Explicit Guidance Solutions for the Lunar Ascent Element of the HERACLES Mission

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***Abstract* - In this paper new explicit algorithms based on optimal control theory for an atmosphereless single stage ascent problem are derived and tested. The different algorithms consist on approximating the remaining burntime estimation by approximating the cosine of the pitch angle by a quadratic function and approximating the thrust acceleration integral in time by a Padé approximant. The sine of the pitch derived from the Lagrane multipliers will also be approximated by linear and quadratic Taylor series expansions from a generic point or Lagrange polynomials, so that an explicit solution can be obtained. The new algorithms are then compared with a previous explicit solution and with the optimal solution for an ascent from the surface of the moon into low lunar orbit using an early design of the Lunar Ascent Element of the HERACLES mission. The linear approximations in particular produce near optimal pitch profiles and merit further studying.**

I. Introduction

In the last issue of the Global Exploration Roadmap[1], the advantages of having human presence in human vicinity for lunar and mars exploration are discussed, and the concept of the DSG (Deep Space Gateway) is explained. This concept introduces the need for manned and non manned ascents from the lunar surface to the DSG. For this reason, the optimal atmosphereless ascent problem, for the moon in particular, becomes very important, as a robust and fuel optimal guidance law is a key factor in having a successful ascent from the lunar surface.

Many ascent laws for exo atmospheric problems have been developed over the past decades, including non optimal schemes, such as Polynomial guidance [2], Powered Explicit Guidance (PEG)[3, 4] and Iterative Guidance Mode (IGM)[5] .Probably the most successful and well known ascent guidance scheme is PEG (Powered Explicit Guidance) as it used was the law used for the space shuttle. It is is based on a linear tangent law for the thrust direction and includes small angle approximations. PEG requires the solution of one equation iteratively to estimate the remaining burntime. Many more similar methods that require the solution of one or more equations numerically exist, such as the ones found in Refs. [6–9]. Other than the numerical solutions, a completely analytical solution for the optimal atmosphereless ascent problem was developed in [10] for a two dimensional ascent and in [11] for a three dimensional ascent by using simplified equations of motion (Eqs. 4) in the LVLH frame considering a non-rotating spherical planet and small angle approximations. This approximation has shown near optimal results and the ability to handle out of plane pertubations[11, 12].In this thesis, new explcit guidance algorithms are developed using the same principles as in [11] but using different small angle approximations.

A. Formulation Of The Problem

In a standard ascent problem in a MCI (Moon Centered Inertial) frame the equations of motion are:

$$\dot{\vec{R}} = \vec{V}, \tag{1a}$$

$$\dot{\vec{V}} = \vec{G} + a_T\hat{F} + \vec{P}, \tag{1b}$$

where $\vec{R}$ is the position vector, $\vec{V}$ is the velocity vector, $\vec{G}$ is the gravity acceleration vector, a_T is the absolute value of the acceleration due to the Thrust, $\hat{F}$ is the Thrust direction and $\vec{P}$. The perturbations have a small enough effect to be ignored for guidance purposes.

B. Optimal Control Approach

The approach to obtain an optimal reference trajectory used in this thesis is the one described in [13], for a constant thrust vehicle. The algorithm consists in assuming a gravitational acceleration that varies linearly with the distance to the center of the moon. This allows the Lagrange multipliers to be represented by elementary transcendental functions (sines and cosines), which facilitates the numerical solution of the system by a huge amount, as the pitch will be defined by elementary functions, removing the need to use an arbitrary pitch profile for the optimization. The approximation keeps the direction of the gravity acceleration vector, and since the amplitude does not change significatively for the ascent trajectories to be tested, using this approximation in a close loop scheme where the pitch profile is calculated every call will provide trajectory that is extremelly close to the absolute optimal trajectory and can be used to evaluate the optimality of the guidance schemes. The pitch is recalculated on a hertz frequency, since increasing the frequency more than this did not have any noticeable effect on the generated trajectory. The position and velocity are calculated each second using *matlab*'s *ode45* function, and the vehicle mass was assumed to change linearly. The Lagrange multipliers, and therefore the pitch profile, are calculated using *matlab*'s *fsolve* function

C. The Reference Trajectory Of LAE

1. Vehicle Features

The vehicle used for testing the algorithms will be the proposed LAE (Lunar Ascent Element) of the HERACLES mission, as described in [14]. The main features of this vehicle can be found on Table 1

Table 1 Main LAE features

Specific Impulse	340 s
Wet Mass	1339 kg
Thrust	6 kN

II. Ascent Guidance

A. Ascent Mission Overview

On Earth launches there are usually two guidance phases. The first one is open loop guidance, which is usually a table of attitude commands, that are optimized before the flight, followed by Powered guidance phase, known as closed loop guidance. This done because there is a portion of the flight where the atmospheric forces are important, and it is very complicated to include these forces in a fast closed loop scheme. A good thing about launching from the surface of the moon is the lack of this problem, since the atmosphere is near non existant and thus the open loop guidance phase can be entirely skipped. The only reason you can't start the closed loop scheme from the surface is to avoid objects like lunar mountains, meaning a vertical flight phase and linear pitchover are necessary [2, 15]. However, in the simulations done in this thesis the vehicle will be considered to be leaving the ground and entering the powered guidance phase right away. This will be done to facilitate the understanding of the simulation results plots because the vertical ascent and linear pitchover would be the same for all the guidance algorithms. Since the ascent problem treated in this thesis is atmosphereless, this assumption will have little effect on the whole trajectory, as the powered ascent phase will constitute about 90% of the total ascent [15].

