and the generalized ones, such that the orthogonal error is minimized (Galerking Projection), with the support of a snapshot matrix.

In order to apply this method, the unknown field $u(\mathbf{x}_n, t)$ has to be considered, where $\mathbf{x}_n$ are the coordinates of the node n of the imposed mesh in a certain domain. The values of $u(\mathbf{x}_n, t)$ are known at the nodes $\mathbf{x}_n$ for the discrete times $t_m = m \cdot \Delta t$, with $n \in \left[1, ..., M\right]$ and $m \in \left[1, ..., P\right]$. The following notation is used to simplify the mathematical formulation: $u(\mathbf{x}_n, t_m) \equiv u^m(\mathbf{x}_n) \equiv u_n^m$, in which $\mathbf{u}_n^m(\mathbf{x})$ is the vector of nodal values $\mathbf{u}_n$ at time t_m. As it was said before, the main goal of the POD is to derive the already mentioned POMs, such that the orthogonal error is minimized. This problem can be formulated as maximizing the scalar quantity presented below [7]

$$\alpha = \frac{\sum_{m=1}^{P} \left[\sum_{n=1}^{P} \phi(\mathbf{x}_n) u^m(\mathbf{x}_n) \right]^2}{\sum_{n=1}^{M} (\phi(\mathbf{x}_n))^2}, \tag{1}$$

this is equivalent to the following eigenproblem

$$\mathbf{c}\boldsymbol{\phi} = \boldsymbol{\alpha}\boldsymbol{\phi} \tag{2}$$

in which the vectors $\boldsymbol{\phi}$ are the POMs with n-component, while α represents the proper orthogonal values (POV) with respect to each POM, the highest values correspond to the modes that best describe the behaviour of the system. Finally the $\mathbf{c}$ matrix is the two-point correlation matrix and can be formulated as

$$c_{ij} = \sum_{m=1}^{P} u^m(\mathbf{x}_i) u^m(\mathbf{x}_j); \quad \mathbf{c} = \sum_{m=1}^{P} \mathbf{u}^m \cdot (\mathbf{u}^m)^T \tag{3}$$

It can be shown that the matrix $\mathbf{c}$ is symmetric and positive definite, so it can relate with the snapshot matrix that is defined by

$$\mathbf{Q} = \begin{pmatrix} u_1^1\sqrt{\alpha_1} & u_1^2\sqrt{\alpha_2} & \cdots & u_1^P\sqrt{\alpha_P} \\ u_2^1\sqrt{\alpha_1} & u_2^2\sqrt{\alpha_2} & \cdots & u_2^P\sqrt{\alpha_P} \\ \vdots & \vdots & \ddots & \vdots \\ u_M^1\sqrt{\alpha_1} & u_M^2\sqrt{\alpha_2} & \cdots & u_M^P\sqrt{\alpha_P} \end{pmatrix} \quad (4)$$

here α_i are the time integration weights. The snapshot matrix can be just a sample of the time iterations of the HDM solution, or it can even contain all the time iterations. The accuracy of the ROM is supposed to increase with the number of snapshots used in the matrix $\mathbf{Q}$. The relation of the snapshot matrix with the matrix $\mathbf{c}$ can be formulated as

$$\mathbf{c} = \mathbf{Q} \cdot \mathbf{Q}^T \quad (5)$$

Then the reduction basis can be defined as

$$\mathbf{B} = \begin{pmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \cdots & \phi_N(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \cdots & \phi_N(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{x}_M) & \phi_2(\mathbf{x}_M) & \cdots & \phi_N(\mathbf{x}_M) \end{pmatrix}, \quad (6)$$

and the number N of POMs used is equal to the order of the ROM.The usual projection procedure into the generalized coordinate space is applied in a similar manner as all the ROM techniques presented in this section

$$\mathbf{M}_r = \mathbf{B}^T\mathbf{M}\mathbf{B}, \quad \mathbf{K}_r = \mathbf{B}^T\mathbf{K}\mathbf{B}, \quad \mathbf{f}_r = \mathbf{B}^T\mathbf{f} \quad (7)$$

always considering the undamped structural HDM used throughout this section, the ROM can be defined as

$$\mathbf{M}_r\ddot{\mathbf{X}}_r(t) + \mathbf{K}_r\mathbf{X}_r(t) = \mathbf{f}_r(t) \quad (8)$$