B. Trajectory solutions

1. Equations of Motion for Guidance

For the developent of a guidance law, the LVLH frame was used . The xy plane is the orbit insertion plane (assuming keplerian orbits) The quantities x, y, u, v are the in-plane downrange, altitude, horizontal velocity component, and vertical velocity component. z is the out of plane distance and w is the out of plane component of the velocity. The pitch (θ) and yaw (ψ) angles are, as usual, defined in relation to the local horizontal. The equations of movement for the above mentioned coordinates are (assuming spherical moon):

$$\dot{x} = \frac{R_\oplus}{R_\oplus + y} u, \tag{2a}$$

$$\dot{y} = v, \tag{2b}$$

$$\dot{z} = \frac{R_\oplus}{R_\oplus + y} w, \tag{2c}$$

$$\dot{u} = a_T \hat{f}_1 + \frac{uw}{R_\oplus + y} \tan\left(\frac{z}{R_\oplus}\right) - \frac{uv}{R_\oplus + y}, \tag{2d}$$

$$\dot{v} = a_T \hat{f}_2 - g + \frac{u^2}{R_\oplus + y} + \frac{w^2}{R_\oplus + y}, \tag{2e}$$

$$\dot{w} = a_T \hat{f}_3 - \frac{u^2}{R_\oplus + y} \tan\left(\frac{z}{R_\oplus}\right) - \frac{vw}{R_\oplus + y}, \tag{2f}$$

where $\hat{f} = [\hat{f}_1, \hat{f}_2, \hat{f}_3]$ is the normalized thrust vector. Assuming the perturbations in the z direction are small, the terms $\frac{uw}{R_\oplus + y}$, $\frac{vw}{R_\oplus + y}$, $\frac{w^2}{R_\oplus + y}$ and $\tan\left(\frac{z}{R_\oplus}\right)$ are all small in comparison to the other terms in the equation. Also for low altitude ascent trajectories $\frac{R_\oplus}{R_\oplus + y} \approx 1$ and $\frac{uv}{r} << a_T \hat{f}_1$, according to [11]. The equations can then be reduced to:

$$\dot{x} = u, \tag{3a}$$
$$\dot{y} = v, \tag{3b}$$
$$\dot{z} = w, \tag{3c}$$
$$\dot{u} = a_T \hat{f}_1, \tag{3d}$$
$$\dot{v} = a_T \hat{f}_2 - g + \frac{u^2}{R_\bigoplus + y}, \tag{3e}$$
$$\dot{w} = a_T \hat{f}_3. \tag{3f}$$

For each iteration of the guidance problem we can approximate the term $g - \frac{u^2}{R_\bigoplus + y}$ as constant and we end up with the following:

$$\dot{x} = u, \tag{4a}$$
$$\dot{y} = v, \tag{4b}$$
$$\dot{z} = w, \tag{4c}$$
$$\dot{u} = a_T \hat{f}_1, \tag{4d}$$
$$\dot{v} = a_T \hat{f}_2 - C_y, \tag{4e}$$
$$\dot{w} = a_T \hat{f}_3, \tag{4f}$$

where $C_y = g - \frac{u^2}{R_\bigoplus + y}$.

2. Optimal control

In [11] the optimal control equations for a quasi-planar ascent lunar trajectory were developed. The derivation in this thesis is slightly different and simpler, but leads to the same guidance algorithm.

Since the value of thrust will be considered constant we can say that $a_T = T/m$ and $m = m_0 - \beta t$, where β is the contant mass flow rate and $T = I_{sp} g_0 \beta$. g_0 is the modulus of the acceleration due to gravity at the surface of the moon. Since we have a constant mass flow rate, the way to spend less fuel will be minimizing the time. In order to calculate the optimal thrust direction inputs, the Hamiltonian of the system is necessary [16], and is in this case

$$H = \lambda^T \dot{X} + 1, \tag{5}$$

where λ is the Lagrange multiplier and the X is the state vector. Expanding Eq. 5

$$H = \lambda_x u + \lambda_y v + \lambda_z w + \lambda_u (a_T \hat{f}_1 - C_x) + \lambda_v (a_T \hat{f}_2 - C_y) + \lambda_w (a_T \hat{f}_3) + 1. \tag{6}$$

The differential equations for the Lagrange multipliers are $\dot{\lambda}_i = -\frac{\partial H}{\partial i}$. The solution is:

$$\lambda_u = -\lambda_x t + A_1, \tag{7a}$$
$$\lambda_v = -\lambda_y t + A_2, \tag{7b}$$
$$\lambda_w = -\lambda_z t + A_3, \tag{7c}$$

where $\lambda_{x,y,z}$ are constant and $A_{1,2,3}$ are integration constants. The optimality condition leads to the thrust having the same direction as the velocity Lagrange multiplier [16], that is

$$\hat{f}_{1,2,3} = \frac{\pm \lambda_{u,v,w}}{\sqrt{\lambda_u^2 + \lambda_v^2 + \lambda_w^2}} \tag{8}$$

The signal ambiguity does not matter, since the two constants need to be determined, so any choice of signal will work. For simplicity the positive sign will be chosen The boundary conditions for $t = 0$ (inital conditions) are $X(0) = [x_0; y_0; z_0; u_0; v_0; w_0]$ and the boundary conditions for $t = t_f$ (insertion time) are $X(t_f) = [x_f; y_f; z_f; u_f; v_f; w_f]$ and The boundary condition for t_f is

$$H_f = 0. \tag{9}$$

This will lead to:

$$x_f = x_0 + u_0 t + \int_0^{t_f} \int_0^t a_T(s) \hat{f}_1 ds dt, \tag{10a}$$
$$y_f = y_0 + v_0 t - \frac{C_y t^2}{2} + \int_0^{t_f} \int_0^t a_T(s) \hat{f}_2 ds dt, \tag{10b}$$
$$z_f = z_0 + \int_0^{t_f} \int_0^t a_T(s) \hat{f}_3 ds dt, \tag{10c}$$
$$u_f = u_0 + \int_0^{t_f} a_T(t) \hat{f}_1 dt, \tag{10d}$$
$$v_f = v_0 + \int_0^{t_f} a_T(t) \hat{f}_2 dt - C_y t, \tag{10e}$$
$$w_f = w_0 + \int_0^{t_f} a_T(t) \hat{f}_3 dt, \tag{10f}$$
$$H_f = 0, \tag{10g}$$

where s is a dummy variable for the integration. The thrust direction can be expressed in terms of pitch(θ) and yaw(ψ) angles as

$$\hat{f} = [\cos(\theta)\cos(\psi); \sin(\theta); \cos(\theta)\sin(\psi)]. \tag{11}$$

From Eqs. 8 and 11 and assuming that $-\pi/2 < \theta < \pi/2$:

$$\sin(\theta) = \frac{\lambda_v}{\sqrt{\lambda_u^2 + \lambda_v^2 + \lambda_w^2}}. \tag{12a}$$
$$\cos(\theta) = \frac{\sqrt{\lambda_u^2 + \lambda_w^2}}{\sqrt{\lambda_u^2 + \lambda_v^2 + \lambda_w^2}}, \tag{12b}$$
$$\sin(\psi) = \frac{\lambda_w}{\sqrt{\lambda_u^2 + \lambda_w^2}}, \tag{12c}$$
$$\cos(\psi) = \frac{\lambda_u}{\sqrt{\lambda_u^2 + \lambda_w^2}}. \tag{12d}$$

The typical orbit insertion conditions will not require a specific value of downrange, so $\lambda_x = 0$. This leaves a constant value for $\lambda_u = A_1$. With this in mind, we can see that

$$\hat{f}_{2,3} = \frac{\lambda_{v,w}}{\sqrt{A_1^2 + \lambda_v^2 + \lambda_w^2}} = \frac{\frac{\lambda_{v,u}}{A_1}}{\sqrt{1 + \left(\frac{\lambda_v}{A_1}\right)^2 + \left(\frac{\lambda_w}{A_1}\right)^2}}. \tag{13}$$

It can be seen from Eq. 13 that the thrust direction only depends on the Lagrange multiplier ratios with A_1 and therefore it is not necessary to calculate A_1 if we introduce the following new variables to the system:

$$\bar{\lambda}_v = \frac{\lambda_v}{A_1} = \frac{-\lambda_y t + A_2}{A_1} = -\bar{\lambda}_y t + \bar{A}_2, \tag{14a}$$

$$\bar{\lambda}_w = \frac{\lambda_w}{A_1} = \frac{-\lambda_z t + A_3}{A_1} = -\bar{\lambda}_z t + \bar{A}_3. \tag{14b}$$

The thrust directions are then defined as:

$$\hat{f_{2,3}} = \frac{\bar{\lambda}_{v,w}}{\sqrt{1 + \bar{\lambda}_v^2 + \bar{\lambda}_w^2}}, \tag{15}$$

and so we can define the cosines and sines of the pitch and yaw angles as such:

$$sin(\theta) = \frac{\bar{\lambda}_v}{\sqrt{1 + \bar{\lambda}_v^2 + \bar{\lambda}_w^2}}, \tag{16a}$$

$$cos(\theta) = \frac{\sqrt{1 + \bar{\lambda}_w^2}}{\sqrt{1 + \bar{\lambda}_v^2 + \bar{\lambda}_w^2}}, \tag{16b}$$

$$sin(\psi) = \frac{\bar{\lambda}_w}{\sqrt{1 + \bar{\lambda}_w^2}}, \tag{16c}$$

$$cos(\psi) = \frac{1}{\sqrt{1 + \bar{\lambda}_w^2}}. \tag{16d}$$

Since we removed two variables, we can remove Eqs. 10a and 10g from the system of Eqs. 10 and we get:

$$y_f = y_0 + v_0 t - \frac{C_y t^2}{2} + \int_0^{t_f} \int_0^t a_T(s) \hat{f}_2 ds dt \tag{17a}$$

$$z_f = z_0 + \int_0^{t_f} \int_0^t a_T(s) \hat{f}_3 ds dt \tag{17b}$$

$$u_f = u_0 + \int_0^{t_f} a_T(t) \hat{f}_1 dt \tag{17c}$$

$$v_f = v_0 + \int_0^{t_f} a_T(t) \hat{f}_2 dt - C_y t \tag{17d}$$

$$w_f = w_0 + \int_0^{t_f} a_T(t) \hat{f}_3 dt \tag{17e}$$

The system can be simplified and solved explicitly if some approximations are taken into account.

C. Explicit Approximate Solutions [for guidance]

As a first approximation, since the perturbations in the z axis are supposed to be small in comparison to it can be said that $sin(\psi) \approx \bar{\lambda}_w$ and $\cos(\psi) \approx 1$. The same can be done for the pitch angle, even though it is only accurate for insertions in low altitude. So, $sin(\theta) \approx \bar{\lambda}_v$ and $\cos(\theta) \approx 1$. This approximation is viable [11], and it is solvable explicitly, requiring no iterations. It can however be improved. These approximations will lead to

$$y_f = y_0 + v_0 t - \frac{C_y t^2}{2} + c_0 \bar{A}_2 - c_1 \bar{\lambda}_y, \tag{18a}$$