There are several ways of developing the POD, like the ones referenced in [30, 15], nevertheless, the common problem to be solved in this method is mainly concerned with the procedure to solve the eigenproblem presented in 1. The three main variants to approach this problem are: Principal Component Analysis (PCA), Karhunen-Love Decomposition (KLD), Singular Value Decomposition (SVD) [32]. There are also some techniques using AI in the computation of the POD method, more specifically, auto-associative neural networks [17]. Since the only variants that are going to be implemented are the PCA and the SVD, because these are the ones with more relevance and proved application in structural dynamic analysis, the next sections will only focus on these two.

2.2. Least Mean Square Method

The least mean square method comes as an alternative for the Galerkin projection scheme, in which the residual error is orthogonalized with respect to the reduced space [16]. In the case of the least mean square method, rather than projecting the HDM space in such a way that the residual is orthogonalized, the method attempts to minimize the residual error related to the reduction basis, this can be defined for the static model as follows[9]:

$$q = arg\min_q |f(\mathbf{V}\mathbf{q})|, \quad (9)$$

where V is the reduction basis and q is the displacement vector of the HDM. With this being said, the reason why all the previous methods used the Galerkin projection as the link between the full order space and the reduced one is because most of the literature, that presents their respective formulations, use the Galerkin projection most of the time. Just in some few exceptions was the least mean square preferred to the projection alternative, when it comes to structural analysis, like in the POD developed in [9] and [6], where the POD (SVD variant) is applied with least mean square instead of using Galerking projection. As it was cited in [9], for dynamical analysis the POD with least mean square scheme is likely to be dissipative and stable [31], on the other side, the POD using the Galerkin projection scheme can generate unstable models [25]. The Matlab least mean square solver algorithm is analytically equivalent to the standard method of conjugated gradients and it can be found in [21] and it will be the one used in the benchmarks presented in chapter 3 of the thesis.

2.3. Error estimation and control

Since all methods mentioned above are all based on the derivation of a base, which is responsible for the projection of the physical coordinates in the reduced space of the generalized ones, the error of this process can be given by the total error e_{tot}, which can be decomposed in two components: the orthogonal error ($e_\perp$) and the colinear error ($e_\parallel$) [9]. Their expressions are presented below.

$$e_{tot} = \mathbf{X} - \mathbf{V}\mathbf{q} = e_\perp + e_\parallel \quad (10)$$

$$e_\perp = e_{ortho} = \mathbf{X}(\mathbf{I} - \mathbf{V}\mathbf{V}^T) \quad (11)$$

$$e_\parallel = e_{colin} = \mathbf{V}(\mathbf{V}^T\mathbf{X} - \mathbf{q}) \quad (12)$$

in which $\mathbf{X}$ is the vector of the HDM variables, $\mathbf{V}$ is the reduction basis and $\mathbf{q}$ is the vector of the ROM variables.

3. Benchmark Reduction

The following section of this paper will explore the implementation of the methods introduced in section 2. This implementation will consist in testing the accuracy of the referred techniques on structural benchmark models taken from [29], which is an online website dedicated to MOR where the community can exchange ideas and test cases. These models will consist in the typical dynamical equations of equilibrium, recurrently mentioned on the formulations of the MOR techniques presented in the previous section; these include the mass, damping and stiffness matrices. The data of the models are given in Matrix Market format, allowing for the dynamic simulation of the systems.

The methods which are tested in the thesisare:

- Condensation method
- Mode displacement method
- Static Ritz vector method
- POD-Galerkin Projection (PCA variant)
- POD-Galerkin Projection (SVD variant)
- POD-Least Mean Squares (SVD variant)

In order to determine the specific characteristics of each one of the methods, there will be different types of time intervals that will be taken into account. They are enumerated and defined as follows:

- Sample time - Defined by the duration of the sampling phase of the method in question. Only valid for POD variants, because of their *a priori* formulation.
- Selection time - Time interval only valid for the condensation method, where the master and slave degrees of freedom are chosen.
- Basis time - Consists in the time duration relative to the derivation of the reduction basis.
- Solve time - Measures the time period related to the actual solving process of the equations obtained in the ROM.
- Reduction time - Time interval where the reduced matrices are derived.
- Total time - Consists in the aggregation of all the time intervals mentioned before.