$$z_f = z_0 + c_0 \bar{A}_3 - c_1 \bar{\lambda}_z, \tag{18b}$$

$$u_f = u_0 + b_0, \tag{18c}$$

$$v_f = v_0 + b_0 \bar{A}_2 - b_1 \bar{\lambda}_y - C_y t, \tag{18d}$$

$$w_f = w_0 + b_0 \bar{A}_3 - b_1 \bar{\lambda}_z. \tag{18e}$$

Eq. 18c can be solved explicitly for t_f and

$$t_f = \tau\left(1 - e^{\frac{u_0 - u_f}{v_e}}\right). \tag{19}$$

With Eqs. 18a and 18d we can solve for $\bar{\lambda}_y$ and $\bar{A}_2$ and with Eqs. 18b and 18e we can solve for $\bar{\lambda}_z$ and $\bar{A}_3$

$$\begin{bmatrix} \bar{A}_2 \\ \bar{\lambda}_y \end{bmatrix} = \frac{1}{c_1 b_0 - b_1 c_0} \begin{bmatrix} -b_1 & c_1 \\ -b_0 & c_0 \end{bmatrix} \begin{bmatrix} Y \\ V \end{bmatrix}, \tag{20a}$$

$$\begin{bmatrix} \bar{A}_3 \\ \bar{\lambda}_z \end{bmatrix} = \frac{1}{c_1 b_0 - b_1 c_0} \begin{bmatrix} -b_1 & c_1 \\ -b_0 & c_0 \end{bmatrix} \begin{bmatrix} Z \\ W \end{bmatrix}, \tag{20b}$$

where $Y = y_f - y_0 - v_0 t_f + C_y \frac{t_f^2}{2}$, $Z = z_f - z_0 - w_0 t_f$, $V = v_f - v_0 + C_y t_f$, $W = w_f - w_0$ and b_0, b_1, c_0 and c_1 are the well known thrust acceleration integrals expressed by:

$$b_0 = \int_0^{t_f} a_T(t) dt = -v_e log\left(1 - \frac{t_f}{\tau}\right), \tag{21a}$$

$$b_1 = \int_0^{t_f} t a_T(t) dt = -b_0 \tau - v_e t_f, \tag{21b}$$

$$c_0 = \int_0^{t_f} \int_0^t a_T(s) ds dt = b_0 t_f - b_1, \tag{21c}$$

$$c_1 = \int_0^{t_f} \int_0^t s a_T(s) ds dt = c_0 \tau - \frac{v_e t_f^2}{2}, \tag{21d}$$

$$\tag{21e}$$

III. Enhanced closed-form solutions

A. Introduction

In subsection II.C an fully explicit solution for a free downrange, free ascent time is explained. The approximations used for the cosine and sine of the pitch angle can be changed to improve the algorithm, while keeping a close form solution. The cosine will be approximated by a quadratic function, instead of just $\cos(\theta) = 1$, and because of this, the acceleration integral will be calculated using a Padé approximation, in order to keep the solution closed-form. The sine will be approximated in four different ways: by a linear Taylor series expansion (around a general point, instead of just using 0), by a quadratic Taylor series expansion (once again, around a general point) and by a linear or quadratic Lagrange polynomial. All of these aproximations still allow for a closed form-solution, even though the second order approximations for the sine require the solution of a fourth degree polynomial.

B. Improved approximation of the co-sine

The cosine of the pitch angle is assumed to be of the form

$$\cos(\theta) = at^2 + bt + c. \tag{22}$$

Using Eq. 22, the acceleration integral becomes:

$$\int_0^t a_T \cos(\theta) dt = \int_0^{t_f} a_T (at^2 + bt + c) dt, \tag{23a}$$

$$\int_0^t a_T (at^2 + bt + c) dt = \int_0^{t_f} \frac{T(at^2 + bt + c)}{m_0(1 - \frac{t}{\tau})} dt, \tag{23b}$$

$$\begin{aligned}\int_0^t \frac{T(at^2 + bt + c)}{m_0(1 - \frac{t}{\tau})} dt = \frac{T}{m_0}\Big[& a \log\left(1 - \frac{t}{\tau}\right)\tau^3 - at\tau^2 \\ & - \frac{at^2\tau}{2} - b\log\left(1 - \frac{t}{\tau}\right)\tau^2 \\ & - bt\tau - c\log\left(1 - \frac{t}{\tau}\right)\tau\Big].\end{aligned} \tag{23c}$$

The values of the parameters of the quadratic approximation of the cosine can be calculated as follows. Since the objective is to use this as in loop guidance, the value for c should be the value of the cosine of the pitch angle from the previous guidance call.

The values for a and b require some other assumptions. The maximum value of the parabola must be 1, as this is attempting to approximate a cosine. This value is typically obtained close to the end of the ascent, thus:

$$a\alpha t_f^2 + b\alpha t_f + c = 1, \tag{24}$$

As this is a maximum, the derivative at this point must be null, so

$$2a\alpha t_f + b = 0, \tag{25}$$

For simplicity $\alpha = 1$ for the rest of this work. From Eqs.24 and 25 an expression for a and b in relation to t_f can be obtained:

$$a = \frac{c-1}{\alpha^2 t_f^2}, \tag{26a}$$

$$b = \frac{2(1-c)}{\alpha t_f}. \tag{26b}$$

Replacing Eqs. 22, 26b and 26a in Eq. 23c

$$\begin{aligned} f(t_f) &= \int_0^{t_f} \frac{T(at^2 + bt + c)}{m_0(1 - \frac{t}{\tau})} dt \\ &= \frac{T}{m_0}\Big[\frac{c-1}{\alpha^2 t_f^2}\log\left(1 - \frac{t_f}{\tau}\right)\tau^3 \\ &- \frac{c-1}{\alpha^2 t_f}\tau^2 - \frac{\frac{c-1}{\alpha^2}\tau}{2} \\ &- \frac{2(1-c)}{\alpha t_f}\log\left(1 - \frac{t_f}{\tau}\right)\tau^2 \\ &- \frac{2(1-c)}{\alpha}\tau - c\log\left(1 - \frac{t_f}{\tau}\right)\tau\Big] \end{aligned} \tag{27}$$