Other parameters of interest will be the errors introduced in the ROM, when compared with the HDM. In the case of the condensation and the POD - Least Mean Square the only error that is considered will be the relative output error, which is the relative error between the values of the HDM and the ROM solutions, consequently defined by

$$e_{out} = \frac{|\mathbf{q}_{HDM} - \mathbf{q}_{ROM}|}{|\mathbf{q}_{ROM}|} \qquad (13)$$

in which the $\mathbf{q}_{HDM}$ and $\mathbf{q}_{ROM}$ are the HDM and ROM solutions, respectively.

Another benchmark parameter is the relative time reduction, which consists in the following formulation:

$$t_r(\%) = \frac{t_{HDM} - t_{ROM}}{t_{HDM}} \times 100, \qquad (14)$$

where the t_{HDM} and t_{ROM} are the HDM and ROM total computation time, respectively.

For the projection based generalized coordinate methods the output error will also be taken into account, as well as the orthogonal, colinear and total errors, which were described in the last section. These errors do not apply to the condensation and least mean square methods, because no projection is made.

The benchmarks that are going to be used in the chapter 3 of the thesis are: Butterfly Gyroscope [5]; Circular Piston [22]; Car Windscreen [19]

All the enumerated benchmarks take into consideration the damping effects of their structures, but since the damping effects are usually a minor issue in a dynamic structural analysis, and since the MOR methods presented previously also neglect these effects, the following analysis will also ignore them, but always bearing in mind that the results will have an error introduced by this assumption. The solver used in the numerical simulation was the Newmark-β method presented in [12] and verified in [9].

Due to the maximum extent allowed in this article, the windscreen and circular piston benchmarks will be suppressed. In this benchmark a parametric analysis of a windscreen model is explored, in order to draw conclusion on the performance of the ROM in the case of a analysis of this kind. The circular piston benchmark was suppressed because none of the methods tested in it are tested in the case study of the thesis.

3.1. Butterfly Gyroscope

The system that will be firstly analyzed is the Butterfly Gyroscope, which has been developed by Imego Institute in a project with Saab Bofors Dynamics AB. This system is used in micro electromechanical systems (MEMS) and it consists of a vibrating micro-mechanical gyro that has a potential to be used in inertial navigation applications. The data given in [5] was generated by modelling and semi-discretizing the system in Ansys, resulting in the form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{E}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{b}u, \qquad (15)$$

in which u is the nodal force applied at the centers of the excitation electrodes and can be represented as:

$$u = 0.055\sin[2384(\text{Hz}) \times t] \quad (\mu\text{N}) \tag{16}$$

where $\mathbf{b}$ is the load vector, which contains unitary values in the positions respective to the degrees of freedom of the nodes at which the nodal force is applied.

One problem with this benchmark is that the units of the matrices are not given, so it was assumed that they were derived in μm, since the unit presented in the nodal force is the μN. In all the simulations presented here, the initial conditions of all the variables were set to be null. The time interval chosen is defined by $t \in [0; 3 \times 10^{-3}](s)$ and the time step is equal to $1 \times 10^{-4}(s)$. As it was stated in the introduction of this section, the damping of the model will be neglected, so the actual HDM that will serve as a reference in this benchmark has the following form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{B}u. \tag{17}$$

This benchmark will focus on the implementation of the POD methods, not only in the Galerkin projection scheme, but also with the least mean square alternative. Therefore the methods that will be applied here are the:

- POD - Galerkin Projection (PCA variant);
- POD - Galerkin Projection (SVD variant);
- POD - Least Mean Squares (SVD variant);

Having a benchmark focused only on the POD method will ease the analysis of the results, since it is the only method where snapshots are needed. Therefore, having this focus will simplify the study of the influence of the number of snapshots and the order of the ROMs in the accuracy of the solutions obtained by the MOR methods.

This benchmark was also used in [9], but the only method implemented there was the SVD variant of the POD in the Galerkin projection scheme. In the thesis the other alternatives of the POD are going to be explored as well, with the goal of comparing their performance.