This is the expression that will be approximated by the Padé approximant [17]. Two options will be presented, order 2 by 2 and order 1 by 1 since neither demands a lot of computational power to obtain an explicit solution. To obtain the approximant of order 2 by 2, a Taylor series expansion of Eq. 27 on a generic point is necessary, up until order 4. In order to choose which approximation should be used and which expansion point should be chosen, the exact function must be compared with each of the approximations. This is done by using the expansion point as a parameter and minimizing integral the square of the absolute error,

$$\int_0^{t_f} [f(t) - g(t)]^2 | dt, \tag{28}$$

where f is the exact function and g is the approximation. This is done for values of c, the initial cosine of the pitch, ranging from 0 to 1, to make sure the approximation is good at any point in the trajectory. The value of t_f will go up to 400s, since the ascent time of none of the trajectories to be tested is expected to be larger than that. In Fig. 1 the

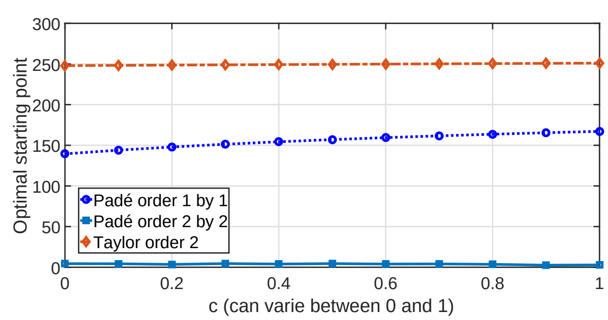

Fig. 1 Minimum integral of the absolute error of the approximations for each value of c

values of the optimal expansion point for each value of c can be seen. It can be seen that for all the approximations, the value does not change drastically with c and thus a single approximation can be used for all the values of c. Fig. 2 shows the integral of the absolute error of each

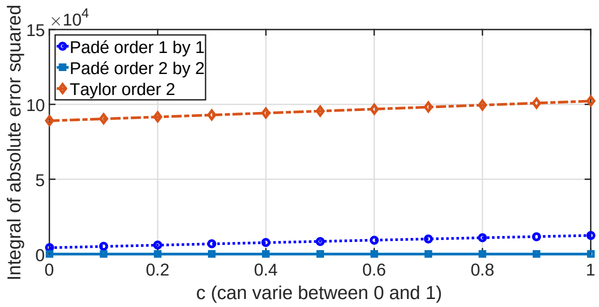

Fig. 2 Minimum integral of the absolute error of the approximations for each value of c

approximation for the values of c and the optimal values for the expansion point shown in Fig. 1. The order 2 by 2 Padé is an excellent approximation for any value of c, and so, even though it requires the solution of a quadratic equation, it will be the one used. An expansion point of 3.81 will be used (average of optimal points). As an example, the exact functions and the approximations for $c = 0$, $c = 0.5$ and $c = 1$ are shown in Fig. 3 For the Padé 2 by 2 the

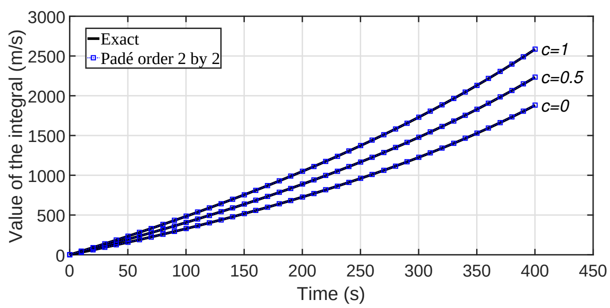

Fig. 3 Value of the exact integral compared with the Padé 2 by 2 approximation for expansion point 3.81

remaining burntime can be calculated using the quadratic formula. Since there are two solutions, given by Eq. 30, some attention must be paid to the choice of the correct solution.

$$a = p_2 + q_2 \frac{(u_f - u_0)\tau}{v_e}, \tag{29a}$$

$$b = p_1 + q_1 \frac{(u_f - u_0)\tau}{v_e}, \tag{29b}$$

$$c = p_0 + q_0 \frac{(u_f - u_0)\tau}{v_e}, \tag{29c}$$

$$t_f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{30}$$

For this case the Padé approximation of order 2 by 2 was chosen, but if in some other case the Padé approximation of order 1 by 1 is prefered, the burntime is given by equation Eq. 31

$$t_f = \frac{v_e p_0 - q_0 \tau (u_f - u_0)}{\tau q1 (u_f - u_0) - v_e p_1}. \tag{31}$$

C. Improved approximations for the sine

In [11] a Taylor series expansion for Eq. 16a at the origin is used (assuming that $\bar{\lambda}_w$ is negletable, i.e the perturbations out of plane are small). In this subsection four different approximation options for the sine of the pitch will be provided. The first one and second ones are simply linear and quadratic Taylor series expansions around an arbitrary point. The third and fourth ones are using first and second order Lagrange polynomials to approximate the function. The linear options only require the solution of a linear system, while the second and third options will require solving a system of two quadratic equations, that can be transformed in a quartic equation, still completely analytically solvable. For all the approximations described, one or more parameters must be chosen *a priori*. For the Taylor series expansions, the initial point needs to be chosen, and for the Lagrange polynomial approximations, the points must be selected. In order to minimize large deviations from the exact function the approximation parameters can be chosen by minimizing the integral of the square of absolute error between a selected interval, i.e. between two selected angles

$$\int_a^b \left[\frac{x}{\sqrt{x^2 + 1}} - g(x) \right]^2 dx, \tag{32}$$

where g is the sine approximation chosen, and a and b can be calculated from the angle interval chosen using the exact function. The minimization needs to be done for each of the approximations and can be done easily using a numerical minimization tool, such as *matlab*'s intrinsic function *fmincon*.

The linear order approximation,

$$\sin(\theta) = \frac{\bar{\lambda}_v}{\sqrt{1 + \bar{\lambda}_v^2 + \bar{\lambda}_w^2}} \approx \frac{\bar{\lambda}_v}{\sqrt{1 + \bar{\lambda}_v^2}} \approx n_0 + n_1 \bar{\lambda}_v, \tag{33}$$

will be solved first, since it is a lot simpler. Replacing Eq. 33 in Eqs. 17a and 17d

$$\frac{Y - c_0 n_0}{n_1} = c_0 A_2 - c_1 \lambda_v, \tag{34a}$$

$$\frac{V - b_0 n_0}{n_1} = b_0 A_2 - b_1 \lambda_v. \tag{34b}$$

The solution to system of Eqs. 34 is

$$\begin{bmatrix} \bar{A}_2 \\ \bar{\lambda}_y \end{bmatrix} = \frac{1}{c_1 b_0 - b_1 c_0} \begin{bmatrix} -b_1 & c_1 \\ -b_0 & c_0 \end{bmatrix} \begin{bmatrix} \frac{Y - c_0 n_0}{n_1} \\ \frac{V - b_0 n_0}{n_1} \end{bmatrix}. \tag{35}$$

The algorithm for the quadratic approximation was created with the same type of reasoning, by saying

$$\sin(\theta) = \frac{\bar{\lambda}_v}{\sqrt{1+\bar{\lambda}_v^2+\bar{\lambda}_w^2}} \approx \frac{\bar{\lambda}_v}{\sqrt{1+\bar{\lambda}_v^2}} \approx n_0 + n_1\bar{\lambda}_v + n_2\bar{\lambda}_v^2. \tag{36}$$

By replacing Eq. 36 in Eqs. 17a and 17d we get quadratic a system of 2 equations of the following form:

$$a_{00} + a_{10}\bar{A}_2 + a_{20}\bar{A}_2^2 + a_{01}\bar{\lambda}_y + a_{02}\bar{\lambda}_y^2 + a_{11}\bar{\lambda}_y\bar{A}_2 = 0, \tag{37a}$$

$$b_{00} + b_{10}\bar{A}_2 + b_{20}\bar{A}_2^2 + b_{01}\bar{\lambda}_y + b_{02}\bar{\lambda}_y^2 + b_{11}\bar{\lambda}_y\bar{A}_2 = 0. \tag{37b}$$

The system of Eqs. 37 can then be turned in a quartic equation, which can still be solved explicitly. Two proposed methods for solving the quartic equation are the methods found in [18] and in [19], but any other general method for solving a quartic equation can be used. A comparison between the sine approximations and the exact function and Hull's approximation can be found in Figs. 4, 5,6 and 7.

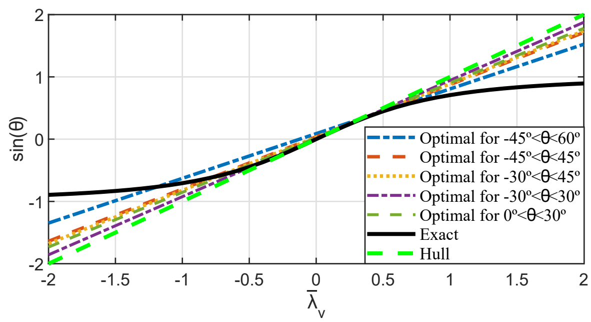

Fig. 4 Optimal linear series expansion approximations for various angle intervals

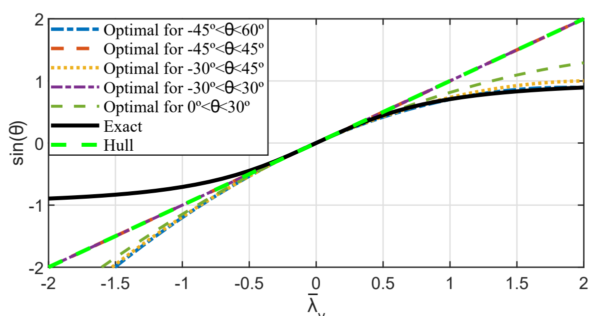

Fig. 5 Optimal quadratic series expansion approximations for various angle intervals

One thing that can be seen right away in Figs. 5 and 7 is that due to the fact that quadratic funtions are non injective (not strictly growing, unlike the case with

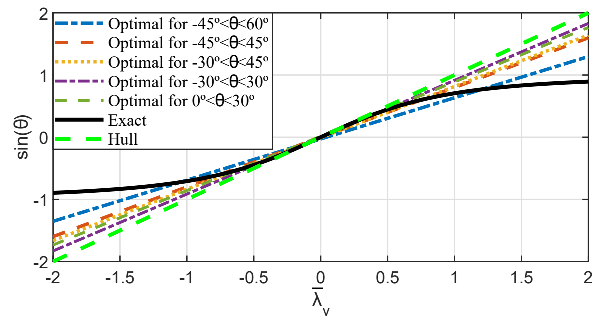

Fig. 6 Optimal linear Lagrange polynomial approximations for various angle intervals