The next tables and figure represent a summary of the error estimation and time performance of the MOR methods. All the tables and figures of the error estimation and time performance of the MOR methods can be consulted in detail on the thesis.

ΔS	e_{out}	t_r (%)
1	4.109e-4	97.547
5	0.023	99.020

Table 1: Time analysis and error estimation for the sixth order ROM obtained via POD-Galerkin (SVD)

ΔS	e_{out}	t_r (%)
1	4.890e-7	97.547
5	0.024	98.879

Table 2: Time analysis and error estimation for the sixth order ROM obtained via POD-Galerkin (PCA)

ΔS	e_{out}	t_r (%)
1	6.976e-4	97.544
5	1.86e-2	99.016

Table 3: Time analysis and error estimation for the tenth order ROM obtained via POD-LSQR (SVD)

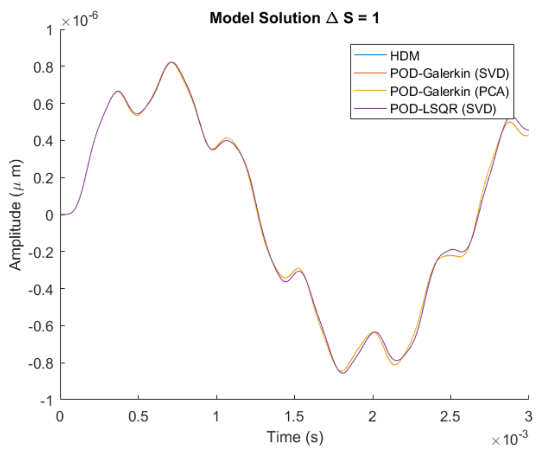


Figure 1: Gyroscope system response for $\Delta S = 1$

4. Structural Analysis and Reduction

In this section, the reduction of a structural model with respect to a wing of a commercial aircraft will be explored. This option was made with the purpose of testing the MOR methods with a much higher number of degrees of freedom than the benchmark models, so that it can be shown that there can be a practical use of these methods in the aerospace industry. The main goal of this thesis is to study the computational time reduction achieved with model order reduction, consequently the structural design and stress analysis of the wing will not be fully explored. Nevertheless the geometric modulation and the results of the stress analysis resulting from a FEM analysis of the model will have characteristics and convergence checks so that it could be upgraded to a more realistic and detailed model.

To delve in the question of the reduction of the computational time of this model, a dynamic analysis will be performed using the dynamic equilibrium equations, which have been the core element of the reduction methods presented in the thesis. These equations are constituted by the mass and stiffness matrices and load vector that are going to be reduced. The MOR methods implemented in this section will be the POD-Galerkin (SVD variant) and the POD using the least mean square method. The FEM will be used to formulate the dynamic equilibrium equations, as it was done for the benchmarks models in the last chapter.

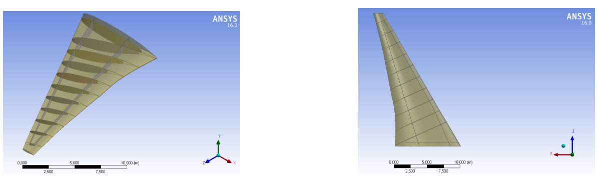
This section will begin with the formulation and analysis of the structural model in question and after that the MOR methods mentioned before will be applied to the model.

4.1. Model Formulation

The wing structure that was chosen to be the target for the MOR studies of this thesis is inspired in the wing of the Boeing 777 commercial aircraft. It has to be emphasized that several components of the wing structure were simplified, for example, the ribs and spars are assumed as thin plates with no holes and no caps, the total number of ribs was reduced from 40 to 10 ribs, all the stringers were suppressed as well as all the joints and fittings that should be connecting all these structural elements.

Another important assumption, which reduces significantly the complexity of this structure, is that only one material was considered for the whole model. In practice, each element of a wing can have a different material, because variables like the mechanical strength, ductility, fatigue resistance and processes of manufacture can influence the mechanical proprieties of the materials and consequently effecting the mechanical behaviour of the airframe elements.

The respective geometric modulation is illustrated in figure 2. The wing was modeled with no torsion, a wing span of 20 meters and the airfoil chosen is the supercritical airfoil NASA SC(2)-0714.