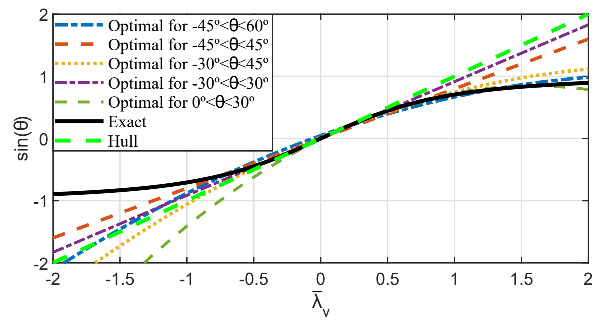

Fig. 7 Optimal quadratic Lagrange polynomial approximations for various angle intervals

linear functions with positive slope), some of the quadratic approximations (Taylor series and Lagrange polynomials) do not completely cover the full codomain of the sine function, i.e. some approximations do not reach a value of 1 and thus have no real solution for certain higher values of $\sin(\theta)$ becoming somewhat risky to use. A way to solve this to only choose approximations where this does not happen. Another problem with a quadratic approximation for an odd bounded function is the fact that for some values the approximation will be worse than any linear approximation. From Table 2 it can be seen all the approximations have a smaller integral of absolute error for every angle interval tested. This will be a general property for any angle interval, since Hull's approximation is a specific case of any of the other approximations. This means that it is likely to obtain trajectories closer to optimal using the new approximations than using Hull's approximation. In section IV all the new sine approximation options will be tested in conjunction with the new cosine approximation and compared with both Hull's and the Optimal trajectory obtained with the method described in I.B

Table 2 Integral of the absolute error of each approximation for the selected angle interval

	Integral of the square of the absolute error in the interval				
	Hull	Linear	Quadratic	Quadratic Lagrange	Linear Lagrange
$-45° < \theta < 60°$	0.2849	0.0379	0.0430	0.0283	0.0087
$-45° < \theta < 45°$	0.0302	0.0071	0.0302	0.0033	0.0033
$-30° < \theta < 45°$	0.0156	0.0061	0.0016	0.0023	6.5×10^{-4}
$-30° < \theta < 30°$	0.0011	3.0×10^{-4}	0.0011	1.5×10^{-4}	1.5×10^{-4}
$0° < \theta < 30°$	5.4×10^{-4}	9.2×10^{-5}	1.4×10^{-5}	4.0×10^{-5}	5.0×10^{-7}

IV. Testing the new guidance solutions: The HERACLES Mission

A. Mission profile

In order to understand the advantages and disadvantages of each algorithm, the various algorithms will be compared using simulations with no perturbations, i.e. in a spherical non rotating moon environment, with no third body influences. Also, in this scenario the vertical flight and linear pitchover phases will not be simulated and the powered flight phase is assumed to start at $t = 0$. This simulations will be done in subsection IV.B. The cosine approximation and the four proposed sine approximations will be tested, using both approximations or only one at a time, against the optimal and Hull's solution for five a 15km), at periapsis. The apoapsis will be at 100km. For all the guidance algorithms a sample and hold scheme updated on a 1Hz frequency is used for the powered guidance phase.

B. Guidance trajectories using the explicit solutions

From Figs. 8,9,10 and 11 it can be seen that the new linear Taylor series and the Lagrange polynomial approximations for the sine of the pitch outperform produce almost optimal pitch profiles, leading to a decrease in ascent time of about 0.1 s. Surprisingly the cosine approximation alone does not have a positive impact on some of the trajectories tested, namely the lower altitude insertions. This was unexpected at first, but it does in fact make sense, because Hull's approximation for the sine of the angle is an overestimate, i.e, for all values of λ_v the approximation has a bigger absolute value than the actual function, and thus having an estimation for the remaining burntime smaller than the real one counters this effect. This is actually good news, because Hull's remaining burntime approximation is simpler and requires less calculations. Since both the quadratic Lagrange polynomial and the linear approximations have similar performance using the linear Taylor series is prefered, since it does not require solving a fourth degree polynomial.

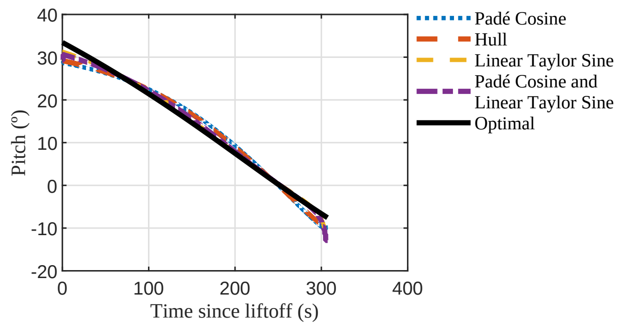

Fig. 8 Comparison of pitch angle for 15 km altitude insertion using a linear Taylor series expansion to approximate the sine of the pitch

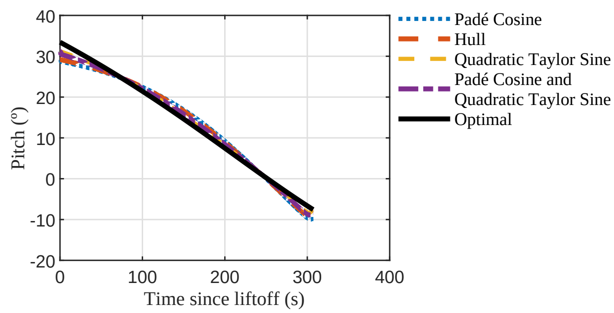

Fig. 9 Comparison of pitch angle for 15 km altitude insertion using a quadratic Taylor series expansion to approximate the sine of the pitch