(a) Isometric View

(b) Top View

Figure 2: Geometric design of the structural model.

Since the MOR methods will need the mass and stiffness matrices, along with the load vector, a modal analysis is also needed to obtain this data.

The cruise conditions of this wing's aircraft can be simulated by a CFD analysis, thus allowing for a better understanding of the behaviour of the structural model and consequently having a basic validation of the respective model. Therefore to obtain a precise result of the pressure distribution caused by the aerodynamic load, a computational fluid dynamic (CFD) analysis of the wing is performed, also using the Ansys software. Therefore two kind of analyses will also be performed: a CFD analysis for the determination of the load/input vector; and a structural analysis using that input for the determination of the displacements of the structure.

The boundary conditions for the structural model are the aerodynamic pressure distribution derived in the CFD model and applied on the skin panels of the wing structure. The constraints are defined at the root of the wing by restraining every degree of freedom of each node that is in the root plane of the wing.

In the case of the CFD model the domain's geometry consists in a half cylinder with one spherical base and a cavity, which is inserted in the middle of the domain, with the exterior shape of the wing. The inlet boundary condition is defined in both the spherical base and the curved lateral face of this geometry. The inlet condition has a value of 251.39 m/s for the velocity magnitude and has an angle of attack of 3. The flat plane that cuts the cylinder in half has a symmetry boundary condition, its flat base is the outlet condition at constant atmospheric pressure and the wall representing the exterior surface of the wing has a no slip condition.

4.2. Convergence Analysis

In order to validate the case study model used in this chapter, a convergence analysis and result check will be performed in this section.

Since the discretization of the domain and constraints of the model are defined in the modal analysis, the natural frequencies of the structure will be one of the outputs of this analysis and can serve as reference for the convergence of this model. In table 4 the relative errors of the meshes used in the convergence analysis are shown. These errors were derived in relation to the most refined mesh, which has 41406 elements. It can be concluded from table 4 that the model's natural frequencies converge to a constant value as the number of elements increases. The convergence graphs are presented in detail in the thesis.

The reference outputs for the convergence study of the CFD model are the lift coefficient (C_L) and drag coefficient (C_D). The convergence plots are presented in detail in the thesis.

With the convergence study of both the CFD and

	Number of Elements		
	1681	5908	9153
$e_{r_{w1}}(\%)$	1.520	1.104	0.122
$e_{r_{w2}}(\%)$	1.735	1.459	0.116
$e_{r_{w3}}(\%)$	2.210	1.202	0.599
$e_{r_{w4}}(\%)$	2.789	4.821	0.240
$e_{r_{w5}}(\%)$	18.633	7.319	3.721
$e_{r_{w6}}(\%)$	30.776	4.674	2.605

Table 4: Relative error of the meshes used in the structural convergence analysis, in respect to the most refined one.

	Number of Elements		
	729326	1223155	2206993
$e_{C_L}(\%)$	2.275	1.506	0.487
$e_{C_D}(\%)$	14.143	11.273	4.378

Table 5: Relative error of the meshes used in the CFD convergence analysis, in respect to the most refined one. Only the three more refined meshes are here presented, a more detailed table is present in the thesis

the structural models done successfully and despite the roughness of the model, a check on the behaviour of the structure was made. The main goal of this check is to make sure that at least the maximum Von-Mises stress derived by the FEM, when multiplied by a safety factor of 1.5, would not be greater then the yielding stress of the material, thus guaranteeing that the model remains in the linear regime. The parameter which controlled the Von-Mises values was the thickness of the structural elements (spar, ribs and skin panels). It is known that there are more structural failure modes then just the ones considered here, but since the analysis in question targets only the linear static behaviour of this model, the material strength failure mode here considered can be seen as sufficient.

The final values of the thickness of each structural element are shown in table 7. These were the values used in the results present in table 6, which considering the roughness of the model, are acceptable, that is, the maximum equivalent stress value multiplied by the safety factor of 1.5 is not greater then the yielding stress of the material (Al 2024-T4) which corresponds to 324 MPa [4]. Despite the unrealistic values of the thickness defined for each structural element, these excessive values have to be considered as an assumption for this model, so that the discussion of this thesis does not delve in topics like material selection and structural optimization.