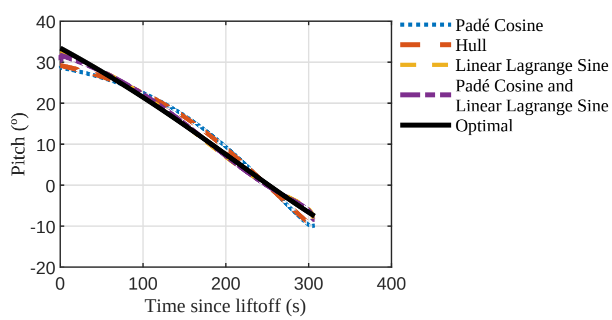

Fig. 10 Comparison of pitch angle for 15 km altitude insertion using a linear Lagrange polynomial to approximate the sine of the pitch

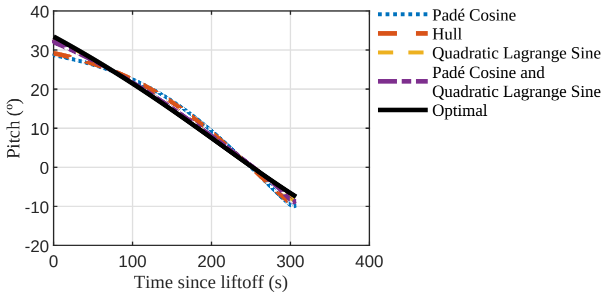

Fig. 11 Comparison of pitch angle for 15 km altitude insertion using a quadratic Lagrange polynomial to approximate the sine of the pitch

C. Multiple simulations with perturbed parameters

In order to make sure that the selected approximation is able to handle out of plane perturbations, a 15km altitude insertion is tested, with different out of plane initial conditions. The pitch angle and the vertical velocity in these cases are indistinguishable from the two dimensional trajectory, hence they will not be plotted. Hence, only the yaw angle and the out of plane velocity will be plotted and compared with Hull's approximation and the optimal value.

Lookin at Figs. 12, 13 and 14 it can be seen that the yaw angle is almost exactly the same for both approximations and the optimal value, thus confirming that changing the approximation of the sine of the pitch angle did not have any effect on the hability of the approximation to handle out of plane perturbations.

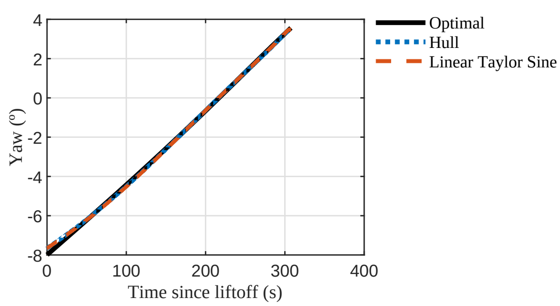

Fig. 12 Comparison of Yaw angle for 15 km altitude insertion with 50 $\mathrm{m\,s^{-1}}$ out of plane initial velocity

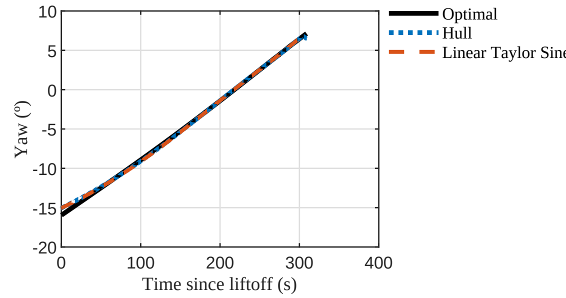

Fig. 13 Comparison of Yaw angle for 15 km altitude insertion with 100 $\mathrm{m\,s^{-1}}$ out of plane initial velocity

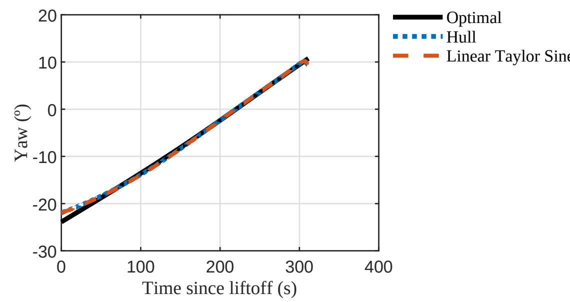

Fig. 14 Comparison of Yaw angle for 15 km altitude insertion with 150 $\mathrm{m\,s^{-1}}$ out of plane initial velocity

V. Conclusion

The intention of this work was to create new explicit guidance laws that could outperform Hull's based on new approximations for a single stage constant thrust atmosphereless ascent. A variety of options were presented and tested, and in the end the objective was achieved. Two of the approximations require the same amount of computational power as Hull's explicit approximation. These are the linear Taylor series approximation around a generic point and the linear Lagrange polynomal approximation for the sine of the pitch angle, while keeping the same approximation as Hull for the cosine of the pitch angle ($\cos(\theta) = 1$). Surprisingly, the linear sine approximations approximate the optimal solution as well as or better than the quadratic approximations proposed. Another interesting result is the fact that the cosine approximation does not improve the results for typical altitudes of insertion, it actually deteriorates them. The new guidance laws can obtain close to optimal trajectories for a wider range of altitude insertions, making them usable in more scenarios. Another important conclusion that might not be too obvious but might be usefull in the future for other reasearch is the fact that a low order Padé approximation can be used to approximate the thrust acceleration integral almost exactly. There is no research using this and a lot more can be done.

There is still some future work to do, as the algorithms were only tested in optimal conditions. The guidance laws should be tested more thoroughly in the future and simulated using real attitude control and perturbed scenarios. Also, different methods can be tested for picking the approximation parameters, like using the unperturbed simulations and minimizing the insertion time in relation to the approximations parameters. This method will take longer than simply minimizing the integral of the square of the absolute error for the sine approximation, but since it can be done before the ascent, this should not be a problem.

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