Therefore the check for a basic validation of this model can be obtained successfully with the results obtained by the analysis presented in this section. The plots of this results and the figures with the domain discretization of the respective models can be seen with more detail in the thesis.

Max. Displacement in y-direction (m)
0.617
Max. Von-Mises Stress (MPa)
199.586

Table 6: Summary of structural analysis results.

Strtuctural Elements	Thickness (mm)
Skin	10.0
Spars	30.0
Ribs	10.0

Table 7: Thickness of structural elements used in the model.

5. Structural Model Reduction

Since the structural model of the wing is already formulated, a dynamic analysis of the system can now be done and it will be the target of the MOR methods, as it was done for the benchmark examples. The dynamic equilibrium equation is represented by:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}, \quad (18)$$

The mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are the ones used in the Ansys solver result from the domain discretization made in that software. The load vector $\mathbf{f}$ for this example is a result of an oscillatory force imposed at the nodes of the wing tip and two other oscillatory forces applied at the nodes which connect the wing spars to its skin panels. The purpose of this last two forces is to mimic the torsion behaviour of the wing, that is why they have the same characteristic, but have an opposite direction in order to induce the binary necessary so that there can be a torsion motion of the structure. The three loads can be represented by the following equations:

$$f(t)_1 = 100000\sin(0.314(\text{rad/s}) \times t) \quad (N), \quad (19)$$

$$f(t)_2 = 500000\sin(0.314(\text{rad/s}) \times t) \quad (N), \quad (20)$$

$$f(t)_3 = -500000\sin(0.314(\text{rad/s}) \times t) \quad (N), \quad (21)$$

where $f(t)_1$ is respective to the force applied at the wing tip and $f(t)_2$ and $f(t)_3$ are respective to the forces of the torsion binary imposed along the wing spars. All the three loads are applied in the y-axis direction.The magnitude of these forces was chosen such that the structural behaviour of the wing remains linear, thus the maximum Von-Mises stress derived by the Ansys solver and multiplied by a safety factor of 1.5 is below a tensile yielding stress of the material. The frequency chosen for this loads was determined so that an harmonic displacement as the one represented in figure 3 could be accomplished.

The dynamic analysis will be done in a time interval defined by $t \in [0; 10]$ s, with a time step of 0.2 seconds. The results obtained in the Matlab solver, which was used in the benchmark models, were verified by the results obtained using Ansys, thus allowing for the reduction of the model and its analysis using Matlab. Figure 3 shows the vertical displacement (the displacement along the y-axis) of one of the degrees of freedom which is placed in the leading edge of the wing's tip. The time duration for solving the HDM with the Matlab solver is presented in table is 253.939 seconds.

The MOR methods which are going to be explored is this section are: POD - Galerkin Projection (SVD variant) (POD - Galerkin); POD - Least Mean Squares. (POD - LMSQ).

The implementation of other MOR methods, like the POD-PCA or the Ritz method, was attempted. Although their algorithms require the multiplication of large matrices, which are not sparsed, in order to obtain the reduced model, their implementation was not successful since the RAM memory required for these operations exceeded the available computer RAM memory (32 Gb). The methods which are going to be explored in this section were ran in the same computer as the benchmark models, using the Matlab software. The POD-LMSQ also uses the SVD as the basis method, like it was done in the benchmarks presented in the thesis.

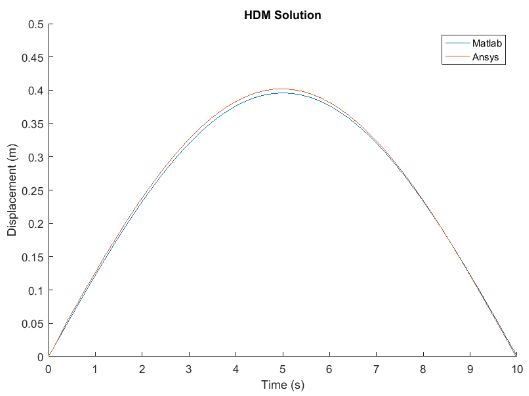


Figure 3: HDM solution - Vertical Displacement of a node placed in the leading edge of the wing tip.

The time performance and error estimation are represented in the figures and tables below. More detailed information is present in the thesis, like the error evolution with respect to the ROM's order and the several time interval duration for each method.

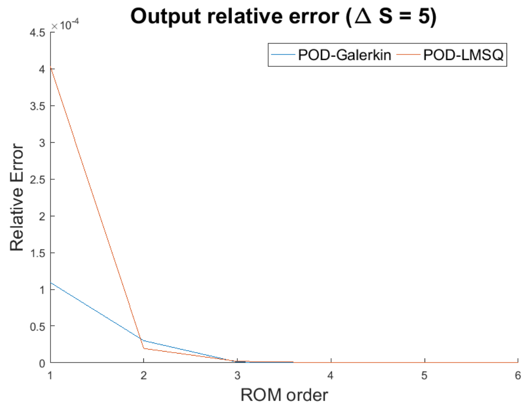


Figure 4: Output error estimation for both POD methods ($\Delta S = 5$).

ΔS	t_r (%)
1	99.506
5	99.915
10	98.938

Table 8: Time analysis for the sixth order ROM obtained via POD-Galerkin.

ΔS	t_r (%)
1	98.972
5	99.041
10	98.872

Table 9: Time analysis for the sixth order ROM obtained via POD-LMSQ.

In figure 3 both the solutions for the already mentioned degree of freedom are shown. These solutions are respective to Ansys and the Matlab solver.The precision of the methods here presented are shown in figures 4 and 5

6. Conclusions

It was shown in the thesis that there can be a reduction on structural analysis computational time, through MOR methods, with a conservation of the solution's quality. The background research done on chapter 2 consisted in presenting the state-of-the-art related to the MOR techniques which apply to the computational study of structural systems formulated by ODE models.

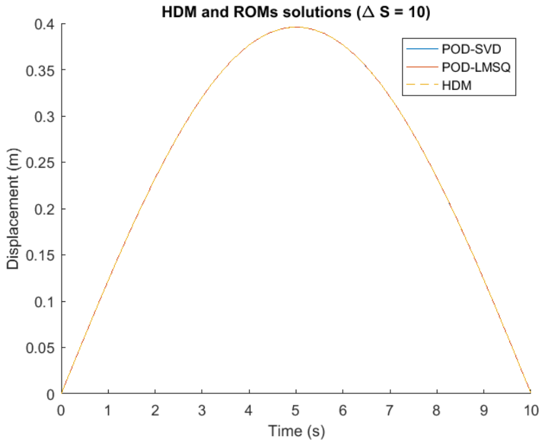


Figure 5: HDM and ROM solution plot - Vertical Displacement of a node placed in the leading edge of the wing tip.

In order to test the methods presented in chapter 2, several benchmark models were reduced.After the benchmark reductions, a finite element model of a simplified wing structure was made, so that it could be verified that the MOR techniques here presented could be applied to an actual airframe structure. In order to validate this simplified model a CFD analysis simulating the cruising conditions of this wing's aircraft was done. It was verified that the wing had an acceptable deflection and that the Von-Mises stress felt by the components of the structure was not higher than the material's ultimate stress. Lastly, the MOR methods were tested using the already mentioned airframe structure. The MOR methods applied successfully in the benchmark models were both the POD-Galerkin variants (SVD and PCA), POD-LMSQ, the static Ritz vector method and the mode displacement method. Only the condensation method struggled in this implementation phase. All the other reduction methods had their error estimations below 5% and a time reduction approximately equal to 99% relative to the HDM computation time. The same was verified in the wing structure model for which the POD-Galerkin (SVD variant) and the POD-LMSQ were applied, proving that these methods can be easily applied in the computational analysis of airframe structures.

With the research made in the thesis, one can conclude that there are many topics which should be using MOR in the study field of computational mechanics, more specifically in structural analysis. These topics can be: structural optimization using MOR methods [33]; ROM interpolation[16, 24, 2, 3]; the study of other methods which have not be fully explored like the PGD [8, 1, 28] or MOR techiques using AI [27, 20] or even hybrid coordinates methods [18, 23] can also be a target for future work. Lastly the airframe structure model here presented can be uploaded in the MOR benchmark database used in this thesis [29], so that it could contribute to the research in this area of study.

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